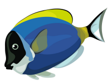
Chapter 2
Insurance Risk

Insurance risk techniques allow insurance companies to estimate the amount of coverage needed for their operations by modeling the payment of claims relating to insured event. In this chapter we present the construction of stochastic processes used for the modeling of insurance risk, such as processes with jumps and independent increments, including the Poisson and compound Poisson processes. We also cover some basic topics in risk theory such as the construction of claim and reserve processes, and the computation of ruin probabilities in the Cramér-Lundberg model.

2.1 The Poisson Process

The most elementary and useful jump process is the *standard Poisson process* $(N_t)_{t\in\mathbb{R}_+}$ which is a *counting process*, *i.e.* $(N_t)_{t\in\mathbb{R}_+}$ has jumps of size $+1$ only, and its paths are constant in between two jumps. In addition, the standard Poisson process starts at $\underline{N_0 = 0}$.



The Poisson process can be used to model discrete arrival times such as claim dates in insurance, or connection logs.

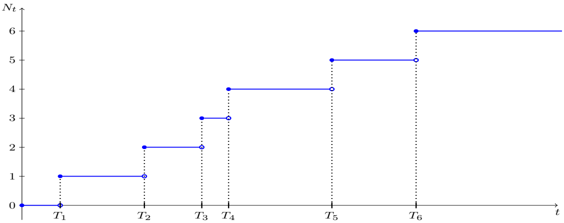


Fig. 2.1: Sample path of a Poisson process $(N_t)_{t\in\mathbb{R}_+}$.

In other words, the value N_t at time t is given by

$$N_t = \sum_{k\geqslant 1} \mathbb{1}_{[T_k,\infty)}(t), \qquad t\in\mathbb{R}_+, \tag{2.1}$$

where

$$\mathbb{1}_{[T_k,\infty)}(t) = \begin{cases} 1 \text{ if } t \geqslant T_k, \\ \\ 0 \text{ if } 0 \leqslant t < T_k, \end{cases}$$

$k\geqslant 1$, and $(T_k)_{k\geqslant 1}$ is the increasing family of jump times of $(N_t)_{t\in\mathbb{R}_+}$ such that

$$\lim_{k\to\infty} T_k = +\infty.$$

In addition, the Poisson process $(N_t)_{t\in\mathbb{R}_+}$ is assumed to satisfy the following conditions:

1. Independence of increments: for all $0\leqslant t_0 < t_1 < \cdots < t_n$ and $n\geqslant 1$ the increments

   $$N_{t_1} - N_{t_0}, \ldots, \ N_{t_n} - N_{t_{n-1}},$$

   are mutually independent random variables.

2. Stationarity of increments: $N_{t+h} - N_{s+h}$ has the same distribution as $N_t - N_s$ for all $h > 0$ and $0 \leqslant s \leqslant t$.

The meaning of the above stationarity condition is that for all fixed $k\in\mathbb{N}$ we have

$$\mathbb{P}(N_{t+h} - N_{s+h} = k) = \mathbb{P}(N_t - N_s = k),$$

for all $h > 0$, *i.e.*, the value of the probability

$$\mathbb{P}(N_{t+h} - N_{s+h} = k)$$

does not depend on $h > 0$, for all fixed $0 \leqslant s \leqslant t$ and $k\in\mathbb{N}$.

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Based on the above assumption, given $T > 0$ a time value, a natural question arises:

what is the probability distribution of the random variable N_T?

We already know that N_t takes values in $\mathbb{N}$ and therefore it has a discrete distribution for all $t \in \mathbb{R}_+$.

It is a remarkable fact that the distribution of the increments of $(N_t)_{t\in\mathbb{R}_+}$, can be completely determined from the above conditions, as shown in the following theorem.

As seen in the next result, cf. Theorem 4.1 in Bosq and Nguyen (1996), the Poisson increment $N_t - N_s$ has the Poisson distribution with parameter $(t-s)\lambda$.

Theorem 2.1. *Assume that the counting process $(N_t)_{t\in\mathbb{R}_+}$ satisfies the above independence and stationarity Conditions 1 and 2 on page 30. Then for all fixed $0 \leqslant s \leqslant t$ the increment $N_t - N_s$ follows the Poisson distribution with parameter $(t-s)\lambda$, i.e. we have*

$$\mathbb{P}(N_t - N_s = k) = \mathrm{e}^{-(t-s)\lambda}\frac{((t-s)\lambda)^k}{k!}, \qquad k \geqslant 0, \tag{2.2}$$

for some constant $\lambda > 0$.

The parameter $\lambda > 0$ is called the intensity of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$ and it is given by

$$\lambda := \lim_{h\to 0}\frac{1}{h}\mathbb{P}(N_h = 1). \tag{2.3}$$

The proof of the above Theorem 2.1 is technical and not included here, cf. *e.g.* Bosq and Nguyen (1996) for details, and we could in fact take this distribution property (2.2) as one of the hypotheses that define the Poisson process.

Precisely, we could restate the definition of the standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity $\lambda > 0$ as being a stochastic process defined by (2.1), which is assumed to have independent increments distributed according to the Poisson distribution, in the sense that for all $0 \leqslant t_0 \leqslant t_1 < \cdots < t_n$,

$$(N_{t_1} - N_{t_0}, \ldots, N_{t_n} - N_{t_{n-1}})$$

is a vector of independent Poisson random variables with respective parameters

$$((t_1 - t_0)\lambda, \ldots, (t_n - t_{n-1})\lambda).$$

In particular, N_t has the Poisson distribution with parameter λt, *i.e.*,

$$\mathbb{P}(N_t = k) = \frac{(\lambda t)^k}{k!} \mathrm{e}^{-\lambda t}, \qquad t > 0.$$

The *expected value* $\mathbb{E}[N_t]$ and variance of N_t can be computed as

$$\mathbb{E}[N_t] = \mathrm{Var}[N_t] = \lambda t, \tag{2.4}$$

see Exercise A.1. As a consequence, the *dispersion index* of the Poisson process is

$$\frac{\mathrm{Var}[N_t]}{\mathbb{E}[N_t]} = 1, \qquad t \in \mathbb{R}_+. \tag{2.5}$$

Short time behaviour

From (2.3) above we deduce the short time asymptotics*

$$\begin{cases} \mathbb{P}(N_h = 0) = \mathrm{e}^{-h\lambda} = 1 - h\lambda + o(h), \qquad h \to 0, \\ \\ \mathbb{P}(N_h = 1) = h\lambda \mathrm{e}^{-h\lambda} \simeq h\lambda, \qquad h \to 0. \end{cases}$$

By stationarity of the Poisson process we also find more generally that

$$\begin{cases} \mathbb{P}(N_{t+h} - N_t = 0) = \mathrm{e}^{-h\lambda} = 1 - h\lambda + o(h), \qquad h \to 0, \\ \\ \mathbb{P}(N_{t+h} - N_t = 1) = h\lambda \mathrm{e}^{-h\lambda} \simeq h\lambda, \qquad h \to 0, \\ \\ \mathbb{P}(N_{t+h} - N_t = 2) \simeq h^2 \dfrac{\lambda^2}{2} = o(h), \qquad h \to 0, \qquad t > 0, \end{cases}$$

for all $t > 0$. This means that within a "short" interval $[t, t+h]$ of length h, the increment $N_{t+h} - N_t$ behaves like a Bernoulli random variable with parameter λh. This fact can be used for the random simulation of Poisson process paths.

