Chapter 7
Reduced-Form Approach to Credit Risk

This chapter deals with the reduced-form approach to credit risk, which relies on a failure rate process and an exogeneous random variable, resulting into an enlarged filtration that incorporates the knowledge of the default event.

7.1 Survival Probabilities

Given \( t > 0 \), let \( \mathbb{P}(\tau > t) \) denote the probability that a random system with lifetime \( \tau \) survives at least \( t \) years. Assuming that survival probabilities \( \mathbb{P}(\tau > t) \) are strictly positive for all \( t > 0 \), we can compute the conditional probability for that system to survive up to time \( T \), given that it was still functioning at time \( t \in [0, T] \), as

\[
\mathbb{P}(\tau > T \mid \tau > t) = \frac{\mathbb{P}(\tau > T \text{ and } \tau > t)}{\mathbb{P}(\tau > t)} = \frac{\mathbb{P}(\tau > T)}{\mathbb{P}(\tau > t)}, \quad 0 \leq t \leq T,
\]

with

\[
\mathbb{P}(\tau \leq T \mid \tau > t) = 1 - \mathbb{P}(\tau > T \mid \tau > t)
\]

\[
= \frac{\mathbb{P}(\tau > t) - \mathbb{P}(\tau > T)}{\mathbb{P}(\tau > t)}
\]

\[
= \frac{\mathbb{P}(\tau \leq T) - \mathbb{P}(\tau \leq t)}{\mathbb{P}(\tau > t)}
\]

\[
= \frac{\mathbb{P}(t < \tau \leq T)}{\mathbb{P}(\tau > t)}, \quad 0 \leq t \leq T,
\]
and the conditional survival probability distribution

\[
P(\tau \in dx \mid \tau > t) = P(x < \tau \leq x + dx \mid \tau > t) \\
= P(\tau \leq x + dx \mid \tau > t) - P(\tau \leq x \mid \tau > t) \\
= \frac{P(\tau \leq x + dx) - P(\tau \leq x)}{P(\tau > t)} \\
= \frac{1}{P(\tau > t)} dP(\tau \leq x) \\
= -\frac{1}{P(\tau > t)} dP(\tau > x), \quad x > t.
\]

Such survival probabilities are typically found in life or mortality tables:

<table>
<thead>
<tr>
<th>Age t</th>
<th>( P(\tau \leq t + 1 \mid \tau &gt; t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.0894%</td>
</tr>
<tr>
<td>30</td>
<td>0.1008%</td>
</tr>
<tr>
<td>40</td>
<td>0.2038%</td>
</tr>
<tr>
<td>50</td>
<td>0.4458%</td>
</tr>
<tr>
<td>60</td>
<td>0.9827%</td>
</tr>
</tbody>
</table>

Table 7.1: Mortality table.

**Proposition 7.1.** The failure rate function, defined as

\[
\lambda(t) := \frac{P(\tau \leq t + dt \mid \tau > t)}{dt},
\]

satisfies

\[
P(\tau > t) = \exp \left( - \int_0^t \lambda(u)du \right), \quad t \in \mathbb{R}_+.
\] (7.1)

*Proof.* We have

\[
\lambda(t) := \frac{P(\tau \leq t + dt \mid \tau > t)}{dt} \\
= \frac{1}{P(\tau > t)} \frac{P(t < \tau \leq t + dt)}{dt} \\
= \frac{1}{P(\tau > t)} \frac{P(\tau > t) - P(\tau > t + dt)}{dt} \\
= -\frac{d}{dt} \log P(\tau > t) \\
= -\frac{1}{P(\tau > t)} \frac{d}{dt} P(\tau > t), \quad t > 0,
\]
and the differential equation
\[ \frac{d}{dt} \mathbb{P}(\tau > t) = -\lambda(t) \mathbb{P}(\tau > t), \]
which can be solved as in (7.1) under the initial condition \( \mathbb{P}(\tau > 0) = 1 \).

