Chapter 7
Reduced-Form Approach to Credit Risk

The reduced-form approach to credit risk modeling focuses on modeling default probabilities as stochastic processes, in contrast to the structural approach in which bankruptcy is modeled from the firm's asset value. The modeling of default risk using failure rate processes and exogeneous random variables results into the use of enlarged filtration that can incorporate information on default events.

7.1 Survival Probabilities

Given $t > 0$, let $\mathbb{P}(\tau > t)$ denote the probability that a random system with lifetime τ survives at least t years. Assuming that survival probabilities $\mathbb{P}(\tau > t)$ are strictly positive for all $t > 0$, we can compute the conditional probability for that system to survive up to time T, given that it was still functioning at time $t \in [0, T]$, as

$$\mathbb{P}(\tau > T \mid \tau > t) = \frac{\mathbb{P}(\tau > T \text{ and } \tau > t)}{\mathbb{P}(\tau > t)} = \frac{\mathbb{P}(\tau > T)}{\mathbb{P}(\tau > t)}, \qquad 0 \leqslant t \leqslant T,$$

with

$$\begin{aligned}\mathbb{P}(\tau \leqslant T \mid \tau > t) &= 1 - \mathbb{P}(\tau > T \mid \tau > t) \\ &= \frac{\mathbb{P}(\tau > t) - \mathbb{P}(\tau > T)}{\mathbb{P}(\tau > t)} \\ &= \frac{\mathbb{P}(\tau \leqslant T) - \mathbb{P}(\tau \leqslant t)}{\mathbb{P}(\tau > t)}\end{aligned}$$

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$$= \frac{\mathbb{P}(t < \tau \leqslant T)}{\mathbb{P}(\tau > t)}, \qquad 0 \leqslant t \leqslant T,$$

and the conditional survival probability distribution

$$\begin{aligned}\mathbb{P}(\tau \in dx \mid \tau > t) &= \mathbb{P}(x < \tau \leqslant x + dx \mid \tau > t) \\ &= \mathbb{P}(\tau \leqslant x + dx \mid \tau > t) - \mathbb{P}(\tau \leqslant x \mid \tau > t) \\ &= \frac{\mathbb{P}(\tau \leqslant x + dx) - \mathbb{P}(\tau \leqslant x)}{\mathbb{P}(\tau > t)} \\ &= \frac{1}{\mathbb{P}(\tau > t)} d\mathbb{P}(\tau \leqslant x) \\ &= -\frac{1}{\mathbb{P}(\tau > t)} d\mathbb{P}(\tau > x), \qquad x > t.\end{aligned}$$

Such survival probabilities are typically found in life (or mortality) tables:

Age t	$\mathbb{P}(\tau \leqslant t+1 \mid \tau > t)$
20	0.0894%
30	0.1008%
40	0.2038%
50	0.4458%
60	0.9827%

Table 7.1: Mortality table.

Proposition 7.1. *The* failure rate *function, defined as*

$$\lambda(t) := \frac{\mathbb{P}(\tau \leqslant t + dt \mid \tau > t)}{dt},$$

satisfies

$$\mathbb{P}(\tau > t) = \exp\left(-\int_0^t \lambda(u) du\right), \qquad t \in \mathbb{R}_+. \tag{7.1}$$

Proof. We have

$$\begin{aligned}\lambda(t) &:= \frac{\mathbb{P}(\tau \leqslant t + dt \mid \tau > t)}{dt} \\ &= \frac{1}{\mathbb{P}(\tau > t)} \frac{\mathbb{P}(t < \tau \leqslant t + dt)}{dt} \\ &= \frac{1}{\mathbb{P}(\tau > t)} \frac{\mathbb{P}(\tau > t) - \mathbb{P}(\tau > t + dt)}{dt} \\ &= -\frac{d}{dt} \log \mathbb{P}(\tau > t)\end{aligned}$$

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$$= -\frac{1}{\mathbb{P}(\tau > t)} \frac{d}{dt}\mathbb{P}(\tau > t), \qquad t > 0,$$

and the differential equation

$$\frac{d}{dt}\mathbb{P}(\tau > t) = -\lambda(t)\mathbb{P}(\tau > t),$$

which can be solved as in (7.1) under the initial condition $\mathbb{P}(\tau > 0) = 1$. □