More generally, for $k \geqslant 1$ we have

$$\mathbb{P}(N_{t+h} - N_t = k) \simeq h^k \frac{\lambda^k}{k!}, \qquad h \to 0, \qquad t > 0.$$

The intensity of the Poisson process can in fact be made time-dependent (*e.g.* by a time change), in which case we have

* The notation $f(h) = o(h^k)$ means $\lim_{h \to 0} f(h)/h^k = 0$, and $f(h) \simeq h^k$ means $\lim_{h \to 0} f(h)/h^k = 1$.

$$\mathbb{P}(N_t - N_s = k) = \exp\left(-\int_s^t \lambda(u)du\right)\frac{\left(\int_s^t \lambda(u)du\right)^k}{k!}, \qquad k = 0, 1, 2, \ldots.$$

In this case, we have in particular

$$\mathbb{P}(N_{t+dt} - N_t = k) = \begin{cases} \mathrm{e}^{-\lambda(t)dt} = 1 - \lambda(t)dt + o(dt), & k = 0, \\ \\ \lambda(t)\mathrm{e}^{-\lambda(t)dt}dt = \lambda(t)dt + o(dt), & k = 1, \\ \\ o(dt), & k \geqslant 2. \end{cases}$$

The intensity process $(\lambda(t))_{t\in\mathbb{R}_+}$ can also be made random, as in the case of Cox processes.

Poisson process jump times

In order to determine the distribution of the first jump time T_1 we note that we have the equivalence

$$\{T_1 > t\} \Longleftrightarrow \{N_t = 0\},$$

which implies

$$\mathbb{P}(T_1 > t) = \mathbb{P}(N_t = 0) = \mathrm{e}^{-\lambda t}, \qquad t \in \mathbb{R}_+,$$

i.e., T_1 has an exponential distribution with parameter $\lambda > 0$.

In order to prove the next proposition we note that more generally, we have the equivalence

$$\{T_n > t\} \Longleftrightarrow \{N_t \leqslant n - 1\},$$

for all $n \geqslant 1$. This allows us to compute the distribution of the random jump time T_n with its probability density function. It coincides with the *gamma* distribution with integer parameter $n \geqslant 1$, also known as the Erlang distribution in queueing theory.

Proposition 2.2. *For all $n \geqslant 1$ the probability distribution of T_n has the gamma probability density function*

$$t \longmapsto \lambda^n \mathrm{e}^{-\lambda t}\frac{t^{n-1}}{(n-1)!}$$

on $\mathbb{R}_+$, i.e., for all $t > 0$ the probability $\mathbb{P}(T_n \geqslant t)$ is given by

$$\mathbb{P}(T_n \geqslant t) = \lambda^n \int_t^\infty \mathrm{e}^{-\lambda s}\frac{s^{n-1}}{(n-1)!}ds.$$

Proof. We have

$$\mathbb{P}(T_1 > t) = \mathbb{P}(N_t = 0) = \mathrm{e}^{-\lambda t}, \qquad t \in \mathbb{R}_+,$$

and by induction, assuming that

$$\mathbb{P}(T_{n-1} > t) = \lambda \int_t^\infty \mathrm{e}^{-\lambda s} \frac{(\lambda s)^{n-2}}{(n-2)!} ds, \qquad n \geqslant 2,$$

we obtain

$$\begin{aligned}
\mathbb{P}(T_n > t) &= \mathbb{P}(T_n > t \geqslant T_{n-1}) + \mathbb{P}(T_{n-1} > t) \\
&= \mathbb{P}(N_t = n-1) + \mathbb{P}(T_{n-1} > t) \\
&= \mathrm{e}^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} + \lambda \int_t^\infty \mathrm{e}^{-\lambda s} \frac{(\lambda s)^{n-2}}{(n-2)!} ds \\
&= \lambda \int_t^\infty \mathrm{e}^{-\lambda s} \frac{(\lambda s)^{n-1}}{(n-1)!} ds, \qquad t \in \mathbb{R}_+,
\end{aligned}$$

where we applied an integration by parts to derive the last line. □

In particular, for all $n \in \mathbb{Z}$ and $t \in \mathbb{R}_+$, we have

$$\mathbb{P}(N_t = n) = p_n(t) = \mathrm{e}^{-\lambda t} \frac{(\lambda t)^n}{n!},$$

i.e., $p_{n-1} : \mathbb{R}_+ \to \mathbb{R}_+$, $n \geqslant 1$, is the probability density function of the random jump time T_n.

In addition to Proposition 2.2 we could show the following proposition which relies on the *strong Markov property*, see *e.g.* Theorem 6.5.4 of Norris (1998).

Proposition 2.3. *The (random) interjump times*

$$\tau_k := T_{k+1} - T_k$$

spent at state $k \in \mathbb{N}$, with $T_0 = 0$, form a sequence of independent identically distributed random variables having the exponential distribution with parameter $\lambda > 0$, i.e.,

$$\mathbb{P}(\tau_0 > t_0, \ldots, \tau_n > t_n) = \mathrm{e}^{-(t_0+t_1+\cdots+t_n)\lambda}, \qquad t_0, t_1, \ldots, t_n \in \mathbb{R}_+.$$

As the expectation of the exponentially distributed random variable τ_k with parameter $\lambda > 0$ is given by

$$\mathbb{E}[\tau_k] = \lambda \int_0^\infty x \mathrm{e}^{-\lambda x} dx = \frac{1}{\lambda},$$

we can check that the nth jump time $T_n = \tau_0 + \cdots + \tau_{n-1}$ has the mean

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$$\mathbb{E}[T_n] = \frac{n}{\lambda}, \qquad n \geqslant 1.$$

Consequently, the higher the intensity $\lambda > 0$ is (*i.e.*, the higher the probability of having a jump within a small interval), the smaller the time spent in each state $k \in \mathbb{N}$ is on average.

