Chapter 9
Volatility Estimation

While the market parameters $r, t, S_t, T,$ and $K$ used to price an option via the Black-Scholes formula can be easily obtained from market data, estimating the volatility coefficient $\sigma$ can be a more difficult task. Several estimation methods are considered in this chapter, together with examples on how the Black-Scholes formula can be fitted to market data. In particular we cover historical, implied, and local volatility estimation and the VIX® volatility index.

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9.1 Historical Volatility

We consider the problem of estimating the parameters $\mu$ and $\sigma$ from market data in the stock price model

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t. \quad (9.1)$$

Historical trend estimation

By discretization of (9.1) along a family $t_0, t_1, \ldots, t_N$ of observation times as

$$\frac{S_{t_{k+1}} - S_{t_k}}{S_{t_k}} = (t_{k+1} - t_k)\mu + (B_{t_{k+1}} - B_{t_k})\sigma, \quad k = 0, 1, \ldots, N - 1, \quad (9.2)$$

a natural estimator for the trend parameter $\mu$ can be constructed as
\[ \widehat{\mu}_N := \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{t_{k+1} - t_k} \left( \frac{S_{tk+1}^M - S_{tk}^M}{S_{tk}^M} \right), \] (9.3)

where \((S_{tk+1}^M - S_{tk}^M)/S_{tk}^M, k = 0, 1, \ldots, N - 1\) denotes market returns observed at discrete times \(t_0, t_1, \ldots, t_N\) on the market.

**Historical log-return estimation**

Alternatively, observe that, replacing (9.3) by the log-returns

\[ \log \left( 1 + \frac{S_{tk+1} - S_{tk}}{S_{tk}} \right) = \log \frac{S_{tk+1}}{S_{tk}} = \log S_{tk+1} - \log S_{tk} \simeq \frac{S_{tk+1} - S_{tk}}{S_{tk}}, \]

with \(t_{k+1} - t_k = T/N, k = 0, 1, \ldots, N - 1\), one can replace (9.3) with the simpler telescoping estimate*

\[ \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{t_{k+1} - t_k} \left( \log S_{tk+1} - \log S_{tk} \right) = \frac{1}{T} \log \frac{S_T}{S_0}. \]

**Historical volatility estimation**

The volatility parameter \(\sigma\) can be estimated by writing, from (9.2),

\[ \sigma^2 \sum_{k=0}^{N-1} (B_{tk+1} - B_{tk})^2 = \sum_{k=0}^{N-1} \left( \frac{S_{tk+1} - S_{tk}}{S_{tk}} - (t_{k+1} - t_k)\mu \right)^2, \]

which yields the (unbiased) realized variance estimator

\[ \widehat{\sigma}_N^2 := \frac{1}{N - 1} \sum_{k=0}^{N-1} \frac{1}{t_{k+1} - t_k} \left( \frac{S_{tk+1} - S_{tk}}{S_{tk}} - (t_{k+1} - t_k)\widehat{\mu}_N \right)^2. \]

---

* Note that strictly speaking, the Itô formula reads \(d \log S_t = dS_t/S_t - (dS_t)^2/(2S_t^2)\).
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```
stock = Ad('0005.HK')
returns = (stock - lag(stock))/stock
returns = diff(log(stock)); times = index(returns)
returns <- as.vector(returns)
n = sum(is.na(returns)) + sum(!is.na(returns))
plot(times, returns, pch=19, cex=0.05, col="blue", ylab="returns", xlab="n", main = "")
segments(x0 = times, x1 = times, cex=0.05, y0 = 0, y1 = returns, col="blue")
abline(seq(1, n), 0, FALSE)
dt = 1.0/365
mu = mean(returns, na.rm=TRUE)/dt
sigma = sd(returns, na.rm=TRUE)/sqrt(dt)
mu; sigma
```

(a) Underlying asset price.

(b) Log returns.

Fig. 9.1: Graph of underlying asset price vs log returns.

```
library(PerformanceAnalytics)
library(quantmod)
returns <- exp(CalculateReturns(stock, method="compound");1 - 1
returns[1,] <- 0
histvol <- rollapply(returns, width = 30, FUN=sd.annualized)
myTheme <- chart_theme()
myTheme$col$line.col <- "blue"
chart_Series(stock, name="0005.HK", theme=myTheme)
add_TA(histvol, name="Historical Volatility")
```

The next Figure 9.2 presents a historical volatility graph with a 30 days rolling window.
Parameter estimation based on historical data usually requires a lot of samples and it can only be valid on a given time interval, or as a moving average. Moreover, it can only rely on past data, which may not reflect future data.

9.2 Implied Volatility

Recall that when $h(x) = (x - K)^+$, the solution of the Black-Scholes PDE is given by

$$Bl(t, x, K, \sigma, r, T) = x\Phi(d_+(T - t)) - K e^{-(T-t)r} \Phi(d_-(T - t)),$$

where

* Scorsese (2013) Click on the figure to play the video (works in Acrobat Reader on the entire pdf file).
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\[ \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy, \quad x \in \mathbb{R}, \]

and

\[
\begin{align*}
    d_+(T - t) &= \frac{\log(x/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}, \\
    d_-(T - t) &= \frac{\log(x/K) + (r - \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}.
\end{align*}
\]

In contrast with the historical volatility, the computation of the implied volatility can be done at a fixed time and requires much less data. Equating the Black-Scholes formula

\[ \text{Bl}(t, S_t, K, \sigma, r, T) = M \] (9.4)

to the observed value \(M\) of a given market price allows one to infer a value of \(\sigma\) when \(t, S_t, r, T\) are known, as in \(e.g.\) Figure 6.20.

![Option price as a function of volatility](https://www.ntu.edu.sg/home/nprivault/indext.html)

This value of \(\sigma\) is called the implied volatility, and it is denoted here by \(\sigma_{\text{imp}}(K, T)\), cf. \(e.g.\) Exercise 6.6. Various algorithms can be implemented to solve (9.4) numerically for \(\sigma_{\text{imp}}(K, T)\), such as the bisection method and the Newton-Raphson method.*

* Download the corresponding **R code** or the **IPython notebook** that can be run [here](https://www.ntu.edu.sg/home/nprivault/indext.html).
The implied volatility value can be used as an alternative way to quote the option price, based on the knowledge of the remaining parameters (such as underlying asset price, time to maturity, interest rate, and strike price). For example, option price data provided by the Hong Kong stock exchange includes implied volatility computed by inverting the Black-Scholes formula, cf. Figure S.20.

**Option chain data in R**

