Chapter 5
Correlation and Dependence

This chapter deals with uncertainty and dependence structures, starting from simple Bernoulli and Gaussian examples, and the construction of copulas.

5.1 Joint Bernoulli Distribution

Given a choice of modeling based on the distributions of two random variables \(X\) and \(Y\), it is natural to consider a dependence structure between \(X\) and \(Y\).

Consider two Bernoulli random variables \(X\) and \(Y\), with

\[
p_X = \mathbb{P}(X = 1) = \mathbb{E}[\mathbb{1}_{\{X=1\}}] \quad \text{and} \quad p_Y = \mathbb{P}(Y = 1) = \mathbb{E}[\mathbb{1}_{\{Y=1\}}]
\]

and correlation

\[
\rho := \frac{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}{\sqrt{\text{Var}[X]\text{Var}[Y]}} = \frac{\mathbb{P}(X = 1 \text{ and } Y = 1) - p_Xp_Y}{\sqrt{p_X(1-p_X)p_Y(1-p_Y)}}.
\]

We note that in that case the joint distribution \(\mathbb{P}(X = i \text{ and } Y = j), i, j \in \{0,1\}\) is fully determined by \(\mathbb{P}(X = 1), \mathbb{P}(Y = 1)\) and the correlation \(\rho \in [-1,1]\), as
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\[
\begin{align*}
\mathbb{P}(X = 1 \text{ and } Y = 1) &= \mathbb{E}[XY] \\
&= p_X p_Y + \rho \sqrt{p_X (1 - p_X) p_Y (1 - p_Y)}, \\
\mathbb{P}(X = 0 \text{ and } Y = 1) &= \mathbb{P}(Y = 1) - \mathbb{P}(X = 1 \text{ and } Y = 1) \\
&= (1 - p_X) p_Y - \rho \sqrt{p_X (1 - p_X) p_Y (1 - p_Y)}, \\
\mathbb{P}(X = 1 \text{ and } Y = 0) &= \mathbb{P}(X = 1) - \mathbb{P}(X = 1 \text{ and } Y = 1) \\
&= p_X (1 - p_Y) - \rho \sqrt{p_X (1 - p_X) p_Y (1 - p_Y)}, \\
\mathbb{P}(X = 0 \text{ and } Y = 0) &= \mathbb{E}[(1 - X)(1 - Y)] \\
&= (1 - p_X)(1 - p_Y) + \rho \sqrt{p_X (1 - p_X) p_Y (1 - p_Y)}.
\end{align*}
\]

### 5.2 Joint Gaussian Distribution

Consider now two centered Gaussian random variables \( X \sim \mathcal{N}(0, \sigma^2) \) and \( Y \sim \mathcal{N}(0, \eta^2) \) with probability density functions

\[
f_X(x) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-x^2/(2\sigma^2)} \quad \text{and} \quad f_Y(x) = \frac{1}{\sqrt{2\pi \eta^2}} e^{-x^2/(2\eta^2)}, \quad x \in \mathbb{R}.
\]

Let

\[
\rho = \text{corr}(X, Y) := \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}[X]\text{Var}[Y]}} = \frac{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}{\sqrt{\text{Var}[X]\text{Var}[Y]}}.
\]

When the covariance matrix

\[
\Sigma := \begin{bmatrix} \mathbb{E}[X^2] & \mathbb{E}[XY] \\ \mathbb{E}[XY] & \mathbb{E}[Y^2] \end{bmatrix} = \begin{bmatrix} \sigma^2 & \rho \sigma \eta \\ \rho \sigma \eta & \eta^2 \end{bmatrix}
\]

with determinant

\[
\text{det} \Sigma = \mathbb{E}[X^2] \mathbb{E}[Y^2] - (\mathbb{E}[XY])^2 \\
= \mathbb{E}[X^2] \mathbb{E}[Y^2](1 - (\text{corr}(X, Y))^2) \\
\geq 0,
\]

is invertible, there exists a probability density function

\[
f_{\Sigma}(x, y) = \frac{1}{\sqrt{2\pi \text{det} \Sigma}} \exp \left( -\frac{1}{2} \begin{bmatrix} x \\ y \end{bmatrix}^\top \Sigma^{-1} \begin{bmatrix} x \\ y \end{bmatrix} \right) \\
= \frac{1}{\sqrt{2\pi \text{det} \Sigma}} \exp \left( -\frac{1}{2} \begin{bmatrix} x \\ y \end{bmatrix}^\top \begin{bmatrix} \mathbb{E}[X^2] & \mathbb{E}[XY] \\ \mathbb{E}[XY] & \mathbb{E}[Y^2] \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} \right),
\]

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with respective marginals $\mathcal{N}(0, \sigma^2)$ and $\mathcal{N}(0, \eta^2)$.