Proposition 7.1 allows us to rewrite the (conditional) survival probability as
\[ \mathbb{P}(\tau > T \mid \tau > t) = \frac{\mathbb{P}(\tau > T)}{\mathbb{P}(\tau > t)} = \exp \left( -\int_t^T \lambda(u) du \right), \quad 0 \leq t \leq T, \]
with
\[ \mathbb{P}(\tau > t + h \mid \tau > t) = e^{-\lambda(t)h} \simeq 1 - \lambda(t)h, \quad [h \downarrow 0], \]
and
\[ \mathbb{P}(\tau \leq t + h \mid \tau > t) = 1 - e^{-\lambda(t)h} \simeq \lambda(t)h, \quad [h \downarrow 0], \]
as \( h \) tends to 0. When the failure rate \( \lambda(t) = \lambda > 0 \) is a constant function of time, Relation (7.1) shows that
\[ \mathbb{P}(\tau > T) = e^{-\lambda T}, \quad T \geq 0, \]
i.e. \( \tau \) has the exponential distribution with parameter \( \lambda \). Note that given \( (\tau_n)_{n \geq 1} \) a sequence of i.i.d. exponentially distributed random variables, letting
\[ T_n = \tau_1 + \tau_2 + \cdots + \tau_n, \quad n \geq 1, \]
defines the sequence of jump times of a standard Poisson process with intensity \( \lambda > 0 \).

### 7.2 Stochastic Default

When the random time \( \tau \) is a stopping time with respect to \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\) we have
\[ \{\tau > t\} \in \mathcal{F}_t, \quad t \in \mathbb{R}_+, \]
i.e. the knowledge of whether default or bankruptcy has already occurred at time \( t \) is contained in \( \mathcal{F}_t, t \in \mathbb{R}_+ \), cf. e.g. Section ?? of Privault (2014). As a consequence, we can write
\[ \mathbb{P}(\tau > t \mid \mathcal{F}_t) = \mathbb{E} \left[ 1_{\{\tau > t\}} \mid \mathcal{F}_t \right] = 1_{\{\tau > t\}}, \quad t \in \mathbb{R}_+. \]
In the sequel we will not assume that \( \tau \) is an \( \mathcal{F}_t \)-stopping time, and by analogy with (7.1) we will write \( \mathbb{P}(\tau > t \mid \mathcal{F}_t) \) as

\[ \mathbb{P}(\tau > t \mid \mathcal{F}_t) \]

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https://www.ntu.edu.sg/home/nprivault/index.html
\[ p(\tau > t \mid F_t) = \exp \left( - \int_0^t \lambda_u du \right), \quad t > 0, \quad (7.2) \]

where the failure rate function \((\lambda_t)_{t \in \mathbb{R}_+}\) is modeled as a random process adapted to a filtration \((F_t)_{t \in \mathbb{R}_+}\).

The process \((\lambda_t)_{t \in \mathbb{R}_+}\) can also be chosen among the classical mean-reverting diffusion processes, including jump-diffusion processes. In Lando (1998), the process \((\lambda_t)_{t \in \mathbb{R}_+}\) is constructed as

\[ \lambda_t = h(X_t), \quad t \in \mathbb{R}_+, \]

where \(h\) is a nonnegative function and \((X_t)_{t \in \mathbb{R}_+}\) is a stochastic process generating the filtration \((F_t)_{t \in \mathbb{R}_+}\). The default time \(\tau\) is then defined as

\[ \tau := \inf \left\{ t \in \mathbb{R}_+ : \int_0^t h(X_u) du \geq L \right\}, \]

where \(L\) is an exponentially distributed random variable independent of \((F_t)_{t \in \mathbb{R}_+}\). In this case, as \(\tau\) is not an \((F_t)_{t \in \mathbb{R}_+}\)-stopping time, we have

\[ p(\tau > t \mid F_t) = p \left( \int_0^t h(X_u) du \geq L \mid F_t \right) = \exp \left( - \int_0^t h(X_u) du \right) = \exp \left( - \int_0^t \lambda_u du \right), \quad t \in \mathbb{R}_+. \]

