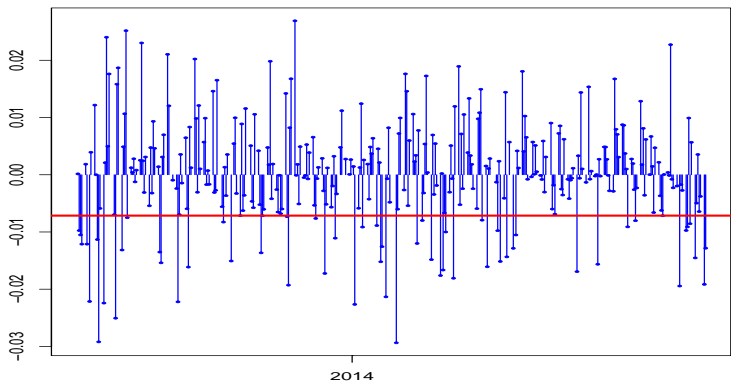


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Notes on
Financial Risk and Analytics



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<https://personal.ntu.edu.sg/nprivault/index.html>

Preface

The topics covered in these notes include an introduction to stochastic modeling with discrete-valued stochastic processes, a basic coverage of Value at Risk and expected shortfall, as well as structures of random dependence. Various types of risk [Gourieroux and Jasiak \(2001\)](#) can be classified into market risk, liquidity risk, credit risk, counterparty risk, model risk, estimation risk. For insurance businesses, a more detailed classification can be set as follows.

a) Financial risk

Investment risk

Credit risk,

Market risk (*e.g.* depreciation),

Counterparty risk.

Liability risk

Catastrophe risk,

Non-catastrophe risk (*e.g.* claim volatility).

b) Operational risk

Business risk (*e.g.* lower production),

Event risk (*e.g.* system failure).

Financial, investment, market and non-catastrophe risks are covered in Chapters 1, 3 and 4 on time series, Value at Risk and expected shortfall.

Credit risk is covered in Chapters 6, 7 and 8 on the structural and reduced-form approaches to credit risk and valuation, which require a basic knowledge of stochastic calculus in continuous time. Credit default is treated via defaultable bonds, Credit Default Swaps (CDS) and collateralized debt obligations (CDOs).

Chapter 9 is devoted to credit scoring, using discriminant analysis and logistic regression. Liability, catastrophe and operational risks such as business or event risk are covered in Chapter 2 on risk theory. Basic risk Theory and

credit scoring are presented with illustrative examples in R.

This material has been used for teaching in the Masters of Science in Financial Engineering (MFE) and in Analytics (MSA) at the Nanyang Technological University in Singapore. The pdf file contains external links and 91 figures, including 2 animated figures that may require using Acrobat Reader for viewing on the complete pdf file.

This text also includes 43 exercises with solutions. Clicking on an exercise number inside the solution section will send to the original problem text inside the file. Conversely, clicking on the problem number sends the reader to the corresponding solution, however this feature should not be misused.

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