![Joint Gaussian probability density](image)

**Fig. 5.1:** Joint Gaussian probability density.

The probability density function (5.2) is called the centered joint (bivariate) Gaussian density with covariance matrix $\Sigma$.

Note that when $\rho = \text{corr}(X,Y) = \pm 1$ we have $\det \Sigma = 0$ and the joint density function $f_\Sigma(x,y)$ is not defined.

More generally, a random vector $(X_1, \ldots, X_n)$ has a multivariate centered Gaussian distribution if every linear combination $Y = a_1 X_1 + \ldots + a_n X_n$ is centered Gaussian, and in this case the probability density function of $(X_1, \ldots, X_n)$ takes the form

$$f_\Sigma(x_1, \ldots, x_n) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp \left( -\frac{1}{2} (x_1, \ldots, x_n)^T \Sigma^{-1} (x_1, \ldots, x_n) \right),$$

$(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, where $\Sigma$ is the covariance matrix

$$\Sigma = \begin{bmatrix}
\text{Var}[X_1] & \text{Cov}(X_1, X_2) & \cdots & \text{Cov}(X_1, X_{n-1}) & \text{Cov}(X_1, X_n) \\
\text{Cov}(X_2, X_1) & \text{Var}[X_2] & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \text{Var}[X_{n-1}] & \text{Cov}(X_{n-1}, X_n) \\
\text{Cov}(X_1, X_n) & \text{Cov}(X_2, X_n) & \cdots & \text{Cov}(X_{n-1}, X_n) & \text{Var}[X_n^2]
\end{bmatrix}.$$

The next remark plays an important role in the modeling of joint default probabilities, see here for a detailed discussion.

**Remark 5.1.** There exist couples $(X, Y)$ with of random variables with Gaussian marginals $\mathcal{N}(0, \sigma^2)$ and $\mathcal{N}(0, \eta^2)$, such that

i) $(X, Y)$ does not have the bivariate Gaussian distribution with density $f_\Sigma(x,y)$, where $\Sigma$ is the covariance matrix (5.1) of $(X, Y)$.

ii) the random variable $X + Y$ is not even Gaussian.
5.3 Copulas and Dependence Structures

The word copula derives from the Latin noun for a “link” or “tie” that connects two different objects or concepts.

**Definition 5.2.** A two-dimensional copula is any joint cumulative distribution function

\[ C : [0, 1] \times [0, 1] \rightarrow [0, 1] \]

\[ (u, v) \mapsto C(u, v) \]

of two uniform \([0, 1]\)-valued random variables.

In other words, any copula function \(C(u, v)\) can be written as

\[ C(u, v) = P(U \leq u \text{ and } V \leq v), \quad u, v \in [0, 1], \]

where \(U\) and \(V\) are uniform \([0, 1]\)-valued random variables.

**Examples.**

i) The copula corresponding to independent uniform random variables \((U, V)\) is given by

\[ C(u, v) = P(U \leq u \text{ and } V \leq v) \]

\[ = P(U \leq u)P(V \leq v) \]

\[ = uv, \quad u, v \in [0, 1]. \]

ii) The copula corresponding to the fully correlated case \(U = V\) is given by

\[ C(u, v) = P(U \leq u \text{ and } V \leq v) \]

\[ = P(U \leq \min(u, v)) \]

\[ = \min(u, v), \quad u, v \in [0, 1]. \]

iii) The copula corresponding to the fully anticorrelated case \(U = 1 - V\) is given by

\[ C(u, v) := P(U \leq u \text{ and } V \leq v) \]

\[ = P(U \leq u \text{ and } 1 - U \leq v) \]

\[ = P(1 - v \leq U \leq u) \]

\[ = (u + v - 1)^+, \quad u, v \in [0, 1]. \]
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The next lemma is well known and can be used to generate random samples of a cumulative distribution function \( F_X \) based on uniformly distributed samples.

