Chapter 3
Value at Risk

Value at risk (VaR) is a measure of risk which is used to estimate the amount that can potentially be lost on an investment within a certain time range. This chapter presents the concept of risk measure, including quantile risk measures and Value at Risk, together with experiments based on financial data sets.

3.1 Financial Data with R

Package Quantmod for financial data

The R package quantmod is able to fetch financial data from various sources such as Yahoo! Finance or the Federal Reserve Bank of St. Louis (FRED). It can be installed and run via the following command.

```
1 install.packages("quantmod")
2 library(quantmod)
3 getSymbols("DEXJPUS",src="FRED") # Japan/U.S. Foreign Exchange Rate
4 getSymbols("CPIAUCNS",src="FRED") # Consumer Price Index
5 getSymbols("GOOG",src="yahoo") # Google Stock Price
```
The adjusted close price \( \text{Ad}(\cdot) \) is the closing price after adjustments for applicable splits and dividend distributions.

![Cumulative stock returns](image1)

**Fig. 3.1:** Cumulative stock returns.

![Stock returns](image2)

**Fig. 3.2:** Stock returns.

### 3.2 Risk Measures

The potential loss associated to any of the above risks will will be modeled via a random variable \( X \).

**Definition 3.1.** A risk measure is a mapping that assigns a value \( V_X \) to a given random variable \( X \).
Value at Risk

For insurance companies, which need to hold a capital in order to meet future liabilities, the capital $C_X$ required to face the risk induced by a potential loss $X$ can be defined as

$$C_X := V_X - L_X,$$

where

a) $V_X$ stands for a upper “reasonable” estimate of the potential loss associated to $X$.

b) $L_X$ represents the liabilities of the company.

When $L_X < 0$ the amount $-L_X > 0$ corresponds to a debt owed by the company, while $L_X > 0$ corresponds to positive liabilities such as deferred revenue or to a debt owed to the company.

When estimating the liabilities of the company by $\mathbb{E}[X]$, the required capital is given by

$$C_X = V_X - \mathbb{E}[X].$$

The above R code is used in the next figure to estimate liabilities using the conditional mean

$$\mathbb{E}[X \mid X < 0] = \frac{\mathbb{E}[X 1_{X<0}]}{\mathbb{P}(X < 0)}.$$

![Figure 3.3: Estimating liabilities by the conditional mean $\mathbb{E}[X \mid X < 0]$ over 346 returns.](image)

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https://www.ntu.edu.sg/home/nprivault/index.html
Example: Guaranteed Maturity Benefits

Variable annuity benefits offered by insurance companies are usually protected via different mechanisms such as Guaranteed Minimum Maturity Benefits (GMMBs) or Guaranteed Minimum Death Benefits (GMDBs). The computation of the corresponding risk measures is an important issue for the practitioner in risk management.

Given a fund value process \( (F_t)_{t \in \mathbb{R}^+} \), an insurer is continuously charging annualized mortality and expense fees at the rate \( m \) from the account of variable annuities, resulting into a margin offset income \( M_t \) given by

\[
M_t := m F_t \quad t \in \mathbb{R}^+.
\]

Denoting by \( \tau_x \) the future lifetime of a policyholder at the age \( x \), the future payment made by the insurer at maturity \( T \) is

\[
(G - F_T)^+ \mathbb{1}_{\{\tau_x > T\}}
\]

where \( G \) is the guarantee level expressed as a percentage of the initial fund value \( F_0 \), \( \delta \) is a roll-up rate according to which the guarantee increases up to the payment time. In this case, the random variable \( X \) is taken equal to

\[
X := e^{-rT} (G - F_T)^+ \mathbb{1}_{\{\tau_x > T\}} - \int_0^{\min(T, \tau_x)} e^{-rs} M_s ds.
\]

Coherent risk measures

**Definition 3.2.** A risk measure \( V \) is said to be coherent if it satisfies the following four properties, for any two random variables \( X, Y \):

i) Monotonicity: 
\[
X \leq Y \implies V_X \leq V_Y,
\]

ii) (Positive) homogeneity:
\[
V_{\lambda X} = \lambda V_X, \quad \text{for constant } \lambda > 0,
\]

iii) Translation invariance:
\[
V_{X + \mu} = \mu + V_X, \quad \text{for constant } \mu > 0,
\]

iv) Subadditivity:
\[
V_{X + Y} \leq V_X + V_Y.
\]

Subadditivity means that the combined risk of several portfolios is lower than the sum of risks of those portfolios, as happens usually through portfolio diversification. For example, one person traveling might insure the un-
likely loss of her phone for $V_X = \$100$. However, two people traveling together might want to insure the phone loss event at a level $V_X + V_Y < \$100 + \$100$ as the simultaneous loss of both phones during a same trip seems even more unlikely.

