Chapter 8
Credit Derivatives

Credit derivatives are option contracts that offer protection against default risk in a creditor/debtor relationship by transferring risk to a third party. The credit derivatives considered in this chapter are Collateralized Debt Obligations (CDOs) and Credit Default Swaps (CDSs) that may be used as a protection against default risk. We also deal with counterparty default risk via Credit Valuation Adjustments (CVAs).

8.1 Credit Default Swaps (CDS)

According to the Bank for International Settlements, the outstanding notional amounts of credit default swap (CDS) contracts has decreased from $61.2 trillion at year-end 2007 to $7.6 trillion at year-end 2019.

We work with a tenor structure $\{t = T_i < \cdots < T_j = T\}$. Here, τ is a default time and the filtration $\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\tau)$ contains the additional information on τ, as defined in (7.3).

A Credit Default Swap (CDS) is a contract consisting in

- *A premium leg*: the buyer is purchasing protection at time t against default at time T_k, $k = i+1, \ldots, j$, and has to make a fixed spread payment $S_t^{i,j}$ at times $T_{i+1}, \ldots, T_j$ between t and T in compensation.

  The discounted value at time t of the premium leg is

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$$
\begin{aligned}
V^p(t,T) &= \mathbb{E}\left[\sum_{k=i}^{j-1} S_t^{i,j}\delta_k \mathbb{1}_{\{\tau>T_{k+1}\}} \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \;\Big|\; \mathcal{G}_t\right] \\
&= \sum_{k=i}^{j-1} S_t^{i,j}\delta_k\, \mathbb{E}\left[\mathbb{1}_{\{\tau>T_{k+1}\}} \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \;\Big|\; \mathcal{G}_t\right] \\
&= S_t^{i,j} \sum_{k=i}^{j-1} \delta_k P(t,T_{k+1}) \\
&= S_t^{i,j} P(t,T_i,T_j),
\end{aligned}
\tag{8.1}
$$

where $\delta_k = T_{k+1} - T_k$,

$$
P(t,T_k) = \mathbb{1}_{\{\tau>t\}}\,\mathbb{E}\left[\exp\left(-\int_t^{T_k}(r_s+\lambda_s)ds\right) \;\Big|\; \mathcal{F}_t\right], \qquad 0 \leqslant t \leqslant T_k,
$$

is the defaultable bond price with maturity T_k, $k = i, \ldots, j-1$, see Lemma 7.3, and

$$
P(t,T_i,T_j) = \sum_{k=i}^{j-1} \delta_k P(t,T_{k+1})
$$

is the (default) annuity numéraire, cf. *e.g.* Relation (12.10) in Privault (2014),

For simplicity we have ignored a possible accrual interest term over the time interval $[T_k,\tau]$ when $\tau \in [T_k,T_{k+1}]$ in the above value of the premium leg.

- *A protection leg*: the seller or issuer of the contract makes a compensation payment $1-\xi_{k+1}$ to the buyer in case default occurs at time T_{k+1}, $k = i, \ldots, j-1$, where ξ_{k+1} is the *recovery rate.*

  The value at time t of the protection leg is

$$
V^d(t,T) := \mathbb{E}\left[\sum_{k=i}^{j-1} \mathbb{1}_{(T_k,T_{k+1}]}(\tau)(1-\xi_{k+1}) \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \;\Big|\; \mathcal{G}_t\right],
\tag{8.2}
$$

where ξ_{k+1} is the recovery rate associated with the maturity T_{k+1}, $k = i, \ldots, j-1$.

In the case of a non-random recovery rate ξ_k, the value of the protection leg becomes

$$\sum_{k=i}^{j-1}(1-\xi_{k+1})\,\mathbb{E}\left[\mathbb{1}_{(T_k,T_{k+1}]}(\tau)\exp\left(-\int_t^{T_{k+1}} r_s ds\right) \Bigm| \mathcal{G}_t\right].$$

The spread $S_t^{i,j}$ is computed by equating the values of the premium (8.1) and protection (8.2) legs, *i.e.* from the relation

$$\begin{aligned} V^p(t,T) &= S_t^{i,j} P(t,T_i,T_j) \\ &= \mathbb{E}\left[\sum_{k=i}^{j-1} \mathbb{1}_{(T_k,T_{k+1}]}(\tau)(1-\xi_{k+1})\exp\left(-\int_t^{T_{k+1}} r_s ds\right) \Bigm| \mathcal{G}_t\right], \end{aligned}$$

which yields

$$S_t^{i,j} = \frac{1}{P(t,T_i,T_j)}\sum_{k=i}^{j-1}\mathbb{E}\left[\mathbb{1}_{(T_k,T_{k+1}]}(\tau)(1-\xi_{k+1})\exp\left(-\int_t^{T_{k+1}} r_s ds\right) \Bigm| \mathcal{G}_t\right]. \tag{8.3}$$

The spread $S_t^{i,j}$, which is quoted in basis points per year and paid at regular time intervals, gives protection against defaults on payments of \$1. For a notional amount N the premium payment will become $N \times S_t^{i,j}$.