Proposition 7.1 allows us to rewrite the (conditional) survival probability as

$$\mathbb{P}(\tau > T \mid \tau > t) = \frac{\mathbb{P}(\tau > T)}{\mathbb{P}(\tau > t)} = \exp\left(-\int_t^T \lambda(u)du\right), \qquad 0 \leqslant t \leqslant T,$$

with

$$\mathbb{P}(\tau > t + h \mid \tau > t) = \mathrm{e}^{-\lambda(t)h} \simeq 1 - \lambda(t)h, \qquad [h \searrow 0],$$

and

$$\mathbb{P}(\tau \leqslant t + h \mid \tau > t) = 1 - \mathrm{e}^{-\lambda(t)h} \simeq \lambda(t)h, \qquad [h \searrow 0],$$

as h tends to 0. When the failure rate $\lambda(t) = \lambda > 0$ is a constant function of time, Relation (7.1) shows that

$$\mathbb{P}(\tau > T) = \mathrm{e}^{-\lambda T}, \qquad T \geqslant 0,$$

i.e. τ has the exponential distribution with parameter λ. Note that given $(\tau_n)_{n\geqslant 1}$ a sequence of *i.i.d.* exponentially distributed random variables, letting

$$T_n = \tau_1 + \tau_2 + \cdots + \tau_n, \qquad n \geqslant 1,$$

defines the sequence of jump times of a standard Poisson process with intensity $\lambda > 0$.

7.2 Stochastic Default

When the random time τ is a *stopping time* with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ we have

$$\{\tau > t\} \in \mathcal{F}_t, \qquad t \in \mathbb{R}_+,$$

i.e. the knowledge of whether default or bankruptcy has already occurred at time t is contained in $\mathcal{F}_t$, $t \in \mathbb{R}_+$, cf. *e.g.* Section 9.3 of Privault (2014). As a consequence, we can write

$$\mathbb{P}(\tau > t \mid \mathcal{F}_t) = \mathbb{E}\left[\mathbb{1}_{\{\tau > t\}} \mid \mathcal{F}_t\right] = \mathbb{1}_{\{\tau > t\}}, \qquad t \in \mathbb{R}_+.$$

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In the sequel we will not assume that τ is an $\mathcal{F}_t$-stopping time, and by analogy with (7.1) we will write $\mathbb{P}(\tau > t \mid \mathcal{F}_t)$ as

$$\mathbb{P}(\tau > t \mid \mathcal{F}_t) = \exp\left(-\int_0^t \lambda_u du\right), \qquad t > 0, \tag{7.2}$$

where the failure rate function $(\lambda_t)_{t\in\mathbb{R}_+}$ is modeled as a random process adapted to a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$.

The process $(\lambda_t)_{t\in\mathbb{R}_+}$ can also be chosen among the classical mean-reverting diffusion processes, including jump-diffusion processes. In Lando (1998), the process $(\lambda_t)_{t\in\mathbb{R}_+}$ is constructed as $\lambda_t := h(X_t)$, $t \in \mathbb{R}_+$, where h is a nonnegative function and $(X_t)_{t\in\mathbb{R}_+}$ is a stochastic process generating the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. The default time τ is then defined as

$$\tau := \inf\left\{t \in \mathbb{R}_+ \ : \ \int_0^t h(X_u)du \geqslant L\right\},$$

where L is an exponentially distributed random variable independent of $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. In this case, as τ is not an $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-stopping time, we have

$$\begin{aligned}\mathbb{P}(\tau > t \mid \mathcal{F}_t) &= \mathbb{P}\left(\int_0^t h(X_u)du \geqslant L \ \middle| \ \mathcal{F}_t\right)\\ &= \exp\left(-\int_0^t h(X_u)du\right)\\ &= \exp\left(-\int_0^t \lambda_u du\right), \qquad t \in \mathbb{R}_+.\end{aligned}$$

Definition 7.2. *Let $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$ be the filtration defined by $\mathcal{G}_\infty := \mathcal{F}_\infty \vee \sigma(\tau)$ and*

$$\mathcal{G}_t := \big\{B \in \mathcal{G}_\infty \ : \ \exists A \in \mathcal{F}_t \text{ such that } A \cap \{\tau > t\} = B \cap \{\tau > t\}\big\}, \tag{7.3}$$

with $\mathcal{F}_t \subset \mathcal{G}_t$, $t \in \mathbb{R}_+$.