The *expectation* of random variables

$$V_X := \mathbb{E}[X]$$

is an example of a coherent risk measure (also called *pure premium* risk measure) satisfying the above conditions (i)-(iv), and is additive.

**Distortion risk measures**

More generally, any risk measure of the form

$$M_X = \mathbb{E}[X f_X(X)],$$

cf. e.g. (4.5) below, where $f_X$ is a non-decreasing *distortion function* satisfying

$$f_X(\lambda x) = f_X(x) \quad \text{and} \quad f_X(\mu + x) = f_X(x), \quad x \in \mathbb{R}, \quad \lambda, \mu > 0,$$

will be monotone, homogeneous, and translation invariant.

### 3.3 Quantile Risk Measures

The *Cumulative Distribution Function* (CDF) of a random variable $X$ is the function

$$F_X : \mathbb{R} \rightarrow [0,1]$$

defined by

$$F_X(x) := \mathbb{P}(X \leq x), \quad x \in \mathbb{R}.$$ 

Any cumulative distribution function $F_X$ satisfies the following properties:

i) $x \mapsto F_X(x)$ is non-decreasing,

ii) $x \mapsto F_X(x)$ is right-continuous,

iii) $\lim_{x \to \infty} F_X(x) = 1$,

iv) $\lim_{x \to -\infty} F_X(x) = 0$.

In addition, any cumulative distribution function $F_X$ admits left limits in the following sense.

**Proposition 3.3.** For any non-decreasing sequence $(x_n)_{n \geq 1}$ converging to $x \in \mathbb{R}$, we have
\[
\lim_{n \to \infty} F_X(x_n) = \lim_{n \to \infty} P(X \leq x_n) = P(X < x). 
\] (3.2)

\textit{Proof.} By (10.10), we have

\[
P(X < x) = P(X \in (-\infty, x))
= P\left( X \in \bigcup_{n \geq 1} (-\infty, x_n] \right)
= \lim_{n \to \infty} P(X \in (-\infty, x_n])
= \lim_{n \to \infty} F_X(x_n).
\]

\hfill \square

The next figure shows the continuous Cumulative Distribution Function of a Gaussian \( \mathcal{N}(0,1) \) random variable.

![Gaussian CDF](image)

**Fig. 3.4:** Gaussian Cumulative distribution function.

On the other hand, a random variable \( X \) may have a discontinuous cumulative distribution function, as illustrated in Figure 3.5 with

\[
P(X = 0) = P(X \leq 0) - P(X < 0) = 0.25 > 0.
\]
More generally, the discontinuity of a CDF at the point $x \in \mathbb{R}$, if it exists, is given by

$$P(X = x) = P(X \leq x) - P(X < x) = F_X(x) - \lim_{y \to x} F_X(y).$$

**Definition 3.4.** Given $X$ a random variable with cumulative distribution function $F_X : \mathbb{R} \to [0, 1]$ and a level $p \in (0, 1)$, the $p$-quantile $q^p_X$ of $X$ is defined by

$$q^p_X := \inf \{ x \in \mathbb{R} : P(X \leq x) \geq p \}.$$  

**Performance analytics in R - quantiles of known distributions**

The quantiles of various distributions can be obtained in R. For example, the command

```r
qnorm(.95, mean=.5, sd=1)
```

shows that the 95%-quantile of a $\mathcal{N}(0, 1)$ Gaussian random variable is 1.644854.

![Gaussian quantile and CDF.](image1)

![Gaussian quantile and CDF.](image2)

**Fig. 3.6:** Gaussian quantile $q^p_Z = 1.644854$ at $p = 0.95$.

* Picture taken from [https://www.probabilitycourse.com/](https://www.probabilitycourse.com/).
On the other hand, the instruction

\[
q_{\exp}(0.95, 1)
\]

displays the 95%-quantile of an exponentially distributed random variable with parameter 1, which is 2.995732.

