Chapter 8
Stochastic Volatility

We consider the pricing of options on realized variance in stochastic volatility models, including realized variance swaps and realized variance options. The methods applied include moment matching approximations, PDE arguments, and perturbation analysis.

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8.1 Stochastic Volatility Models

Time-dependent stochastic volatility

The next Figure 8.1 refers to the EURO/SGD exchange rate, and shows some spikes that cannot be generated by Gaussian returns with constant variance.
This type data shows that, in addition to jump models that are commonly used to take into account the slow decrease of probability tails observed in market data, other tools should be implemented in order to model a possibly random and time-varying volatility.

We consider an asset price driven by the stochastic differential equation

\[ dS_t = rS_t dt + S_t \sqrt{\nu_t} dB_t \]  

under the risk-neutral probability measure \( \mathbb{P}^* \), with solution

\[ S_T = S_t \exp \left( r(T - t) + \int_t^T \sqrt{v(s)} dB_s - \frac{1}{2} \int_t^T v(s) ds \right) \]  

where \( (\nu_t)_{t \in \mathbb{R}^+} \) is a (possibly random) squared volatility process adapted to the filtration \( \mathcal{F}_t^{(1)} \) generated by \( (B_t)_{t \in \mathbb{R}^+} \).

**Time-dependent deterministic volatility**

When the volatility \( (\nu(t))_{t \in \mathbb{R}^+} \) is a deterministic function of time, the solution (8.2) of (8.1) is a lognormal random variable at time \( T \) with conditional log-variance

\[ \int_t^T v(s) ds \]

given \( \mathcal{F}_t \). In particular, a European call option on \( S_T \) can be priced by the Black-Scholes formula as

\[ e^{-(T-t)r} \mathbb{E}^* [(S_T - K)^+ | \mathcal{F}_t] = \text{Bl}(S_t, K, r, T - t, \sqrt{\nu(t)}) \],

with integrated squared volatility parameter.
Independent (stochastic) volatility

When the volatility \((v_t)_{t \in \mathbb{R}_+}\) is a random process generating a filtration \(\mathcal{F}^{(2)}_t\) independent of the filtration \(\mathcal{F}^{(1)}_t\) generated by the driving Brownian motion \((B_t^{(1)})_{t \in \mathbb{R}_+}\) under \(\mathbb{P}^*\), the equation (8.1) can still be solved as

\[
S_T = S_t \exp \left( r(T-t) + \int_t^T \sqrt{v_s} dB_s^{(1)} - \frac{1}{2} \int_t^T v_s ds \right),
\]

and, given \(\mathcal{F}^{(2)}_T\), the asset price \(S_T\) is a lognormal random variable with random variance \(\int_t^T v_s ds\).

In this case, taking

\[
\mathcal{F}_t := \mathcal{F}^{(1)}_t \lor \mathcal{F}^{(2)}_t, \quad 0 \leq t \leq T,
\]

where \((\mathcal{F}^{(1)}_t)_{t \in \mathbb{R}_+}\) is the filtration generated by \((B_t^{(1)})_{t \in \mathbb{R}_+}\), we can still price an option with payoff \(\phi(S_T)\) on the underlying asset price \(S_T\) using the tower property

\[
\mathbb{E}^*[\phi(S_T) \mid \mathcal{F}_t] = \mathbb{E}^* \left[ \mathbb{E}^*[\phi(S_T) \mid \mathcal{F}^{(1)}_t \lor \mathcal{F}^{(2)}_T] \mid \mathcal{F}^{(1)}_t \lor \mathcal{F}^{(2)}_t \right].
\]

As an example, a European call option on \(S_T\) can be priced by averaging the Black-Scholes formula as follows:

\[
e^{-r(T-t)} \mathbb{E}^* [(S_T - K)^+ \mid \mathcal{F}_t] = e^{-r(T-t)} \mathbb{E}^* \left[ \mathbb{E}^*[\phi(S_T) \mid \mathcal{F}^{(1)}_t \lor \mathcal{F}^{(2)}_T] \mid \mathcal{F}^{(1)}_t \lor \mathcal{F}^{(2)}_t \right].
\]

\[
e^{-r(T-t)} \mathbb{E}^* \left[ \text{Bl} \left( S_t, K, r, T-t, \sqrt{\int_t^T v_s ds} \right) \mid \mathcal{F}^{(1)}_t \lor \mathcal{F}^{(2)}_t \right] \mid x = S_t,
\]

which represents an averaged version of Black-Scholes prices, with the random integrated volatility

\[
\tilde{v}(t, T) := \frac{1}{T-t} \int_t^T v_s ds, \quad 0 \leq t \leq T.
\]
On the other hand, when \((v_t)_{t \in \mathbb{R}_+}\) is a geometric Brownian motion, the probability distribution of the time integral \(\int_t^T v_s ds\) given \(\mathcal{F}_t^{(2)}\) can be computed using integral expressions, cf. Yor (1992) and Proposition 13.1.