In other words, $\mathcal{G}_t$ contains insider information on whether default at time τ has occurred or not before time t, and τ is a $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$-stopping time. Note that this information on τ may not be available to a generic user who has only access to the smaller filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. The next key Lemma 7.3 allows us to price a contingent claim given the information in the larger filtration $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$, by only using information in $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and factoring in the default rate factor $\exp\left(-\int_t^T \lambda_u du\right)$.

Lemma 7.3. *(Guo et al. (2007)) For any $\mathcal{F}_T$-measurable integrable random variable F, we have*

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$$\mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \mid \mathcal{G}_t\right] = \mathbb{1}_{\{\tau>t\}}\,\mathbb{E}\left[F\exp\left(-\int_t^T \lambda_u du\right) \;\Big|\; \mathcal{F}_t\right].$$

Proof. By (7.2) we have

$$\frac{\mathbb{P}(\tau > T \mid \mathcal{F}_T)}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} = \frac{\mathrm{e}^{-\int_0^T \lambda_u du}}{\mathrm{e}^{-\int_0^t \lambda_u du}} = \exp\left(-\int_t^T \lambda_u du\right),$$

hence, since F is $\mathcal{F}_T$-measurable,

$$\begin{aligned}
\mathbb{1}_{\{\tau>t\}}\,\mathbb{E}\left[F\exp\left(-\int_t^T \lambda_u du\right) \;\Big|\; \mathcal{F}_t\right] &= \mathbb{1}_{\{\tau>t\}}\,\mathbb{E}\left[F\frac{\mathbb{P}(\tau > T \mid \mathcal{F}_T)}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} \;\Big|\; \mathcal{F}_t\right] \\
&= \frac{\mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)}\,\mathbb{E}\left[F\,\mathbb{E}[\mathbb{1}_{\{\tau>T\}} \mid \mathcal{F}_T] \mid \mathcal{F}_t\right] \\
&= \frac{\mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)}\,\mathbb{E}\left[\mathbb{E}[F\mathbb{1}_{\{\tau>T\}} \mid \mathcal{F}_T] \mid \mathcal{F}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \,\big|\, \mathcal{F}_t\right]}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} \\
&= \mathbb{1}_{\{\tau>t\}}\,\mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \,\big|\, \mathcal{G}_t\right] \\
&= \mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \,\big|\, \mathcal{G}_t\right], \qquad 0 \leqslant t \leqslant T.
\end{aligned}$$

In the last step of the above argument we used the key relation

$$\mathbb{1}_{\{\tau>t\}}\,\mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \;\Big|\; \mathcal{G}_t\right] = \frac{\mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)}\,\mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \mid \mathcal{F}_t\right],$$

cf. Relation (75.2) in § XX-75 page 186 of Dellacherie et al. (1992), Theorem VI-3-14 page 371 of Protter (2004), and Lemma 3.1 of Elliott et al. (2000), under the conditional probability measure $\mathbb{P}_{|\mathcal{F}_t}$, $0 \leqslant t \leqslant T$. Indeed, according to (7.3), for any $B \in \mathcal{G}_t$ we have, for some event $A \in \mathcal{F}_t$,