**Definition 7.2.** Let \((G_t)_{t \in \mathbb{R}_+}\) be the filtration defined by \(G_\infty := F_\infty \vee \sigma(\tau)\) and

\[ G_t := \{ B \in G_\infty : \exists A \in F_t \text{ such that } A \cap \{ \tau > t \} = B \cap \{ \tau > t \} \}, \quad (7.3) \]

with \(F_t \subset G_t, \ t \in \mathbb{R}_+.\)

In other words, \(G_t\) contains insider information on whether default at time \(\tau\) has occurred or not before time \(t\), and \(\tau\) is a \((G_t)_{t \in \mathbb{R}_+}\)-stopping time. Note that this information on \(\tau\) may not be available to a generic user who has only access to the smaller filtration \((F_t)_{t \in \mathbb{R}_+}\). The next key Lemma 7.3 allows us to price a contingent claim given information in the larger filtration \((G_t)_{t \in \mathbb{R}_+}\), by only using information in \((F_t)_{t \in \mathbb{R}_+}\) and factoring in the default rate factor

\[ \exp \left( - \int_t^T \lambda_u du \right). \]

**Lemma 7.3.** (Guo et al. (2007)) For any \(F_T\)-measurable integrable random variable \(F\) we have

\[ \mathbb{E} \left[ F \mathbb{1}_{\{ \tau > T \}} \mid G_t \right] = \mathbb{1}_{\{ \tau > t \}} \mathbb{E} \left[ F \exp \left( - \int_t^T \lambda_u du \right) \mid F_t \right]. \]

**Proof.** By (7.2) we have
\[
\frac{\mathbb{P}(\tau > T \mid \mathcal{F}_T)}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} = \frac{e^{-\int_0^T \lambda_u du}}{e^{-\int_0^t \lambda_u du}} = \exp \left( - \int_t^T \lambda_u du \right),
\]

hence, since \( F \) is \( \mathcal{F}_T \)-measurable,

\[
\mathbb{1}_{\{\tau > t\}} \mathbb{E} \left[ F \exp \left( - \int_t^T \lambda_u du \right) \mid \mathcal{F}_t \right] = \mathbb{1}_{\{\tau > t\}} \mathbb{E} \left[ F \frac{\mathbb{P}(\tau > T \mid \mathcal{F}_T)}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} \mid \mathcal{F}_t \right]
\]

\[
= \mathbb{1}_{\{\tau > t\}} \mathbb{E} \left[ F \mathbb{E}[\mathbb{1}_{\{\tau > T\}} \mid \mathcal{F}_T] \mid \mathcal{F}_t \right]
\]

\[
= \mathbb{1}_{\{\tau > t\}} \mathbb{E} \left[ \mathbb{E}[F\mathbb{1}_{\{\tau > T\}} \mid \mathcal{F}_T] \mid \mathcal{F}_t \right]
\]

\[
= \mathbb{1}_{\{\tau > t\}} \mathbb{E} \left[ F \mathbb{1}_{\{\tau > T\}} \mid \mathcal{F}_t \right]
\]

\[
= \mathbb{1}_{\{\tau > t\}} \mathbb{E} \left[ F \mathbb{1}_{\{\tau > T\}} \mid \mathcal{G}_t \right], \quad 0 \leq t \leq T.
\]

In the last step of the above argument we used the key relation

\[
\mathbb{1}_{\{\tau > t\}} \mathbb{E} \left[ F \mathbb{1}_{\{\tau > T\}} \mid \mathcal{G}_t \right] = \frac{\mathbb{1}_{\{\tau > t\}}}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} \mathbb{E} \left[ F \mathbb{1}_{\{\tau > T\}} \mid \mathcal{F}_t \right],
\]

cf. Relation (75.2) in § XX-75 page 186 of Dellacherie et al. (1992), Theorem VI-3-14 page 371 of Protter (2004), and Lemma 3.1 of Elliott et al. (2000), under the probability measure \( \mathbb{P}_{\mathcal{F}_t} \), \( 0 \leq t \leq T \). Indeed, according to (7.3), for any \( B \in \mathcal{G}_t \) we have, for some event \( A \in \mathcal{F}_t \),

\[
\mathbb{E} \left[ \mathbb{1}_B \mathbb{1}_{\{\tau > t\}} F \mathbb{1}_{\{\tau > T\}} \right] = \mathbb{E} \left[ \mathbb{1}_{B \cap \{\tau > t\}} F \mathbb{1}_{\{\tau > T\}} \right]
\]