```r
install.packages("quantmod")
library(quantmod)
getSymbols("^GSPC", src = "yahoo", from = as.Date("2018-01-01"), to = as.Date("2018-03-01"))
head(GSPC)
# Only the front-month expiry
GSPC.OPT <- getOptionChain("^GSPC")
# All expiries
GSPC.OPTS <- getOptionChain("^GSPC", NULL)
# All 2018 to 2020 expiries
GSPC.OPTS <- getOptionChain("^GSPC", "2018/2020")
# Only the front-month expiry
AAPL.OPT <- getOptionChain("AAPL")
# All expiries
AAPL.OPTS <- getOptionChain("AAPL", NULL)
# All 2018 to 2020 expiries
AAPL.OPTS <- getOptionChain("AAPL", "2018/2020")
```

**Exporting option price data**

```r
write.table(AAPL.OPT$puts, file = "AAPL.puts")
write.csv(AAPL.OPT$puts, file = "AAPL.puts.csv")
install.packages("xlsx")
library(xlsx)
```
Volatility smiles

Given two European call options with strike prices $K_1$, resp. $K_2$, maturities $T_1$, resp. $T_2$, and prices $C_1$, resp. $C_2$, on the same stock $S$, this procedure should yield two estimates $\sigma_{\text{imp}}(K_1, T_1)$ and $\sigma_{\text{imp}}(K_2, T_2)$ of implied volatilities according to the following equations.

\[
\begin{align*}
\text{Bl}(t, S_t, K_1, \sigma_{\text{imp}}(K_1, T_1), r, T_1) &= M_1, \\
\text{Bl}(t, S_t, K_2, \sigma_{\text{imp}}(K_2, T_2), r, T_2) &= M_2,
\end{align*}
\]

Clearly, there is no reason a priori for the implied volatilities $\sigma_{\text{imp}}(K_1, T_1)$, $\sigma_{\text{imp}}(K_2, T_2)$ solutions of (9.5a)-(9.5b) to coincide across different strike prices and different maturities. However, in the standard Black-Scholes model the value of the parameter $\sigma$ should be unique for a given stock $S$. This contradiction between a model and market data is a reason for the development of more sophisticated stochastic volatility models.
```r
# Maturity to be updated as needed
maturity <- as.Date("2019-12-20", format="%Y-%m-%d")
CHAIN <- getOptionChain("^GSPC", maturity)
# Last trading day (may require update)
today <- as.Date(Sys.Date(), format="%Y-%m-%d")
T <- as.numeric((maturity - today)/365); r = 0.02; ImpVol < 1:1
getSymbols("^GSPC", from=today-1, to=today, src="yahoo")
S = as.numeric(Ad(GSPC))
for (i in 1:length(CHAIN$calls$Strike)) {ImpVol[i] = implied.vol(S, CHAIN$calls$Strike[i], T, r, CHAIN$calls$Last[i])}
plot(CHAIN$calls$Strike[!is.na(ImpVol)], ImpVol[!is.na(ImpVol)], xlab = "Strike price", ylab = "Implied volatility", lwd = 3, type = "l", col = "blue")
fit4 <- lm(ImpVol[!is.na(ImpVol)] ~ poly(CHAIN$calls$Strike[!is.na(ImpVol)], 4, raw = TRUE))
lines(CHAIN$calls$Strike[!is.na(ImpVol)], predict(fit4, data.frame(x = CHAIN$calls$Strike[!is.na(ImpVol)])), col = "red", lwd = 3)
```

Fig. 9.5: S&P500 option prices plotted against strike prices.

Plotting the different values of the implied volatility $\sigma$ as a function of $K$ and $T$ will yield a three-dimensional plot called the volatility surface. Figure 9.6 presents an estimation of implied volatility for Asian options whose underlying asset is the price of light sweet crude oil futures traded on the New York Mercantile Exchange (NYMEX), based on contract specifications and market data obtained from the Chicago Mercantile Exchange.
Fig. 9.6: Implied volatility surface of Asian options on light sweet crude oil futures.*

As observed in Figure 9.6, the volatility surface can exhibit a *smile* phenomenon, in which implied volatility is higher at a given end (or at both ends) of the range of strike price values.

**Black-Scholes Formula vs Market Data**

On July 28, 2009 a call warrant has been issued by Merrill Lynch on the stock price $S$ of Cheung Kong Holdings (0001.HK) with strike price $K=109.99$, Maturity $T = December 13, 2010$, and entitlement ratio 100.

The market price of the option (17838.HK) on September 28 was $12.30$, as obtained from [https://www.hkex.com.hk/eng/dwrc/search/listsearch.asp](https://www.hkex.com.hk/eng/dwrc/search/listsearch.asp).

The next graph in Figure 9.8 shows the evolution of the market price of the option over time. One sees that the option price is much more volatile than the underlying asset price.

* © Tan Yu Jia.
In Figure 9.9 we have fitted the path

\[ t \mapsto g_c(t, S_t) \]

of the Black-Scholes price to the data of Figure 9.8 using the stock price data of Figure 9.7, by varying the values of the volatility \( \sigma \).

Another example

Let us consider the stock price of HSBC Holdings (0005.HK) over one year:
Next, we consider the graph of the price of the call option issued by Societe Generale on 31 December 2008 with strike price $K=63.704$, maturity $T =$ October 05, 2009, and entitlement ratio 100, cf. page 8.

As above, in Figure 9.12 we have fitted the path $t \rightarrow g_c(t, S_t)$ of the Black-Scholes option price to the data of Figure 9.11 using the stock price data of Figure 9.10.

In this case the option is in the money at maturity. We can also check that the option is worth $100 \times 0.2650 = $26.650 at that time, which, according to absence of arbitrage, is very close to the actual value $90 - 63.703 = $26.296 of its payoff.

For one more example, consider the graph of the price of a put option issued by BNP Paribas on 04 November 2008 on the underlying asset HSBC, with strike price $K=77.667$, maturity $T =$ October 05, 2009, and entitlement ratio 92.593.

One checks easily that at maturity, the price of the put option is worth $0.01 (a market price cannot be lower), which almost equals the option payoff $0$, by absence of arbitrage opportunities. Figure 9.14 is a fit of the Black-Scholes
Fig. 9.12: Graph of the Black-Scholes call option price on HSBC Holdings.

Fig. 9.13: Graph of the (market) put option price on HSBC Holdings.