Similarly, the instruction

\[
q_{t}(0.90, df=5)
\]

displays the 90%-quantile of a Student t-distributed random variable with 5 degrees of freedom, which is 1.475884.
Value at Risk

Performance analytics in R - empirical CDF

The *empirical Cumulative Distribution Function* can be estimated as

\[
F_N(x) := \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\{x_i \leq x\}}, \quad x \in \mathbb{R}.
\]

```r
getSymbols("^STI", from="1990-01-03", to="2015-02-01", src="yahoo")
getSymbols("1800.HK", from="1990-01-03", to="2015-02-01", src="yahoo")
stock.rtn <- as.vector(diff(log(Ad(1800.HK))))
stock.ecdf <- ecdf(as.vector(stock.rtn))
plot(stock.ecdf, xlab = "Sample Quantiles", ylab = "", lwd = 2, main = "Empirical CDF", col="blue")
```

Fig. 3.9: Empirical cumulative distribution function.

Note that the empirical distribution function has a visible discontinuity, or gap, at 0, whose height 0.03967611 is given by

```r
sum(!is.na(stock.rtn[stock.rtn==0]))/sum(!is.na(stock.rtn))
```

3.4 Value at Risk (VaR)

Value at Risk has two objectives:

i) to provide a measure for risk, and

ii) to determine an adequate level of capital reserves that matches the current level of risk.
In other words, managing risk means here determining a level $V_X$ of provision or capital requirement that will not be “too much” exceeded by $X$.

In this respect, the probability $\mathbb{P}(X > V)$ that $X$ exceeds the level $V$ is of a capital importance. Setting $V$ such that for example

$$\mathbb{P}(X \leq V) \geq 0.95, \quad \text{i.e.} \quad \mathbb{P}(X > V) \leq 0.05,$$

means that insolvency will occur with probability less than 5%.

The 95%-quantile risk measure is the smallest value of $V$ such that

$$\mathbb{P}(X \leq V) \geq 0.95, \quad \text{i.e.} \quad \mathbb{P}(X > V) \leq 0.05.$$

More precisely, we have the following definition.

**Definition 3.5.** The Value at Risk $V^p_X$ of a random variable $X$ at the level $p \in (0, 1)$ is the $p$-quantile of $X$ defined by

$$V^p_X := \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq p\}. \quad (3.3)$$

In other words, for some decreasing sequence $(x_n)_{n \geq 1}$ such that

$$\mathbb{P}(X \leq x_n) \geq p \quad \text{for all} \quad n \geq 1,$$

we have

$$V^p_X := \lim_{n \to \infty} x_n. \quad (3.4)$$

On the other hand, the Value at Risk $V^p_X$ does not contain any information on how large losses can be beyond $V^p_X$.

**Proposition 3.6.** The function $p \mapsto V^p_X$ is a non-decreasing, left-continuous function of $p \in [0, 1]$, and it admits limits on the right.

**Proof.** By (3.3), the function $p \mapsto V^p_X$ is the generalized inverse of the Cumulative Distribution Function $x \mapsto F_X(x) := \mathbb{P}(X \leq x), \quad x \in \mathbb{R}$.

of $X$. By e.g. Proposition 2.3-(2) in Embrechts and Hofert (2013) since $F_X(x)$ is non-decreasing in $x \in \mathbb{R}$, its generalized inverse $p \mapsto V^p_X$ is non-decreasing, left-continuous, and it admits limits on the right. \hfill \Box

In addition, if $F_X$ is continuous and strictly increasing it admits an inverse $F_X^{-1}$, and in this case we have

$$V_X(p) = F_X^{-1}(p), \quad p \in (0, 1).$$

The next proposition follows from the Definition 3.5 of $V^p_X$. It shows in particular that, with probability at least $p$, the value of $X$ is always lower than
Value at Risk

\( V_X^p \), i.e. we have

\[
F_X(V_X^p) = \mathbb{P}(X \leq V_X^p) \geq p \quad \text{and} \quad \mathbb{P}(X > V_X^p) \leq 1 - p. \quad (3.5)
\]

**Proposition 3.7.** a) For all \( x \in \mathbb{R} \) we have

\[
V_X^p \leq x \iff \mathbb{P}(X \leq x) \geq p. \quad (3.6)
\]

b) If \( \mathbb{P}(X = V_X^p) = 0 \), then we have \( p = \mathbb{P}(X \leq V_X^p) \).