$$\begin{aligned}
\mathbb{E}\left[\mathbb{1}_B\mathbb{1}_{\{\tau>t\}}F\mathbb{1}_{\{\tau>T\}}\right] &= \mathbb{E}\left[\mathbb{1}_{B\cap\{\tau>t\}}F\mathbb{1}_{\{\tau>T\}}\right] \\
&= \mathbb{E}\left[\mathbb{1}_{A\cap\{\tau>t\}}F\mathbb{1}_{\{\tau>T\}}\right] \\
&= \mathbb{E}\left[\mathbb{1}_A\mathbb{1}_{\{\tau>t\}}F\mathbb{1}_{\{\tau>T\}}\right] \\
&= \mathbb{E}\left[\mathbb{1}_A\mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}[\mathbb{1}_{\{\tau>t\}} \mid \mathcal{F}_t]}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)}F\mathbb{1}_{\{\tau>T\}}\right] \\
&= \mathbb{E}\left[\frac{\mathbb{E}[\mathbb{1}_A\mathbb{1}_{\{\tau>t\}} \mid \mathcal{F}_t]}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)}F\mathbb{1}_{\{\tau>T\}}\right] \\
&= \mathbb{E}\left[\frac{\mathbb{E}[\mathbb{1}_A\mathbb{1}_{\{\tau>t\}} \mid \mathcal{F}_t]}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)}\,\mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \mid \mathcal{F}_t\right]\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left[\frac{\mathbb{1}_A \mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau>t \mid \mathcal{F}_t)} \mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \mid \mathcal{F}_t\right]\right] \\
&= \mathbb{E}\left[\frac{\mathbb{E}[\mathbb{1}_A \mathbb{1}_{\{\tau>t\}} \mid \mathcal{F}_t]}{\mathbb{P}(\tau>t \mid \mathcal{F}_t)} \mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \mid \mathcal{F}_t\right]\right] \\
&= \mathbb{E}\left[\frac{\mathbb{1}_A \mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau>t \mid \mathcal{F}_t)} \mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \mid \mathcal{F}_t\right]\right] \\
&= \mathbb{E}\left[\frac{\mathbb{1}_B \mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau>t \mid \mathcal{F}_t)} \mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \mid \mathcal{F}_t\right]\right],
\end{aligned}
$$

hence by a standard characterization of conditional expectations, see *e.g.* Relation (10.22), we have

$$
\mathbb{E}\left[\mathbb{1}_{\{\tau>t\}} F \mathbb{1}_{\{\tau>T\}} \mid \mathcal{G}_t\right] = \frac{\mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau>t \mid \mathcal{F}_t)} \mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \mid \mathcal{F}_t\right]
$$

□

Taking $F = 1$ in Lemma 7.3 allows one to write the survival probability up to time T, given the information known up to time t, as

$$
\begin{aligned}
\mathbb{P}(\tau>T \mid \mathcal{G}_t) &= \mathbb{E}\left[\mathbb{1}_{\{\tau>T\}} \mid \mathcal{G}_t\right] \qquad (7.4) \\
&= \mathbb{1}_{\{\tau>t\}} \mathbb{E}\left[\exp\left(-\int_t^T \lambda_u du\right) \,\Big|\, \mathcal{F}_t\right], \quad 0 \leqslant t \leqslant T.
\end{aligned}
$$

In particular, applying Lemma 7.3 for $t = T$ and $F = 1$ shows that

$$
\mathbb{E}\left[\mathbb{1}_{\{\tau>t\}} \mid \mathcal{G}_t\right] = \mathbb{1}_{\{\tau>t\}},
$$

which shows that $\{\tau > t\} \in \mathcal{G}_t$ for all $t > 0$, and recovers the fact that τ is a $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$-stopping time, while in general, τ is not $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-stopping time.

The computation of $\mathbb{P}(\tau > T \mid \mathcal{G}_t)$ according to (7.4) is then similar to that of a bond price, by considering the failure rate $\lambda(t)$ as a "virtual" short-term interest rate. In particular the failure rate $\lambda(t,T)$ can be modeled in the HJM framework, cf. *e.g.* Chapter 11.4 of Privault (2014), and

$$
\mathbb{P}(\tau>T \mid \mathcal{G}_t) = \mathbb{E}\left[\exp\left(-\int_t^T \lambda(t,u) du\right) \,\Big|\, \mathcal{F}_t\right]
$$

can then be computed by applying HJM bond pricing techniques.