In addition, conditionally to $\{N_T = n\}$, the n jump times on $[0, T]$ of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$ are independent uniformly distributed random variables on $[0, T]^n$, cf. *e.g.* § 12.1 of Privault (2018). This fact can be useful for the random simulation of the Poisson process.

As a consequence of Propositions 2.2 and 2.2, random samples of Poisson process jump times can be generated using the following *R* code.

```
lambda = 0.6;n = 20;Z<-cumsum(c(0,rep(1,n)))
tau_n <- rexp(n,rate=lambda); Tn <- cumsum(tau_n)
plot(stepfun(Tn,Z),xlim =c(0,10),ylim=c(0,8),xlab="t",ylab="Nt",pch=1, cex=0.8,
      col="blue", lwd=2, main="")
```

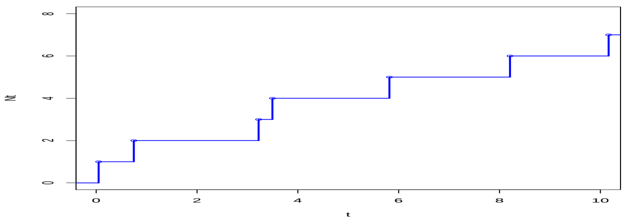

Fig. 2.2: Sample path of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$.

Compensated Poisson martingale

From (2.4) above we deduce that

$$\mathbb{E}[N_t - \lambda t] = 0, \tag{2.6}$$

i.e., the compensated Poisson process $(N_t - \lambda t)_{t\in\mathbb{R}_+}$ has *centered increments*.

```
lambda = 0.6;n = 20;Z<-cumsum(c(0,rep(1,n)));
tau_n <- rexp(n,rate=lambda); Tn <- cumsum(tau_n)
N <- function(t) {return(stepfun(Tn,Z)(t))};t <- seq(0,10,0.01)
plot(t,N(t)-lambda*t,xlim = c(0,10),ylim =
      c(-2,2),xlab="t",ylab="Nt-t",type="l",lwd=2,col="blue",main="", xaxs = "i", yaxs =
      "i", xaxs = "i", yaxs = "i")
abline(h = 0, col="black", lwd =2)
points(Tn,N(Tn)-lambda*Tn,pch=1,cex=0.8,col="blue",lwd=2)
```

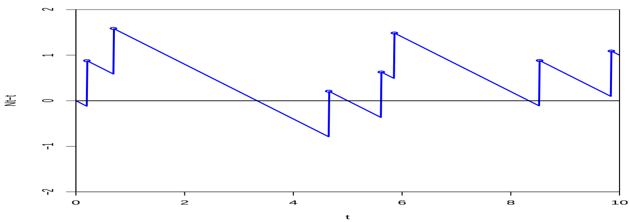

Fig. 2.3: Sample path of the compensated Poisson process $(N_t - \lambda t)_{t\in\mathbb{R}_+}$.

Since in addition $(N_t - \lambda t)_{t\in\mathbb{R}_+}$ also has independent increments, we get the following proposition. We let

$$\mathcal{F}_t := \sigma\big(N_s \ : \ s \in [0,t]\big), \qquad t \in \mathbb{R}_+,$$

denote the *filtration* generated by the Poisson process $(N_t)_{t\in\mathbb{R}_+}$.

Proposition 2.4. *The compensated Poisson process*

$$(N_t - \lambda t)_{t\in\mathbb{R}_+}$$

is a martingale *with respect* $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$.

Extensions of the Poisson process include Poisson processes with time-dependent intensity, and with random time-dependent intensity (Cox processes). Poisson processes belong to the family of *renewal processes* which are counting processes of the form

$$N_t = \sum_{n\geqslant 1} \mathbb{1}_{[T_n,\infty)}(t), \qquad t \in \mathbb{R}_+,$$

for which $\tau_k := T_{k+1} - T_k$, $k \geqslant 0$, is a sequence of independent identically distributed random variables.

2.2 Compound Poisson Process

The Poisson process itself appears to be too limited to develop realistic price models as its jumps are of constant size. Therefore there is some interest in considering jump processes that can have random jump sizes.

Let $(Z_k)_{k\geqslant 1}$ denote an *i.i.d.* sequence of square-integrable random variables distributed as the common random variable Z with the probability distribution $\nu(dy) = \varphi(y)dy$ on $\mathbb{R}$, independent of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$. We have

$$\mathbb{P}(Z \in [a,b]) = \nu([a,b]) = \int_a^b \nu(dy) = \int_a^b \varphi(y)dy, \quad -\infty < a \leqslant b < \infty, \quad k \geqslant 1.$$

Definition 2.5. *The process* $(Y_t)_{t\in\mathbb{R}_+}$ *given by the random sum*

$$Y_t := Z_1 + Z_2 + \cdots + Z_{N_t} = \sum_{k=1}^{N_t} Z_k, \qquad t \in \mathbb{R}_+, \tag{2.7}$$

is called a compound Poisson process.[*]

Letting Y_{t^-} denote the left limit

$$Y_{t^-} := \lim_{s \nearrow t} Y_s, \qquad t > 0,$$

we note that the jump size

$$\Delta Y_t := Y_t - Y_{t^-}, \qquad t \in \mathbb{R}_+,$$

of $(Y_t)_{t\in\mathbb{R}_+}$ at time t is given by the relation

$$\Delta Y_t = Z_{N_t} \Delta N_t, \qquad t \in \mathbb{R}_+, \tag{2.8}$$

where

$$\Delta N_t := N_t - N_{t^-} \in \{0,1\}, \qquad t \in \mathbb{R}_+,$$

denotes the jump size of the standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$, and N_{t^-} is the left limit

$$N_{t^-} := \lim_{s \nearrow t} N_s, \qquad t > 0,$$

For a typical example of a compound Poisson process we can assume that jump sizes are Gaussian distributed with mean δ and variance η^2, in which case $\nu(dy)$ is given by

$$\nu(dy) = \frac{1}{\sqrt{2\pi\eta^2}} \mathrm{e}^{-(y-\delta)^2/(2\eta^2)} dy.$$

[*] We use the convention $\sum_{k=1}^{n} Z_k = 0$ if $n = 0$, so that $Y_0 = 0$.