**Lemma 5.3.** Given \( X \) a random variable with distribution function 
\[
F_X(x) := \mathbb{P}(X \leq x), \quad x \in \mathbb{R},
\]
the random variable 
\[
U := F_X(X)
\]
is uniformly distributed on \([0, 1]\).

**Proof.** We have 
\[
F_U(u) = \mathbb{P}(U \leq u) = \mathbb{P}(F_X(X) \leq u) = \mathbb{P}(X \leq F_X^{-1}(u)) = F_X(F_X^{-1}(u)) = u, \quad u \in [0, 1].
\]

As a consequence of Lemma 5.3, given \((X, Y)\) a couple of random variables with joint cumulative distribution function
\[
F_{(X,Y)}(x,y) := \mathbb{P}(X \leq x \text{ and } Y \leq y), \quad x, y \in \mathbb{R},
\]
and cumulative distribution functions 
\[
F_X(x) = F_{(X,Y)}(x, \infty) = \mathbb{P}(X \leq x) \quad \text{and} \quad F_Y(y) = F_{(X,Y)}(\infty, y) = \mathbb{P}(Y \leq y),
\]
we not the following points.

i) The random variables
\[
U := F_X(X) \quad \text{and} \quad V := F_Y(Y)
\]
are uniformly distributed on \([0, 1]\).

ii) The copula \((u,v) \mapsto C_{(X,Y)}(u,v) : \mathbb{P}(U \leq u \text{ and } V \leq v)\) satisfies
\[
C_{(X,Y)}(u,v) := \mathbb{P}(U \leq u \text{ and } V \leq v) = \mathbb{P}(F_X(X) \leq u \text{ and } F_Y(Y) \leq v) = \mathbb{P}(X \leq F_X^{-1}(u) \text{ and } Y \leq F_Y^{-1}(v)) = F_{(X,Y)}(F_X^{-1}(u), F_Y^{-1}(v)), \quad u, v \in [0, 1],
\]
is a copula.

iii) The joint cumulative distribution function of $(X, Y)$ can be recovered as

$$
P(X \leq x \text{ and } Y \leq y) = P(F_X(X) \leq F_X(x) \text{ and } F_Y(Y) \leq F_Y(y))
= P(U \leq F_X(x) \text{ and } V \leq F_Y(y))
= C_{(X,Y)}(F_X(x), F_Y(y)), \quad x, y \in \mathbb{R}.
$$

Higher dimensional copulas

**Definition 5.4.** An $n$-dimensional copula is any joint cumulative distribution function

$$
C : [0, 1] \times \cdots \times [0, 1] \to [0, 1]
(u_1, \ldots, u_n) \mapsto C(u_1, \ldots, u_n)
$$

of $n$ uniform $[0, 1]$-valued random variables.

Consider the joint cumulative distribution function

$$
F_{(X_1, \ldots, X_n)}(x_1, \ldots, x_n) := P(X_1 \leq x_1, \ldots, X_n \leq x_n)
$$

of a family $(X_1, \ldots, X_n)$ of random variables with marginal cumulative distribution functions

$$
F_{X_i}(x) = F_{(X_1, \ldots, X_n)}(+\infty, \ldots, +\infty, x, +\infty, \ldots, +\infty), \quad x \in \mathbb{R},
$$

$i = 1, 2, \ldots, n$. The copula defined in Sklar’s theorem encodes the dependence structure of the vector $(X_1, \ldots, X_n)$.

**Theorem 5.5.** Sklar’s theorem (Sklar (1959)*, Sklar (2010)). Given a joint cumulative distribution function $F_{(X_1, \ldots, X_n)}$ there exists an $n$-dimensional copula $C(u_1, \ldots, u_n)$ such that

$$
F_{(X_1, \ldots, X_n)}(x_1, x_2, \ldots, x_n) = C(F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_n}(x_n)),
$$

$x_1, x_2, \ldots, x_n \in \mathbb{R}$.

The following proposition is a consequence of Sklar’s Theorem 5.5.