The normalized market data graph in Figure 9.15 shows how the option price can track the values of the underlying asset price. Note that the range of values $[26.55, 26.90]$ for the underlying asset price corresponds to $[0.675, 0.715]$ for the option price, meaning $1.36\%$ vs $5.9\%$ in percentage. This is a European call option on the ALSTOM underlying asset with strike price $K = €20$, maturity March 20, 2015, and entitlement ratio 10.

9.3 Local Volatility

As the constant volatility assumption in the Black-Scholes model does not appear to be satisfactory due to the existence of volatility smiles, it can make more sense to consider models of the form

$$ t \mapsto g_p(t, S_t) $$

to Figure 9.13 as a function of the stock price data of Figure 9.12. Note that the Black-Scholes price at maturity is strictly equal to 0 while the corresponding market price cannot be lower than one cent.
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![Graph of the Black-Scholes put option price on HSBC Holdings.](image1)

**Fig. 9.14:** Graph of the Black-Scholes put option price on HSBC Holdings.

![ALSTO 20TL 0315Z / ALSTOM](image2)

**Fig. 9.15:** Call option price vs underlying asset price.

\[
\frac{dS_t}{S_t} = rdt + \sigma_t dB_t
\]

where \( \sigma_t \) is a random process. Such models are called stochastic volatility models.

A particular class of stochastic volatility models can be written as

\[
\frac{dS_t}{S_t} = rdt + \sigma(t, S_t) dB_t
\]  

(9.6)

where \( \sigma(t, x) \) is a deterministic function of time \( t \) and of the underlying asset price \( x \). Such models are called local volatility models. The corresponding Black-Scholes PDE for the option prices

\[
g(t, x, K) := e^{-(T-t)r} \mathbb{E}[(S_T - K)^+ | S_t = x],
\]

(9.7)

where \( (S_t)_{t \in \mathbb{R}^+} \) is defined by (9.6), can be written as
\[
\begin{aligned}
rg(t, x, K) &= \frac{\partial g}{\partial t}(t, x, K) + rx \frac{\partial g}{\partial x}(t, x, K) + \frac{1}{2}x^2 \sigma^2(t, x) \frac{\partial^2 g}{\partial x^2}(t, x, K), \\
g(T, x, K) &= (x - K)^+, 
\end{aligned}
\]
with terminal condition \(g(T, x, K) = (x - K)^+,\) i.e. we consider European call options.

**Lemma 9.1.** (Relation (1) in Breeden and Litzenberger (1978)). Consider a family \((C^M(T, K))_{T,K>0}\) of market call option prices with maturities \(T\) and strike prices \(K\) given at time 0. Then the probability density function \(\varphi_t(y)\) of \(S_t, t \in [0, T]\), is given by

\[
\varphi_T(K) = e^{rT} \frac{\partial^2 C^M}{\partial K^2}(T, K), \quad K > 0. \tag{9.9}
\]

**Proof.** Assume that the market prices \(C^M(T, K)\) match the Black-Scholes prices \(e^{-rT} \mathbb{E}[(S_T - K)^+]\), \(K > 0\). Letting \(\varphi_T(y)\) denote the probability density function of \(S_T\), Condition (9.12) can be written at \(t = 0\) as

\[
C^M(T, K) = e^{-rT} \mathbb{E}[(S_T - K)^+] = e^{-rT} \int_0^\infty (y - K)^+ \varphi_T(y) dy = e^{-rT} \int_K^\infty (y - K) \varphi_T(y) dy = e^{-rT} \int_K^\infty y \varphi_T(y) dy - K e^{-rT} \int_K^\infty \varphi_T(y) dy - K e^{-rT} \mathbb{P}(S_T \geq K). \tag{9.10}
\]

By differentiation of (9.10) with respect to \(K\), one gets

\[
\frac{\partial C^M}{\partial K}(T, K) = -e^{-rT} K \varphi_T(K) - e^{-rT} \int_K^\infty \varphi_T(y) dy + e^{-rT} K \varphi_T(K) = -e^{-rT} \int_K^\infty \varphi_T(y) dy,
\]

which yields (9.9) by twice differentiation of \(C^M(T, K)\) with respect to \(K\).

In order to implement a stochastic volatility model such as (9.6), it is important to first calibrate the local volatility function \(\sigma(t, x)\) to market data.

In principle, the Black-Scholes PDE could allow one to recover the value of \(\sigma(t, x)\) as a function of the option price \(g(t, x, K)\), as

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\[
\sigma(t, x) = \sqrt{\frac{2rg(t, x, K) - 2\frac{\partial g}{\partial t}(t, x, K) - 2rx\frac{\partial g}{\partial x}(t, x, K)}{x^2 \frac{\partial^2 g}{\partial x^2}(t, x, K)}}, \quad x, t > 0,
\]

however, this formula requires the knowledge of the option price for different values of the underlying asset price \(x\), in addition to the knowledge of the strike price \(K\).

The Dupire (1994) formula brings a solution to the local volatility calibration problem by providing an estimator of \(\sigma(t, x)\) as a function \(\sigma(t, K)\) based on the values of the strike price \(K\).

**Proposition 9.2.** (Dupire (1994), Derman and Kani (1994)) Consider a family \((C^M(T, K))_{T, K>0}\) of market call option prices with maturities \(T\) and strike prices \(K\) given at time 0 with \(S_0 = x\), and define the volatility function \(\sigma(t, y)\) by

\[
\sigma(t, y) := \sqrt{\frac{2\frac{\partial C^M}{\partial t}(t, y) + 2ry\frac{\partial C^M}{\partial y}(t, y)}{y^2 \frac{\partial^2 C^M}{\partial y^2}(t, y)}} = \sqrt{\frac{\partial C^M}{\partial t}(t, y) + ry\frac{\partial C^M}{\partial y}(t, y)}\frac{y e^{-rT/2} \sqrt{\varphi_t(y)/2}}{y^2 \frac{\partial^2 C^M}{\partial y^2}(t, y)}},
\]

where \(\varphi_t(y)\) denotes the probability density function of \(S_t, t \in [0, T]\). Then the prices generated from the Black-Scholes PDE (9.8) will be compatible with the market option prices \(C^M(T, K)\) in the sense that

\[
C^M(T, K) = e^{-rT} \mathbb{E}[(S_T - K)^+], \quad K > 0.
\]

**Proof.** For any sufficiently smooth function \(f \in C^\infty_0(\mathbb{R})\), with \(\lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} f(x) = 0\), using the Itô formula we have

\[
\mathbb{E}[f(S_T)] = \mathbb{E} \left[ f(S_0) + r \int_0^T S_t f'(S_t) dt + \int_0^T S_t f''(S_t) \sigma(t, S_t) dB_t + \frac{1}{2} \int_0^T S_t^2 \sigma^2(t, S_t) dt \right]
\]

\[
= f(S_0) + \mathbb{E} \left[ r \int_0^T S_t f'(S_t) dt + \frac{1}{2} \int_0^T S_t^2 \sigma^2(t, S_t) dt \right]
