Chapter 7
Reduced-Form Approach to Credit Risk

This chapter deals with the reduced-form approach to credit risk, which relies on a failure rate process and an exogeneous random variable, resulting into an enlarged filtration that incorporates the knowledge of the default event.

7.1 Survival Probabilities

Given \( t > 0 \), let \( P(\tau > t) \) denote the probability that a random system with lifetime \( \tau \) survives at least \( t \) years. Assuming that survival probabilities \( P(\tau > t) \) are strictly positive for all \( t > 0 \), we can compute the conditional probability for that system to survive up to time \( T \), given that it was still functioning at time \( t \in [0, T] \), as

\[
P(\tau > T \mid \tau > t) = \frac{P(\tau > T \text{ and } \tau > t)}{P(\tau > t)} = \frac{P(\tau > T)}{P(\tau > t)}, \quad 0 \leq t \leq T,
\]

with

\[
P(\tau \leq T \mid \tau > t) = 1 - P(\tau > T \mid \tau > t)
= \frac{P(\tau > t) - P(\tau > T)}{P(\tau > t)}
= \frac{P(\tau \leq T) - P(\tau \leq t)}{P(\tau > t)}
= \frac{P(t < \tau \leq T)}{P(\tau > t)}, \quad 0 \leq t \leq T,
\]
and the conditional survival probability distribution
\[
P(\tau \in dx \mid \tau > t) = P(x < \tau \leq x + dx \mid \tau > t) \\
= P(\tau \leq x + dx \mid \tau > t) - P(\tau \leq x \mid \tau > t) \\
= \frac{P(\tau \leq x + dx) - P(\tau \leq x)}{P(\tau > t)} \\
= \frac{1}{P(\tau > t)} dP(\tau \leq x) \\
= -\frac{1}{P(\tau > t)} dP(\tau > x), \quad x > t.
\]

Such survival probabilities are typically found in life or mortality tables:

<table>
<thead>
<tr>
<th>Age t</th>
<th>P(\tau \leq t + 1 \mid \tau &gt; t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.0894%</td>
</tr>
<tr>
<td>30</td>
<td>0.1008%</td>
</tr>
<tr>
<td>40</td>
<td>0.2038%</td>
</tr>
<tr>
<td>50</td>
<td>0.4458%</td>
</tr>
<tr>
<td>60</td>
<td>0.9827%</td>
</tr>
</tbody>
</table>

Table 7.1: Mortality table.

**Proposition 7.1.** The failure rate function, defined as
\[
\lambda(t) := \frac{P(\tau \leq t + dt \mid \tau > t)}{dt},
\]
satisfies
\[
P(\tau > t) = \exp \left( -\int_0^t \lambda(u)du \right), \quad t \in \mathbb{R}_+.
\]  $(7.1)$

**Proof.** We have
\[
\lambda(t) := \frac{P(\tau \leq t + dt \mid \tau > t)}{dt} \\
= \frac{1}{P(\tau > t)} \frac{P(t < \tau \leq t + dt)}{dt} \\
= \frac{1}{P(\tau > t)} \frac{P(\tau > t) - P(\tau > t + dt)}{dt} \\
= -\frac{d}{dt} \log P(\tau > t) \\
= -\frac{1}{P(\tau > t)} \frac{d}{dt} P(\tau > t), \quad t > 0,
\]
and the differential equation
\[ \frac{d}{dt} P(\tau > t) = -\lambda(t) P(\tau > t), \]
which can be solved as in (7.1) under the initial condition \( P(\tau > 0) = 1. \)

Proposition 7.1 allows us to rewrite the (conditional) survival probability as
\[
P(\tau > T \mid \tau > t) = \frac{P(\tau > T)}{P(\tau > t)} = \exp \left(-\int_t^T \lambda(u) du\right), \quad 0 \leq t \leq T,
\]
with
\[
P(\tau > t + h \mid \tau > t) = e^{-\lambda(t)h} \simeq 1 - \lambda(t)h, \quad [h \searrow 0],
\]
and
\[
P(\tau \leq t + h \mid \tau > t) = 1 - e^{-\lambda(t)h} \simeq \lambda(t)h, \quad [h \searrow 0],
\]
as \( h \) tends to 0. When the failure rate \( \lambda(t) = \lambda > 0 \) is a constant function of time, Relation (7.1) shows that
\[
P(\tau > T) = e^{-\lambda T}, \quad T \geq 0,
\]
\textit{i.e.} \( \tau \) has the exponential distribution with parameter \( \lambda \). Note that given \((\tau_n)_{n \geq 1}\) a sequence of \( i.i.d. \) exponentially distributed random variables, letting
\[
T_n = \tau_1 + \tau_2 + \cdots + \tau_n, \quad n \geq 1,
\]
defines the sequence of jump times of a standard Poisson process with intensity \( \lambda > 0 \).