\[
= \mathbb{E} \left[ \mathbb{1}_{A \cap \{\tau > t\}} F \mathbb{1}_{\{\tau > T\}} \right]
\]

\[
= \mathbb{E} \left[ \mathbb{1}_{A} \mathbb{1}_{\{\tau > t\}} F \mathbb{1}_{\{\tau > T\}} \right]
\]

\[
= \mathbb{E} \left[ \mathbb{1}_A \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}[\mathbb{1}_{\{\tau > T\}} \mid \mathcal{F}_T]}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} F \mathbb{1}_{\{\tau > T\}} \right]
\]

\[
= \mathbb{E} \left[ \frac{\mathbb{E}[\mathbb{1}_A \mathbb{1}_{\{\tau > t\}} \mid \mathcal{F}_T]}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} F \mathbb{1}_{\{\tau > T\}} \right]
\]

\[
= \mathbb{E} \left[ \frac{\mathbb{E}[\mathbb{1}_A \mathbb{1}_{\{\tau > t\}} \mid \mathcal{F}_T]}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} \mathbb{E} \left[ F \mathbb{1}_{\{\tau > T\}} \mid \mathcal{F}_T \right] \right]
\]

\[
= \mathbb{E} \left[ \frac{\mathbb{1}_A \mathbb{1}_{\{\tau > t\}}}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} \mathbb{E} \left[ F \mathbb{1}_{\{\tau > T\}} \mid \mathcal{F}_T \right] \right]
\]

\[
= \mathbb{E} \left[ \frac{\mathbb{1}_A \mathbb{1}_{\{\tau > t\}}}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} \mathbb{E} \left[ F \mathbb{1}_{\{\tau > T\}} \mid \mathcal{F}_T \right] \right]
\]
hence by a standard characterization of conditional expectations, see e.g. § ?? of Privault (2018), we have

\[
\mathbb{E} \left[ \mathbbm{1}_{\{\tau > t\}} \mathbbm{1}_{\{\tau > T\}} \mid G_t \right] = \frac{\mathbbm{1}_{\{\tau > t\}}}{\mathbb{P}(\tau > t \mid F_t)} \mathbb{E} \left[ \mathbbm{1}_{\{\tau > T\}} \mid F_t \right],
\]

Taking \( F = 1 \) in Lemma 7.3 allows one to write the survival probability up to time \( T \), given information known up to \( t \), as

\[
\mathbb{P}(\tau > T \mid G_t) = \mathbb{E} \left[ \mathbbm{1}_{\{\tau > T\}} \mid G_t \right] = \mathbbm{1}_{\{\tau > t\}} \mathbb{E} \left[ \exp \left( - \int_t^T \lambda_u du \right) \mid F_t \right], \quad 0 \leq t \leq T.
\]

In particular, applying Lemma 7.3 for \( t = T \) and \( F = 1 \) shows that

\[
\mathbb{E} \left[ \mathbbm{1}_{\{\tau > T\}} \mid G_t \right] = \mathbbm{1}_{\{\tau > t\}},
\]

which shows that \( \{\tau > t\} \in G_t \) for all \( t > 0 \), and recovers the fact that \( \tau \) is a \((G_t)_{t \in \mathbb{R}^+}\)-stopping time, while in general, \( \tau \) is not \((F_t)_{t \in \mathbb{R}^+}\)-stopping time.

The computation of \( \mathbb{P}(\tau > T \mid G_t) \) according to (7.4) is then similar to that of a bond price, by considering the failure rate \( \lambda(t) \) as a “virtual” short-term interest rate. In particular the failure rate \( \lambda(t, T) \) can be modeled in the HJM framework, cf. e.g. Chapter ?? of Privault (2014), and

\[
\mathbb{P}(\tau > T \mid G_t) = \mathbb{E} \left[ \exp \left( - \int_t^T \lambda(t, u) du \right) \mid F_t \right]
\]
can then be computed by applying HJM bond pricing techniques.