\]

\[
= f(S_0) + r \int_0^T \mathbb{E}[S_t f'(S_t)] dt + \frac{1}{2} \int_0^T \mathbb{E}[S_t^2 f''(S_t) \sigma^2(t, S_t)] dt
\]

\[
= f(S_0) + r \int_{-\infty}^\infty y f'(y) \int_0^T \varphi_t(y) dt dy
\]
hence, after differentiating both sides of the equality with respect to $T$, 
\[ \int_{-\infty}^{\infty} f(y) \frac{\partial \varphi_T}{\partial T}(y) dy = r \int_{-\infty}^{\infty} y f'(y) \varphi_T(y) dy + \frac{1}{2} \int_{-\infty}^{\infty} y^2 f''(y) \sigma^2(T, y) \varphi_T(y) dy. \]
Integrating by parts in the above relation yields 
\[ \int_{-\infty}^{\infty} \frac{\partial \varphi_T}{\partial T}(y) f(y) dy = -r \int_{-\infty}^{\infty} f(y) \frac{\partial}{\partial y} (y \varphi_T(y)) dy + \frac{1}{2} \int_{-\infty}^{\infty} f(y) \frac{\partial^2}{\partial y^2} \left( y^2 \sigma^2(T, y) \varphi_T(y) \right) dy, \]
for all smooth functions $f(y)$ with compact support, hence 
\[ \frac{\partial \varphi_T}{\partial T}(y) = -r \frac{\partial}{\partial y} (y \varphi_T(y)) + \frac{1}{2} \frac{\partial^2}{\partial y^2} \left( y^2 \sigma^2(T, y) \varphi_T(y) \right), \quad y \in \mathbb{R}. \]
Making use of Relation (9.9) in Lemma 9.1, we have 
\[ \frac{\partial \varphi_T}{\partial T}(K) = r e^{rT} \frac{\partial^2 C^M}{\partial K^2} (T, K) + e^{rT} \frac{\partial^3 C^M}{\partial T \partial K^2} (T, K), \]
hence we get 
\[ -r \frac{\partial^2 C^M}{\partial y^2} (T, y) - \frac{\partial^3 C^M}{\partial T \partial y^2} (T, y) \]
\[ = r \frac{\partial}{\partial y} \left( y \frac{\partial^2 C^M}{\partial y^2} (T, y) \right) - \frac{1}{2} \frac{\partial^2}{\partial y^2} \left( y^2 \sigma^2(T, y) \frac{\partial^2 C^M}{\partial y^2} (T, y) \right), \quad y \in \mathbb{R}. \]
After a first integration with respect to $y$ under the boundary condition 
\[ \lim_{y \to +\infty} C^M(T, y) = 0, \]
we obtain 
\[ -r \frac{\partial C^M}{\partial y} (T, y) - \frac{\partial}{\partial T} \frac{\partial C^M}{\partial y} (T, y) \]
\[ = r y \frac{\partial^2 C^M}{\partial y^2} (T, y) - \frac{1}{2} \frac{\partial}{\partial y} \left( y^2 \sigma^2(T, y) \frac{\partial^2 C^M}{\partial y^2} (T, y) \right), \]
i.e. 
\[ -r \frac{\partial C^M}{\partial y} (T, y) - \frac{\partial}{\partial T} \frac{\partial C^M}{\partial y} (T, y) \]
\[ = r \frac{\partial}{\partial y} \left( y \frac{\partial C^M}{\partial y} (T, y) \right) - r \frac{\partial C^M}{\partial y} (T, y) - \frac{1}{2} \frac{\partial}{\partial y} \left( y^2 \sigma^2(T, y) \frac{\partial^2 C^M}{\partial y^2} (T, y) \right), \]
or
\[- \frac{\partial}{\partial y} \frac{\partial C^M}{\partial T}(T, y) = r \frac{\partial}{\partial y} \left( y \frac{\partial C^M}{\partial y}(T, y) \right) - \frac{1}{2} \frac{\partial}{\partial y} \left( y^2 \sigma^2(T, y) \frac{\partial^2 C^M}{\partial y^2}(T, y) \right).\]

Integrating one more time with respect to \(y\) yields
\[- \frac{\partial C^M}{\partial T}(T, y) = r y \frac{\partial C^M}{\partial y}(T, y) - \frac{1}{2} y^2 \sigma^2(T, y) \frac{\partial^2 C^M}{\partial y^2}(T, y), \quad y \in \mathbb{R},\]
which conducts to (9.11) and is called the Dupire Dupire (1994) PDE. \(\Box\)

Partial derivatives in time can be approximated using forward finite difference approximations as
\[
\frac{\partial C}{\partial t}(t_i, y_j) \simeq \frac{C(t_{i+1}, y_j) - C(t_i, y_j)}{\Delta t}, \quad (9.13)
\]
or, using backward finite difference approximations, as
\[
\frac{\partial C}{\partial t}(t_i, y_j) \simeq \frac{C(t_i, y_j) - C(t_{i-1}, y_j)}{\Delta t}. \quad (9.14)
\]

First order spatial derivatives can be approximated as
\[
\frac{\partial C}{\partial y}(t, y_j) \simeq \frac{C(t, y_j) - C(t, y_{j-1})}{\Delta y}, \quad \frac{\partial C}{\partial y}(t, y_{j+1}) \simeq \frac{C(t, y_{j+1}) - C(t, y_j)}{\Delta y}. \quad (9.15)
\]

Reusing (9.15), second order spatial derivatives can be similarly approximated as
\[
\frac{\partial^2 C}{\partial y^2}(t, y_j) \simeq \frac{1}{\Delta y} \left( \frac{\partial C}{\partial y}(t, y_{j+1}) - \frac{\partial C}{\partial y}(t, y_j) \right) \simeq \frac{C(t, y_{j+1}) + C(t, y_{j-1}) - 2C(t, y_j)}{(\Delta y)^2}. \quad (9.16)
\]

Figure 9.16* presents an estimation of local volatility by the finite differences (9.13)-(9.16), based on Boeing NYSE:BA option price data. See Achdou and Pironneau (2005) and in particular Figure 8.1 therein for numerical methods applied to local volatility estimation using spline functions instead of the discretization (9.13)-(9.16).

The attached R code © Abhishek Vijayakumar) plots a local volatility estimate for a given stock.

* © Yu Zhi Yu.
Based on (9.11), the local volatility $\sigma(t, y)$ can also be estimated by computing $C^M(T, y)$ from the Black-Scholes formula, based on a value of the implied volatility $\sigma$.

**Local volatility from put option prices**

Note that by the call-put parity relation

$$C^M(T, y) = P^M(T, y) + x - y e^{-rT}, \quad y, T > 0,$$

where $S_0 =$, cf. (6.18), we have

\[
\begin{aligned}
\frac{\partial C^M}{\partial T}(T, y) &= ry e^{-rT} + \frac{\partial P^M}{\partial T}(T, y), \\
\frac{\partial P^M}{\partial y}(t, y) &= e^{-rT} + \frac{\partial C^M}{\partial y}(t, y), \\
\varphi_T(K) &= e^{rT} \frac{\partial^2 C^M}{\partial y^2}(T, y) = e^{rT} \frac{\partial^2 P^M}{\partial y^2}(T, y),
\end{aligned}
\]

and

$$\frac{\partial C^M}{\partial T}(T, y) + ry \frac{\partial C^M}{\partial y}(T, y) = \frac{\partial P^M}{\partial T}(T, y) + ry \frac{\partial P^M}{\partial y}(T, y).$$

Consequently, the local volatility in Proposition 9.2 can be rewritten in terms of market put prices as