**Proposition 5.6.** Assume that the marginal distribution functions $F_{X_i}$ are continuous and strictly increasing. Then the joint cumulative distribution function $F_{(X_1, \ldots, X_n)}$ defines a $n$-dimensional copula

* "The author considers continuous nondecreasing functions $C_n$ on the $n$-dimensional cube $[0, 1]^n$ with $C_n(0, \ldots, 0) = 0$, $C_n(1, \ldots, 1, \alpha, 1, \ldots, 1) = \alpha$. Several theorems are stated relating $n$-dimensional distribution functions and their marginals in terms of functions $C_n$. No proofs are given." M. Loève, Math. Reviews MR0125600.
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\[ C(u_1, \ldots, u_n) := F_{(X_1, \ldots, X_n)}(F_{X_1}^{-1}(u_1), \ldots, F_{X_n}^{-1}(u_n)), \quad (5.3) \]

\( u_1, u_2, \ldots, u_n \in [0, 1] \), which encodes the dependence structure of the vector \((X_1, \ldots, X_n)\).

Proof. Indeed it can be checked as in Lemma 5.3 that \( C(u_1, \ldots, u_n) \) has uniform marginal distributions on \([0, 1]\), as

\[
\begin{align*}
C(1, \ldots, 1, u, 1, \ldots, 1) &= F_{(X_1, \ldots, X_n)}(F_{X_1}^{-1}(1), \ldots, F_{X_{i-1}}^{-1}(1), F_{X_i}^{-1}(u), F_{X_{i+1}}^{-1}(1), \ldots, F_{X_n}^{-1}(1)) \\
&= F_{(X_1, \ldots, X_n)}(+\infty, \ldots, +\infty, F_{X_i}^{-1}(u), +\infty, \ldots, +\infty) \\
&= F_{\tilde{X}_i}(F_{\tilde{X}_i}^{-1}(u)) \\
&= u, \quad u \in [0, 1].
\end{align*}
\]

□

Proposition 5.7. Given a family \((\tilde{X}_1, \ldots, \tilde{X}_n)\) of random variables with marginal cumulative distribution functions \(F_{\tilde{X}_1}, \ldots, F_{\tilde{X}_n}\) and a multidimensional copula \(C(u_1, \ldots, u_n)\), the function

\[
F^C_{(\tilde{X}_1, \ldots, \tilde{X}_n)}(x_1, \ldots, x_n) := C(F_{\tilde{X}_1}(x_1), \ldots, F_{\tilde{X}_n}(x_n)), \quad x_1, x_2, \ldots, x_n \in \mathbb{R},
\]

defines joint cumulative distribution function with marginals \(\tilde{X}_1, \ldots, \tilde{X}_n\).

Proof. We note that the marginal distributions generated by \(F^C_{(\tilde{X}_1, \ldots, \tilde{X}_n)}(x_1, \ldots, x_n)\) coincide with the respective marginals of \((\tilde{X}_1, \ldots, \tilde{X}_n)\), as we have

\[
\begin{align*}
F^C_{(\tilde{X}_1, \ldots, \tilde{X}_n)}(+\infty, \ldots, +\infty, u, +\infty, \ldots, +\infty) &= C(F_{\tilde{X}_1}(+\infty), \ldots, F_{\tilde{X}_{i-1}}(+\infty), F_{\tilde{X}_i}(u), F_{\tilde{X}_{i+1}}(+\infty), \ldots, F_{\tilde{X}_n}(+\infty)) \\
&= C(1, \ldots, 1, F_{\tilde{X}_i}(u), 1, \ldots, 1) \\
&= F_{\tilde{X}_i}(u), \quad u \in [0, 1].
\end{align*}
\]

□

5.4 Examples of Copulas

Gaussian copulas

The choice of \((5.2)\) above as joint density function actually induces a particular dependence structure between the Gaussian random variables \(X\) and \(Y\), and corresponding to the joint cumulative distribution function
\[ \Phi_\Sigma(x, y) := \mathbb{P}(X \leq x \text{ and } Y \leq y) \]
\[ = \frac{1}{\sqrt{2\pi \det \Sigma}} \int_{-\infty}^{x} \int_{-\infty}^{y} \exp \left( -\frac{1}{2} \left\langle \begin{bmatrix} u \\ v \end{bmatrix}, \Sigma^{-1} \begin{bmatrix} u \\ v \end{bmatrix} \right\rangle \right) \, du \, dv, \]