7.2 Stochastic Default

When the random time \( \tau \) is a \textit{stopping time} with respect to \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\) we have
\[
\{\tau > t\} \in \mathcal{F}_t, \quad t \in \mathbb{R}_+,
\]
\textit{i.e.} the knowledge of whether default or bankruptcy has already occurred at time \( t \) is contained in \( \mathcal{F}_t, \) \( t \in \mathbb{R}_+ \), cf. \textit{e.g.} Section ?? of Privault (2014). As a consequence, we can write
\[
P(\tau > t \mid \mathcal{F}_t) = \mathbb{E} [1_{\{\tau > t\}} \mid \mathcal{F}_t] = 1_{\{\tau > t\}}, \quad t \in \mathbb{R}_+.
\]
In the sequel we will not assume that \( \tau \) is an \( \mathcal{F}_t \)-stopping time, and by analogy with (7.1) we will write \( P(\tau > t \mid \mathcal{F}_t) \) as
\[ \mathbb{P}(\tau > t \mid \mathcal{F}_t) = \exp \left( - \int_0^t \lambda_u du \right), \quad t > 0, \quad (7.2) \]

where the failure rate function \((\lambda_t)_{t \in \mathbb{R}_+}\) is modeled as a random process adapted to a filtration \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\).

The process \((\lambda_t)_{t \in \mathbb{R}_+}\) can also be chosen among the classical mean-reverting diffusion processes, including jump-diffusion processes. In Lando (1998), the process \((\lambda_t)_{t \in \mathbb{R}_+}\) is constructed as \(\lambda_t := h(X_t), t \in \mathbb{R}_+\), where \(h\) is a nonnegative function and \((X_t)_{t \in \mathbb{R}_+}\) is a stochastic process generating the filtration \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\). The default time \(\tau\) is then defined as

\[ \tau := \inf \left\{ t \in \mathbb{R}_+ : \int_0^t h(X_u) du \geq L \right\}, \]

where \(L\) is an exponentially distributed random variable independent of \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\). In this case, as \(\tau\) is not an \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\)-stopping time, we have

\[ \mathbb{P}(\tau > t \mid \mathcal{F}_t) = \mathbb{P} \left( \int_0^t h(X_u) du \geq L \mid \mathcal{F}_t \right) \]

\[ = \exp \left( - \int_0^t h(X_u) du \right) \]

\[ = \exp \left( - \int_0^t \lambda_u du \right), \quad t \in \mathbb{R}_+. \]

**Definition 7.2.** Let \((\mathcal{G}_t)_{t \in \mathbb{R}_+}\) be the filtration defined by \(\mathcal{G}_\infty := \mathcal{F}_\infty \cup \sigma(\tau)\) and

\[ \mathcal{G}_t := \{ B \in \mathcal{G}_\infty : \exists A \in \mathcal{F}_t \text{ such that } A \cap \{ \tau > t \} = B \cap \{ \tau > t \} \}, \quad (7.3) \]

with \(\mathcal{F}_t \subset \mathcal{G}_t, t \in \mathbb{R}_+\).

In other words, \(\mathcal{G}_t\) contains insider information on whether default at time \(\tau\) has occurred or not before time \(t\), and \(\tau\) is a \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\)-stopping time. Note that this information on \(\tau\) may not be available to a generic user who has only access to the smaller filtration \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\). The next key Lemma 7.3 allows us to price a contingent claim given information in the larger filtration \((\mathcal{G}_t)_{t \in \mathbb{R}_+}\), by only using information in \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\) and factoring in the default rate factor \(\exp \left( - \int_0^T \lambda_u du \right)\).

