Chapter 6
Structural Approach to Credit Risk

The structural approach to credit risk modeling focuses on modeling bankruptcy from a firm’s asset value, in contrast to the reduced form approach in which default probabilities are modeled as stochastic processes. Here, the credit default event occurs when the assets of a firm drop below a certain pre-defined level. This chapter also considers the modeling of correlation and dependence between multiple default times.

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6.1 Merton Model

The Merton (1974) credit risk model reframes corporate debt as an option on a firm’s underlying value. Precisely the value $S_t$ of a firm’s asset is modeled by a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

under the historical (or physical) measure $\mathbb{P}$. Recall that $S_t$ is modeled as

$$dS_t = r S_t dt + \sigma S_t d\hat{B}_t$$

under the risk-neutral probability measure $\mathbb{P}^*$. The company debt is represented by an amount $K$ in bonds to be paid at maturity $T$, cf. § 4.1 of Grasselli and Hurd (January 3, 2010).

Default occurs if $S_T < K$ with probability $\mathbb{P}(S_T < K)$, the bond holder will receive the recovery value $S_T$. Otherwise, if $S_T \geq K$ the bond holder
receives $K$ and the equity holder is entitled to receive $S_T - K$, which can be represented as $(S_T - K)^+$ in general.

**Proposition 6.1.** The default probability $\mathbb{P}(S_T < K \mid \mathcal{F}_t)$ can be computed from the lognormal distribution of $S_T$ as

$$\mathbb{P}(S_T < K \mid \mathcal{F}_t) = \Phi \left( -\frac{(\mu - \sigma^2/2)(T-t) + \log(S_t/K)}{\sigma \sqrt{T-t}} \right),$$

where $\Phi$ is the cumulative distribution function of the standard normal distribution, and

$$d^\mu := \frac{(\mu - \sigma^2/2)(T-t) + \log(S_t/K)}{\sigma \sqrt{T-t}}.$$

**Proof.** The default probability $\mathbb{P}(S_T < K \mid \mathcal{F}_t)$ can be computed from the lognormal distribution of $S_T$ as

$$\mathbb{P}(S_T < K \mid \mathcal{F}_t) = \mathbb{P}(S_0 e^{\sigma B_T + (\mu - \sigma^2/2)T} < K \mid \mathcal{F}_t)$$

$$= \mathbb{P}(B_T < (-\mu - \sigma^2/2)T + \log(K/S_0)/\sigma \mid \mathcal{F}_t)$$

$$= \mathbb{P}(B_T - B_t + y < (-\mu - \sigma^2/2)T + \log(K/S_0)/\sigma \mid y = B_t)$$

$$= \frac{1}{\sqrt{2(T-t)\pi}} \int_{-\infty}^{(-\mu - \sigma^2/2)(T-t) + \log(K/S_t)/\sigma} e^{-x^2/(2(T-t))} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(-\mu - \sigma^2/2)(T-t) + \log(K/S_t)/(\sigma \sqrt{T-t})} e^{-x^2/2} dx$$

$$= 1 - \Phi \left( \frac{(\mu - \sigma^2/2)(T-t) + \log(S_t/K)}{\sigma \sqrt{T-t}} \right)$$

$$= 1 - \Phi(d^\mu)$$

$$= \Phi(-d^\mu)$$

$$= \Phi \left( -\frac{(\mu - \sigma^2/2)(T-t) + \log(S_t/K)}{\sigma \sqrt{T-t}} \right).$$

\[\square\]

Note that under the risk-neutral probability measure $\mathbb{P}^*$ we have, replacing $\mu$ with $r$,

$$\mathbb{P}^*(S_T < K \mid \mathcal{F}_t) = \Phi(-d^r_\mu),$$

with

$$d^r_\mu = \frac{(r - \sigma^2/2)(T-t) + \log(S_t/K)}{\sigma \sqrt{T-t}},$$

which implies the relation

$$d^r_\mu = d^\mu - \frac{\mu - r}{\sigma} \sqrt{T-t}.$$
or

$$\Phi^{-1}(P(S_T < K \mid F_t)) = -\frac{\mu - r}{\sigma} \sqrt{T - t} + \Phi^{-1}(P^*(S_T < K \mid F_t)).$$