The next Figure 2.4 represents a sample path of a compound Poisson process, with here $Z_1 = 0.9$, $Z_2 = -0.7$, $Z_3 = 1.4$, $Z_4 = 0.6$, $Z_5 = -2.5$, $Z_6 = 1.5$, $Z_7 = -0.5$, with the relation

$$Y_{T_k} = Y_{T_k^-} + Z_k, \qquad k \geqslant 1.$$

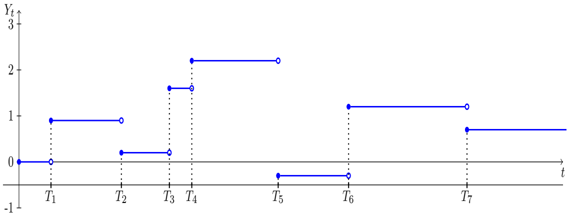

Fig. 2.4: Sample path of a compound Poisson process $(Y_t)_{t\in\mathbb{R}_+}$.

```
N<-50
x<-cumsum(rexp(N,rate=0.5))
y<-cumsum(c(0,rexp(N,rate=0.5)))
plot(stepfun(x,y),xlim = c(0,10),do.points = F,main="L=0.5",col="blue")
y<-cumsum(c(0,rnorm(N,mean=0,sd=1)))
plot(stepfun(x,y),xlim = c(0,10),do.points = F,main="L=0.5",col="blue")
```

Given that $\{N_T = n\}$, the n jump sizes of $(Y_t)_{t\in\mathbb{R}_+}$ on $[0, T]$ are independent random variables which are distributed on $\mathbb{R}$ according to $\nu(dx)$. Based on this fact, the next proposition allows us to compute the moment generating function (MGF) of the increment $Y_T - Y_t$.

Proposition 2.6. *For any $t \in [0, T]$ we have*

$$\mathbb{E}\left[e^{\alpha(Y_T - Y_t)}\right] = \exp\left((T-t)\lambda \int_{-\infty}^{\infty}(e^{\alpha y} - 1)\nu(dy)\right), \quad \alpha \in \mathbb{R}. \qquad (2.9)$$

Proof. Since N_t has a Poisson distribution with parameter $t > 0$ and is independent of $(Z_k)_{k\geqslant 1}$, for all $\alpha \in \mathbb{R}$ we have, by conditioning on the value of $N_T - N_t = n$,

$$\mathbb{E}\left[e^{\alpha(Y_T - Y_t)}\right] = \mathbb{E}\left[\exp\left(\alpha \sum_{k=N_t+1}^{N_T} Z_k\right)\right]$$

$$= \mathbb{E}\left[\exp\left(\alpha \sum_{k=1}^{N_T - N_t} Z_k\right)\right]$$

$$= \sum_{n\geqslant 0} \mathbb{E}\left[\exp\left(\alpha \sum_{k=1}^{n} Z_k\right) \Bigg| N_T - N_t = n\right] \mathbb{P}(N_T - N_t = n)$$

$$
\begin{aligned}
&= e^{-(T-t)\lambda} \sum_{n \geqslant 0} \frac{\lambda^n}{n!} (T-t)^n \, \mathbb{E}\left[\exp\left(\alpha \sum_{k=1}^n Z_k \right) \right] \\
&= e^{-(T-t)\lambda} \sum_{n \geqslant 0} \frac{\lambda^n}{n!} (T-t)^n \prod_{k=1}^n \mathbb{E}\left[e^{\alpha Z_k} \right] \\
&= e^{-(T-t)\lambda} \sum_{n \geqslant 0} \frac{\lambda^n}{n!} (T-t)^n \left(\mathbb{E}\left[e^{\alpha Z} \right] \right)^n \\
&= \exp\left((T-t)\lambda \left(\mathbb{E}\left[e^{\alpha Z} \right] - 1 \right) \right) \\
&= \exp\left((T-t)\lambda \int_{-\infty}^{\infty} e^{\alpha y} \nu(dy) - (T-t)\lambda \int_{-\infty}^{\infty} \nu(dy) \right) \\
&= \exp\left((T-t)\lambda \int_{-\infty}^{\infty} (e^{\alpha y} - 1) \nu(dy) \right),
\end{aligned}
$$

since the probability distribution $\nu(dy)$ of Z satisfies

$$
\mathbb{E}\left[e^{\alpha Z} \right] = \int_{-\infty}^{\infty} e^{\alpha y} \nu(dy) \quad \text{and} \quad \int_{-\infty}^{\infty} \nu(dy) = 1.
$$

□

From the moment generating function (2.9) we can compute the expectation of Y_t for fixed t as the product of the mean number of jump times $\mathbb{E}[N_t] = \lambda t$ and the mean jump size $\mathbb{E}[Z]$, *i.e.*,

$$
\mathbb{E}[Y_t] = \frac{\partial}{\partial \alpha} \mathbb{E}[e^{\alpha Y_t}]_{|\alpha=0} = \lambda t \int_{-\infty}^{\infty} y \nu(dy) = \mathbb{E}[N_t] \, \mathbb{E}[Z] = \lambda t \, \mathbb{E}[Z]. \tag{2.10}
$$

Note that the above identity requires to exchange the differentiation and expectation operators, which is possible when the moment generating function (2.9) takes finite values for all α in a certain neighborhood $(-\varepsilon, \varepsilon)$ of 0.

Relation (2.10) states that the mean value of Y_t is the mean jump size $\mathbb{E}[Z]$ times the mean number of jumps $\mathbb{E}[N_t]$. It can be directly recovered using series summations, as

$$
\begin{aligned}
\mathbb{E}[Y_t] &= \mathbb{E}\left[\sum_{k=1}^{N_t} Z_k \right] \\
&= \sum_{n \geqslant 1} \mathbb{E}\left[\sum_{k=1}^{n} Z_k \,\middle|\, N_t = n \right] \mathbb{P}(N_t = n) \\
&= e^{-\lambda t} \sum_{n \geqslant 1} \frac{\lambda^n t^n}{n!} \mathbb{E}\left[\sum_{k=1}^{n} Z_k \,\middle|\, N_t = n \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{e}^{-\lambda t} \sum_{n\geqslant 1} \frac{\lambda^n t^n}{n!} \mathbb{E}\left[\sum_{k=1}^n Z_k\right] \\
&= \lambda t \mathrm{e}^{-\lambda t} \mathbb{E}[Z] \sum_{n\geqslant 1} \frac{(\lambda t)^{n-1}}{(n-1)!} \\
&= \lambda t \, \mathbb{E}[Z] \\
&= \mathbb{E}[N_t] \, \mathbb{E}[Z].
\end{aligned}
$$