**Proof.** (a) If \( \mathbb{P}(X \leq x) \geq p \) then we have

\[
V_X^p = \inf\{y \in \mathbb{R} : \mathbb{P}(X \leq y) \geq p\} \leq x.
\]

On the other hand, if \( V_X^p \leq x \) then there exists a strictly decreasing sequence \( (x_n)_{n \geq 1} \) such that

\[
\lim_{n \to \infty} x_n = V_X^p \quad \text{and} \quad \mathbb{P}(X \leq x_n) \geq p, \quad n \geq 1.
\]

Therefore, by right continuity of the cumulative distribution function \( F_X(x) = \mathbb{P}(X \leq x) \), we have

\[
\mathbb{P}(X \leq V_X^p) = \lim_{n \to \infty} \mathbb{P}(X \leq x_n) \geq p.
\]

(b) Assume that \( V_X^p = x \). By (3.5) we have \( \mathbb{P}(X \leq x) \geq p \). If \( \mathbb{P}(X \leq V_X^p) > p \), we choose a strictly increasing sequence \( (x_n)_{n \geq 1} \) such that

\[
\lim_{n \to \infty} x_n = V_X^p \quad \text{and} \quad \mathbb{P}(X \leq x_n) < p, \quad n \geq 1.
\]

Since the cumulative distribution function \( F_X(x) = \mathbb{P}(X \leq x) \) admits left limits by (3.2), we have

\[
\mathbb{P}(X < V_X^p) = \lim_{n \to \infty} \mathbb{P}(X \leq x_n) \leq p < \mathbb{P}(X \leq V_X^p),
\]

which contradicts \( \mathbb{P}(X = V_X^p) = \mathbb{P}(X \leq V_X^p) - \mathbb{P}(X < V_X^p) = 0 \), and therefore we have \( \mathbb{P}(X \leq V_X^p) = p \). \( \square \)

In particular, if \( \mathbb{P}(X = V_X^p) = 0 \) then 1 - \( p = \mathbb{P}(X > V_X^p) \), and if \( p < \mathbb{P}(X \leq V_X^p) \) or \( 1 - p > \mathbb{P}(X > V_X^p) \)

then we have \( \mathbb{P}(X = V_X^p) > 0 \).

**Proposition 3.8.** Assume that the cumulative distribution function \( F_X \) is continuous and strictly increasing. Then we have \( V_X^p = -V_X^{1-p} \).

**Proof.** We have

\[
F_{-X}(x) = \mathbb{P}(-X \leq x)
\]
N. Privault

\[ \begin{align*}
\mathbb{P}(X \geq -x) &= \mathbb{P}(X < -x) \\
&= 1 - \mathbb{P}(X \leq -x) \\
&= 1 - F_X(-x),
\end{align*} \]

hence

\[ p = F_X(F_X^{-1}(p)) = 1 - F_X(-F_X^{-1}(p)), \]

which yields

\[ V^p_X = F_X^{-1}(p) = -F_X^{-1}(1-p) = -V_X^{1-p}, \quad p \in (0, 1). \]

\[ \square \]

The next Figure 3.10 shows that the continuity of \( F_X \) is necessary in order to ensure the symmetry property of Proposition 3.8.

![Fig. 3.10: Symmetric and nonsymmetric VaR.](image)

Next, we check the properties of Value at Risk.

a) **Monotonicity.** Value at Risk is a monotone risk measure.

**Proof.** If \( X \leq Y \) then

\[ \mathbb{P}(Y \leq x) = \mathbb{P}(X \leq Y \leq x) \leq \mathbb{P}(X \leq x), \quad x \in \mathbb{R}, \]

hence

\[ \mathbb{P}(Y \leq x) \geq p \implies \mathbb{P}(X \leq x) \geq p, \quad x \in \mathbb{R}, \]

which shows that

\[ V_X^p \leq V_Y^p \]

by (3.3).

\[ \square \]

b) **Positive homogeneity and translation invariance.** Value at Risk satisfies the positive homogeneity and translation invariance properties.

**Proof.** For all \( a > 0 \) and \( b \in \mathbb{R} \) we have
Value at Risk

\[ V^p_{a+bX} = \inf \{ x \in \mathbb{R} : P(a+bX \leq x) \geq p \} \]
\[ = \inf \{ x \in \mathbb{R} : P(X \leq (x-a)/b) \geq p \} \]
\[ = \inf \{ a+by \in \mathbb{R} : P(X \leq y) \geq p \} \]
\[ = a + b \inf \{ y \in \mathbb{R} : P(X \leq y) \geq p \} \]
\[ = a + bV^p_X. \]

\( \square \)

c) **Subadditivity and coherence.** Although Value at Risk satisfies the monotonicity, positive homogeneity and translation invariance properties, it is *not* subadditive in general. Namely, the Value at Risk \( V^p_{X+Y} \) of \( X+Y \) may be larger than the sum \( V^p_X + V^p_Y \). Therefore, Value at Risk is *not* a coherent risk measure.