In the case of a constant recovery rate ξ, we find

$$S_t^{i,j} = \frac{1-\xi}{P(t,T_i,T_j)}\sum_{k=i}^{j-1}\mathbb{E}\left[\mathbb{1}_{(T_k,T_{k+1}]}(\tau)\exp\left(-\int_t^{T_{k+1}} r_s ds\right) \Bigm| \mathcal{G}_t\right],$$

and if τ is constrained to take values in the tenor structure $\{t = T_i, \ldots, T_j\}$, we get

$$S_t^{i,j} = \frac{1-\xi}{P(t,T_i,T_j)}\,\mathbb{E}\left[\mathbb{1}_{(t,T]}(\tau)\exp\left(-\int_t^{\tau} r_s ds\right) \Bigm| \mathcal{G}_t\right].$$

The buyer of a Credit Default Swap (CDS) is purchasing protection at time t against default at time T_k, $k = i+1, \ldots, j$, by making a fixed payment $S_t^{i,j}$ (the premium leg) at times $T_{i+1}, \ldots, T_j$. On the other hand, the issuer of the contract makes a payment $1-\xi_{k+1}$ to the buyer in case default occurs at time T_{k+1}, $k = i, \ldots, j-1$.

The contract is priced in terms of the swap rate $S_t^{i,j}$ (or spread) computed by equating the values $V^d(t,T)$ and $V^p(t,T)$ of the protection and premium legs, and acts as a compensation that makes the deal fair to both parties. Recall that from (8.3) and Lemma 7.3, we have

$$
\begin{aligned}
S_t^{i,j} &= \frac{1}{P(t,T_i,T_j)} \sum_{k=i}^{j-1} \mathbb{E}\left[\mathbb{1}_{(T_k,T_{k+1}]}(\tau)(1-\xi_{k+1}) \exp\left(-\int_t^{T_{k+1}} r_s ds \right) \Bigm| \mathcal{G}_t \right] \\
&= \frac{1}{P(t,T_i,T_j)} \sum_{k=i}^{j-1} \mathbb{E}\left[(\mathbb{1}_{\{T_k<\tau\}} - \mathbb{1}_{\{T_{k+1}<\tau\}})(1-\xi_{k+1}) \exp\left(-\int_t^{T_{k+1}} r_s ds \right) \Bigm| \mathcal{G}_t \right] \\
&= \frac{\mathbb{1}_{\{\tau>t\}}}{P(t,T_i,T_j)} \sum_{k=i}^{j-1} \mathbb{E}\left[(1-\xi_{k+1}) \left(\exp\left(-\int_t^{T_k} \lambda_s ds \right) - \exp\left(-\int_t^{T_{k+1}} \lambda_s ds \right) \right) \right. \\
&\quad \left. \times \exp\left(-\int_t^{T_{k+1}} r_s ds \right) \Bigm| \mathcal{F}_t \right].
\end{aligned}
$$

Estimating a deterministic failure rate

In case the rates $r(s)$, $\lambda(s)$ and the recovery rate ξ_{k+1} are deterministic, the above spread can be written as

$$
\begin{aligned}
S_t^{i,j} P(t,T_i,T_j) = \mathbb{1}_{\{\tau>t\}} \sum_{k=i}^{j-1} (1-\xi_k) \exp\left(-\int_t^{T_{k+1}} r(s) ds \right) \\
\times \left(\exp\left(-\int_t^{T_k} \lambda_s ds \right) - \exp\left(-\int_t^{T_{k+1}} \lambda_s ds \right) \right).
\end{aligned}
$$

Given that

$$
P(t,T_i,T_j) = \sum_{k=i}^{j-1} (T_{k+1}-T_k) P(t,T_{k+1}), \qquad t \in [T_i,T_j],
$$

we can write

$$
\begin{aligned}
& S_t^{i,j} \sum_{k=i}^{j-1} (T_{k+1}-T_k) \exp\left(-\int_t^{T_{k+1}} (r(s)+\lambda(s)) ds \right) \\
&= \mathbb{1}_{\{\tau>t\}} \sum_{k=i}^{j-1} (1-\xi_k) \exp\left(-\int_t^{T_{k+1}} r(s) ds \right) \left(\exp\left(-\int_t^{T_k} \lambda(s) ds \right) - \exp\left(-\int_t^{T_{k+1}} \lambda(s) ds \right) \right).
\end{aligned}
$$

In particular, when $r(t)$ and $\lambda(t)$ are written as in (7.6) and assuming that $\xi_k = \xi$, $k = i,\ldots,j$, we get, with $t = T_i$ and writing $\delta_k = T_{k+1} - T_k$, $k = i,\ldots,j-1$,

$$
\begin{aligned}
& S_{T_i}^{i,j} \sum_{k=i}^{j-1} \delta_{k+1} \exp\left(-\sum_{p=i}^{k} \delta_p (r_p + \lambda_p) \right) \\
&= \mathbb{1}_{\{\tau>t\}} (1-\xi) \sum_{k=i}^{j-1} \exp\left(-\sum_{p=i}^{k} \delta_p (r_p + \lambda_p) \right) (1 - \mathrm{e}^{-\delta_k \lambda_k}).
\end{aligned}
$$