Regarding the variance, we have

$$
\begin{aligned}
\mathbb{E}\left[Y_t^2\right] &= \frac{\partial^2}{\partial \alpha^2} \mathbb{E}[\mathrm{e}^{\alpha Y_t}]_{|\alpha=0} \\
&= \lambda t \int_{-\infty}^{\infty} y^2 \nu(dy) + (\lambda t)^2 \left(\int_{-\infty}^{\infty} y \nu(dy)\right)^2 \\
&= \lambda t \, \mathbb{E}\left[Z^2\right] + (\lambda t \, \mathbb{E}[Z])^2,
\end{aligned}
$$

which yields

$$
\mathrm{Var}\,[Y_t] = \lambda t \int_{-\infty}^{\infty} y^2 \nu(dy) = \lambda t \, \mathbb{E}\left[|Z|^2\right] = \mathbb{E}[N_t] \, \mathbb{E}\left[|Z|^2\right]. \tag{2.11}
$$

As a consequence, the *dispersion index* of the compound Poisson process

$$
\frac{\mathrm{Var}\,[Y_t]}{\mathbb{E}[Y_t]} = \frac{\mathbb{E}\left[|Z|^2\right]}{\mathbb{E}[Z]}, \qquad t \in \mathbb{R}_+.
$$

is the dispersion index of the random jump size Z. By a multivariate version of Theorem 10.15, the above identity can be used to show the next proposition.

Proposition 2.7. *The compound Poisson process*

$$
Y_t = \sum_{k=1}^{N_t} Z_k, \qquad t \in \mathbb{R}_+,
$$

has independent increments, i.e. for any finite sequence of times $t_0 < t_1 < \cdots < t_n$*, the increments*

$$
Y_{t_1} - Y_{t_0}, \; Y_{t_2} - Y_{t_1}, \ldots, \; Y_{t_n} - Y_{t_{n-1}}
$$

are mutually independent random variables.

Proof. This result relies on the fact that the result of Proposition 2.6 can be extended to sequences $0 \leqslant t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$, as

$$
\mathbb{E}\left[\prod_{k=1}^n \mathrm{e}^{i\alpha_k (Y_{t_k} - Y_{t_{k-1}})}\right] = \mathbb{E}\left[\exp\left(i \sum_{k=1}^n \alpha_k (Y_{t_k} - Y_{t_{k-1}})\right)\right]
$$

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$$= \exp\left(\lambda \sum_{k=1}^{n} (t_k - t_{k-1}) \int_{-\infty}^{\infty} (e^{i\alpha_k y} - 1)\nu(dy)\right) \tag{2.12}$$

$$= \prod_{k=1}^{n} \exp\left((t_k - t_{k-1})\lambda \int_{-\infty}^{\infty} (e^{i\alpha_k y} - 1)\nu(dy)\right)$$

$$= \prod_{k=1}^{n} \mathbb{E}\left[e^{i\alpha_k (Y_{t_k} - Y_{t_{k-1}})}\right].$$

□

Since the compensated compound Poisson process also has independent and centered increments by (2.6) we have the following counterpart of Proposition 2.4.

Proposition 2.8. *The compensated compound Poisson process*

$$M_t := Y_t - \lambda t\, \mathbb{E}[Z], \qquad t \in \mathbb{R}_+,$$

is a martingale.

By construction, compound Poisson processes only have a *finite* number of jumps on any interval. They belong to the family of *Lévy processes* which may have an infinite number of jumps on any finite time interval, see *e.g.* § 4.4.1 of Cont and Tankov (2004).

The stochastic integral of a deterministic function $f(t)$ with respect to $(Y_t)_{t\in\mathbb{R}_+}$ is defined as

$$\int_0^T f(t)dY_t = \sum_{k=1}^{N_T} Z_k f(T_k).$$

Relation (2.12) can be used to show that, more generally, the moment generating function of $\int_0^T f(t)dY_t$ is given by

$$\mathbb{E}\left[\exp\left(\int_0^T f(t)dY_t\right)\right] = \exp\left(\lambda \int_0^T \int_{\mathbb{R}} (e^{yf(t)} - 1)\nu(dy)dt\right)$$
$$= \exp\left(\lambda \int_0^T \left(\mathbb{E}\left[e^{f(t)Z}\right] - 1\right)dt\right).$$

We also have

$$\log \mathbb{E}\left[\exp\left(\int_0^T f(t)dY_t\right)\right] = \lambda \int_0^T \int_{\mathbb{R}} (e^{yf(t)} - 1)\nu(dy)dt$$
$$= \lambda \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^T \int_{\mathbb{R}} y^n f^n(t)\nu(dy)dt$$

$$= \lambda \sum_{n=1}^{\infty} \frac{1}{n!} \mathbb{E}[Z^n] \int_0^T f^n(t) dt,$$

hence the *cumulant* of order $n \geqslant 1$ of $\int_0^T f(t) dY_t$ is given by

$$\kappa_n = \lambda \mathbb{E}[Z^n] \int_0^T f^n(t) dt,$$

which recovers (2.10) and (2.11) by taking $f(t) = \mathbb{1}_{[0,T]}(t)$ when $n = 1, 2$.

2.3 Claim and Reserve Processes

We consider

- a number N_t of claims made until $t \geqslant 0$, which is modeled by an homogeneous Poisson process $(N_t)_{t \geqslant 0}$ with intensity $\lambda > 0$,
- a sequence $(W_k)_{k \geqslant 1}$ of non-negative independent, identically-distributed random variables, which represent the claim amounts.

We assume that the claim amounts $(W_k)_{k \geqslant 1}$ and the process of arrivals $(N_t)_{t \geqslant 0}$ are independent. In the next definition we use the convention $S(t) = 0$ if $N_t = 0$.

Definition 2.9. *The* aggregate claim amount *up to time t is defined as the compound Poisson process*

$$S(t) = \sum_{k=1}^{N_t} W_k.$$

The aggregate claim amount $(S(t))_{t \in \mathbb{R}_+}$ can also be written as

$$S(t) = Y_{N_t}, \qquad t \in \mathbb{R}_+,$$

where $(Y_k)_{k \geqslant 1}$ is the sequence of random variables independent of $(N_t)_{t \in \mathbb{R}_+}$ given by

$$Y_k = \sum_{j=1}^{k} W_j, \qquad k \in \mathbb{N},$$

with $Y_0 = 0$. In the next definition, $f : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function mapping $t > 0$ to the premium income $f(t)$ received between time 0 and time t, with $f(0) = 0$.