The probability of default of the firm at a time $\tau$ before $T$ can be defined as the probability that the level of its assets falls below the level $K$ at time $T$. In this case the conditional distribution of $\tau$ is given by

$$P(\tau < T \mid F_t) := P(S_T < K \mid F_t)$$

$$= \Phi \left( -\frac{(\mu - \sigma^2/2)(T - t) + \log(S_t/K)}{\sigma \sqrt{T - t}} \right), \quad T \geq t. \quad (6.1)$$

We also have

$$P(\tau < T \mid F_t) = P(S_T < K \mid F_t)$$

$$= \Phi \left( \Phi^{-1}(P^*(S_T < K \mid F_t)) - \frac{\mu - r}{\sigma} \sqrt{T - t} \right)$$

and

$$P^*(\tau < T \mid F_t) = P^*(S_T < K \mid F_t)$$

$$= \Phi \left( \frac{-(r - \sigma^2/2)(T - t) + \log(S_t/K)}{\sigma \sqrt{T - t}} \right)$$

$$= \Phi \left( \Phi^{-1}(P(S_T < K \mid F_t)) + \frac{\mu - r}{\sigma} \sqrt{T - t} \right)$$

$$= \Phi \left( \Phi^{-1}(P(\tau < T \mid F_t)) + \frac{\mu - r}{\sigma} \sqrt{T - t} \right). \quad (6.2)$$

Note that when $\mu < r$ we have

$$P(\tau < T \mid F_t) > P^*(\tau < T \mid F_t),$$

whereas when $\mu > r$ we get

$$P(\tau < T \mid F_t) < P^*(\tau < T \mid F_t),$$

as illustrated in the next Figure 6.1.
Fig. 6.1: Function $x \mapsto \Phi\left(\Phi^{-1}(x) - (\mu - r)\sqrt{T}/\sigma\right)$ for $\mu > r$, $\mu = r$, and $\mu < r$.

The discounted expected cash flow $e^{-(T-t)r} \mathbb{E}^*[(S_T - K)^+ | \mathcal{F}_t]$ received by the equity holder can be estimated at time $t \in [0, T]$ as the price of a European call option from the Black-Scholes formula

$$e^{-(T-t)r} \mathbb{E}^*[(S_T - K)^+ | \mathcal{F}_t] = S_t \Phi\left(\frac{(r + \sigma^2/2)(T-t) + \log(S_t/K)}{\sigma \sqrt{T-t}}\right) - Ke^{-(T-t)r} \Phi\left(\frac{(r - \sigma^2/2)(T-t) + \log(S_t/K)}{\sigma \sqrt{T-t}}\right), \quad 0 \leq t \leq T.$$

In the following proposition we price at time $t \in [0, T]$ the amount $\min(S_T, K)$ received by the bond holder (or junior creditor) at maturity, based on the recovery value $S_T$. This price can interpreted at the price $P(t,T)$ at time $t \in [0, T]$ of a default bond with face value $1$, maturity $T$ and recovery value $\min(S_T/K, 1)$.

**Proposition 6.2.** The amount received by the bond holder (or junior creditor) at maturity is priced at time $t \in [0, T]$ as

$$e^{-(T-t)r} \mathbb{E}^*\left[\min(S_T, K) | \mathcal{F}_t\right] = Ke^{-(T-t)r}\Phi(d^-_r) - S_t\Phi(-d^+_r), \quad 0 \leq t \leq T.$$

**Proof.** Using the Black-Scholes put option pricing formula we have

$$e^{-(T-t)r} \mathbb{E}^*\left[\min(S_T, K) | \mathcal{F}_t\right] = e^{-(T-t)r}K - e^{-(T-t)r} \mathbb{E}^*\left[(K - S_T)^+ | \mathcal{F}_t\right]$$

$$= e^{-(T-t)r}K - \left(S_t\Phi(-d^+_r) - Ke^{-(T-t)r}\Phi(-d^-_r)\right)$$

$$= Ke^{-(T-t)r}\Phi(d^-_r) - S_t\Phi(-d^+_r).$$

Writing

$$P(t,T) = e^{-(T-t)y_{t,T}}$$

$$= \frac{1}{K} e^{-(T-t)r} \mathbb{E}^*\left[\min(S_T, K) | \mathcal{F}_t\right]$$