**Lemma 7.3.** (Guo et al. (2007)) For any \(\mathcal{F}_T\)-measurable integrable random variable \(F\) we have

\[ \mathbb{E} \left[ F \mathbb{1}_{\{ \tau > T \}} \mid \mathcal{G}_t \right] = \mathbb{1}_{\{ \tau > t \}} \mathbb{E} \left[ F \exp \left( - \int_t^T \lambda_u du \right) \mid \mathcal{F}_t \right]. \]

**Proof.** By (7.2) we have

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\[ \frac{\mathbb{P}(\tau > T \mid \mathcal{F}_T)}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} = \frac{e^{-\int_0^T \lambda_u du}}{e^{-\int_0^t \lambda_u du}} = \exp \left( - \int_t^T \lambda_u du \right), \]

hence, since \( F \) is \( \mathcal{F}_T \)-measurable,

\[
\mathbb{1}_{\{\tau > t\}} \mathbb{E} \left[ F \exp \left( - \int_t^T \lambda_u du \right) \mid \mathcal{F}_t \right] = \mathbb{1}_{\{\tau > t\}} \mathbb{E} \left[ F \frac{\mathbb{P}(\tau > T \mid \mathcal{F}_T)}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} \mid \mathcal{F}_t \right] = \frac{\mathbb{1}_{\{\tau > t\}}}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} \mathbb{E} \left[ F \mathbb{1}_{\{\tau > T\}} \mid \mathcal{F}_t \right] = \mathbb{1}_{\{\tau > t\}} \mathbb{E} \left[ F \mathbb{1}_{\{\tau > T\}} \mid \mathcal{G}_t \right] = \mathbb{E} \left[ F \mathbb{1}_{\{\tau > T\}} \mid \mathcal{G}_t \right], \quad 0 \leq t \leq T.
\]

In the last step of the above argument we used the key relation

\[
\mathbb{1}_{\{\tau > t\}} \mathbb{E} \left[ F \mathbb{1}_{\{\tau > T\}} \mid \mathcal{G}_t \right] = \frac{\mathbb{1}_{\{\tau > t\}}}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} \mathbb{E} \left[ F \mathbb{1}_{\{\tau > T\}} \mid \mathcal{F}_t \right],
\]

cf. Relation (75.2) in § XX-75 page 186 of Dellacherie et al. (1992), Theorem VI-3-14 page 371 of Protter (2004), and Lemma 3.1 of Elliott et al. (2000), under the probability measure \( \mathbb{P}_{\mathcal{F}_t}, 0 \leq t \leq T \). Indeed, according to (7.3), for any \( B \in \mathcal{G}_t \) we have, for some event \( A \in \mathcal{F}_t \),

\[
\mathbb{E} \left[ \mathbb{1}_B \mathbb{1}_{\{\tau > T\}} \right] = \mathbb{E} \left[ \mathbb{1}_{B \cap \{\tau > T\}} \mathbb{1}_{\{\tau > T\}} \right] = \mathbb{E} \left[ \mathbb{1}_{A \cap \{\tau > T\}} \mathbb{1}_{\{\tau > T\}} \right] = \mathbb{E} \left[ \mathbb{1}_A \mathbb{1}_{\{\tau > T\}} \mathbb{1}_{\{\tau > T\}} \right] = \mathbb{E} \left[ \mathbb{1}_A \mathbb{1}_{\{\tau > T\}} \mid \mathcal{F}_t \right] \frac{\mathbb{1}_{\{\tau > t\}}}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} \mathbb{E} \left[ F \mathbb{1}_{\{\tau > T\}} \mid \mathcal{F}_t \right] = \mathbb{E} \left[ \frac{\mathbb{1}_A \mathbb{1}_{\{\tau > T\}}}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} \mathbb{E} \left[ F \mathbb{1}_{\{\tau > T\}} \mid \mathcal{F}_t \right] \right],
\]

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\[
= \mathbb{E}\left[ \frac{1_A \mathbb{1}_{\{\tau > t\}}}{\mathbb{P}(\tau > t \mid F_t)} \mathbb{E}\left[ F \mathbb{1}_{\{\tau > T\}} \mid F_t \right] \right]
\]

\[
= \mathbb{E}\left[ \frac{1_B \mathbb{1}_{\{\tau > t\}}}{\mathbb{P}(\tau > t \mid F_t)} \mathbb{E}\left[ F \mathbb{1}_{\{\tau > T\}} \mid F_t \right] \right],
\]

hence by a standard characterization of conditional expectations, see \textit{e.g.} § ?? of Privault (2018), we have

\[
\mathbb{E}\left[ \mathbb{1}_{\{\tau > t\}} F \mathbb{1}_{\{\tau > T\}} \mid G_t \right] = \frac{1_{\{\tau > t\}}}{\mathbb{P}(\tau > t \mid F_t)} \mathbb{E}\left[ F \mathbb{1}_{\{\tau > T\}} \mid F_t \right]
\]