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\[ \sigma(t, y) := \sqrt{\frac{2 \frac{\partial P^M}{\partial t}(t, y) + 2ry \frac{\partial P^M}{\partial y}(t, y)}{y^2 \frac{\partial^2 P^M}{\partial y^2}(t, y)}} = \sqrt{\frac{\frac{\partial P^M}{\partial t}(t, y) + ry \frac{\partial P^M}{\partial y}(t, y)}{ye^{-rT/2} \sqrt{\varphi(y)/2}}}, \]

which is formally identical to (9.11) after replacing market call option prices \( C^M(T, K) \) with market put option prices \( P^M(T, K) \).

### 9.4 The VIX® Index

Other ways to estimate market volatility include the CBOE Volatility Index® (VIX®) for the S&P 500 stock index, cf. e.g. §3.1.1 of Papanicolaou and Sircar (2014). Let the asset price process \((S_t)_{t \in \mathbb{R}_+}\) be given as

\[ dS_t = rS_t dt + \sigma_t S_t dB_t \]

where, as in Section 8.2, \((\sigma_t)_{t \in \mathbb{R}_+}\) is a stochastic volatility process which is independent of the Brownian motion \((B_t)_{t \in \mathbb{R}_+}\).

The next Proposition 9.3, cf. Friz and Gatheral (2005), shows that the VIX® Volatility Index defined as

\[ \text{VIX}_t := \sqrt{\frac{2}{\tau} \mathbb{E}^* \left[ \int_0^{F_t,t+\tau} P(t, t+\tau, K) \frac{dK}{K^2} \right] + \int_0^{\infty} C(t, t+\tau, K) \frac{dK}{K^2} } \]

at time \( t \) can be interpreted as an average of future volatility values. Here, \( \tau = 30 \) days, \( F_{t,t+\tau} := \mathbb{E}^*[S_{t+\tau} | \mathcal{F}_t] \) is a future price, and \( P(t, t+\tau, K) \) and \( C(t, t+\tau, K) \) are out of the money put option prices, and out of the money call option prices with strike price \( K \) and maturity \( t+\tau \), cf. §3.1.1 of Papanicolaou and Sircar (2014).

**Proposition 9.3.** The value of the VIX® Volatility Index at time \( t \geq 0 \) is given from the averaged realized variance option as

\[ \text{VIX}_t := \sqrt{\frac{1}{\tau} \mathbb{E}^* \left[ \int_t^{t+\tau} \sigma_u^2 du \right] + 2 \frac{e^{r\tau} - r\tau - 1}{\tau}}. \]

**Proof.** For simplicity we take \( t = 0 \). We start by showing that for every \( \lambda > 0 \) we have

\[ \frac{1}{\lambda} \mathbb{E} \left[ \exp \left( r\lambda T + \lambda \int_0^T \sigma_t^2 dt \right) - 1 \right] = \frac{2e^{rT}}{S_0^{p\lambda}} \left( \int_0^{S_0} P(T, K) \frac{dK}{K^{2-p\lambda}} + \int_{S_0}^{\infty} C(T, K) \frac{dK}{K^{2-p\lambda}} \right) + \frac{p\lambda}{\lambda} (e^{rT} - 1). \]
By Lemma 8.1 we have
\[
\frac{1}{\lambda} \mathbb{E} \left[ \exp \left( r \lambda T + \lambda \int_0^T \sigma_t^2 dt \right) - 1 \right] = \frac{1}{\lambda} \mathbb{E} \left[ \left( \frac{S_T}{S_0} \right)^{p\lambda} - 1 \right]
\]

Next, by integration by parts over the intervals [0, S₀] and [S₀, ∞) and using the boundary conditions
\[
P(T, 0) = C(T, \infty) = 0, \quad \frac{\partial P}{\partial K}(T, 0) = \frac{\partial C}{\partial K}(T, \infty) = 0,
\]
with
\[
P(T, S₀) - C(T, S₀) = S₀(1 - e^{-rT}),
\]
and
\[
\frac{\partial P}{\partial K}(T, K) - \frac{\partial C}{\partial K}(T, K) - e^{-rT} = 0,
\]
we have
\[
\frac{1}{\lambda} \mathbb{E} \left[ \exp \left( r \lambda T + \lambda \int_0^T \sigma_t^2 dt \right) - 1 \right] = \frac{1}{\lambda} \mathbb{E} \left[ \left( \frac{S_T}{S_0} \right)^{p\lambda} - 1 \right]
\]
\[
= \frac{1}{\lambda S₀^{p\lambda}} \mathbb{E} \left[ Sₜ^{p\lambda} - S₀^{p\lambda} \right]
\]
\[
= \frac{1}{\lambda S₀^{p\lambda}} \left( \int_0^{S₀} K^{p\lambda} \varphi_T(K) dK + \int_{S₀}^\infty K^{p\lambda} \varphi_T(K) dK - S₀^{p\lambda} \right)
\]
\[
= \frac{1}{\lambda S₀^{p\lambda}} \left( e^{rT} \int_0^{S₀} K^{p\lambda} \frac{\partial^2 P}{\partial K^2}(T, K) dK + e^{rT} \int_{S₀}^\infty K^{p\lambda} \frac{\partial^2 C}{\partial K^2}(T, K) dK - S₀^{p\lambda} \right)
\]
\[
= \frac{1}{\lambda S₀^{p\lambda}} \left( e^{rT} S₀^{p\lambda} \frac{\partial P}{\partial K}(T, S₀) - p_\lambda e^{rT} \int_0^{S₀} K^{p\lambda-1} \frac{\partial P}{\partial K}(T, K) dK - e^{rT} S₀^{p\lambda} \frac{\partial C}{\partial K}(T, S₀) - p_\lambda e^{rT} \int_{S₀}^\infty K^{p\lambda-1} \frac{\partial C}{\partial K}(T, K) dK + S₀^{p\lambda} \right)
\]
\[
= \frac{p_\lambda e^{rT}}{\lambda S₀^{p\lambda}} \left( \int_0^{S₀} K^{p\lambda-1} P(T, S₀) + (p_\lambda - 1) \int_0^{S₀} K^{p\lambda-2} P(T, K) dK + (p_\lambda - 1) \int_{S₀}^\infty K^{p\lambda-2} C(T, K) dK \right)
\]
\[
= \frac{p_\lambda (p_\lambda - 1)}{\lambda S₀^{p\lambda}} e^{rT} \left( \frac{S₀^{p\lambda} - e^{-rT}}{p_\lambda - 1} + \int_0^{S₀} K^{p\lambda-2} P(T, K) dK + \int_{S₀}^\infty K^{p\lambda-2} C(T, K) dK \right)
\]
\[
= \frac{2 e^{rT}}{S₀^{p\lambda}} \left[ \int_0^{S₀} P(T, K) dK + \int_{S₀}^\infty C(T, K) dK \right] + \frac{p_\lambda}{\lambda} (e^{rT} - 1)
\]
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\[ 2 e^{rT} \left( \int_{S_0}^{S_T} P(T, K) \frac{dK}{K^2-p_\lambda} + \int_{S_0}^{\infty} C(T, K) \frac{dK}{K^2-p_\lambda} \right) + \frac{2}{p_\lambda - 1} (e^{rT} - 1). \]

Finally, taking \( p_\lambda := p_\lambda^- \) and letting \( \lambda \) tend to zero, we find

\[-2rT + \mathbb{E} \left[ \int_{0}^{T} \sigma_t^2 dt \right] = \lim_{\lambda \to 0} \frac{1}{\lambda} \mathbb{E} \left[ \exp \left( r p_\lambda T + \lambda \int_{0}^{T} \sigma_t^2 dt \right) - 1 \right] + 2(1 - e^{rT})
\]

\[= \lim_{\lambda \to 0} \frac{2 e^{rT}}{S_0^p} \left( \int_{S_0}^{S_T} P(T, K) \frac{dK}{K^2-p_\lambda} + \int_{S_0}^{\infty} C(T, K) \frac{dK}{K^2-p_\lambda} \right) + 2(1 - e^{rT}).\]

\[= 2 e^{rT} \left( \int_{S_0}^{S_T} P(T, K) \frac{dK}{K^2} + \int_{S_0}^{\infty} C(T, K) \frac{dK}{K^2} \right) + 2(1 - e^{rT}),\]

hence

\[\mathbb{E} \left[ \int_{0}^{T} \sigma_t^2 dt \right] = 2 e^{rT} \left( \int_{S_0}^{S_T} P(T, K) \frac{dK}{K^2} + \int_{S_0}^{\infty} C(T, K) \frac{dK}{K^2} \right) - 2(e^{rT} - rT - 1).\]

The following code allows us to estimate the VIX\textsuperscript{®} based on the discretization of (9.17) and market option prices on the S&P 500.