**Proof.** We show that Value at Risk is *not subadditive* by considering two independent Bernoulli random variables \( X, Y \in \{0, 1\} \) with the distribution
\[
\begin{align*}
P(X = 1) &= P(Y = 1) = 2\%, \\
P(X = 0) &= P(Y = 0) = 98%.
\end{align*}
\]
hence \( V^0.975_X = V^0.975_Y = 0. \)

On the other hand, we have
\[
\begin{align*}
P(X+Y = 2) &= (0.02)^2 = 0.04\%, \\
P(X+Y = 1) &= 2 \times 0.02 \times 0.98 = 3.92\%, \\
P(X+Y = 0) &= (0.98)^2 = 96.04%,
\end{align*}
\]
hence
\[ V^0.975_{X+Y} = 1 > V^0.975_X + V^0.975_Y = 0. \]
Proposition 3.9. Gaussian Value at Risk. Given $X \approx \mathcal{N}(\mu_X, \sigma_X^2)$, we have

$$V_p^X = \mu_X + \sigma_X q_p^Z$$

(3.7)

where the normal quantile $q_p^Z = V_p^Z$ at the level $p$ satisfies

$$P(Z \leq q_p^Z) = p \quad \text{for} \quad Z \approx \mathcal{N}(0,1).$$

Proof. We represent the random variable $X \approx \mathcal{N}(\mu_X, \sigma_X^2)$ as

$$X = \mu_X + \sigma_X Z,$$

where $Z \approx \mathcal{N}(0,1)$ is a standard normal random variable, and use the relation

$$p = P(X \leq V_p^X)$$

$$= P(\mu_X + \sigma_X Z \leq V_p^X)$$

$$= P(Z \leq (V_p^X - \mu_X)/\sigma_X)$$

$$= P(Z \leq q_p^Z).$$

Remark 3.10. Although Value at Risk is not sub-additive in general, it is sub-additive (and therefore coherent) on (not necessarily independent) Gaussian random variables.

Proof. By (3.7), for any two random variables $X$ and $Y$ we have

$$\sigma_{X+Y}^2 = \text{Var}[X + Y]$$

$$= \mathbb{E}[(X + Y)^2] - \left(\mathbb{E}[X + Y]\right)^2$$

$$= \mathbb{E}[X^2] + \mathbb{E}[Y^2] + 2\mathbb{E}[XY] - \mathbb{E}[X]^2 - \mathbb{E}[Y]^2 - \mathbb{E}[X] \mathbb{E}[Y]$$

$$= \text{Var}[X] + \text{Var}[Y] + 2(\mathbb{E}[XY] - \mathbb{E}[X] \mathbb{E}[Y])$$

$$= \text{Var}[X] + \text{Var}[Y] + 2 \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$$

(3.8)
Value at Risk

\[
\text{Var}[X] + \text{Var}[Y] + 2\sqrt{\mathbb{E}[(X - \mathbb{E}[X])^2]} \sqrt{\mathbb{E}[(Y - \mathbb{E}[Y])^2]} \quad (3.9)
\]

\[
= \left(\sqrt{\text{Var}[X]} + \sqrt{\text{Var}[Y]}\right)^2,
\]

where, from (3.8) to (3.9) we applied the Cauchy-Schwarz inequality. Hence we have

\[
\sigma_{X+Y} \leq \sigma_X + \sigma_Y.
\]

Assuming that \( X \) and \( Y \) are Gaussian, by (3.7) we find

\[
V_{X+Y}^p = \mu_{X+Y} + \sigma_{X+Y}q_z^p
\]

\[
= \mu_X + \mu_Y + \sigma_{X+Y}q_z^p
\]

\[
\leq \mu_X + \mu_Y + (\sigma_X + \sigma_Y)q_z^p
\]

\[
= V_X^p + V_Y^p.
\]