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Assuming further that $\lambda_k = \lambda$, $k = i, \ldots, j$, we have

$$S_{T_i}^{i,j} \sum_{k=i}^{j-1} \delta_k \exp\left(- \sum_{p=i}^{k} \delta_p r_p - \lambda \sum_{p=i}^{k} \delta_p \right) \tag{8.4}$$
$$= (1-\xi) \sum_{k=i}^{j-1} (1 - \mathrm{e}^{-\lambda \delta_k}) \exp\left(- \sum_{p=i}^{k} \delta_p r_p - \lambda \sum_{p=i}^{k} \delta_p \right),$$

which can be solved numerically for λ, cf. Sections 4 and 5 of Castellacci (2008) for the JP Morgan model, and Exercise 8.1. Note that, as λ tends to ∞, the ratio

$$\frac{S_{T_i}^{i,j} \sum_{k=i}^{j-1} \delta_k \exp\left(- \sum_{p=i}^{k} \delta_p r_p - \lambda \sum_{p=i}^{k} \delta_p \right)}{(1-\xi) \sum_{k=i}^{j-1} (1 - \mathrm{e}^{-\lambda \delta_k}) \exp\left(- \sum_{p=i}^{k} \delta_p r_p - \lambda \sum_{p=i}^{k} \delta_p \right)}$$

converges to $S_{T_i}^{i,j} \delta_k / (1-\xi) \leqslant 1$, while it tends to $+\infty$ as λ goes to 0. Therefore, the equation (8.4) admits a numerical solution as we normally have $S_{T_i}^{i,j} \delta_k / (1-\xi) \leqslant 1$ under standard market conditions, see Exercise 8.1.

8.2 Collateralized Debt Obligations (CDO)

Consider a portfolio consisting of $N = j - i$ bonds with default times $\tau_k \in (T_k, T_{k+1}]$, $k = i, \ldots, j-1$, and recovery rates $\xi_k \in [0,1]$, $k = i+1, \ldots, j$.

A synthetic CDO is a structured investment product constructed by splitting the above portfolio into n ordered tranches numbered $i = 1, 2, \ldots, n$, where tranche n°i represents a percentage $p_i\%$ of the total portfolio value. We let

$$\alpha_l = p_1 + p_2 + \cdots + p_l, \qquad l = 1, 2, \ldots, n,$$

with $\alpha_n = p_1 + p_2 + \cdots + p_n = 100\%$ and $\alpha_0 = 0$.

The tranches are ordered according to increasing default risk, tranche n°1 being the riskiest one ("equity tranche"), and tranche n°n being the safest one ("senior tranche"), while the intermediate tranches are referred to as "mezzanine tranches". In practice, losses occur first to the "equity" tranches, next to the "mezzanine" tranche holders, and finally to "senior" tranches.

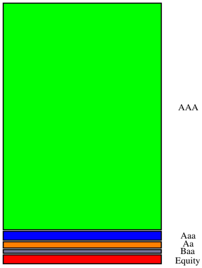

Fig. 8.1: A representation of CDO tranches.

CDOs can attract different types of investors.

- Unfunded investors (usually for the higher tranches) are receiving premiums and make payments in case of default.
- Funded investors (usually in the lower tranches) are investing in risky bonds to receive principal payments at maturity, and they are the first in line to incur losses.
- A CDO can also be used as a Credit Default Swap (CDS) for the "short investors" who make premium payments in exchange for credit protection in case of default.

The market for synthetic CDOs has been significantly reduced since the 2006-2008 subprime crisis.

Synthetic CDOs are based on $N = j - i$ bonds that can potentially generate a cumulative loss

$$L_t := \sum_{l=i}^{j-1} (1 - \xi_{l+1}) \mathbb{1}_{\{\tau_l \leqslant t\}} \in [0, N],$$

at time $t \in [T_i, T_j]$, based on the default time τ_l and recovery rate ξ_{l+1} of each involved CDS, $k = i, \ldots, j-1$, with $N = j - i$.

When the first loss occurs, tranche n°1 is the first in line, and it loses the amount

$$L_t^1 = L_t \mathbb{1}_{\{L_t \leqslant p_1 N\}} + N p_1 \mathbb{1}_{\{L_t > p_1 N\}} = N \min(L_t / N, p_1).$$

In case $L_t > p_1 N$, then tranche n°2 takes the remaining loss up to the amount $N p_2$, that means the loss L_t^2 of tranche n°2 is

$$\begin{aligned} L_t^2 &= (L_t - N p_1) \mathbb{1}_{\{p_1 N < L_t \leqslant (p_1 + p_2) N\}} + N p_2 \mathbb{1}_{\{L_t > (p_1 + p_2) N\}} \\ &= (L_t - N p_1) \mathbb{1}_{\{p_1 N < L_t \leqslant \alpha_2 N\}} + N p_2 \mathbb{1}_{\{L_t > \alpha_2 N\}} \end{aligned}$$

$$= (L_t - Np_1)^+ \mathbb{1}_{\{L_t \leqslant \alpha_2 N\}} + Np_2 \mathbb{1}_{\{L_t > \alpha_2 N\}}$$
$$= \min((L_t - Np_1)^+, Np_2)$$
$$= \max(\min(L_t, Np_1 + Np_2) - Np_1, 0)$$
$$= \max(\min(L_t, N\alpha_2) - Np_1, 0).$$

By induction, the potential loss taken by tranche n°i is given by

$$\begin{aligned} L_t^i &= (L_t - N\alpha_{i-1}) \mathbb{1}_{\{\alpha_{i-1} N < L_t \leqslant \alpha_i N\}} + Np_i \mathbb{1}_{\{L_t > \alpha_i N\}} \\ &= (L_t - N\alpha_{i-1})^+ \mathbb{1}_{\{L_t \leqslant \alpha_i N\}} + Np_i \mathbb{1}_{\{L_t > \alpha_i N\}} \\ &= \min((L_t - N\alpha_{i-1})^+, Np_i) \\ &= \max(\min(L_t, N\alpha_i) - N\alpha_{i-1}, 0), \end{aligned}$$

where $\alpha_i := p_1 + p_2 + \cdots + p_i$, $i = 1, 2, \ldots, n$.