```r
library(quantmod)
date = "2019-02-15"
getSymbols("^GSPC", src = "yahoo", from = date)
S0 = as.vector(Ad(GSPC)[1])
GSPC.OPTS <- getOptionChain("^GSPC", "2021-12-17")
Call <- as.data.frame(GSPC.OPTS$calls)
Put <- as.data.frame(GSPC.OPTS$puts)
K0 = max(Put[Put$Strike<S0,]$Strike, Call[Call$Strike<S0,]$Strike)
Call_OTM <- Call[Call$Strike>=K0,];Call_OTM$dif = c(S0-K0, diff(Call_OTM$Strike))
Put_OTM <-Put[Put$Strike<=K0,];Put_OTM$dif = c(diff(Put_OTM$Strike),S0-K0)
T = 30/365;r=0.02
VIX_imp = 100*sqrt((2*exp(r*T)/T)*(sum(Call_OTM$Last/(Call_OTM$Strike^2)*Call_OTM$dif)+sum(Put_OTM$Last/(Put_OTM$Strike^2)*Put_OTM$dif)))
getSymbols("^VIX", src = "yahoo", from = date)
VIX_market = as.vector(Ad(VIX)[1])
c("Estimated VIX"= VIX_imp, "market VIX"=VIX_market)
VIX.OPTS <- getOptionChain("^VIX")
```

The following code is fetching VIX\textsuperscript{®} data using the quantmod R package.