\]

Taking \( F = 1 \) in Lemma 7.3 allows one to write the survival probability up to time \( T \), given information known up to \( t \), as

\[
\mathbb{P}(\tau > T \mid G_t) = \mathbb{E}\left[ \mathbb{1}_{\{\tau > T\}} \mid G_t \right] = \mathbb{1}_{\{\tau > t\}} \mathbb{E}\left[ \exp\left( - \int_t^T \lambda_u du \right) \mid F_t \right], \quad 0 \leq t \leq T.
\]

In particular, applying Lemma 7.3 for \( t = T \) and \( F = 1 \) shows that

\[
\mathbb{E}\left[ \mathbb{1}_{\{\tau > t\}} \mid G_t \right] = \mathbb{1}_{\{\tau > t\}},
\]

which shows that \( \{\tau > t\} \in G_t \) for all \( t > 0 \), and recovers the fact that \( \tau \) is a \((G_t)_{t \in \mathbb{R}^+}\)-stopping time, while in general, \( \tau \) is not \((F_t)_{t \in \mathbb{R}^+}\)-stopping time.

The computation of \( \mathbb{P}(\tau > T \mid G_t) \) according to (7.4) is then similar to that of a bond price, by considering the failure rate \( \lambda(t) \) as a “virtual” short-term interest rate. In particular the failure rate \( \lambda(t, T) \) can be modeled in the HJM framework, cf. \textit{e.g.} Chapter ?? of Privault (2014), and

\[
\mathbb{P}(\tau > T \mid G_t) = \mathbb{E}\left[ \exp\left( - \int_t^T \lambda(t, u) du \right) \mid F_t \right]
\]

can then be computed by applying HJM bond pricing techniques.

The computation of expectations given \( G_t \) as in Lemma 7.3 can be useful for pricing under insider trading, in which the insider has access to the augmented filtration \( G_t \) while the ordinary trader has only access to \( F_t \), therefore generating two different prices \( \mathbb{E}^*[F \mid F_t] \) and \( \mathbb{E}^*[F \mid G_t] \) for the same claim payoff \( F \) under the same risk-neutral probability measure \( \mathbb{P}^* \). This leads to the issue of computing the dynamics of the underlying asset price by decomposing it using a \( F_t \)-martingale \( \mathcal{M} \) vs a \( G_t \)-martingale \( \mathcal{M}' \) instead of using different forward measures as in \textit{e.g.} § ?? of Privault (2014). This can be obtained
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### 7.3 Defaultable Bonds

Bond pricing models are generally based on the terminal condition $P(T, T) = 1$ according to which the bond payoff at maturity is always equal to $1$, and default does not occur. In this chapter we allow for the possibility of default at a random time $\tau$, in which case the terminal payoff of a bond is allowed to vanish at maturity.

The price $P_d(t, T)$ at time $t$ of a default bond with maturity $T$, (random) default time $\tau$ and (possibly random) recovery rate $\xi \in [0, 1]$ is given by

$$P_d(t, T) = \mathbb{E}^* \left[ \mathbb{1}_{\{\tau > T\}} \exp \left( - \int_t^T r_u du \right) \bigg| \mathcal{G}_t \right] + \mathbb{E}^* \left[ \xi \mathbb{1}_{\{\tau \leq T\}} \exp \left( - \int_t^T r_u du \right) \bigg| \mathcal{G}_t \right], \quad 0 \leq t \leq T.$$

**Proposition 7.4.** The default bond with maturity $T$ and default time $\tau$ can be priced at time $t \in [0, T]$ as

$$P_d(t, T) = \mathbb{1}_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T (r_u + \lambda_u) du \right) \bigg| \mathcal{F}_t \right] + \mathbb{E}^* \left[ \xi \mathbb{1}_{\{\tau \leq T\}} \exp \left( - \int_t^T r_u du \right) \bigg| \mathcal{G}_t \right], \quad 0 \leq t \leq T.$$