In the end, tranche n°n will take the loss

$$L_t^n = (L_t - N\alpha_{n-1}) \mathbb{1}_{\{\alpha_{n-1} N < L_t\}} = (L_t - N\alpha_{n-1})^+.$$

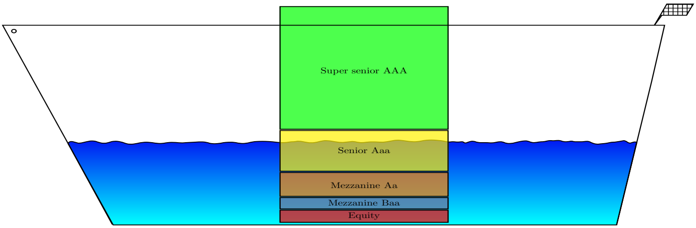

Fig. 8.2: A Titanic-style representation of cumulative tranche losses.

The CDO tranche n°l, $l = 1, 2, \ldots, n$, can be decomposed into:

- A premium leg: the short investor in tranche n°l is purchasing protection at time t against default at time T_k, $k = i+1, \ldots, j$, by making fixed payments $S_t^{i,j}$ at times $T_{i+1}, \ldots, T_j$ between t and T in compensation. This premium can also be received by the unfunded investor.

  The discounted value at time t of the premium leg for the tranche n°l is

$$V_l^p(t,T) = \mathbb{E}\left[\sum_{k=i}^{j-1} S_t^l \delta_k (Np_l - L_{T_{k+1}}^l) \exp\left(- \int_t^{T_{k+1}} r_s ds \right) \ \Big| \ \mathcal{G}_t \right]$$

$$= S_t^l \sum_{k=i}^{j-1} \delta_k \, \mathbb{E}\left[(Np_l - L^l_{T_{k+1}}) \exp\left(- \int_t^{T_{k+1}} r_s ds \right) \ \Big| \ \mathcal{G}_t \right] \tag{8.5}$$

$l = 1, 2, \ldots, N$, where the premium spread S_t^l is quoted as a proportion of the compensation $Np_l - L^l_{T_{k+1}}$ and paid at each time T_{k+1} until $k = j-1$ or $L_{T_{k+1}} = 100\%$, whichever comes first, with $\delta_k = T_{k+1} - T_k$, $k = i, \ldots, j-1$.

- A protection leg: the short investor receives protection against default from the premium leg, which can also be paid by the unfunded investors. Noting that at each default time $\tau_k \in (T_k, T_{k+1}]$, $k = i, \ldots, j-1$, the loss L_t^l taken by tranche n°l jumps by the amount $\Delta L^l_{\tau_k} = L^l_{\tau_k} - L^l_{\tau_k^-}$, the value at time t of the protection leg is

$$\begin{aligned} V_l^d(t,T) &= \mathbb{E}\left[\sum_{k=i}^{j-1} \mathbb{1}_{[T_i,T_j]}(\tau_k) \Delta L^l_{\tau_k} \exp\left(- \int_t^{\tau_k} r_u du \right) \ \Big| \ \mathcal{G}_t \right] \\ &= \mathbb{E}\left[\int_{T_i}^{T_j} \exp\left(- \int_t^s r_u du \right) dL_s^l \Big| \mathcal{G}_t \right] \qquad (8.6) \\ &= \mathbb{E}\left[\exp\left(- \int_t^{T_j} r_u du \right) L^l_{T_j} - \exp\left(- \int_t^{T_i} r_u du \right) L^l_{T_i} \Big| \mathcal{G}_t \right] \\ &\quad + \mathbb{E}\left[\int_{T_i}^{T_j} r_s \exp\left(- \int_t^s r_u du \right) L_s^l ds \Big| \mathcal{G}_t \right] \\ &= \mathbb{E}\left[\exp\left(- \int_t^{T_j} r_u du \right) L^l_{T_j} \Big| \mathcal{G}_t \right] + \mathbb{E}\left[\int_{T_i}^{T_j} r_s \exp\left(- \int_t^s r_u du \right) L_s^l ds \Big| \mathcal{G}_t \right], \end{aligned}$$

where we applied integration by parts on $[T_i, T_j]$ and used the fact that $L_{T_i} = 0$.

The spread S_t^l paid by tranche n°l is computed by equating the values $V_l^p(t,T) = V_l^d(t,T)$ of the protection and premium legs in (8.5) and (8.6), which yields

$$\begin{aligned} S_t^l &= \frac{\mathbb{E}\left[\int_{T_i}^{T_j} \exp\left(- \int_t^s r_u du \right) dL_s^l \Big| \mathcal{G}_t \right]}{\sum_{k=i}^{j-1} \delta_k \, \mathbb{E}\left[(Np_l - L^l_{T_{k+1}}) \exp\left(- \int_t^{T_{k+1}} r_s ds \right) \ \Big| \ \mathcal{G}_t \right]} \\ &= \frac{\mathbb{E}\left[\exp\left(- \int_t^{T_j} r_u du \right) L^l_{T_j} \Big| \mathcal{G}_t \right] + \mathbb{E}\left[\int_{T_i}^{T_j} r_s \exp\left(- \int_t^s r_u du \right) L_s^l ds \Big| \mathcal{G}_t \right]}{\sum_{k=i}^{j-1} \delta_k \, \mathbb{E}\left[(Np_l - L^l_{T_{k+1}}) \exp\left(- \int_t^{T_{k+1}} r_s ds \right) \ \Big| \ \mathcal{G}_t \right]} \\ &\geqslant 0, \end{aligned}$$

$l = 1, 2, \ldots, n$.