\( x, y \in \mathbb{R} \). In case \((X, Y)\) are normalized centered Gaussian random variables with unit variance, \( \Sigma \) is given by

\[ \Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}, \]

with correlation parameter \( \rho \in [-1, 1] \). Letting

\[ F_X(x) := \mathbb{P}(X \leq x) \quad \text{and} \quad F_Y(y) := \mathbb{P}(Y \leq y), \]

denote the cumulative distribution functions of \( X \) and \( Y \), the random variables \( F_X(X) \) and \( F_Y(Y) \) are known to be uniformly distributed on \([0, 1]\), and \((F_X(X), F_Y(Y))\) is a \([0, 1] \times [0, 1]\)-valued random variable with joint cumulative distribution function

\[ C_\Sigma(u, v) := \mathbb{P}(F_X(X) \leq u \text{ and } F_Y(Y) \leq v) \]
\[ = \mathbb{P}(X \leq F_X^{-1}(u) \text{ and } Y \leq F_Y^{-1}(v)) \]
\[ = \Phi_\Sigma(F_X^{-1}(u), F_Y^{-1}(v)), \quad u, v \in [0, 1]. \quad (5.4) \]

The function \( C_\Sigma(u, v) \), which is the joint cumulative distribution function of a couple of uniformly distributed \([0, 1]\)-valued random variables, is called the Gaussian copula generated by the jointly Gaussian distribution of \((X, Y)\) with covariance matrix \( \Sigma \).

Fig. 5.2: Different Gaussian copula graphs for \( \rho = 0 \), \( \rho = 0.85 \) and \( \rho = 1 \).

The above leftmost figure corresponds to independent uniformly distributed \([0, 1]\)-valued random variables \( U, V \), i.e. to the copula
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\[ C(u, v) := \mathbb{P}(U \leq u \text{ and } V \leq v) = \mathbb{P}(U \leq u)\mathbb{P}(U \leq u) = uv, \quad u, v \in [0, 1]. \]

On the other hand the rightmost figure corresponds to equal uniformly distributed \([0, 1]\)-valued random variables \(U = V\), i.e. to the copula

\[
C(u, v) := \mathbb{P}(U \leq u \text{ and } V \leq v)
= \mathbb{P}(U \leq u \text{ and } U \leq v)
= \mathbb{P}(U \leq \min(u, v))
= \min(u, v), \quad u, v \in [0, 1],
\]

The middle figure corresponds to an intermediate dependence level given by a Gaussian copula, cf. (5.4) below.

The leftmost figure above represents a uniform (product) probability density on the square \([0, 1] \times [0, 1]\), which corresponds to two independent uniformly distributed \([0, 1]\)-valued random variables \(U, V\). The rightmost figure shows the probability distribution of the fully correlated couple \((U, U)\), which does not admit a probability density on the square \([0, 1] \times [0, 1]\).

The Gaussian copula \(C_{\Sigma}(u, u)\) admits a probability density function on \([0, 1] \times [0, 1]\) given by

\[
c_{\Sigma}(u, v) = \frac{\partial^2 C_{\Sigma}}{\partial u \partial v}(u, v)
= \frac{\partial^2}{\partial u \partial v} \Phi_{\Sigma}(F_X^{-1}(u), F_Y^{-1}(v))
= \frac{\partial}{\partial u} \left( \frac{1}{F'_Y(F_Y^{-1}(v))} \frac{\partial \Phi_{\Sigma}}{\partial y} (F_X^{-1}(u), F_Y^{-1}(v)) \right)
= \frac{\partial}{\partial u} \left( \frac{1}{f_Y(F_Y^{-1}(v))} \frac{\partial \Phi_{\Sigma}}{\partial y} (F_X^{-1}(u), F_Y^{-1}(v)) \right)
\]
hence the Gaussian copula $C_\Sigma(u, v)$ can be computed as

$$C_\Sigma(u, v) = \int_0^u \int_0^v c_\Sigma(a, b) \, dadb$$

$$= \int_0^u \int_0^v \frac{f_\Sigma(F_X^{-1}(a), F_Y^{-1}(b))}{f_X(F_X^{-1}(a))f_Y(F_Y^{-1}(b))} \, dadb, \quad u, v \in [0, 1].$$

The joint cumulative distribution function $F_{(X,Y)}(x, y)$ of $(X, Y)$ can be recovered from Proposition 5.6 as

$$F_{(X,Y)}(x, y) = C_\Sigma(F_X(x), F_Y(y)), \quad x, y \in \mathbb{R}. \quad (5.5)$$

from the Gaussian copula $C_\Sigma(x, y)$ and the respective cumulative distribution functions $F_X(x), F_Y(y)$ of $X$ and $Y$.