The computation of expectations given $\mathcal{G}_t$ as in Lemma 7.3 can be useful for pricing under insider trading, in which the insider has access to the augmented filtration $\mathcal{G}_t$ while the ordinary trader has only access to $\mathcal{F}_t$, therefore generating two different prices $\mathbb{E}^*[F \mid \mathcal{F}_t]$ and $\mathbb{E}^*[F \mid \mathcal{G}_t]$ for the same claim payoff F under the same risk-neutral probability measure $\mathbb{P}^*$. This leads to the issue of computing the dynamics of the underlying asset price by decomposing it using a $\mathcal{F}_t$-martingale *vs* a $\mathcal{G}_t$-martingale instead of using different forward measures as in *e.g.* § 12.1 of Privault (2014). This can be obtained by the technique of enlargement of filtration, cf. Jeulin (1980), Elliott and Jeanblanc (1999), Jacod (1985), Yor (1985).

7.3 Defaultable Bonds

Bond pricing models are generally based on the terminal condition $P(T,T) =$ \$1 according to which the bond payoff at maturity is always equal to \$1, and default does not occurs. In this chapter we allow for the possibility of default at a random time τ, in which case the terminal payoff of a bond is allowed to vanish at maturity.

The price $P_d(t,T)$ at time t of a default bond with maturity T, (random) default time τ and (possibly random) recovery rate $\xi \in [0,1]$ is given by

$$P_d(t,T) = \mathbb{E}^*\left[\mathbb{1}_{\{\tau>T\}} \exp\left(-\int_t^T r_u du\right) \,\Big|\, \mathcal{G}_t\right] + \mathbb{E}^*\left[\xi \mathbb{1}_{\{\tau\leqslant T\}} \exp\left(-\int_t^T r_u du\right) \,\Big|\, \mathcal{G}_t\right], \qquad 0 \leqslant t \leqslant T.$$

Proposition 7.4. *The default bond with maturity T and default time τ can be priced at time $t \in [0,T]$ as*

$$P_d(t,T) = \mathbb{1}_{\{\tau>t\}} \mathbb{E}^*\left[\exp\left(-\int_t^T (r_u+\lambda_u) du\right) \,\Big|\, \mathcal{F}_t\right] + \mathbb{E}^*\left[\xi \mathbb{1}_{\{\tau\leqslant T\}} \exp\left(-\int_t^T r_u du\right) \,\Big|\, \mathcal{G}_t\right], \quad 0 \leqslant t \leqslant T.$$

Proof. We take $F = \exp\left(-\int_t^T r_u du\right)$ in Lemma 7.3, which shows that

$$\mathbb{E}^*\left[\mathbb{1}_{\{\tau>T\}} \exp\left(-\int_t^T r_u du\right) \,\Big|\, \mathcal{G}_t\right] = \mathbb{1}_{\{\tau>t\}} \mathbb{E}^*\left[\exp\left(-\int_t^T (r_u+\lambda_u) du\right) \,\Big|\, \mathcal{F}_t\right],$$

cf. *e.g.* Lando (1998), Duffie and Singleton (2003), Guo et al. (2007). □

In the case of complete default (zero-recovery) we have $\xi = 0$ and

$$P_d(t,T) = \mathbb{1}_{\{\tau>t\}} \mathbb{E}^*\left[\exp\left(-\int_t^T (r_s+\lambda_s) ds\right) \,\Big|\, \mathcal{F}_t\right], \quad 0 \leqslant t \leqslant T. \tag{7.5}$$

From the above expression (7.5) we note that the effect of the presence of a default time τ is to decrease the bond price, which can be viewed as an increase of the short rate by the amount λ_u. In a simple setting where the interest rate $r > 0$ and failure rate $\lambda > 0$ are constant, the default bond price becomes

$$P_d(t,T) = \mathbb{1}_{\{\tau>t\}} \mathrm{e}^{-(r+\lambda)(T-t)}, \qquad 0 \leqslant t \leqslant T.$$

Finally, from *e.g.* Proposition 12.1 of Privault (2014) the bond price (7.5) can also be expressed under the forward measure $\widehat{\mathbb{P}}$ with maturity T, as

$$\begin{aligned} P_d(t,T) &= \mathbb{1}_{\{\tau>t\}} \mathbb{E}^* \left[\exp\left(-\int_t^T (r_s + \lambda_s) ds \right) \Bigm| \mathcal{F}_t \right] \\ &= \mathbb{1}_{\{\tau>t\}} \mathbb{E}^* \left[\exp\left(-\int_t^T r_s ds \right) \Bigm| \mathcal{F}_t \right] \widehat{\mathbb{E}} \left[\exp\left(-\int_t^T \lambda_s ds \right) \Bigm| \mathcal{F}_t \right] \\ &= \mathbb{1}_{\{\tau>t\}} N_t \widehat{\mathbb{P}}(\tau > T \mid \mathcal{G}_t), \end{aligned}$$