Performance analytics in R - Value at Risk

We are using the PerformanceAnalytics R package, which can be installed via the commands

```r
install.packages("PerformanceAnalytics")
getSymbols("0700.HK", from="2010-01-03", to="2018-02-01", src="yahoo")
stock=Ad(\text{get.ts(0700.HK))})
chartSeries(stock,up.col="blue",theme="white")
stock.rtn=diff(log(Ad(\text{get.ts(0700.HK))}))
chart.CumReturns(stock.rtn,main="Cumulative Returns")
library(PerformanceAnalytics)
var=VaR(stock.rtn, p=.95, method="historical")
sum(!is.na(stock.rtn[stock.rtn<var[1]])/sum(!is.na(stock.rtn))

The historical 95\%-Value at Risk over \( N \) samples \( (x_i)_{i=1,2,...,N} \) can be estimated by inverting the empirical cumulative distribution function \( F_N(x) \), \( x \in \mathbb{R} \). It is found equal to \( V_X^{95\%} = -0.03157435 \).

```r
VaR(stock.rtn, p=.95, method="gaussian")
```

The Gaussian 95\%-Value at Risk is estimated from (3.7) by

\[
V_X^{95\%} = \mu + \sigma q_z^p,
\]

where \( \mu = \mathbb{E}[X] \) and \( \sigma^2 = \text{Var}[X] \), and is found equal to \( V_X^{95\%} = -0.03115105 \). It can be recovered up to approximation as
which yields $-0.0311592$. Note that here we are concerned about large negative returns, which explains the negative sign in $m-s^*q$.

The next lemma is useful for random simulation purposes, and it will also be used in the proof of Propositions 4.4 and 4.9 below.

**Lemma 3.11.** Any random variable $X$ can be represented as $X = V^U$ where $U$ is uniformly distributed on $[0, 1]$.

**Proof.** It suffices to note that by (3.6) we have

$$P(V^U_X \leq x) = P(U \leq P(X \leq x)) = P(X \leq x) = F_X(x), \quad x \in \mathbb{R}.$$ 

□

**Exercises**

Exercise 3.1 Consider a random variable $X$ having the Pareto distribution with probability density function

$$f_X(x) = \frac{\gamma \theta^{\gamma}}{(\theta + x)^{\gamma+1}}, \quad x \in \mathbb{R}_+.$$ 

a) Compute the cumulative distribution function

$$F_X(x) := \int_0^x f_X(y)dy, \quad x \in \mathbb{R}_+.$$ 

b) Compute the value at risk $V^p_X$ at the level $p$ for any $\theta$ and $\gamma$, and then for $p = 99\%$, $\theta = 40$ and $\gamma = 2$.

Exercise 3.2 Consider a random variable $X$ with the cumulative distribution function

$$F_X(x) := \int_0^x f_X(y)dy, \quad x \in \mathbb{R}_+.$$
Value at Risk

Fig. 3.13: Cumulative distribution function of $X$.

a) Give the value of $P(X = 100)$.

b) Give the value of $V_q^X$ for all $q$ in the interval $[0.97, 0.99]$.

c) Compute the value of $V_q^X$ for all $q$ in the interval $[0.99, 1]$.

Hint: We have

$$F_X(x) = P(X \leq x) = 0.99 + 0.01 \times (x - 100)/50, \quad x \in [100, 150].$$

Exercise 3.3 Discrete distribution. Consider $X \in \{10, 100, 110\}$ with the distribution

$$P(X = 10) = 90\%, \quad P(X = 100) = 9.5\%, \quad P(X = 110) = 0.5\%.$$  

Compute $V_{X}^{99\%}$.

Exercise 3.4 Exponential distribution. Assume that $X$ has an exponential distribution with parameter $\lambda > 0$ and mean $1/\lambda$, i.e.

$$P(X \leq x) = 1 - e^{-\lambda x}, \quad x \geq 0.$$

a) Compute

$$V_p^X := \inf \{x \in \mathbb{R} : P(X \leq x) \geq p\}$$

and $V_{X}^{95\%}$.

b) Assuming that the liabilities of a company are estimated by $\mathbb{E}[X]$, compute the amount of required capital $C_X$ from (3.1).

Exercise 3.5 Estimating risk probabilities from moments.

a) Using the Chebyshev inequality, show that for every $r > 0$

$$V_p^X \leq \left( \frac{\mathbb{E}[X^r]}{1 - p} \right)^{1/r} = \frac{\|X\|_{L^r}(\Omega)}{(1 - p)^{1/r}}.$$
b) Give an upper bound for $V_{X}^{95\%}$ when $p = 95\%$ and $r = 1$. 