The computation of expectations given \( G_t \) as in Lemma 7.3 can be useful for pricing under insider trading, in which the insider has access to the augmented filtration \( G_t \) while the ordinary trader has only access to \( F_t \), therefore generating two different prices \( \mathbb{E}^* [F \mid G_t] \) and \( \mathbb{E}^* [F \mid F_t] \) for the same claim payoff \( F \) under the same risk-neutral probability measure \( \mathbb{P}^* \). This leads to the issue of computing the dynamics of the underlying asset price by decomposing it using a \( F_t \)-martingale vs a \( G_t \)-martingale instead of using different forward measures as in e.g. § ?? of Privault (2014). This can be obtained...

### 7.3 Defaultable Bonds

Bond pricing models are generally based on the terminal condition $P(T, T) = 1$ according to which the bond payoff at maturity is always equal to $1$, and default does not occur. In this chapter we allow for the possibility of default at a random time $\tau$, in which case the terminal payoff of a bond is allowed to vanish at maturity.

The price $P_d(t, T)$ at time $t$ of a default bond with maturity $T$, (random) default time $\tau$ and (possibly random) recovery rate $\xi \in [0, 1]$ is given by

$$P_d(t, T) = \mathbb{E}^* \left[ \mathbb{1}_{\{\tau > T\}} \exp \left( - \int_t^T r_u du \right) \mid \mathcal{G}_t \right] + \mathbb{E}^* \left[ \xi \mathbb{1}_{\{\tau \leq T\}} \exp \left( - \int_t^T r_u du \right) \mid \mathcal{G}_t \right], \quad 0 \leq t \leq T.$$

**Proposition 7.4.** The default bond with maturity $T$ and default time $\tau$ can be priced at time $t \in [0, T]$ as

$$P_d(t, T) = \mathbb{1}_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T (r_u + \lambda_u) du \right) \mid \mathcal{F}_t \right] + \mathbb{E}^* \left[ \xi \mathbb{1}_{\{\tau \leq T\}} \exp \left( - \int_t^T r_u du \right) \mid \mathcal{G}_t \right], \quad 0 \leq t \leq T.$$

**Proof.** We take $F = \exp \left( - \int_t^T r_u du \right)$ in Lemma 7.3, which shows that

$$\mathbb{E}^* \left[ \mathbb{1}_{\{\tau > T\}} \exp \left( - \int_t^T r_u du \right) \mid G_t \right] = \mathbb{1}_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T (r_u + \lambda_u) du \right) \mid F_t \right],$$

cf. e.g. Lando (1998), Guo et al. (2007), Duffie and Singleton (2003). 

In the case of complete default (zero-recovery) we have $\xi = 0$ and

$$P_d(t, T) = \mathbb{1}_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T (r_s + \lambda_s) ds \right) \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T. \quad (7.5)$$

From the above expression (7.5) we note that the effect of the presence of a default time $\tau$ is to decrease the bond price, which can be viewed as an increase of the short rate by the amount $\lambda_u$. In a simple setting where the interest rate $r > 0$ and failure rate $\lambda > 0$ are constant, the default bond price becomes

$$P_d(t, T) = \mathbb{1}_{\{\tau > t\}} e^{-(r + \lambda)(T - t)}, \quad 0 \leq t \leq T.$$
Finally, from *e.g.* Proposition ?? of Privault (2014) the bond price (7.5) can also be expressed under the forward measure \( \hat{P} \) with maturity \( T \), as

\[
P_d(t, T) = \mathbb{1}_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T (r_s + \lambda_s) ds \right) \bigg| \mathcal{F}_t \right]
= \mathbb{1}_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T r_s ds \right) \bigg| \mathcal{F}_t \right] \mathbb{E} \left[ \exp \left( - \int_t^T \lambda_s ds \right) \bigg| \mathcal{F}_t \right]
= \mathbb{1}_{\{\tau > t\}} N_t \hat{P}(\tau > T \mid G_t),
\]

where \((N_t)_{t \in \mathbb{R}_+}\) is the numéraire process

\[
N_t := P(t, T) = \mathbb{E}^* \left[ \exp \left( - \int_t^T r_s ds \right) \bigg| \mathcal{F}_t \right], \quad 0 \leq t \leq T,
\]