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= e^{-(T-t)r} \Phi(d'_-) - \frac{S_t}{K} \Phi(-d'_+) ,

gives the default bond yield

\[ y_{t,T} = -\frac{1}{T-t} \log(P(t,T)) \]

\[ = -\frac{1}{T-t} \log \left( e^{-(T-t)r} \mathbb{E}^* \left[ \min \left( \frac{1}{K}, \frac{S_T}{K} \right) \bigg| \mathcal{F}_t \right] \right) \]

\[ = r - \frac{1}{T-t} \log \left( \mathbb{E}^* \left[ \min \left( \frac{1}{K}, \frac{S_T}{K} \right) \bigg| \mathcal{F}_t \right] \right) \]

\[ = r - \frac{1}{T-t} \log \left( \frac{1}{K} \mathbb{E}^* \left[ \min \left( K, \frac{S_T}{K} \right) \bigg| \mathcal{F}_t \right] \right) \]

\[ = r - \frac{1}{T-t} \log \left( \Phi(d'_-) - \frac{S_t}{K} e^{(T-t)r} \Phi(-d'_+) \right) > r. \]

### 6.2 Black-Cox Model

In the Black and Cox (1976) model the firm has to maintain an account balance above the level \( K \) throughout time, therefore default occurs at the first time the process \( S_t \) hits the level \( K \), cf. § 4.2 of Grasselli and Hurd (January 3, 2010). The default time \( \tau_K \) is therefore the first hitting time

\[ \tau_K := \inf \left\{ t \geq 0 : S_t := S_0 e^{\sigma B_t + (\mu - \sigma^2/2)t} \leq K \right\}, \]

of the level \( K \) by

\[ (S_t)_{t \in \mathbb{R}} = (S_0 e^{\sigma B_t + (\mu - \sigma^2/2)t})_{t \in \mathbb{R}}. \]

after starting from \( S_0 > K \).

**Proposition 6.3.** The probability distribution function of the default time \( \tau_K \) is given by

\[ P(\tau_K \leq T) = P(S_T \leq K) + \left( S_0 \right)^{1-2\mu/\sigma^2} \Phi \left( \frac{\log(K/S_0) + (\mu - \sigma^2/2)T}{\sigma \sqrt{T}} \right), \]

with \( S_0 \geq K \).

**Proof.** By e.g. Corollary 7.2.2 and pages 297-299 of Shreve (2004), or from Relation (8.7) in Privault (2014), we have
\[ \mathbb{P}(\tau_K \leq T) = \mathbb{P}\left( \min_{t \in [0,T]} S_t \leq K \right) \]

\[ = \mathbb{P}\left( \min_{t \in [0,T]} e^{\sigma B_t + (\mu - \sigma^2/2)t} \leq \frac{K}{S_0} \right) \]

\[ = \mathbb{P}\left( \min_{t \in [0,T]} \left( B_t + \frac{(\mu - \sigma^2/2)t}{\sigma} \right) \leq \frac{1}{\sigma} \log \left( \frac{K}{S_0} \right) \right) \]

\[ = \Phi \left( \frac{\log(K/S_0) - (\mu - \sigma^2/2)T}{\sigma \sqrt{T}} \right) \]

\[ + \left( \frac{S_0}{K} \right)^{1 - 2\mu/\sigma^2} \Phi \left( \frac{\log(K/S_0) + (\mu - \sigma^2/2)T}{\sigma \sqrt{T}} \right) \quad (6.3) \]

with \( S_0 \geq K \).

The cash flow

\[ (S_T - K)^+ \mathbb{1}_{\{\tau_K > T\}} = (S_T - K)^+ \mathbb{1}_{\left\{ \min_{t \in [0,T]} S_t > K \right\}} \]

received at maturity \( T \) by the equity holder can be priced at time \( t \in [0, T] \) as a down-and-out barrier call option with strike price \( K \) and barrier level \( K \) is priced in the next proposition.