In that sense, the Gaussian copula $C_\Sigma(x, y)$ encodes the Gaussian dependence structure of the covariance matrix $\Sigma$. Moreover, the Gaussian copula $C_\Sigma(x, y)$ can be used to generate a joint distribution function $F_{C,(X,Y)}(x, y)$ by letting

$$F_{C,(X,Y)}(x, y) := C_\Sigma(F_X(x), F_Y(y)), \quad x, y \in \mathbb{R}, \quad (5.6)$$

based on other, possibly non-Gaussian cumulative distribution functions $F_X(x), F_Y(y)$ of two random variables $X$ and $Y$. In this case we note that the marginals of the joint cumulative distribution function $F_{C,(X,Y)}(x, y)$ are $F_X(x)$ and $F_Y(y)$ because $C_\Sigma(x, y)$ has uniform marginals on $[0, 1]$.

**Gumbel copula**

The Gumbel copula is given by

$$C(u, v) = \exp \left( - \left( \frac{-\log u}{\theta} + \frac{-\log v}{\theta} \right)^{1/\theta} \right), \quad u, v \in [0, 1],$$

with $\theta \geq 1$, and $C(u, v) = uv$ when $\theta = 1$. The following is an R code for Gaussian, Student, and Gumbel copulas with uniform marginals as illustrated in Figure 5.4.

```R
install.packages("copula")
install.packages("gumbel")
library(copula);library(gumbel)
```

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```r
norm.cop <- normalCopula(0.35); norm.cop
persp(norm.cop, pCopula, n.grid = 51, xlab="u", ylab="v", zlab="C(u,v)", main="", sub="")
persp(norm.cop, dCopula, n.grid = 51, xlab="u", ylab="v", zlab="c(u,v)", main="", sub="")
norm <- rCopula(4000, normalCopula(0.7))
plot(norm[,1], norm[,2], cex=3, pch='.', col='blue')
points(rCopula(4000, tCopula(0.5, dim=2, df=1))
points(norm[,1], norm[,2], cex=3, pch='.', col='red')
gumb <- rgumbel(4000, 4)
points(gumb[,1], gumb[,2], cex=3, pch='.', col='green')
```

(a) Gaussian copula.  
(b) Student copula.  
(c) Gumbel copula.

Fig. 5.4: Samples with uniform marginals and given copulas.

The following R code plots the histograms of Figure 5.4.

```r
joint_hist <- function(u) {x <- u[1]; y <- u[2]
xhist <- hist(x, breaks=40, plot=FALSE); yhist <- hist(y, breaks=40, plot=FALSE)
top <- max(c(xhist$count, yhist$count))
nf <- layout(matrix(c(2,0,1,3),2,2, byrow=TRUE), c(3,1, c(1,3), TRUE)
par(mar=c(3,3,1,1))
plot(x, y, xlab="", ylab="", col="blue", pch=19, cex=0.4)
points(x[1:50], -0.01+rep(min(y),50), xlab="", ylab="", col="black", pch=18, cex=0.8)
points(-0.01+rep(min(x),50), y[1:50], xlab="", ylab="", col="black", pch=18, cex=0.8)
par(mar=c(0,3,1,1))
barplot(xhist$count, axes=FALSE, ylim=c(0, top), space=0, col="purple")
par(mar=c(3,0,1,1))
barplot(yhist$count, axes=FALSE, xlim=c(0, top), space=0, horiz=TRUE, col="purple")
joint_hist(norm); joint_hist(norm); joint_hist(gumb)
```

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The following is an R code for Gaussian, Student, and Gumbel copulas with gaussian marginals as illustrated in Figure 5.5.

```r
set.seed(100); N=10000
gaussMVD<-mvdc(normalCopula(0.8), margins=c("norm","norm"), paramMargins=list(list(mean=0, sd=1), list(mean=0, sd=1)))
norm <- rMvdc(N,gaussMVD)
studentMVD<-mvdc(tCopula(0.5,dim=2,df=1), margins=c("norm","norm"), paramMargins=list(list(mean=0, sd=1), list(mean=0, sd=1)))
stud <- rMvdc(N,studentMVD)
gumbelMVD<-mvdc(gumbelCopula(param=4, dim=2), margins=c("norm","norm"), paramMargins=list(list(mean=0, sd=1), list(mean=0, sd=1)))
gumb <- rMvdc(N,gumbelMVD)
plot(norm[,1],norm[,2],cex=3,pch=".",col="blue")
plot(stud[,1],stud[,2],cex=3,pch=".",col="blue")
plot(gumb[,1],gumb[,2],cex=3,pch=".",col="blue")
joint_hist(norm);joint_hist(stud);joint_hist(gumb)
```

Fig. 5.5: Samples with Gaussian marginals and given copulas.

The following is an R code for joint densities with Gaussian marginals and given copulas as illustrated in Figure 5.6.