and by (7.4),

\[
\hat{P}(\tau > T \mid G_t) = \mathbb{1}_{\{\tau > t\}} \mathbb{E} \left[ \exp \left( - \int_t^T \lambda_s ds \right) \bigg| \mathcal{F}_t \right]
\]

is the survival probability under the forward measure \( \hat{P} \) defined as

\[
\frac{d\hat{P}}{dP} := \frac{N_T}{N_0} e^{-\int_0^T r_l dt},
\]

cf. Chen and Huang (2001), Chen et al. (2008),

**Estimating the default rates**

Recall that the price of a default bond with maturity \( T \), (random) default time \( \tau \) and (possibly random) recovery rate \( \xi \in [0, 1] \) is given by

\[
P_d(t, T) = \mathbb{1}_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T (r_u + \lambda_u) du \right) \bigg| \mathcal{F}_t \right] + \mathbb{E}^* \left[ \xi \mathbb{1}_{\{\tau \leq T\}} \exp \left( - \int_t^T r_u du \right) \bigg| \mathcal{G}_t \right], \quad 0 \leq t \leq T,
\]

where \( \xi \) denotes the recovery rate. We consider a simplified deterministic step function model with zero recovery rate and tenor structure

\[
\{t = T_0 < T_1 < \cdots < T_n = T\},
\]

where

\[
r(t) = \sum_{l=0}^{n-1} r_l \mathbb{1}_{(T_l, T_{l+1}]}(t) \quad \text{and} \quad \lambda(t) = \sum_{l=0}^{n-1} \lambda_l \mathbb{1}_{(T_l, T_{l+1}]}(t), \quad t \in \mathbb{R}_+. \quad (7.6)
\]
Reduced-Form Approach to Credit Risk

i) Estimating the default rates from default bond prices.

We have

\[ P_d(t, T_k) = \mathbf{1}_{\{\tau > t\}} \exp \left( - \int_t^{T_k} (r(u) + \lambda(u))du \right) = \mathbf{1}_{\{\tau > t\}} \exp \left( - \sum_{l=0}^{k-1} (r_l + \lambda_l)(T_{l+1} - T_l) \right) , \]

\( k = 1, 2, \ldots, n, \) from which we can infer

\[ \lambda_k = -r_k + \frac{1}{T_{k+1} - T_k} \log \frac{P_d(T_{k+1}, T)}{P_d(T_k, T)} > 0, \quad k = 0, 1, \ldots, n - 1. \]

ii) Estimating (implied) default probabilities \( P^*(\tau < T \mid G_t) \) from default rates.

Based on the expression

\[ P^*(\tau > T \mid G_t) = \mathbb{E}^*[\mathbf{1}_{\{\tau > T\}} \mid G_t] \]

\[ = \mathbf{1}_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T \lambda_u du \right) \mid \mathcal{F}_t \right] , \quad 0 \leq t \leq T, \]

of the survival probability up to time \( T, \) and given information known up to \( t, \) in terms of the hazard rate process \( (\lambda_u)_{u \in \mathbb{R}^+} \) adapted to a filtration \( (\mathcal{F}_t)_{t \in \mathbb{R}^+}, \) we find

\[ P(\tau > T \mid G_{T_k}) = \mathbf{1}_{\{\tau > T_k\}} \exp \left( - \int_{T_k}^T \lambda_u du \right) = \mathbf{1}_{\{\tau > t\}} \exp \left( - \sum_{l=k}^{n-1} \lambda_l(T_{l+1} - T_l) \right) , \quad k = 0, 1, \ldots, n - 1, \]

where

\[ G_t = \mathcal{F}_t \lor \sigma(\{\tau \leq u\} : 0 \leq u \leq t), \quad t \in \mathbb{R}^+, \]

i.e. \( G_t \) contains the additional information on whether default at time \( \tau \) has occurred or not before time \( t. \)