Definition 2.10. Classical compound Poisson risk model *The surplus (or reserve) process $(R_x(t))_{t \geqslant 0}$ is defined as*

$$R_x(t) = x + f(t) - S(t), \qquad t \geqslant 0,$$

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where $x \geqslant 0$ is the amount of initial reserves and $f(t)$ is the premium income received between time 0 and time $t > 0$.

In the next Figures 2.5 and 2.6 we take $f(t) := ct$ with $c = 1/2$.

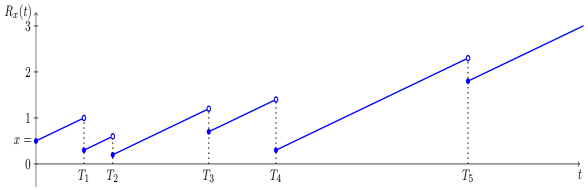

Fig. 2.5: Sample path (without ruin) of a risk process $(R_x(t))_{t\in\mathbb{R}_+}$.

Unlike the above figure, the next Figure 2.6 contains a ruin event.

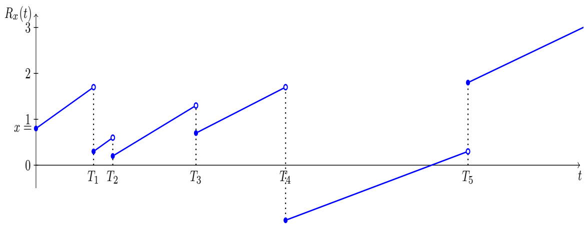

Fig. 2.6: Sample path (with ruin) of a risk process $(R_x(t))_{t\in\mathbb{R}_+}$.

2.4 Ruin Probabilities

We will consider the infinite time ruin probability

$$\Psi(x) = \mathbb{P}\big(\exists\, t > 0 \ : \ R_x(t) < 0\big),$$

and the finite-time ruin probability defined as

$$\Psi_T(x) = \mathbb{P}\big(\exists\, t \in [0, T] \ : \ R_x(t) < 0\big),$$

given $T > 0$ a finite time horizon. The ruin probability $\Psi_T(x)$ can also be written as

$$\Psi_T(x) = \mathbb{P}\left(\mathcal{M}_{[0,T]} < -x\right), \qquad x \geqslant 0,$$

where $\mathcal{M}_{[0,T]}$ is the infimum

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$$\mathcal{M}_{[0,T]} := \inf_{0\leqslant t\leqslant T}(f(t) - S(t)).$$

Cramér-Lundberg Model

In Proposition 2.11 we compute the ruin probability in infinite time starting from the initial reserve $x = 0$, assuming that the claim sizes $(W_k)_{k\geqslant 1}$ for a sequence of independent, exponentially distributed random variables with mean $\mu > 0$ (*i.e.* with parameter $1/\mu$), and that $f(t) = ct$ with the premium rate $c > 0$.

Proposition 2.11. *The ruin probability in infinite time starting from the initial reserve $x \geqslant 0$ is given by*

$$\Psi(x) = \frac{\lambda\mu}{c}\mathrm{e}^{(\lambda/c-1/\mu)x}, \qquad x \geqslant 0, \tag{2.13}$$

provided that $c < \lambda\mu$.

Proof. For simplicity the proof is only presented in the case $x = 0$, and we refer to Exercise 2.2 for the computation of the ruin probability starting from any $x > 0$. Let

$$\Phi(x) := 1 - \Psi(x) = \mathbb{P}\left(R_x(t) \geqslant 0, \ \forall t > 0\right)$$

denote the probability of non-ruin. Since $x, c \geqslant 0$ we have

$$\begin{aligned}
\Phi(x) &= \mathbb{P}\left(R_x(t) \geqslant 0, \ \forall \ t > 0\right) \\
&= \mathbb{P}\left(x + ct - \sum_{k=1}^{N_t} W_k \geqslant 0, \ \forall \ t > 0\right) \\
&= \mathbb{P}\left(x + ct - \sum_{k=1}^{N_t} W_k \geqslant 0, \ \forall \ t > T_1\right) \\
&= \mathbb{E}\left[\mathbb{1}_{\left\{x+cT_1-W_1+c(t-T_1)-\sum_{k=2}^{N_t} W_k\geqslant 0, \ \forall \ t>T_1\right\}}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\left\{x+cT_1-W_1+c(t-T_1)-\sum_{k=2}^{N_t} W_k\geqslant 0, \ \forall \ t>T_1\right\}} \,\Big|\, T_1\right]\right] \\
&= \mathbb{E}\left[\Phi(x + cT_1 - W_1)\right] \\
&= \lambda\int_0^\infty \mathrm{e}^{-\lambda s}\int_0^{x+cs}\Phi(x + cs - z)dF(z)ds \\
&= \frac{\lambda}{c}\mathrm{e}^{\lambda x/c}\int_x^\infty \mathrm{e}^{-\lambda u/c}\int_0^u \Phi(u - z)dF(z)du,
\end{aligned} \tag{2.14}$$

where

$$F(z) := \mathbb{P}(W_1 \leqslant z), \qquad z \in \mathbb{R}_+,$$

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denotes the cumulative distribution function of the claim size W_1. By differentiating (2.14) with respect to x, we find

$$\Phi'(x) = \frac{\lambda}{c}\left(\Phi(x) - \int_0^x \Phi(x-z)dF(z)ds\right), \qquad (2.15)$$

hence by integration by parts with respect to $z \in [0, x]$, we get

$$\begin{aligned}
\Phi(x) &= \Phi(0) + \int_0^x \Phi'(y)dy \\
&= \Phi(0) + \frac{\lambda}{c}\int_0^x \Phi(y)dy - \frac{\lambda}{c}\int_0^x \int_0^y \Phi(y-z)dF(z)dy, \\
&= \Phi(0) + \frac{\lambda}{c}\int_0^x \Phi(x-z)(1-F(z))dz,
\end{aligned}$$