where $(N_t)_{t\in\mathbb{R}_+}$ is the numéraire process

$$N_t := P(t,T) = \mathbb{E}^* \left[\exp\left(-\int_t^T r_s ds \right) \Bigm| \mathcal{F}_t \right], \qquad 0 \leqslant t \leqslant T,$$

and by (7.4),

$$\widehat{\mathbb{P}}(\tau > T \mid \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \widehat{\mathbb{E}} \left[\exp\left(-\int_t^T \lambda_s ds \right) \Bigm| \mathcal{F}_t \right]$$

is the survival probability under the forward measure $\widehat{\mathbb{P}}$ defined as

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} := \frac{N_T}{N_0} \mathrm{e}^{-\int_0^T r_t dt},$$

cf. Chen and Huang (2001), Chen et al. (2008),

Estimating the default rates

Recall that the price of a default bond with maturity T, (random) default time τ and (possibly random) recovery rate $\xi \in [0,1]$ is given by

$$\begin{aligned} P_d(t,T) &= \mathbb{1}_{\{\tau>t\}} \mathbb{E}^* \left[\exp\left(-\int_t^T (r_u + \lambda_u) du \right) \Bigm| \mathcal{F}_t \right] \\ &\quad + \mathbb{E}^* \left[\xi \mathbb{1}_{\{\tau \leqslant T\}} \exp\left(-\int_t^T r_u du \right) \Bigm| \mathcal{G}_t \right], \qquad 0 \leqslant t \leqslant T, \end{aligned}$$

where ξ denotes the recovery rate. We consider a simplified deterministic step function model with zero recovery rate and tenor structure

$$\{t = T_0 < T_1 < \cdots < T_n = T\},$$

where

$$r(t) = \sum_{l=0}^{n-1} r_l \mathbb{1}_{(T_l,T_{l+1}]}(t) \quad \text{and} \quad \lambda(t) = \sum_{l=0}^{n-1} \lambda_l \mathbb{1}_{(T_l,T_{l+1}]}(t), \quad t \in \mathbb{R}_+. \quad (7.6)$$

i) Estimating the default rates from default bond prices.

We have

$$\begin{aligned} P_d(t, T_k) &= \mathbb{1}_{\{\tau>t\}} \exp\left(-\int_t^{T_k} (r(u) + \lambda(u))du\right) \\ &= \mathbb{1}_{\{\tau>t\}} \exp\left(-\sum_{l=0}^{k-1} (r_l + \lambda_l)(T_{l+1} - T_l)\right), \end{aligned}$$

$k = 1, 2, \ldots, n$, from which we can infer

$$\lambda_k = -r_k + \frac{1}{T_{k+1} - T_k} \log \frac{P_d(T_{k+1}, T)}{P_d(T_k, T)} > 0, \qquad k = 0, 1, \ldots, n-1.$$

ii) Estimating (implied) default probabilities $\mathbb{P}^*(\tau < T \mid \mathcal{G}_t)$ from default rates.

Based on the expression

$$\begin{aligned} \mathbb{P}^*(\tau > T \mid \mathcal{G}_t) &= \mathbb{E}^*\left[\mathbb{1}_{\{\tau>T\}} \mid \mathcal{G}_t\right] \quad (7.7) \\ &= \mathbb{1}_{\{\tau>t\}} \mathbb{E}^*\left[\exp\left(-\int_t^T \lambda_u du\right) \Bigm| \mathcal{F}_t\right], \qquad 0 \leqslant t \leqslant T, \end{aligned}$$

of the survival probability up to time T, and given the information known up to time t, in terms of the hazard rate process $(\lambda_u)_{u\in\mathbb{R}_+}$ adapted to a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, we find

$$\begin{aligned} \mathbb{P}(\tau > T \mid \mathcal{G}_{T_k}) &= \mathbb{1}_{\{\tau>T_k\}} \exp\left(-\int_{T_k}^T \lambda_u du\right) \\ &= \mathbb{1}_{\{\tau>t\}} \exp\left(-\sum_{l=k}^{n-1} \lambda_l (T_{l+1} - T_l)\right), \quad k = 0, 1, \ldots, n-1, \end{aligned}$$

where

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau \leqslant u\} : 0 \leqslant u \leqslant t), \quad t \in \mathbb{R}_+,$$

i.e. $\mathcal{G}_t$ contains the additional information on whether default at time τ has occurred or not before time t.