```r
plot(normalCopula(0.8), margins=c("norm","norm"), paramMargins=list(list(mean=0, sd=1), list(mean=0, sd=1)))
plot(tCopula(0.5,dim=2,df=1), margins=c("norm","norm"), paramMargins=list(list(mean=0, sd=1), list(mean=0, sd=1)))
plot(gumbelCopula(param=4, dim=2), margins=c("norm","norm"), paramMargins=list(list(mean=0, sd=1), list(mean=0, sd=1)))
```

Fig. 5.6: Joint densities with Gaussian marginals and given copulas.

The following is an R code for joint densities with Gaussian marginals and given copulas as illustrated in Figure 5.6.
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The following R code for contour plots with Gaussian marginals and given copulas as illustrated in Figure 5.7.

```r
contour(gaussMVD, dMvdc, xlim = c(-3,3), ylim = c(-3,3), nlevels = 10, xlab = "X", ylab = "Y", cex.axis = 1.5)
contour(studentMVD, dMvdc, xlim = c(-3,3), ylim = c(-3,3), nlevels = 10, xlab = "X", ylab = "Y", cex.axis = 1.5)
contour(gumbelMVD, dMvdc, xlim = c(-3,3), ylim = c(-3,3), nlevels = 10, xlab = "X", ylab = "Y", cex.axis = 1.5)
```

The following R code generates random samples with exponential marginals according to the Gaussian, Student, and Gumbel copulas as illustrated in Figure 5.8.

```r
library(copula); set.seed(100); N = 4000
gaussMVD <- mvdc(normalCopula(0.7), margins = c("exp","exp"), paramMargins = list(list(rate = 1), list(rate = 1)))
norm <- rMvdc(N, gaussMVD)
studentMVD <- mvdc(tCopula(0.5, dim = 2, df = 1), margins = c("exp","exp"), paramMargins = list(list(rate = 1), list(rate = 1)))
stud <- rMvdc(N, studentMVD)
gumbelMVD <- mvdc(gumbelCopula(param = 4, dim = 2), margins = c("exp","exp"), paramMargins = list(list(rate = 1), list(rate = 1)))
gumb <- rMvdc(N, gumbelMVD)
```

(a) Gaussian copula.  
(b) Student copula.  
(c) Gumbel copula.

Fig. 5.7: Joint density contour plots with Gaussian marginals and given copulas.
Exercises

Exercise 5.1  Copulas. In the sequel, $U$ denotes a uniformly distributed $[0, 1]$-valued random variable.

a) To which couple $(U, V)$ of uniformly distributed $[0, 1]$-valued random variables does the copula function

$$C_M(u, v) = \min(u, v), \quad u, v \in [0, 1],$$

correspond?

b) Show that the function

$$C_m(u, v) := (u + v - 1)^+, \quad u, v \in [0, 1],$$

is the copula on $[0, 1] \times [0, 1]$ corresponding to $(U, V) = (U, 1 - U)$.

c) Show that for any copula function $C(u, v)$ on $[0, 1] \times [0, 1]$ we have

$$C(u, v) \leq C_M(u, v), \quad u, v \in [0, 1].$$

(5.7)

d) Show that for any copula function $C(u, v)$ on $[0, 1] \times [0, 1]$ we also have

$$C_m(u, v) \leq C(u, v), \quad u, v \in [0, 1].$$

(5.8)
Correlation and Dependence

\textit{Hint:} For fixed \( v \in [0, 1] \), let \( h(u) := C(u, v) - (u + v - 1) \) and show that \( h(1) = 0 \) and \( h'(u) \leq 0 \).

Exercise 5.2 Exponential copulas. Consider the random vector \((X, Y)\) of nonnegative random variables, whose joint distribution is given by the survival function

\[ P(X \geq x \text{ and } Y \geq y) := e^{-\lambda x - \mu y - \nu \max(x, y)}, \quad x, y \in \mathbb{R}_+, \]

where \( \lambda, \mu, \nu > 0 \).

a) Find the marginal distributions of \( X \) and \( Y \).

b) Find the joint cumulative distribution function \( F(x, y) := P(X \leq x \text{ and } Y \leq y) \) of \((X, Y)\).

c) Construct an “exponential copula” based on the joint cumulative distribution function of \((X, Y)\).

Exercise 5.3 Gumbel bivariate logistic distribution. Consider the random vector \((X, Y)\) of non-negative random variables, whose joint distribution is given by the joint cumulative distribution function (CDF)

\[ F_{(X,Y)}(x, y) := P(X \leq x \text{ and } Y \leq y) := \frac{1}{1 + e^{-x} + e^{-y}}, \quad x, y \in \mathbb{R}. \]

a) Find the marginal distributions of \( X \) and \( Y \).

b) Construct the copula based on the joint CDF of \((X, Y)\).

Exercise 5.4 Consider the random vector \((X, Y)\) with the joint density

\[ \tilde{f}(x, y) := \frac{1}{\pi \sigma \eta} \mathbb{1}_{\mathbb{R}^-_2}(x, y) e^{-x^2/(2\sigma^2) - y^2/(2\eta^2)} + \frac{1}{\pi \sigma \eta} \mathbb{1}_{\mathbb{R}^+_2}(x, y) e^{-x^2/(2\sigma^2) - y^2/(2\eta^2)}, \]

plotted as a heat map in Figure 5.9b.