Expected tranche loss

The expected cumulative loss can be computed by linearity as

$$\begin{aligned}\mathbb{E}[L_t \mid M = m] &= \sum_{l=i}^{j-1} \mathbb{E}\left[(1-\xi_{l+1})\mathbb{1}_{\{\tau_l \leqslant t\}} \mid M = m\right] \\ &= \sum_{l=i}^{j-1} (1-\xi_{l+1})\mathbb{P}\left(\tau_l \leqslant t \mid M = m\right) \\ &= \sum_{l=i}^{j-1} (1-\xi_{l+1})\Phi\left(\frac{\Phi^{-1}(\mathbb{P}(\tau_l \leqslant t)) + a_k m}{\sqrt{1-a_k^2}}\right),\end{aligned}$$

by (6.5), and the expected cumulative loss can be written as

$$\mathbb{E}[L_t] = \int_{-\infty}^{\infty} \mathbb{E}\left[L_t \mid M = m\right] \phi(m) dm = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbb{E}\left[L_t \mid M = m\right] \mathrm{e}^{-m^2/2} dm.$$

The situation is different for the expected loss of tranche n°k is written as the expected value

$$\mathbb{E}[L_t^k] = \mathbb{E}\left[\min((L_t - N\alpha_{k-1})^+, Np_k)\right], \qquad k = 1, 2, \ldots, n.$$

of the *nonlinear* function $f_k(x) := \min((x - N\alpha_{k-1})^+, Np_k)$ of L_t.

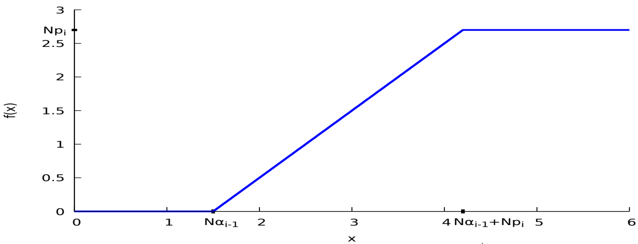

Fig. 8.3: Function $f_k(x) = \min((x - N\alpha_{k-1})^+, Np_k)$.

The expected tranche loss $\mathbb{E}[L_t^k]$ n°k can be estimated by the Monte Carlo method when the default times are generated according to (6.8).

In order to compute expected tranche losses we can use the fact that the cumulative loss L_t is a discrete random variable, with for example

$$\mathbb{P}\left(L_t = N - \sum_{k=i}^{j-1} \xi_{k+1}\right) = \mathbb{P}(\tau_i \leqslant t, \ldots, \tau_{j-1} \leqslant t),$$

and

$$\mathbb{P}(L_t = 0) = \mathbb{P}(\tau_i > t, \ldots, \tau_{j-1} > t),$$

which require the knowledge of the joint distribution of the default times $\tau_i, \ldots, \tau_{j-1}$.

If the $\tau_k' s$ are independent and identically distributed with common cumulative distribution function F_τ and $a_k = a$, $\xi_k = \xi$, $k = i+1, \ldots, j$, then the cumulative loss L_t has a binomial distribution given M, given by

$$\mathbb{P}(L_t = (1-\xi)k \mid M) = \binom{N}{k}(1 - \mathbb{P}(\tau \leqslant T \mid M))^{N-k}(\mathbb{P}(\tau \leqslant T \mid M))^k$$

$$= \binom{N}{k}\left(1 - \Phi\left(\frac{\Phi^{-1}(F_\tau(T)) - aM}{\sqrt{1-a^2}}\right)\right)^{N-k}\left(\Phi\left(\frac{\Phi^{-1}(F_\tau(T)) - aM}{\sqrt{1-a^2}}\right)\right)^k,$$

$k = 0, 1, \ldots, N$. The expected loss of tranche n°k can then be expressed as

$$\begin{aligned}\mathbb{E}[L_t^k] &= \int_{-\infty}^{\infty} \mathbb{E}\left[f_k(L_t) \mid M = m\right] \phi(m) dm \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbb{E}\left[f_k(L_t) \mid M = m\right] \mathrm{e}^{-m^2/2} dm,\end{aligned}$$

$k = 1, 2, \ldots, n$, where $\mathbb{E}\left[f_k(L_t) \mid M = m\right]$ is computed either by the Monte Carlo method, from the distribution of L_t.

In Vašiček (2002), the tranche loss has been approximated by a Gaussian random variable for very large portfolios with $N \to \infty$.

The α-percentile loss of the portfolio can be estimated as

$$\mathbb{E}[L_t \mid M = m] = \sum_{k=i}^{j-1}(1 - \xi_{k+1})\Phi\left(\frac{\Phi^{-1}(\mathbb{P}(\tau_k \leqslant T)) - a_k\Phi^{-1}(\alpha)}{\sqrt{1-a_k^2}}\right),$$

where $m = \Phi^{-1}(\alpha)$.

Such (Gaussian) Merton (1974) and Vašiček (2002) type models have been implemented in the Basel II recommandations on Banking Supervision (2005). Namely in Basel II, banks are expected to hold capital in prevision of unexpected losses in a worst case scenario, according to the Internal Ratings-Based (IRB) formula

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$$\sum_{k=i}^{j-1}(1-\xi_{k+1})\left(\Phi\left(\frac{\Phi^{-1}(\mathbb{P}(\tau_k \leqslant T)) - a_k\Phi^{-1}(\alpha)}{\sqrt{1-a_k^2}}\right) - \mathbb{P}(\tau_k \leqslant T)\right),$$

with confidence level set at $\alpha = 0.999$ *i.e.* $m = \Phi^{-1}(0.999) = 3.09$, cf. Relation (2.4) page 10 of Aas (2005). Recall that the function

$$x \longmapsto \Phi\left(\frac{\Phi^{-1}(x) + a_k\Phi^{-1}(\alpha)}{\sqrt{1-a_k^2}}\right)$$

always lies above the graph of x when $a_k < 0$, as in the next figure.