In Table 7.2, bond ratings are determined according to hazard (or failure) rate thresholds.
<table>
<thead>
<tr>
<th>Bond rating categories</th>
<th>Moody’s Municipal</th>
<th>Moody’s Corporate</th>
<th>S&amp;P Municipal</th>
<th>S&amp;P Corporate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aaa/AAAs</td>
<td>0.00</td>
<td>0.52</td>
<td>0.00</td>
<td>0.60</td>
</tr>
<tr>
<td>Aa/AA</td>
<td>0.06</td>
<td>0.52</td>
<td>0.00</td>
<td>1.50</td>
</tr>
<tr>
<td>A/A</td>
<td>0.03</td>
<td>1.29</td>
<td>0.23</td>
<td>2.91</td>
</tr>
<tr>
<td>Baa/BBB</td>
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<td>4.64</td>
<td>0.32</td>
<td>10.29</td>
</tr>
<tr>
<td>Ba/BB</td>
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<td>1.74</td>
<td>29.93</td>
</tr>
<tr>
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<td>2.09</td>
<td>0.20</td>
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</tr>
<tr>
<td>Non-Invest Grade</td>
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<td>31.37</td>
<td>7.37</td>
<td>42.35</td>
</tr>
<tr>
<td>All</td>
<td>0.10</td>
<td>9.70</td>
<td>0.29</td>
<td>12.98</td>
</tr>
</tbody>
</table>

Table 7.2: Cumulative historic default rates (in percentage). *

Exercises

Exercise 7.1 Consider a standard zero-coupon bond with constant yield $r > 0$ and a defaultable (risky) bond with constant yield $r_d$ and default probability $\alpha \in (0, 1)$. Find a relation between $r$, $r_d$, $\alpha$ and the bond maturity $T$.

Exercise 7.2 A standard zero-coupon bond with constant yield $r > 0$ and maturity $T$ is priced $P(t, T) = e^{-(T-t)r}$ at time $t \in [0, T]$. Assume that the company can get bankrupt at a random time $t + \tau$, and default on its final $\$1$ payment if $\tau < T - t$.

a) Explain why the defaultable bond price $P_d(t, T)$ can be expressed as

$$P_d(t, T) = e^{-(T-t)r} \mathbb{E}^\star [\mathbb{1}_{\{\tau > T-t\}}]. \tag{7.8}$$

b) Assuming that the default time $\tau$ is exponentially distributed with parameter $\lambda > 0$, compute the default bond price $P_d(t, T)$ using (7.8).

c) Find a formula that can estimate the parameter $\lambda$ from the risk-free rate $r$ and the market data $P_M(t, T)$ of the defaultable bond price at time $t \in [0, T]$.

Exercise 7.3 Consider a (random) default time $\tau$ with cumulative distribution function

$$\mathbb{P}(\tau > t \mid \mathcal{F}_t) = \exp \left( - \int_0^t \lambda_u du \right), \quad t \in \mathbb{R}_+,$$

where $\lambda_t$ is a (random) default rate process which is adapted to the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$. Recall that the probability of survival up to time $T$, given

* Source: Moody’s, S&P.
information known up to time $t$, is given by

$$P(\tau > T \mid \mathcal{G}_t) = 1_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T \lambda_u du \right) \mid \mathcal{F}_t \right],$$

where $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau < u \} : 0 \leq u \leq t)$, $t \in \mathbb{R}_+$, is the filtration defined by adding the default time information to the history $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$. In this framework, the price $P(t, T)$ of defaultable bond with maturity $T$, short-term interest rate $r_t$ and (random) default time $\tau$ is given by

$$P(t, T) = \mathbb{E}^* \left[ 1_{\{\tau > T\}} \exp \left( - \int_t^T r_u du \right) \mid \mathcal{G}_t \right] = 1_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T (r_u + \lambda_u) du \right) \mid \mathcal{F}_t \right].$$

In the sequel we assume that the processes $(r_t)_{t \in \mathbb{R}_+}$ and $(\lambda_t)_{t \in \mathbb{R}_+}$ are modeled according to the Vasicek processes

$$\begin{cases}
    dr_t = -ar_t dt + \sigma dB^{(1)}_t, \\
    d\lambda_t = -b\lambda_t dt + \eta dB^{(2)}_t,
\end{cases}$$

where $(B^{(1)}_t)_{t \in \mathbb{R}_+}$ and $(B^{(2)}_t)_{t \in \mathbb{R}_+}$ are two standard $\mathcal{F}_t$-Brownian motions with correlation $\rho \in [-1, 1]$, and $dB^{(1)}_t \cdot dB^{(2)}_s = \rho dt$.