**Proposition 6.4.** We have

\[ \mathbb{E}^* \left( (S_T - K)^+ \mathbb{1}_{\left\{ \min_{0 \leq t \leq T} S_t > K \right\}} \bigg| \mathcal{F}_t \right) = \mathbb{1}_{\left\{ \min_{t \in [0,T]} S_t > B \right\}} g(t, S_t), \]

\[ t \in [0, T], \text{ where} \]

\[ g(t, S_t) = BS_c(S_t, r, T - t, \sigma, K) - S_t \left( \frac{K}{S_t} \right)^{2r/\sigma^2} BS_c(K/S_t, r, T - t, \sigma, 1), \]

\[ 0 \leq t \leq T. \]

**Proof.** By e.g. Chapter 8 in Privault (2014), as

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\[
\mathbb{E}^* \left[ (S_T - K)^+ | \mathcal{F}_t \right] = \mathbb{1} \left\{ \min_{0 \leq t \leq T} S_t > K \right\}
\]

\[
= \mathbb{1} \left\{ \min_{t \in [0, T]} S_t > B \right\} g(t, S_t),
\]

t \in [0, T], where

\[
g(t, S_t) = S_t \Phi \left( \frac{\delta^+_T - (S_t)}{K} \right) - e^{-(T-t)r} K \Phi \left( \frac{\delta^-_T - (S_t)}{K} \right)
\]

\[
- K \left( \frac{K}{S_t} \right)^{2r/\sigma^2} \Phi \left( \frac{\delta^+_T - (K)}{S_t} \right) + e^{-(T-t)r} K \left( \frac{S_t}{K} \right)^{1-2r/\sigma^2} \Phi \left( \frac{\delta^-_T - (K)}{S_t} \right)
\]

\[
= BS_e(S_t, r, T - t, \sigma, K) - K \left( \frac{S_t}{K} \right)^{-2r/\sigma^2} \Phi \left( \frac{\delta^+_T - (K)}{S_t} \right) + e^{-(T-t)r} S_t \left( \frac{S_t}{K} \right)^{-2r/\sigma^2} \Phi \left( \frac{\delta^-_T - (K)}{S_t} \right)
\]

\[
= BS_e(S_t, r, T - t, \sigma, K) - S_t \left( \frac{K}{S_t} \right)^{2r/\sigma^2} BS_e(K/S_t, r, T - t, \sigma, 1),
\]

0 \leq t \leq T, cf. Relation (8.12) and Exercise 8.2 in Privault (2014). \( \Box \)

For \( t \geq 0 \), taking now

\[
\tau_K := \inf \left\{ u \in [t, \infty) : S_u := S_0 e^{\sigma B_u + (\mu-\sigma^2/2)u} \leq K \right\},
\]

the recovery value received by the bond holder at time \( \min(\tau_K, T) \) is \( K \), and it can be priced as in the next proposition.

**Proposition 6.5.** After discounting from time \( \min(\tau_K, T) \) to time \( t \in [0, T] \), we have

\[
\mathbb{E}^* \left[ Ke^{-(\min(\tau_K, T)-t)r} | \mathcal{F}_t \right]
\]

\[
= \mathbb{1}_{\{\tau_K \geq t\}} \int_t^T e^{-(u-t)r} d\mathbb{P}(\tau_K \leq u | \mathcal{F}_t) + Ke^{-(T-t)r} \mathbb{P}^*(\tau_K > T | \mathcal{F}_t).
\]

**Proof.** We have

\[
\mathbb{E}^* \left[ Ke^{-(\min(\tau_K, T)-t)r} | \mathcal{F}_t \right]
\]

\[
= \mathbb{E}^* \left[ Ke^{-(\tau_K-t)r} \mathbb{1}_{\{t \leq \tau_K \leq T\}} \right| \mathcal{F}_t) + Ke^{-(T-t)r} \mathbb{1}_{\{\tau_K > T\}} | \mathcal{F}_t]
\]

\[
= K \mathbb{E}^* \left[ e^{-(\tau_K-t)r} \mathbb{1}_{\{t \leq \tau_K \leq T\}} \right| \mathcal{F}_t] + Ke^{-(T-t)r} \mathbb{P}^*(\tau_K > T | \mathcal{F}_t)
\]

\[
= K \mathbb{1}_{\{\tau_K \geq t\}} \mathbb{E}^* \left[ e^{-(\tau_K-t)r} \mathbb{1}_{\{t \leq \tau_K \leq T\}} \right| \mathcal{F}_t] + Ke^{-(T-t)r} \mathbb{P}^*(\tau_K > T | \mathcal{F}_t)
\]

\[
\bigcirc
\]