**Proof.** We take $F = \exp \left( - \int_t^T r_u du \right)$ in Lemma 7.3, which shows that

$$\mathbb{E}^* \left[ \mathbb{1}_{\{\tau > T\}} \exp \left( - \int_t^T r_u du \right) \bigg| \mathcal{G}_t \right] = \mathbb{1}_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T (r_u + \lambda_u) du \right) \bigg| \mathcal{F}_t \right],$$

cf. e.g. Lando (1998), Guo et al. (2007), Duffie and Singleton (2003). □

In the case of complete default (zero-recovery) we have $\xi = 0$ and

$$P_d(t, T) = \mathbb{1}_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T (r_s + \lambda_s) ds \right) \bigg| \mathcal{F}_t \right], \quad 0 \leq t \leq T. \quad (7.5)$$

From the above expression (7.5) we note that the effect of the presence of a default time $\tau$ is to decrease the bond price, which can be viewed as an increase of the short rate by the amount $\lambda_u$. In a simple setting where the interest rate $r > 0$ and failure rate $\lambda > 0$ are constant, the default bond price becomes

$$P_d(t, T) = \mathbb{1}_{\{\tau > t\}} e^{-(r+\lambda)(T-t)}, \quad 0 \leq t \leq T.$$
Finally, from \( e.g. \) Proposition ? of Privault (2014) the bond price (7.5) can also be expressed under the forward measure \( \hat{P} \) with maturity \( T \), as

\[
P_d(t, T) = \mathbb{1}_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T (r_s + \lambda_s) ds \right) \mid \mathcal{F}_t \right] = \mathbb{1}_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T r_s ds \right) \mid \mathcal{F}_t \right] \hat{E} \left[ \exp \left( - \int_t^T \lambda_s ds \right) \mid \mathcal{F}_t \right] = \mathbb{1}_{\{\tau > t\}} \hat{N}_t \hat{P}(\tau > T \mid \mathcal{G}_t),
\]

where \( (N_t)_{t \in \mathbb{R}_+} \) is the numéraire process

\[
N_t := P(t, T) = \mathbb{E}^* \left[ \exp \left( - \int_t^T r_s ds \right) \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T,
\]

and by (7.4),

\[
\hat{P}(\tau > T \mid \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \hat{E} \left[ \exp \left( - \int_t^T \lambda_s ds \right) \mid \mathcal{F}_t \right]
\]

is the survival probability under the forward measure \( \hat{P} \) defined as

\[
\frac{d\hat{P}}{dP} := \frac{N_T}{N_0} e^{-\int_0^T r_l dt},
\]

cf. Chen and Huang (2001), Chen et al. (2008),

**Estimating the default rates**

Recall that the price of a default bond with maturity \( T \), (random) default time \( \tau \) and (possibly random) recovery rate \( \xi \in [0, 1] \) is given by

\[
P_d(t, T) = \mathbb{1}_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T (r_u + \lambda_u) du \right) \mid \mathcal{F}_t \right] + \mathbb{E}^* \left[ \xi \mathbb{1}_{\{\tau \leq T\}} \exp \left( - \int_t^T r_u du \right) \mid \mathcal{G}_t \right], \quad 0 \leq t \leq T,
\]

where \( \xi \) denotes the recovery rate. We consider a simplified deterministic step function model with zero recovery rate and tenor structure

\[
\{t = T_0 < T_1 < \cdots < T_n = T\},
\]

where

\[
r(t) = \sum_{l=0}^{n-1} r_l \mathbb{1}_{(T_l, T_{l+1})}(t) \quad \text{and} \quad \lambda(t) = \sum_{l=0}^{n-1} \lambda_l \mathbb{1}_{(T_l, T_{l+1})}(t), \quad t \in \mathbb{R}_+.
\]
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i) Estimating the default rates from default bond prices.

We have

\[ P_d(t, T_k) = \mathbb{1}_{\{\tau > t\}} \exp \left( - \int_t^{T_k} (r(u) + \lambda(u))du \right) = \mathbb{1}_{\{\tau > t\}} \exp \left( - \sum_{l=0}^{k-1} (r_l + \lambda_l)(T_{l+1} - T_l) \right), \]

\[ k = 1, 2, \ldots, n, \text{ from which we can infer} \]

\[ \lambda_k = -r_k + \frac{1}{T_{k+1} - T_k} \log \frac{P_d(T_{k+1}, T)}{P_d(T_k, T)} > 0, \quad k = 0, 1, \ldots, n - 1. \]

ii) Estimating (implied) default probabilities \( P^*(\tau < T \mid G_t) \) from default rates.