a) Give a justification for the fact that

$$\mathbb{E}^* \left[ \exp \left( - \int_t^T (r_s + \lambda_s) ds \right) \mid \mathcal{F}_t \right]$$

can be written as a function $F(t, r_t, \lambda_t)$ of $t$, $r_t$ and $\lambda_t$, $t \in [0, T]$.

b) Show that

$$t \mapsto \exp \left( - \int_0^t (r_s + \lambda_s) ds \right) \mathbb{E}^* \left[ \exp \left( - \int_t^T (r_u + \lambda_u) du \right) \mid \mathcal{F}_t \right]$$

is an $\mathcal{F}_t$-martingale under $\mathbb{P}$.

c) Use the Itô formula with two variables to derive a PDE on $\mathbb{R}^2$ for the function $F(t, x, y)$.

d) Taking $r_0 := 0$, show that we have

$$\int_t^T r_s ds = C(a, t, T) r_t + \sigma \int_t^T C(a, s, T) dB^{(1)}_s,$$

and

$$\int_t^T \lambda_s ds = C(b, t, T) \lambda_t + \eta \int_t^T C(b, s, T) dB^{(2)}_s,$$
where
\[ C(a, t, T) = -\frac{1}{a}(e^{-(T-t)a} - 1). \]

e) Show that the random variable
\[
\int_t^T r_s ds + \int_t^T \lambda_s ds
\]
is Gaussian and compute its conditional mean
\[
\mathbb{E}^* \left[ \int_t^T r_s ds + \int_t^T \lambda_s ds \mid \mathcal{F}_t \right]
\]
and variance
\[
\text{Var} \left[ \int_t^T r_s ds + \int_t^T \lambda_s ds \mid \mathcal{F}_t \right],
\]
conditionally to \( \mathcal{F}_t \).
f) Compute \( P(t, T) \) from its expression (7.9) as a conditional expectation.
g) Show that the solution \( F(t, x, y) \) to the 2-dimensional PDE of Question (c) is
\[
F(t, x, y) = \exp \left( -C(a, t, T)x - C(b, t, T)y \right)
\times \exp \left( \frac{\sigma^2}{2} \int_t^T C^2(a, s, T) ds + \frac{\eta^2}{2} \int_t^T C^2(b, s, T) ds \right)
\times \exp \left( \rho \sigma \eta \int_t^T C(a, s, T)C(b, s, T) ds \right).
\]
h) Show that the defaultable bond price \( P(t, T) \) can also be written as
\[
P(t, T) = e^{U(t, T)} \mathbb{P}(\tau > T \mid \mathcal{G}_t) \mathbb{E}^* \left[ \exp \left( -\int_t^T r_s ds \right) \mid \mathcal{F}_t \right],
\]
where
\[
U(t, T) = \frac{\rho \sigma \eta}{ab} \left( T - t - C(a, t, T) - C(b, t, T) + C(a+b, t, T) \right).
\]
i) By partial differentiation of \( \log P(t, T) \) with respect to \( T \), compute the corresponding instantaneous short rate
\[
f(t, T) = -\frac{\partial \log P(t, T)}{\partial T}.
\]
j) Show that \( \mathbb{P}(\tau > T \mid \mathcal{G}_t) \) can be written using an HJM type default rate as
\[
\mathbb{P}(\tau > T \mid \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \exp \left( -\int_t^T f_2(t, u) du \right),
\]
where
\[ f_2(t, u) = \lambda t e^{-(u-t)b} - \frac{\eta^2}{2} C^2(b, t, u). \]

k) Show how the result of Question (h) can be simplified when \((B^{(1)}_t)_{t \in \mathbb{R}_+}\) and \((B^{(2)}_t)_{t \in \mathbb{R}_+}\) are independent.