In Table 7.2, bond ratings are determined according to hazard (or failure) rate thresholds.

Bond Credit Ratings	Moody's		S & P	
	Municipal	Corporate	Municipal	Corporate
Aaa/AAAs	0.00	0.52	0.00	0.60
Aa/AA	0.06	0.52	0.00	1.50
A/A	0.03	1.29	0.23	2.91
Baa/BBB	0.13	4.64	0.32	10.29
Ba/BB	2.65	19.12	1.74	29.93
B/B	11.86	43.34	8.48	53.72
Caa-C/CCC-C	16.58	69.18	44.81	69.19
Investment Grade	0.07	2.09	0.20	4.14
Non-Invest. Grade	4.29	31.37	7.37	42.35
All	0.10	9.70	0.29	12.98

Table 7.2: Cumulative historic default rates (in percentage).*

Exercises

Exercise 7.1 Consider a standard zero-coupon bond with constant yield $r > 0$ and a defaultable (risky) bond with constant yield r_d and default probability $\alpha \in (0, 1)$. Find a relation between r, r_d, α and the bond maturity T.

Exercise 7.2 A standard zero-coupon bond with constant yield $r > 0$ and maturity T is priced $P(t, T) = \mathrm{e}^{-(T-t)r}$ at time $t \in [0, T]$. Assume that the company can get bankrupt at a random time $t + \tau$, and default on its final \$1 payment if $\tau < T - t$.

a) Explain why the defaultable bond price $P_d(t, T)$ can be expressed as

$$P_d(t, T) = \mathrm{e}^{-(T-t)r} \, \mathbb{E}^* \left[\mathbb{1}_{\{\tau > T-t\}} \right]. \tag{7.8}$$

b) Assuming that the default time τ is exponentially distributed with parameter $\lambda > 0$, compute the default bond price $P_d(t, T)$ using (7.8).
c) Find a formula that can estimate the paramater λ from the risk-free rate r and the market data $P_M(t, T)$ of the defaultable bond price at time $t \in [0, T]$.

Exercise 7.3 Consider a (random) default time τ with cumulative distribution function

$$\mathbb{P}(\tau > t \mid \mathcal{F}_t) = \exp\left(- \int_0^t \lambda_u du \right), \qquad t \in \mathbb{R}_+,$$

* Source: Moody's, S&P.

where λ_t is a (random) default rate process which is adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Recall that the probability of survival up to time T, given the information known up to time t, is given by

$$\mathbb{P}(\tau > T \mid \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\, \mathbb{E}^*\left[\exp\left(-\int_t^T \lambda_u du\right) \Bigm| \mathcal{F}_t\right],$$

where $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau < u\} : 0 \leqslant u \leqslant t)$, $t \in \mathbb{R}_+$, is the filtration defined by adding the default time information to the history $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. In this framework, the price $P(t,T)$ of defaultable bond with maturity T, short-term interest rate r_t and (random) default time τ is given by

$$\begin{aligned} P(t,T) &= \mathbb{E}^*\left[\mathbb{1}_{\{\tau>T\}} \exp\left(-\int_t^T r_u du\right) \Bigm| \mathcal{G}_t\right] \qquad (7.9)\\ &= \mathbb{1}_{\{\tau>t\}}\, \mathbb{E}^*\left[\exp\left(-\int_t^T (r_u+\lambda_u) du\right) \Bigm| \mathcal{F}_t\right]. \end{aligned}$$