Based on the expression

\[ P^*(\tau > T \mid G_t) = \mathbb{E}^* \left[ \mathbb{1}_{\{\tau > T\}} \mid G_t \right] \]

\[ = \mathbb{1}_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T \lambda_u du \right) \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T, \tag{7.7} \]

of the survival probability up to time \( T \), and given information known up to \( t \), in terms of the hazard rate process \( (\lambda_u)_{u \in \mathbb{R}^+} \) adapted to a filtration \( (\mathcal{F}_t)_{t \in \mathbb{R}^+} \), we find

\[ P(\tau > T \mid \mathcal{G}_{T_k}) = \mathbb{1}_{\{\tau > T_k\}} \exp \left( - \int_{T_k}^T \lambda_u du \right) \]

\[ = \mathbb{1}_{\{\tau > t\}} \exp \left( - \sum_{l=k}^{n-1} \lambda_l(T_{l+1} - T_l) \right), \quad k = 0, 1, \ldots, n - 1, \]

where

\[ \mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau \leq u\} : 0 \leq u \leq t), \quad t \in \mathbb{R}^+, \]

i.e. \( \mathcal{G}_t \) contains the additional information on whether default at time \( \tau \) has occurred or not before time \( t \).

In Table 7.2, bond ratings are determined according to hazard (or failure) rate thresholds.
### Exercises

Exercise 7.1  Consider a standard zero-coupon bond with constant yield \( r > 0 \) and a defaultable (risky) bond with constant yield \( r_d \) and default probability \( \alpha \in (0, 1) \). Find a relation between \( r, r_d, \alpha \) and the bond maturity \( T \).

Exercise 7.2  A standard zero-coupon bond with constant yield \( r > 0 \) and maturity \( T \) is priced \( P(t, T) = e^{-(T-t)r} \) at time \( t \in [0, T] \). Assume that the company can get bankrupt at a random time \( t + \tau \), and default on its final $1 payment if \( \tau < T - t \).

a) Explain why the defaultable bond price \( P_d(t, T) \) can be expressed as

\[
P_d(t, T) = e^{-(T-t)r} \mathbb{E}^* \left[ \mathbb{1}_{\{\tau > T-t\}} \right]. \tag{7.8}
\]

b) Assuming that the default time \( \tau \) is exponentially distributed with parameter \( \lambda > 0 \), compute the default bond price \( P_d(t, T) \) using (7.8).

c) Find a formula that can estimate the parameter \( \lambda \) from the risk-free rate \( r \) and the market data \( P_M(t, T) \) of the defaultable bond price at time \( t \in [0, T] \).

Exercise 7.3  Consider a (random) default time \( \tau \) with cumulative distribution function

\[
P(\tau > t \mid \mathcal{F}_t) = \exp \left( -\int_0^t \lambda_u du \right), \quad t \in \mathbb{R}_+,
\]

where \( \lambda_t \) is a (random) default rate process which is adapted to the filtration \( (\mathcal{F}_t)_{t \in \mathbb{R}_+} \). Recall that the probability of survival up to time \( T \), given

* Source: Moody’s, S&P.
information known up to time $t$, is given by

$$
P(\tau > T \mid G_t) = \1_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T \lambda_u du \right) \mid \mathcal{F}_t \right],
$$

where $G_t = \mathcal{F}_t \lor \sigma(\{\tau < u : 0 \leq u \leq t\})$, $t \in \mathbb{R}_+$, is the filtration defined by adding the default time information to the history $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$. In this framework, the price $P(t, T)$ of defaultable bond with maturity $T$, short-term interest rate $r_t$ and (random) default time $\tau$ is given by

$$
P(t, T) = \mathbb{E}^* \left[ \1_{\{\tau > T\}} \exp \left( - \int_t^T r_u du \right) \mid G_t \right] 
= \1_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T (r_u + \lambda_u) du \right) \mid \mathcal{F}_t \right].
$$