Noting that $\Phi(\infty) = 1$, we deduce

$$\begin{aligned}
1 &= \Phi(\infty) \\
&= \Phi(0) + \frac{\lambda}{c}\int_0^\infty \Phi(\infty - z)(1-F(z))dz \\
&= \Phi(0) + \frac{\lambda}{c}\int_0^\infty (1-F(z))dz \\
&= \Phi(0) + \frac{\lambda}{c}\int_0^\infty (1-F(z))dz \\
&= \Phi(0) + \frac{\lambda\mu}{c}, \qquad (2.16)
\end{aligned}$$

since

$$\begin{aligned}
\mu &= \mathbb{E}[W_1] \\
&= \int_0^\infty z d\mathbb{P}(W_1 \leqslant z) \\
&= \int_0^\infty \mathbb{P}(W_1 \leqslant z)dz \\
&= \int_0^\infty (1-F(z))dz
\end{aligned}$$

is the average claim size. From (2.16) we conclude that $\Phi(0) = 1 - \lambda\mu/c$, hence the ruin probability in infinite time starting from the initial reserve $x = 0$ is given by

$$\begin{aligned}
\Psi(0) &= 1 - \Phi(0) \\
&= \mathbb{P}\big(\exists\, t > 0 \ : \ R_x(t) < 0\big) \\
&= \frac{\lambda\mu}{c}, \qquad (2.17)
\end{aligned}$$

provided that $\lambda\mu \leqslant c$. □

When $x = 0$, the ruin probability in infinite time starting from the initial reserve $x = 0$ is given by

$$\Psi(0) = \mathbb{P}\big(\exists\, t > 0 \ : \ R_0(t) < 0\big) = \frac{\lambda\mu}{c},$$

provided that $\mu\lambda \leqslant c$.

R simulation[*]

The following R code recovers the formula (2.13) of Proposition 2.11 for the ruin probability in infinite time starting from the initial reserve $x \geqslant 0$, see also (2.19) in Exercise 2.2 by Monte Carlo simulation.

```r
lambda = 0.4; x = 7.5; mu = 4.0;
T=20; # Use T=500 to approximate infinite time
nSim = 21; c = 3.0; N <- rep(Inf, nSim)
for (k in 1:nSim){
tauk <- rexp(T*lambda)/lambda;Ti <- cumsum(tauk)
n=length(Ti[Ti<T]);Wk <- rexp(n,1/mu);Si <- x + Ti*c
Ui <- Si - cumsum(Wk);Si <- c(Si,Ui[n]+c*(T-Ui[n]))
ruin <- !all(Ui[1:n]>=0);color="blue"
if (ruin) {N[k] <- min(which(Ui<0));color="red"}
plot(c(0,rbind(Ti[1:n+1],Ti[1:n+1])),c(x,rbind(Si[1:n+1],Ui[1:n])),xlab="time
      t",xlim=c(0,T*0.99),ylim=c(0,x+c*T),lwd=3,ylab="S(t)",type="l",col=color,
      main=paste(length(N[N<Inf]),"/",k,"=",format(length(N[N<Inf])/k,digits=4)),
      axes=FALSE)
axis(1, pos=0);axis(2, pos=0);Sys.sleep(0.2)}
N <- N[N<Inf];length(N); mean(N); sd(N); max(N)
cat('Theoretical value:',lambda*mu*exp(-x*(1/mu-lambda/c))/c,'\n')
cat('Simulation:',length(N)/nSim,'\n')
```

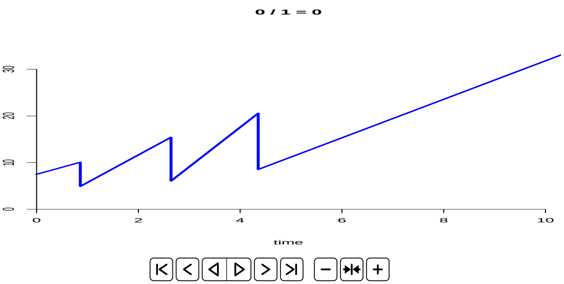

Fig. 2.7: Sample paths of the reserve process $(S_t)_{t\geqslant 0}$.[†]

[*] Kaas et al. (2009) Example 4.3.7

Probability density function

The probability density function of $\mathcal{M}_{[0,T]}$ at $-x < 0$ can be computed as

$$-\frac{\partial \Psi_T}{\partial x}(x).$$

An important practical problem is to obtain numerical values of the sensitivity of the finite-time ruin probability with respect to the initial reserve

$$\frac{\partial \Psi_T}{\partial x}(x),$$

in particular due to new solvency regulations in Europe. In Privault and Wei (2004), Privault and Wei (2007), the Malliavin calculus has been used to provide a way to compute the sensitivity of the probability

$$\mathbb{P}\left(R_x(T) < 0\right)$$

that the terminal surplus is negative with respect to parameters such as the initial reserve or the interest rate of the model.

The problem of computing the corresponding sensitivity for the finite-time ruin probability $\Psi_T(x)$ has been covered in Loisel and Privault (2009) based on multiple integration. Formulas for the finite-time ruin probability

$$\Psi_T(x) = \mathbb{P}\big(\exists\, t \in [0,T] \ : \ R_x(t) < 0\big)$$

have been recently proposed in Picard and Lefèvre (1997), cf. also De Vylder (1999) and Ignatova et al. (2001), Rullière and Loisel (2004).

Integral expressions

Starting from $f(0) := 0$ we clearly we have $\mathcal{M}_{[0,T]} \leqslant 0 = f(0)$ hence the distribution of $\mathcal{M}_{[0,T]}$ is carried by $(-\infty, 0]$. On the other hand, we have $\mathcal{M}_{[0,T]} = 0$ if and only if $N_T = 0$ or $f(T_k) - Y_k > 0$ for all $k \geqslant 1$ such that $T_k \leqslant T$, hence the distribution of $\mathcal{M}_{[0,T]}$ has a Dirac mass at 0 with weight

$$\mathbb{P}(\mathcal{M}_{[0,T]} = 0) = \mathbb{P}(N_T = 0) + \mathbb{P}(\{\mathcal{M}_{[0,T]} \geqslant 0\} \cap \{N_T \geqslant 1\})$$
$$= \mathrm{e}^{-\lambda T} + \mathrm{e}^{-\lambda T}\,\mathbb{E}\left[\sum_{k\geqslant 1} \lambda^k \int_0^T \int_0^{t_k} \cdots \int_0^{t_2} \mathbb{1}_{\{f(t_1)>Y_1\}} \cdots \mathbb{1}_{\{f(t_k)>Y_k\}} dt_1 \cdots dt_k\right],$$

† The animation works in Acrobat Reader on the entire pdf file.

where we used the fact that Poisson jump times are independent uniformly distributed on the square $[0,T]^n$ given that $\{N_T = n\}$.