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\[
= K \mathbb{1}_{\{\tau_K \geq t\}} \int_t^T e^{-(u-t)r} d\mathbb{P}(\tau_K \leq u \mid F_t) + Ke^{-(T-t)r} \mathbb{P}^*(\tau_K > T \mid F_t),
\]

\[0 \leq t \leq T. \]

The probabilities \(\mathbb{P}^*(\tau_K \leq u \mid F_t)\) and \(\mathbb{P}^*(\tau_K > T \mid F_t) = 1 - \mathbb{P}^*(\tau_K \leq T \mid F_t)\) above can be computed from (6.3) as

\[
\mathbb{P}^*(\tau_K \leq u \mid F_t) = \Phi \left( \frac{\log(K/S_t) - (r - \sigma^2/2)(u-t)}{\sigma \sqrt{u-t}} \right)
+ \left(\frac{S_t}{K}\right)^{1-2r/\sigma^2} \Phi \left( \frac{\log(S_t/K) + (r - \sigma^2/2)(u-t)}{\sigma \sqrt{u-t}} \right)
\]

\[
= \mathbb{P}(S_u \leq K \mid F_t) + \left(\frac{S_t}{K}\right)^{1-2r/\sigma^2} \Phi \left( \frac{\log(S_t/K) + (r - \sigma^2/2)(u-t)}{\sigma \sqrt{u-t}} \right),
\]

with \(S_t \geq K\) and \(u > t\), from which the probability density function of the hitting time \(\tau_K\) can be estimated by differentiation with respect to \(u > t\).

Note also that we have

\[
\mathbb{P}^*(\tau_K < \infty \mid F_t) = \lim_{u \to \infty} \mathbb{P}^*(\tau_K \leq u \mid F_t)
= \begin{cases} 
\left(\frac{K}{S_t}\right)^{-1+2r/\sigma^2} & \text{if } r > \sigma^2/2 \\
1 & \text{if } r \leq \sigma^2/2.
\end{cases}
\]

### 6.3 Correlated Default Times

In order to model correlated default and possible “domino effects”, one can regard two given default times \(\tau_1\) and \(\tau_2\) are correlated random variables.

Namely, given \(\tau_1\) and \(\tau_2\) two default times we can consider the correlation

\[
\rho = \frac{\text{Cov}(\tau_1, \tau_2)}{\sqrt{\text{Var}[\tau_1] \text{Var}[\tau_2]}} \in [-1, 1].
\]

When trying to build a dependence structure for the default times \(\tau_1\) and \(\tau_2\), the idea of Li (2000) is to use the normalized Gaussian copula \(C_{\Sigma}(x, y)\), with

\[
\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},
\]

with correlation parameter \(\rho \in [-1, 1]\), and to model the joint default probability \(\mathbb{P}(\tau_1 \leq T \text{ and } \tau_2 \leq T)\) as

\[
\mathbb{P}(\tau_1 \leq T \text{ and } \tau_2 \leq T) := C_{\Sigma}(\mathbb{P}(\tau_1 \leq T), \mathbb{P}(\tau_2 \leq T)),
\]
where $C_\Sigma$ is given by (5.4). Given two default events $A = \{\tau_1 \leq T\}$ and $B = \{\tau_2 \leq T\}$ with probabilities

$$P(\tau_1 \leq T) = 1 - \exp\left(-\int_0^T \lambda_1(s)ds\right) \text{ and } P(\tau_2 \leq T) = 1 - \exp\left(-\int_0^T \lambda_2(s)ds\right)$$

we can also define the default correlation $\rho^D \in [-1, 1]$ as

$$\rho^D = \frac{P(A \cap B) - P(A)P(B)}{\sqrt{P(A)(1 - P(A))\sqrt{P(B)(1 - P(B))}}} = \frac{C_\Sigma(P(\tau_1 \leq T), P(\tau_2 \leq T)) - P(\tau_1 \leq T)P(\tau_2 \leq T)}{\sqrt{P(\tau_1 \leq T)(1 - P(\tau_1 \leq T))\sqrt{P(\tau_2 \leq T)(1 - P(\tau_2 \leq T))}}}.$$ 