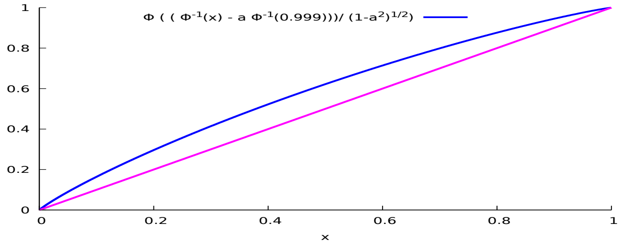

Fig. 8.4: Internal Ratings-Based formula.

8.3 Credit Valuation Adjustment (CVA)

Credit Valuation Adjustments (CVA) aim at estimating the amount of capital required in the event of counterparty default, and are specially relevant to the Basel III regulatory framework. Other credit value adjustments (XVA) include the DVA, FVA, KVA, and MVA. The purpose of XVAs is also to take into account the future value of trades and their associated risks. The real-time estimation of XVA measures is generally highly demanding from a computational point of view.

Net Present Value (NPV) of a CDS

We work with a tenor structure $\{t = T_i < \cdots < T_j = T\}$. Let

$$\Pi(T_l, T_j) := \text{protection\_leg} - \text{premium\_leg} = V^d(T_l, T) - V^p(T_l, T),$$

denote the difference between the remaining protection and premium legs from time T_l until time T_j, where the values $V^d(t, T)$ and $V^p(t, T)$ of the premium and protection legs are given by (8.1) and (8.2). We have

$$\begin{aligned}\Pi(T_l,T_j) &:= \text{protection_leg} - \text{premium_leg}\\
&= \mathbb{E}\left[\sum_{k=l}^{j-1}\mathbb{1}_{(T_k,T_{k+1}]}(\tau)(1-\xi_{k+1})\exp\left(-\int_t^{T_{k+1}}r_sds\right)\ \Big|\ \mathcal{G}_{T_l}\right]\\
&\quad -\mathbb{E}\left[\sum_{k=l}^{j-1}S_t^{i,j}\delta_k\mathbb{1}_{\{\tau>T_{k+1}\}}\exp\left(-\int_t^{T_{k+1}}r_sds\right)\ \Big|\ \mathcal{G}_{T_l}\right]\\
&= (1-\xi)\,\mathbb{E}\left[\sum_{k=l}^{j-1}\mathbb{1}_{(T_k,T_{k+1}]}(\tau)\exp\left(-\int_t^{T_{k+1}}r_sds\right)\ \Big|\ \mathcal{G}_{T_l}\right]\\
&\quad -\mathbb{E}\left[\sum_{k=l}^{j-1}S_t^{i,j}\delta_k\mathbb{1}_{\{\tau>T_{k+1}\}}\exp\left(-\int_t^{T_{k+1}}r_sds\right)\ \Big|\ \mathcal{G}_{T_l}\right]\\
&= \mathbb{E}\left[\sum_{k=l}^{j-1}\left((1-\xi)\mathbb{1}_{(T_k,T_{k+1}]}(\tau)\exp\left(-\int_t^{T_{k+1}}r_sds\right)\right.\right.\\
&\qquad \left.\left. -\delta_k\mathbb{1}_{\{\tau>T_{k+1}\}}\exp\left(-\int_t^{T_{k+1}}r_sds\right)\right)\ \Big|\ \mathcal{G}_{T_l}\right]\end{aligned}$$

Note that we have $\Pi(t,T_j)=0$ by definition of the spread $S_t^{i,j}$.

Definition 8.1. *The Net Present Value (NPV) at time T_l of the CDS is the conditional expected value*

$$\text{NPV}(T_l,T_j) := \mathbb{E}\left[\Pi(T_l,T_j)\mid\mathcal{G}_{T_l}\right]$$

of the difference between the values at time T_l of the remaining protection and premium legs from time T_l until time T_j.

The Net Present Value (NPV) at time T_l of the CDS satisfies

$$\begin{aligned}&\text{NPV}(T_l,T_j) := \mathbb{E}\left[\Pi(T_l,T_j)\mid\mathcal{G}_{T_l}\right]\\
&= \mathbb{E}\left[\sum_{k=l}^{j-1}\mathbb{1}_{(T_k,T_{k+1}]}(\tau)(1-\xi_{k+1})\exp\left(-\int_t^{T_{k+1}}r_sds\right)\ \Big|\ \mathcal{G}_{T_l}\right] \qquad (8.7)\\
&\quad -\mathbb{E}\left[\sum_{k=l}^{j-1}S_t^{i,j}\delta_k\mathbb{1}_{\{\tau>T_{k+1}\}}\exp\left(-\int_t^{T_{k+1}}r_sds\right)\ \Big|\ \mathcal{G}_{T_l}\right]\\
&= (1-\xi)\sum_{k=l}^{j-1}\mathbb{E}\left[\mathbb{1}_{(T_k,T_{k+1}]}(\tau)\exp\left(-\int_t^{T_{k+1}}r_sds\right)\ \Big|\ \mathcal{G}_t\right] - S_t^{i,j}\sum_{k=l}^{j-1}\delta_kP(t,T_{k+1})\\
&= \sum_{k=l}^{j-1}\left((1-\xi)\,\mathbb{E}\left[\mathbb{1}_{(T_k,T_{k+1}]}(\tau)\exp\left(-\int_t^{T_{k+1}}r_sds\right)\ \Big|\ \mathcal{G}_t\right] - S_t^{i,j}\delta_kP(t,T_{k+1})\right)\end{aligned}$$

of the difference between the values at time T_l of the remaining protection and premium legs from time T_l until time T_j.