```r
library(quantmod)
getSymbols("^GSPC", from="2000-01-01",to=Sys.Date(),src="yahoo")
getSymbols("^VIX", from="2000-01-01",to=Sys.Date(),src="yahoo")
myTheme <- chart_theme()
myTheme$col$line.col <- "blue"
chart_Series(Ad("GSPC"),name="S&P500",theme=myTheme)
add_TA(Ad("VIX"), name="VIX")
```

The impact of various events, such as the June 23, 2016 “Brexit” referendum, can be observed on the VIX\textsuperscript{®} index in Figure 9.17.
Figure 9.17: VIX® Index vs the S&P 500.

```r
library(PerformanceAnalytics)
library(quantmod)
getSymbols("^GSPC",from="2000-01-01",to=Sys.Date(),src="yahoo")
getSymbols("^VIX",from="2000-01-01",to=Sys.Date(),src="yahoo")
SP500=Ad(GSPC)
SP500.rtn <- exp(CalculateReturns(SP500,method="compound")) - 1
histvol <- rollapply(SP500.rtn, width = 30, FUN=sd.annualized)
myTheme <- chart_theme()
myTheme$col$line.col <- "blue"
chart_Series(SP500,name="SP500",theme=myTheme)
add_TA(histvol, name="Historical Volatility")
add_TA(Ad("VIX"), name="VIX")
```

Figure 9.18 compares the VIX® estimate to the historical volatility of Section 9.1.

![VIX® Index vs historical volatility for the year 2011.](image)

We note that the variations of the stock index are negatively correlated to the variations of the VIX® index, however the same cannot be said of the correlation to the variations of historical volatility.
Volatility Estimation

(a) Underlying vs the VIX® index.  (b) Underlying vs hist. volatility.

Fig. 9.19: Correlation estimates between GSPC and the VIX®.