In this case, the default correlation $\rho^D$ in (6.4) can be written as

When the default probabilities are specified in the Merton model of credit risk as

$$P(\tau_i \leq T) = P(S_T < K)$$

$$= P\left(e^{\sigma_i B_T + (\mu_i - \sigma_i^2/2)T} < \frac{K}{S_0}\right)$$

$$= P\left(B_T \leq \frac{(\mu_i - \sigma_i^2/2)T}{\sigma_i} + \frac{1}{\sigma_i} \log \frac{K}{S_0}\right)$$

$$= \Phi\left(\frac{\log(K/S_0) - (\mu_i - \sigma_i^2/2)T}{\sigma_i\sqrt{T}}\right), \quad i = 1, 2,$$

where

$$(A^i_t)_{t \in \mathbb{R}^+} := (S_0 e^{\sigma_i B_t + (\mu_i - \sigma_i^2/2)t})_{t \in \mathbb{R}^+}, \quad i = 1, 2,$$

the default correlation $\rho^D$ becomes

$$\rho^D = \frac{P(\tau_1 \leq T \text{ and } \tau_2 \leq T) - P(\tau_1 \leq T)P(\tau_2 \leq T)}{\sqrt{P(\tau_1 \leq T)(1 - P(\tau_1 \leq T))\sqrt{P(\tau_2 \leq T)(1 - P(\tau_2 \leq T))}}} = \Phi_\Sigma\left(\frac{\log(S_0/K) + (\mu_1 - \sigma_1^2/2)T}{\sigma_1\sqrt{T}}, \frac{\log(S_0/K) + (\mu_2 - \sigma_2^2/2)T}{\sigma_2\sqrt{T}}\right) - P(\tau_1 \leq T)P(\tau_2 \leq T)$$

$\Phi_\Sigma$.

In Li (2000) it was suggested to use a single average correlation estimate, see (8.1) page 82 of the Credit Metrics™ Technical Document Gupton et al. (1997), and also the Appendix F therein.
It is worth noting that the outcomes of this methodology have been discussed in a number of magazine articles in recent years, to name a few:

“Recipe for disaster: the formula that killed Wall Street”, *Wired Magazine*, by F. Salmon (2009);

“The formula that felled Wall Street”, *Financial Times Magazine*, by S. Jones (2009);

“Formula from hell”, *Forbes.com*, by S. Lee (2009), see also [here](https://www.ntu.edu.sg/home/nprivault/indext.html).

On the other hand, a more proper definition of the default correlation $\rho^D$ should be

$$
\rho^D := \frac{\mathbb{P}(\tau_1 \leq T \text{ and } \tau_2 \leq T) - \mathbb{P}(\tau_1 \leq T)\mathbb{P}(\tau_2 \leq T)}{\sqrt{\mathbb{P}(\tau_1 \leq T)(1 - \mathbb{P}(\tau_1 \leq T))}\sqrt{\mathbb{P}(\tau_2 \leq T)(1 - \mathbb{P}(\tau_2 \leq T))}},
$$

which requires the actual computation of the joint default probability $\mathbb{P}(\tau_1 \leq T \text{ and } \tau_2 \leq T)$. An exact expression for this joint default probability in the first passage time Black-Cox model, and the associated correlation, have been recently obtained in *Li and Krehbiel (2016)*.

**Multiple default times**

Consider now a sequence $(\tau_k)_{k=1,2,\ldots,n}$ of random default times and, for more flexibility, a standardized random variable $M$ with probability density function $\phi(m)$ and variance $\text{Var}[M] = 1$.

As in the *Merton (1974)* model, cf. § 6.1, a common practice, see *Vašiček (1987), Gibson (2004), Hull and White (2004)* is to parametrize the default probability associated to each $\tau_k$ by the conditioning

$$
\mathbb{P}(\tau_k \leq T \mid M = m) = \Phi\left(\frac{\Phi^{-1}(\mathbb{P}(\tau_k \leq T)) - a_km}{\sqrt{1 - a_k^2}}\right), \quad k = 1, 2, \ldots, n,
$$

see (6.2), where $a_k \in (-1, 1), \ k = 1, 2, \ldots, n$. Note that we have

$$
\mathbb{P}(\tau_k \leq T) = \int_{-\infty}^{\infty} \mathbb{P}(\tau_k \leq T \mid M = m)\phi(m)dm
$$

$$
= \int_{-\infty}^{\infty} \Phi\left(\frac{\Phi^{-1}(\mathbb{P}(\tau_k \leq T)) - a_km}{\sqrt{1 - a_k^2}}\right)\phi(m)dm, \quad (6.6)
$$
and $\phi(m)$ can be typically chosen as a standard normal Gaussian probability density function.