In addition to the credit default time τ we introduce a second stopping time $\nu \in [T_l, T_j]$ representing the possible default time of the party providing the protection leg.

The Net Present Value $\mathrm{NPV}(\nu, T_j)$ is estimated when default occurs at time ν.

i) If $\mathrm{NPV}(\nu, T_j) > 0$ then a payment is due from the party providing the protection leg, and only a fraction $\eta\mathrm{NPV}(\nu, T_j)$ of this payment may be recovered, where $\eta \in [0, 1]$ is the recovery rate of the party providing protection in the CDS.

ii) On the other hand, if $\mathrm{NPV}(\nu, T_j) < 0$ then the original fee payment $-\mathrm{NPV}(\nu, T_j)$ is still due.

As a consequence, in the event of default at time $\nu \in [T_l, T_j]$, the net present value of the CDS at time ν is

$$\begin{aligned}
\eta\mathrm{NPV}(\nu, T_j)\mathbb{1}_{\{\mathrm{NPV}(\nu,T_j)>0\}} &+ \mathrm{NPV}(\nu, T_j)\mathbb{1}_{\{\mathrm{NPV}(\nu,T_j)<0\}} \\
&= \eta\big(\mathrm{NPV}(\nu, T_j)\big)^+ - \big(\mathrm{NPV}(\nu, T_j)\big)^- \\
&= \eta\big(\mathrm{NPV}(\nu, T_j)\big)^+ - \big(-\mathrm{NPV}(\nu, T_j)\big)^+ \\
&= \eta\big(\mathrm{NPV}(\nu, T_j)\big)^+ + \big(\mathrm{NPV}(\nu, T_j) - \big(\mathrm{NPV}(\nu, T_j)\big)^+\big) \\
&= \mathrm{NPV}(\nu, T_j) - (1-\eta)\big(\mathrm{NPV}(\nu, T_j)\big)^+. \qquad (8.8)
\end{aligned}$$

Credit Valuation Adjustment (CVA)

Under the event of counterparty default at a time $\nu \in [T_l, T_j]$, the discounted payment estimated at time T_l becomes

$$\begin{aligned}
&\Pi(T_l, \nu) + \exp\left(-\int_{T_l}^{\nu} r_s ds\right)\left(\eta\big(\mathrm{NPV}(\nu, T_j)\big)^+ - \big(-\mathrm{NPV}(\nu, T_j)\big)^+\right) \\
&= \Pi(T_l, \nu) + \exp\left(-\int_{T_l}^{\nu} r_s ds\right)\left(\mathrm{NPV}(\nu, T_j) - (1-\eta)\big(\mathrm{NPV}(\nu, T_j)\big)^+\right) \\
&= \Pi(T_l, T_j) - (1-\eta)\exp\left(-\int_{T_l}^{\nu} r_s ds\right)\big(\mathrm{NPV}(\nu, T_j)\big)^+,
\end{aligned}$$

since

$$\Pi(T_l, T_j) = \Pi(T_l, \nu) + \exp\left(-\int_{T_l}^{\nu} r_s ds\right)\mathrm{NPV}(\nu, T_j).$$

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More generally, the total discounted payment due at time T_l under counterparty risk rewrites as

$$\begin{aligned}
&\Pi^D(T_l,T_j) = \mathbb{1}_{\{T_j<\nu\}}\Pi(T_l,T_j) \\
&+\mathbb{1}_{\{T_l<\nu\leqslant T_j\}}\left(\Pi(T_l,\nu)+\exp\left(-\int_{T_l}^{\nu} r_s ds\right)\left(\eta\big(\mathrm{NPV}(\nu,T_j)\big)^+ - \big(-\mathrm{NPV}(\nu,T_j)\big)^+\right)\right) \\
&= \mathbb{1}_{\{T_j<\nu\}}\Pi(T_l,T_j) \\
&\quad +\mathbb{1}_{\{T_l<\nu\leqslant T_j\}}\left(\Pi(T_l,T_j)-(1-\eta)\exp\left(-\int_{T_l}^{\nu} r_s ds\right)\big(\mathrm{NPV}(\nu,T_j)\big)^+\right) \\
&= \Pi(T_l,T_j)-\mathbb{1}_{\{T_l<\nu\leqslant T_j\}}(1-\eta)\exp\left(-\int_{T_l}^{\nu} r_s ds\right)\big(\mathrm{NPV}(\nu,T_j)\big)^+, \qquad (8.9)
\end{aligned}$$

see Brigo and Chourdakis (2009), Brigo and Masetti (2006). As a consequence of (8.9), we derive the following result.