In the sequel we assume that the processes $(r_t)_{t \in \mathbb{R}_+}$ and $(\lambda_t)_{t \in \mathbb{R}_+}$ are modeled according to the Vasicek processes

$$
\begin{cases}
    dr_t = -ar_t dt + \sigma dB_t^{(1)}, \\
    d\lambda_t = -b\lambda_t dt + \eta dB_t^{(2)},
\end{cases}
$$

where $(B_t^{(1)})_{t \in \mathbb{R}_+}$ and $(B_t^{(2)})_{t \in \mathbb{R}_+}$ are two standard $\mathcal{F}_t$-Brownian motions with correlation $\rho \in [-1, 1]$, and $dB_t^{(1)} \cdot dB_t^{(2)} = \rho dt$.

a) Give a justification for the fact that

$$
\mathbb{E}^* \left[ \exp \left( - \int_t^T (r_u + \lambda_u) du \right) \mid \mathcal{F}_t \right]
$$

can be written as a function $F(t, r_t, \lambda_t)$ of $t$, $r_t$ and $\lambda_t$, $t \in [0, T]$.

b) Show that

$$
t \mapsto \exp \left( - \int_0^t (r_s + \lambda_s) ds \right) \mathbb{E}^* \left[ \exp \left( - \int_t^T (r_u + \lambda_u) du \right) \mid \mathcal{F}_t \right]
$$
is an $\mathcal{F}_t$-martingale under $\mathbb{P}$.

c) Use the Itô formula with two variables to derive a PDE on $\mathbb{R}^2$ for the function $F(t, x, y)$.

d) Taking $r_0 := 0$, show that we have

$$
\int_t^T r_s ds = C(a, t, T) r_t + \sigma \int_t^T C(a, s, T) dB_s^{(1)},
$$

and

$$
\int_t^T \lambda_s ds = C(b, t, T) \lambda_t + \eta \int_t^T C(b, s, T) dB_s^{(2)},
$$

where $C(a, t, T)$ and $C(b, t, T)$ are functions of $a$, $t$, and $T$. 

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where
\[ C(a, t, T) = -\frac{1}{a}(e^{-a(T-t)}-1). \]

e) Show that the random variable
\[ \int_t^T r_s ds + \int_t^T \lambda_s ds \]
is Gaussian and compute its conditional mean
\[ \mathbb{E}^* \left[ \int_t^T r_s ds + \int_t^T \lambda_s ds \mid \mathcal{F}_t \right] \]
and variance
\[ \text{Var} \left[ \int_t^T r_s ds + \int_t^T \lambda_s ds \mid \mathcal{F}_t \right], \]
conditionally to \( \mathcal{F}_t \).

f) Compute \( P(t, T) \) from its expression (7.9) as a conditional expectation.

g) Show that the solution \( F(t, x, y) \) to the 2-dimensional PDE of Question (c) is
\[ F(t, x, y) = \exp \left( -C(a, t, T)x - C(b, t, T)y \right) \]
\[ \times \exp \left( \frac{\sigma^2}{2} \int_t^T C^2(a, s, T) ds + \frac{\eta^2}{2} \int_t^T C^2(b, s, T) ds \right) \]
\[ \times \exp \left( \rho \sigma \eta \int_t^T C(a, s, T)C(b, s, T) ds \right). \]

h) Show that the defaultable bond price \( P(t, T) \) can also be written as
\[ P(t, T) = e^{U(t, T)} \mathbb{P}(\tau > T \mid \mathcal{G}_t) \mathbb{E}^* \left[ \exp \left( -\int_t^T r_s ds \right) \mid \mathcal{F}_t \right], \]
where
\[ U(t, T) = \rho \frac{\sigma \eta}{ab} (T - t - C(a, t, T) - C(b, t, T) + C(a+b, t, T)). \]

i) By partial differentiation of \( \log P(t, T) \) with respect to \( T \), compute the corresponding instantaneous short rate
\[ f(t, T) = -\frac{\partial \log P(t, T)}{\partial T}. \]

j) Show that \( \mathbb{P}(\tau > T \mid \mathcal{G}_t) \) can be written using an HJM type default rate as
\[ \mathbb{P}(\tau > T \mid \mathcal{G}_t) = \mathbb{I}_{\{\tau > t\}} \exp \left( -\int_t^T f_2(t, u) du \right), \]
where

\[ f_2(t, u) = \lambda_t e^{-(u-t)b} - \frac{\eta^2}{2} C^2(b, t, u). \]

k) Show how the result of Question (h) can be simplified when \((B_{t}^{(1)})_{t \in \mathbb{R}^+}\) and \((B_{t}^{(2)})_{t \in \mathbb{R}^+}\) are independent.