```
1  chart.Correlation(cbind(Ad("GSPC")-lag(Ad("GSPC")),Ad("VIX")-lag(Ad("VIX"))),
2    histogram=TRUE, pch="+")
3  colnames(histvol) <- "HistVol"
4  chart.Correlation(cbind(Ad("GSPC")-lag(Ad("GSPC")),histvol-lag(histvol)),
5    histogram=TRUE, pch="+")
```

The next Figure 9.20 shortens the time range to year 2011 and shows the reactivity of the VIX® in comparison with the moving average of historical volatility.

Fig. 9.20: VIX® Index vs 30 day historical volatility for the S&P 500.

Exercises

Exercise 9.1  Consider the Black-Scholes call pricing formula

\[ C(T - t, x, K) = K f(T - t, x/K) \]
written using the function 

\[ f(\tau, z) := z \Phi \left( \frac{(r + \sigma^2/2)\tau + \log z}{|\sigma|\sqrt{\tau}} \right) - e^{-r\tau} \Phi \left( \frac{(r - \sigma^2/2)\tau + \log z}{|\sigma|\sqrt{\tau}} \right). \]

a) Compute \( \frac{\partial C}{\partial x} \) and \( \frac{\partial C}{\partial K} \) using the function \( f \), and find the relation between 
\( \frac{\partial C}{\partial K}(T - t, x, K) \) and \( \frac{\partial C}{\partial x}(T - t, x, K) \).

b) Compute \( \frac{\partial^2 C}{\partial x^2} \) and \( \frac{\partial^2 C}{\partial K^2} \) using the function \( f \), and find the relation between
\( \frac{\partial^2 C}{\partial K^2}(T - t, x, K) \) and \( \frac{\partial^2 C}{\partial x^2}(T - t, x, K) \).

c) From the Black-Scholes PDE
\[ rC(T - t, x, K) = \frac{\partial C}{\partial t}(T - t, x, K) + rx \frac{\partial C}{\partial x}(T - t, x, K) \]
\[ + \frac{\sigma^2 x^2}{2} \frac{\partial^2 C}{\partial x^2}(T - t, x, K), \]
recover the Dupire PDE for the constant volatility \( \sigma \).

Exercise 9.2 Let \( \sigma_{\text{imp}}(K) \) denote the implied volatility of a call option with strike price \( K \), defined from the relation
\[ M_C(K, S, r, \tau) = C(K, S, \sigma_{\text{imp}}(K), r, \tau), \]
where \( M_C \) is the market price of the call option, \( C(K, S, \sigma_{\text{imp}}(K), r, \tau) \) is the Black-Scholes call pricing function, \( S \) is the underlying asset price, \( \tau \) is the time remaining until maturity, and \( r \) is the risk-free interest rate.

a) Compute the partial derivative 
\[ \frac{\partial M_C}{\partial K}(K, S, r, \tau). \]
using the functions \( C \) and \( \sigma_{\text{imp}} \).

b) Knowing that market call option prices \( M_C(K, S, r, \tau) \) are decreasing in the strike prices \( K \), find an upper bound for the slope \( \sigma'_{\text{imp}}(K) \) of the implied volatility curve.

c) Similarly, knowing that the market put option prices \( M_P(K, S, r, \tau) \) are increasing in the strike prices \( K \), find a lower bound for the slope \( \sigma'_{\text{imp}}(K) \) of the implied volatility curve.
Volatility Estimation

Exercise 9.3 Hagan et al. (2002) Consider a European option priced as $e^{-rT} \mathbb{E}^*[(S_T - K)^+]$ in a local volatility model $dS_t = \sigma_{\text{loc}}(S_t) S_t dB_t$. The implied volatility $\sigma_{\text{imp}}(K, S_0)$, computed from the equation

$$\text{Bl}(S_0, K, T, \sigma_{\text{imp}}(K, S_0), r) = e^{-rT} \mathbb{E}^*[(S_T - K)^+]$$

is known to admit the approximation

$$\sigma_{\text{imp}}(K, S_0) \simeq \sigma_{\text{loc}} \left( \frac{K + S_0}{2} \right).$$

a) Taking a local volatility of the form $\sigma_{\text{loc}}(x) := \sigma_0 + \beta (x - S_0)^2$, estimate the implied volatility $\sigma_{\text{imp}}(K, S)$ when the underlying asset price is at the level $S$.

b) Express the Delta of the Black Scholes call option price given by

$$\text{Bl}(S, K, T, \sigma_{\text{imp}}(K, S), r),$$

using the standard Black-Scholes Delta and the Black-Scholes Vega.