Next, we present a dependence structure which implements of the Gaussian copula correlation method Li (2000) in the case of multiple default times.

**Definition 6.6.** Given $n$ Gaussian samples $X_1, X_2, \ldots, X_n$ defined as

$$X_k := a_k M + \sqrt{1-a_k^2} Z_k, \quad k = 1, 2, \ldots, n, \quad (6.7)$$

conditionally to $M$, where $Z_1, Z_2, \ldots, Z_n$ are normal random variables with same cumulative distribution function $\Phi$, independent of $M$, we let the correlated default times $(\tau_1, \ldots, \tau_n)$ be defined as

$$\tau_k := F_{\tau_k}^{-1}(\Phi(X_k)), \quad k = 1, 2, \ldots, n, \quad (6.8)$$

In the next proposition we compute the joint distribution of the default times $(\tau_1, \ldots, \tau_n)$ according to the above dependence structure.

**Proposition 6.7.** The default times $(\tau_k)_{k=1,2,\ldots,n}$ have the joint distribution

$$P(\tau_1 \leq y_1, \ldots, \tau_n \leq y_n) = C(P(\tau_1 \leq y_1), \ldots, P(\tau_n \leq y_n)), \quad (6.9)$$

where

$$C(x_1, \ldots, x_n) := \int_{-\infty}^{\infty} \Phi \left( \frac{\Phi^{-1}(x_1) - a_1 m}{\sqrt{1-a_1^2}} \right) \cdots \Phi \left( \frac{\Phi^{-1}(x_n) - a_n m}{\sqrt{1-a_n^2}} \right) \phi(m) dm,$$

$x_1, x_2, \ldots, x_n \in [0, 1]$, is a Gaussian copula on $[0, 1]^n$ with covariance matrix

$$\Sigma = \begin{bmatrix} 1 & a_1 a_2 & \cdots & a_1 a_{n-1} & a_1 a_n \\ a_2 a_1 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ a_{n-1} a_1 & a_{n-1} a_2 & \cdots & 1 \\ a_n a_1 & a_n a_2 & \cdots & a_n a_{n-1} & 1 \end{bmatrix}. \quad (6.9)$$

**Proof.** We start by recovering the conditional distribution (6.5) as follows:

$$P(\tau_k \leq T \mid M = m) = P(F_{\tau_k}^{-1}(\Phi(X_k)) \leq T \mid M = m)$$

$$= P(\Phi(X_k) \leq F_{\tau_k}(T) \mid M = m)$$

$$= P(X_k \leq \Phi^{-1}(F_{\tau_k}(T)) \mid M = m)$$

$$= P \left( a_k m + \sqrt{1-a_k^2} Z_k \leq \Phi^{-1}(F_{\tau_k}(T)) \right)$$

\( \blacksquare \)
where the function

\[ \Phi^{-1}(F_{\tau_k}(T)) - a_k m \]

is independent random variables, we can compute the joint distribution

\[ P(\tau \leq T) \]

Note that the above recovers the correct marginal distributions (6.6), i.e. we have

\[ P(\tau_k \leq y_k) = P(\tau_1 \leq \infty, \ldots, \tau_k - 1 \leq \infty, \tau_k \leq y_k, \tau_{k+1} \leq \infty, \ldots, \tau_n \leq \infty) \]

\[ = \int_{-\infty}^{\infty} \Phi \left( \frac{\Phi^{-1}(P(\tau_k \leq y_k)) - a_k m}{\sqrt{1 - a_k^2}} \right) \phi(m) dm \]

\[ = \int_{-\infty}^{\infty} P(\tau_k \leq T | M = m) \phi(m) dm, \quad k = 1, 2, \ldots, n. \]