In the sequel we assume that the processes $(r_t)_{t\in\mathbb{R}_+}$ and $(\lambda_t)_{t\in\mathbb{R}_+}$ are modeled according to the Vasicek processes

$$\begin{cases} dr_t = -ar_t dt + \sigma dB_t^{(1)}, \\ \\ d\lambda_t = -b\lambda_t dt + \eta dB_t^{(2)}, \end{cases}$$

where $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$ and $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$ are standard $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-Brownian motions with correlation $\rho \in [-1,1]$, and $dB_t^{(1)} \bullet dB_t^{(2)} = \rho dt$.

a) Give a justification for the fact that

$$\mathbb{E}^*\left[\exp\left(-\int_t^T (r_u+\lambda_u) du\right) \Bigm| \mathcal{F}_t\right]$$

can be written as a function $F(t, r_t, \lambda_t)$ of t, r_t and λ_t, $t \in [0,T]$.

b) Show that

$$t \longmapsto \exp\left(-\int_0^t (r_s+\lambda_s) ds\right) \mathbb{E}^*\left[\exp\left(-\int_t^T (r_u+\lambda_u) du\right) \Bigm| \mathcal{F}_t\right]$$

is an $\mathcal{F}_t$-martingale under $\mathbb{P}$.

c) Use the Itô formula with two variables to derive a PDE on $\mathbb{R}^2$ for the function $F(t,x,y)$.

d) Taking $r_0 := 0$, show that we have

$$\int_t^T r_s ds = C(a,t,T) r_t + \sigma \int_t^T C(a,s,T) dB_s^{(1)},$$

and

$$\int_t^T \lambda_s ds = C(b,t,T)\lambda_t + \eta \int_t^T C(b,s,T) dB_s^{(2)},$$

where

$$C(a,t,T) = -\frac{1}{a}(\mathrm{e}^{-(T-t)a} - 1).$$

e) Show that the random variable

$$\int_t^T r_s ds + \int_t^T \lambda_s ds$$

is has a Gaussian distribution, and compute its conditional mean

$$\mathbb{E}^*\left[\int_t^T r_s ds + \int_t^T \lambda_s ds \;\Big|\; \mathcal{F}_t\right]$$

and variance

$$\mathrm{Var}\left[\int_t^T r_s ds + \int_t^T \lambda_s ds \;\Big|\; \mathcal{F}_t\right],$$

conditionally to $\mathcal{F}_t$.

f) Compute $P(t,T)$ from its expression (7.9) as a conditional expectation.

g) Show that the solution $F(t,x,y)$ to the 2-dimensional PDE of Question (c) is

$$\begin{aligned} F(t,x,y) &= \exp\left(-C(a,t,T)x - C(b,t,T)y\right) \\ &\quad \times \exp\left(\frac{\sigma^2}{2}\int_t^T C^2(a,s,T)ds + \frac{\eta^2}{2}\int_t^T C^2(b,s,T)ds\right) \\ &\quad \times \exp\left(\rho\sigma\eta \int_t^T C(a,s,T)C(b,s,T)ds\right). \end{aligned}$$

h) Show that the defaultable bond price $P(t,T)$ can also be written as

$$P(t,T) = \mathrm{e}^{U(t,T)} \mathbb{P}(\tau > T \mid \mathcal{G}_t)\, \mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right) \;\Big|\; \mathcal{F}_t\right],$$

where

$$U(t,T) = \rho\frac{\sigma\eta}{ab}\left(T - t - C(a,t,T) - C(b,t,T) + C(a+b,t,T)\right).$$

i) By partial differentiation of $\log P(t,T)$ with respect to T, compute the corresponding instantaneous short rate

$$f(t,T) = -\frac{\partial}{\partial T}\log P(t,T).$$

j) Show that $\mathbb{P}(\tau > T \mid \mathcal{G}_t)$ can be written using an HJM type default rate as

$$\mathbb{P}(\tau > T \mid \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \exp\left(- \int_t^T f_2(t,u) du \right),$$

where

$$f_2(t,u) = \lambda_t e^{-(u-t)b} - \frac{\eta^2}{2} C^2(b,t,u).$$

k) Show how the result of Question (h) can be simplified when $\left(B_t^{(1)}\right)_{t\in\mathbb{R}_+}$ and $\left(B_t^{(2)}\right)_{t\in\mathbb{R}_+}$ are independent.