```r
library(MASS)
Sigma <- matrix(c(1,0,0,1),2,2); N=10000
u<-mvrnorm(n = N, rep(0, 2), Sigma);
j=1
for (i in 1:N){
  if (u[i,1]>0 & u[i,2]>0) {j<-j+1;}
  if (u[i,1]<0 & u[i,2]<0) {j<-j+1;}
}
v<-matrix(nrow=j-1, ncol=2);j=1
for (i in 1:N){
  if (u[i,1]>0 & u[i,2]>0) {v[i,]=u[i,];j<-j+1;}
  if (u[i,1]<0 & u[i,2]<0) {v[i,]=u[i,];j<-j+1;}
}
joint_hist(v)
```

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a) Show that \((X, Y)\) has Gaussian marginals \(N(0, \sigma^2)\) and \(N(0, \eta^2)\).

b) Does the couple \((X, Y)\) have the bivariate Gaussian distribution with density \(f_\Sigma(x, y)\), where \(\Sigma\) is the covariance matrix (5.1) of \((X, Y)\)?

c) Show that the random variable \(X + Y\) is not Gaussian (take \(\sigma = \eta = 1\) for simplicity).

d) Show that under the rotation

\[
R = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix},
\]

of angle \(\theta \in [0, 2\pi]\) the random vector \((X \cos \theta - Y \sin \theta, X \sin \theta + Y \cos \theta)\) can have an arbitrary covariance depending on the value of \(\theta \in [0, 2\pi]\).

Exercise 5.5 Let \(\tau_1, \tau_2\) and \(\tau\) denote three independent exponentially distributed random times with respective parameters \(\lambda_1, \lambda_2, \lambda > 0\). Consider two firms with respective default times \(\tau_1 \wedge \tau = \min(\tau_1, \tau)\) and \(\tau_2 \wedge \tau = \min(\tau_2, \tau)\), where \(\tau\) represents the time of a macro-economic shock.

a) Find the (survival) distribution functions of \(\tau_1 \wedge \tau\) and \(\tau_2 \wedge \tau\).

b) Compute the joint survival probability

\[
P(\tau_1 \wedge \tau > s \text{ and } \tau_2 \wedge \tau > t), \quad s, t \in \mathbb{R}_+.
\]

\textit{Hint:} Use the relation \(\max(s, t) = s + t - \min(s, t)\), \(s, t \in \mathbb{R}_+\).

c) Compute the joint cumulative distribution function

\[
P(\tau_1 \wedge \tau \leq s \text{ and } \tau_2 \wedge \tau \leq t), \quad s, t \in \mathbb{R}_+.
\]

d) Compute the resulting copula

\[
C(u, v) := F_{X,Y}(F_X^{-1}(u), F_Y^{-1}(v)), \quad u, v \in [0, 1].
\]
e) Compute the resulting copula density \( \frac{\partial^2 C}{\partial u \partial v}(u, v) \), \( u, v \in [0, 1] \).