Knowing that, given the sample \( M = m \), the default times \( \tau_k, k = 1, 2, \ldots, n \), are independent random variables, we can compute the joint distribution

\[ P(\tau_1 \leq y_1, \ldots, \tau_n \leq y_n | M = m) \]

\[ = P(\tau_1 \leq y_1 | M = m) \times \cdots \times P(\tau_n \leq y_n | M = m), \]

conditionally to \( M = m \). This yields

\[ P(\tau_1 \leq y_1, \ldots, \tau_n \leq y_n) = \int_{-\infty}^{\infty} P(\tau_1 \leq y_1, \ldots, \tau_n \leq y_n | M = m) \phi(m) dm \]

\[ = \int_{-\infty}^{\infty} P(\tau_1 \leq y_1 | M = m) \cdots P(\tau_n \leq y_n | M = m) \phi(m) dm \]

\[ = \int_{-\infty}^{\infty} \Phi \left( \frac{\Phi^{-1}(P(\tau_1 \leq y_1)) - a_1 m}{\sqrt{1 - a_1^2}} \right) \cdots \Phi \left( \frac{\Phi^{-1}(P(\tau_n \leq y_n)) - a_n m}{\sqrt{1 - a_n^2}} \right) \phi(m) dm. \]

In other words, we have

\[ P(\tau_1 \leq y_1, \ldots, \tau_n \leq y_n) = C(P(\tau_1 \leq y_1), \ldots, P(\tau_n \leq y_n)), \]

where the function

\[ C(x_1, \ldots, x_n) \]

\[ := \int_{-\infty}^{\infty} \Phi \left( \frac{\Phi^{-1}(x_1) - a_1 m}{\sqrt{1 - a_1^2}} \right) \cdots \Phi \left( \frac{\Phi^{-1}(x_n) - a_n m}{\sqrt{1 - a_n^2}} \right) \phi(m) dm, \]

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$x_1, x_2, \ldots, x_n \in [0, 1]$, is a Gaussian copula on $[0, 1]^n$, built as

$$
C(x_1, \ldots, x_n) = F(\Phi^{-1}(x_1), \ldots, \Phi^{-1}(x_n)),
$$

from the Gaussian cumulative distribution function

$$
F(x_1, \ldots, x_n) := \int_{-\infty}^{\infty} \Phi \left( \frac{x_1 - a_1 m}{\sqrt{1 - a_1^2}} \right) \cdots \Phi \left( \frac{x_n - a_n m}{\sqrt{1 - a_n^2}} \right) \phi(m) dm
$$

$$
= \int_{-\infty}^{\infty} \mathbb{P} \left( Z_1 \leq \frac{x_1 - a_1 m}{\sqrt{1 - a_1^2}} \right) \cdots \mathbb{P} \left( Z_n \leq \frac{x_n - a_n m}{\sqrt{1 - a_n^2}} \right) \phi(m) dm
$$

$$
= \int_{-\infty}^{\infty} \mathbb{P}(X_1 \leq x_1, \ldots, X_n \leq x_n | M = m) \phi(m) dm
$$

$$
= \mathbb{P}(X_1 \leq x_1, \ldots, X_n \leq x_n), \quad 0 \leq x_1, x_2, \ldots, x_n \leq 1,
$$
of the vector $(X_1, \ldots, X_n)$, with covariance matrix given by (6.9). \qed

### Exercises

**Exercise 6.1** Compute the conditional probability density function of the default time $\tau$ defined in (6.1).

**Exercise 6.2** Credit Default Contract. The assets of a company are modeled using a geometric Brownian motion $(S_t)_{t \in \mathbb{R}_+}$ with drift $r > 0$ under the risk-neutral probability measure $\mathbb{P}^\blackstar$. A Credit Default Contract pays $1 as soon as the asset $S_t$ hits a level $K > 0$. Price this contract at time $t > 0$ assuming that $S_t > K$.

**Exercise 6.3**

a) Check that the vector $(X_1, X_2, \ldots, X_n)$ defined in (6.7) has the covariance matrix given by (6.9).

b) Show that the vector $(X_1, X_2, \ldots, X_n)$, with covariance matrix (6.9) has standard Gaussian marginals.

c) By computing explicitly the probability density function of $(X_1, \ldots, X_n)$, recover the fact that it is a jointly Gaussian random vector with covariance matrix (6.9).

**Exercise 6.4** Compute the inverse $\Sigma^{-1}$ of the covariance matrix (6.9) in case $n = 2$. 

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