On the other hand, since f is increasing we have

$$\mathcal{M}_{[0,T]} = \inf_{T_k \leqslant T,\ k\geqslant 0} (f(T_k) - Y_k) = \mathbb{1}_{\{N_T \geqslant 1\}} \inf_{T_k \leqslant T,\ k\geqslant 1} (f(T_k) - Y_k),$$

with $T_0 = 0$. Hence we have the integral expression

$$\begin{aligned} & P(\{\mathcal{M}_{[0,T]} \geqslant y\} \cap \{N_T \geqslant 1\}) \qquad (2.18) \\ &= \mathrm{e}^{-\lambda T}\, \mathbb{E}\left[\sum_{k\geqslant 1} \lambda^k \int_0^T \int_0^{t_k} \cdots \int_0^{t_2} \mathbb{1}_{\{y < \inf_{1\leqslant l \leqslant k}(f(t_l)-Y_l)\}} dt_1 \cdots dt_k\right] \\ &= \lambda \mathrm{e}^{-\lambda T}\, \mathbb{E}\left[\sum_{k\geqslant 0} \lambda^k \int_0^T \int_0^{t_{k+1}} \cdots \int_0^{t_2} \mathbb{1}_{\{f(t_1) > Y_1 + y\}} \cdots \mathbb{1}_{\{f(t_{k+1}) > Y_{k+1} + y\}} dt_1 \cdots dt_{k+1}\right] \end{aligned}$$

Analytic expressions for ruin probabilities have been obtained in case $(Y_k)_{k\geqslant 1}$ are independent, exponentially distributed random variables with parameter $\mu > 0$ and $f(t) = ct$ is linear, $c \geqslant 0$, Theorem 4.1 and Relation (4.6) of Dozzi and Vallois (1997) show that

$$\begin{aligned} & \mathbb{P}(\mathcal{M}_{[0,T]} < x) \\ &= \lambda \int_0^T \left(x \sum_{n\geqslant 0} \frac{(\lambda \mu t(x+ct))^n}{(n!)^2} + ct \sum_{n\geqslant 0} \frac{(\lambda \mu t(x+ct))^n}{n!(n+1)!} \right) \frac{\mathrm{e}^{-\mu(x+ct)-\lambda t}}{x+ct} dt. \end{aligned}$$

Random drift

Here we consider the infimum

$$\mathcal{M}_{[0,T]} = \inf_{0\leqslant t \leqslant T} (Z_t - S(t))$$

where $(Z_t)_{t\in\mathbb{R}_+}$ is a stochastic process with independent increments and $Z_0 = 0$, independent of $(S(t))_{t\in\mathbb{R}_+}$, and such that

$$\inf_{t\in[a,b]} Z_t, \qquad 0 \leqslant a < b,$$

has a probability density function denoted by $\phi_{a,b}(x)$. For example, if $(Z_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion then $\phi_{a,b}(x)$ is given by

$$\int_x^\infty \phi_{a,b}(z) dz = \mathbb{P}\left(\inf_{t\in[a,b]} Z_t \geqslant x \right)$$

$$= \mathbb{E}\left[\mathbb{1}_{\{Z_a<x\}}\mathbb{P}\left(\inf_{t\in[a,b]} Z_t \geqslant x \;\middle|\; Z_a\right)\right] + \mathbb{E}\left[\mathbb{1}_{\{Z_a\geqslant x\}}\mathbb{P}\left(\inf_{t\in[a,b]} Z_t \geqslant x \;\middle|\; Z_a\right)\right]$$
$$= \mathbb{E}\left[\mathbb{1}_{\{Z_a<x\}}\mathbb{P}\left(\inf_{t\in[0,b-a]} B_t \geqslant x - Z_a \;\middle|\; Z_a\right)\right] + \mathbb{P}(Z_a \geqslant x)$$
$$= 2\,\mathbb{E}\left[\mathbb{1}_{\{Z_a<x\}}\mathbb{P}\left(B_{b-a} \geqslant x - Z_a \mid Z_a\right)\right] + \mathbb{P}(Z_a \geqslant x)$$
$$= \frac{1}{\pi\sqrt{a(b-a)}}\int_0^\infty \mathrm{e}^{-(x-y)^2/(2a)}\int_y^\infty \mathrm{e}^{-z^2/(2(b-a))}dzdy + \frac{1}{\sqrt{2\pi a}}\int_x^\infty \mathrm{e}^{-z^2/(2a)}dz.$$

We have $\mathcal{M}_{[0,T]} \leqslant Z_0 = 0$ *a.s.*, hence the distribution of $\mathcal{M}_{[0,T]}$ is carried by $(-\infty, 0]$.

Exercises

Exercise 2.1 Consider N a Poisson random variable with distribution

$$\mathbb{P}(N = n) = \mathrm{e}^{-\lambda}\frac{\lambda^n}{n!}, \qquad n = 0, 1, 2, \ldots$$

where $\lambda > 0$, and let $Y := \sum_{k=1}^{N} Z_k$, where $(Z_k)_{k\geqslant 1}$ is a sequence of independent centered $\mathcal{N}(0, \sigma^2)$ Gaussian random variables with variance σ^2 and cumulative distribution function

$$\mathbb{P}(Z_k \leqslant x) = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{x} \mathrm{e}^{-y^2/(2\sigma^2)}dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x/\sigma} \mathrm{e}^{-y^2/2}dy = \Phi(x/\sigma),$$

$x \in \mathbb{R}_+$.

a) Compute $\mathbb{P}(Y \geqslant y)$ using the conditioning

$$\mathbb{P}(Y \geqslant y) = \sum_{n\geqslant 1}\mathbb{P}\left(\sum_{k=1}^{N} Z_k \geqslant y \;\middle|\; N = n\right)\mathbb{P}(N = n) = \cdots$$

b) Find $\mathbb{E}[Y]$.

Exercise 2.2 Show that when the claim size distribution is exponential with mean $\mu > 0$, *i.e.* when $F(z) = 1 - \mathrm{e}^{-z/\mu}$, $z \geqslant 0$, the ruin probability is given by

$$\Psi(x) = \mathbb{P}\big(\exists\, t \in \mathbb{R}_+ \;:\; R_x(t) < 0\big) = \frac{\lambda\mu}{c}\mathrm{e}^{-x(1/\mu-\lambda/c)}, \quad x \geqslant 0, \qquad (2.19)$$

provided that $\lambda\mu \leqslant c$.