Proposition 8.2. *The price at time T_l of the payoff $\Pi^D(T_l,T_j)$ under counterparty risk is given by*

$$\begin{aligned}
&\mathbb{E}\left[\Pi^D(T_l,T_j)\mid\mathcal{F}_{T_l}\right] = \mathbb{E}\left[\Pi(T_l,T_j)\mid\mathcal{F}_{T_l}\right] \\
&\qquad -(1-\eta)\,\mathbb{E}\left[\mathbb{1}_{\{T_l<\nu\leqslant T_j\}}\exp\left(-\int_{T_l}^{\nu} r_s ds\right)\big(\mathrm{NPV}(\nu,T_j)\big)^+ \;\Big|\; \mathcal{F}_{T_l}\right].
\end{aligned}$$

The quantity

$$(1-\eta)\,\mathbb{E}\left[\mathbb{1}_{\{T_l<\nu\leqslant T_j\}}\exp\left(-\int_{T_l}^{\nu} r_s ds\right)\big(\mathrm{NPV}(\nu,T_j)\big)^+ \;\Big|\; \mathcal{F}_{T_l}\right]$$

is called the (positive) Counterparty Risk (CR) Credit Valuation Adjustment (CVA).

Exercises

Exercise 8.1 Credit default swaps. Estimate the first default rate λ_1 and the associated default probability in the framework of (8.4), based on CDS market data, cf. also Castellacci (2008).

Exercise 8.2 We work with a tenor structure $\{t=T_i<\cdots<T_j=T\}$. Let

$$\begin{aligned}
&\sum_{k=i}^{j-1}\mathbb{E}\left[\mathbb{1}_{(T_k,T_{k+1}]}(\tau)(1-\xi_{k+1})\exp\left(-\int_t^{T_{k+1}} r_s ds\right)\;\Big|\;\mathcal{G}_t\right] \\
&=\sum_{k=i}^{j-1}\mathbb{E}\left[\left(\mathbb{1}_{\{T_k<\tau\}}-\mathbb{1}_{\{T_{k+1}<\tau\}}\right)(1-\xi_{k+1})\exp\left(-\int_t^{T_{k+1}} r_s ds\right)\;\Big|\;\mathcal{G}_t\right]
\end{aligned}$$

$$= \mathbb{1}_{\{\tau>t\}} \sum_{k=i}^{j-1} \mathbb{E}\left[(1-\xi_{k+1}) \left(e^{-\int_t^{T_k} \lambda_s ds} - e^{-\int_t^{T_{k+1}} \lambda_s ds} \right) e^{-\int_t^{T_{k+1}} r_s ds} \Big| \mathcal{F}_t \right]$$

$$= \mathbb{1}_{\{\tau>t\}} (1-\xi) \sum_{k=i}^{j-1} e^{-\int_t^{T_{k+1}} r_s ds} \mathbb{E}\left[e^{-\int_t^{T_k} \lambda_s ds} - e^{-\int_t^{T_{k+1}} \lambda_s ds} \Big| \mathcal{F}_t \right]$$

$$= \mathbb{1}_{\{\tau>t\}} (1-\xi) \sum_{k=i}^{j-1} P(t, T_{k+1}) \left(Q(t, T_k) - Q(t, T_{k+1}) \right),$$

denote the discounted value at time t of the protection leg, where

$$P(t, T_k) = \exp\left(-\int_t^{T_k} r_s ds \right) = e^{-(T_k - t) r_k}, \qquad k = i, \ldots, j,$$

is a deterministic discount factor, and

$$Q(t, T_k) = \mathbb{E}\left[\exp\left(-\int_t^{T_k} \lambda_s ds \right) \Big| \mathcal{F}_t \right]$$

is the survival probability. Let

$$V^p(t, T) = S_t^{i,j} \sum_{k=i}^{j-1} \delta_k \mathbb{E}\left[\mathbb{1}_{\{\tau > T_{k+1}\}} \exp\left(-\int_t^{T_{k+1}} r_s ds \right) \Big| \mathcal{G}_t \right]$$

$$= S_t^{i,j} \sum_{k=i}^{j-1} \delta_k \mathbb{E}\left[\mathbb{1}_{\{T_{k+1} < \tau\}} \exp\left(-\int_t^{T_{k+1}} r_s ds \right) \Big| \mathcal{G}_t \right]$$

$$= \mathbb{1}_{\{\tau>t\}} S_t^{i,j} \sum_{k=i}^{j-1} \delta_k \mathbb{E}\left[\exp\left(-\int_t^{T_{k+1}} \lambda_s ds \right) \exp\left(-\int_t^{T_{k+1}} r_s ds \right) \Big| \mathcal{F}_t \right]$$

$$= S_t^{i,j} \mathbb{1}_{\{\tau>t\}} \sum_{k=i}^{j-1} \delta_k \exp\left(-\int_t^{T_{k+1}} r_s ds \right) \mathbb{E}\left[\exp\left(-\int_t^{T_{k+1}} \lambda_s ds \right) \Big| \mathcal{F}_t \right]$$

$$= \mathbb{1}_{\{\tau>t\}} S_t^{i,j} \sum_{k=i}^{j-1} \delta_k P(t, T_{k+1}) Q(t, T_{k+1}),$$

denote the discounted value at time t of the premium leg, where $\delta_k := T_{k+1} - T_k$, $k = i, \ldots, j-1$.

a) By equating the protection and premium legs, find the value of $Q(t, T_{i+1})$ with $Q(t, T_i) = 1$, and derive a recurrence relation between $Q(t, T_{j+1})$ and $Q(t, T_i), \ldots, Q(t, T_j)$.

b) For a given underlying asset, retrieve the corresponding CDS spreads $S_t^{i,j}$ and discount factors $P(t, T_i), \ldots, P(t, T_n)$, and estimate the corresponding survival probabilities $Q(t,T_i),\ldots,Q(t,T_n)$.