Two-factor Stochastic Volatility Models

Evidence based on financial market data, see Figure 9.18, Figure 1 of Papanicolaou and Sircar (2014) or § 2.3.1 in Fouque et al. (2011), shows that the variations in volatility tend to be negatively correlated with the variations of underlying asset prices. For this reason we need to consider an asset price process \((S_t)_{t \in \mathbb{R}_+}\) and a stochastic volatility process \((v_t)_{t \in \mathbb{R}_+}\) driven by

\[
\begin{align*}
\frac{dS_t}{S_t} &= r_t dt + \sqrt{v_t} dB_{1t}, \\
\frac{dv_t}{v_t} &= (\mu(t, v_t) - \lambda v_t) dt + \eta dB_{2t},
\end{align*}
\]

Here, \((B_{1t})_{t \in \mathbb{R}_+}\) and \((B_{2t})_{t \in \mathbb{R}_+}\) are two standard Brownian motions such that

\[
\text{Cov}(B_{1t}, B_{2t}) = \rho t \quad \text{and} \quad dB_{1t} \cdot dB_{2t} = \rho dt,
\]

where the correlation parameter \(\rho\) satisfies \(-1 \leq \rho \leq 1\), and the coefficients \(\mu(t, x)\) and \(\beta(t, x)\) can be chosen e.g. from mean-reverting models (CIR) or geometric Brownian models, as follows. Note that the observed correlation coefficient \(\rho\) is usually negative, cf. e.g. § 2.1 in Papanicolaou and Sircar (2014) and Figures 9.18 and 9.19.

The Heston model

In the Heston (1993) model, the stochastic volatility \((v_t)_{t \in \mathbb{R}_+}\) is chosen to be a Cox et al. (1985) (CIR) process, i.e. we have

\[
\begin{align*}
\frac{dS_t}{S_t} &= r_t dt + \sqrt{v_t} dB_{1t}, \\
\frac{dv_t}{v_t} &= -\lambda (v_t - m) dt + \eta \sqrt{v_t} dB_{2t},
\end{align*}
\]

and \(\mu(t, v) = -\lambda (v_t - v)\) and \(\beta(t, v) = \eta \sqrt{v}\), where \(\lambda, m, \eta > 0\).

Option pricing formulas can be derived in the Heston model using Fourier inversion and complex integrals, cf. (8.18) below.
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The SABR model

In the Stochastic Alpha Beta Rho (SABR) model Hagan et al. (2002), based on the parameters \((\alpha, \beta, \rho)\), the stochastic volatility process \((\sigma_t)_{t \in \mathbb{R}_+}\) is modeled as a geometric Brownian motion with

\[
\begin{cases}
  dF_t = \sigma_t F_t^\beta dB_t^{(1)} \\
  d\sigma_t = \alpha \sigma_t dB_t^{(2)},
\end{cases}
\]

where \((F_t)_{t \in \mathbb{R}_+}\) typically models a forward interest rate. Here, we have \(\alpha > 0\) and \(\beta \in (0, 1]\), and \((B_t^{(1)})_{t \in \mathbb{R}_+}, (B_t^{(2)})_{t \in \mathbb{R}_+}\) are standard Brownian motions with the correlation

\[dB_t^{(1)} \cdot dB_t^{(2)} = \rho dt.\]

This model is typically used for the modeling of LIBOR rates and is not mean-reverting, hence it is preferably used with a short time horizon. It allows in particular for short time asymptotics of Black implied volatilities that can be used for pricing by inputting them into the Black pricing formula, cf. § 3.3 in Rebonato (2009).

8.2 Realized Variance Swaps

Another look at historical volatility

When \(t_k = kT/N, k = 0, 1, \ldots, N\), a natural estimator for the trend parameter \(\mu\) can be written as

\[
\hat{\mu}_N := \frac{1}{N} \sum_{k=1}^{N} \frac{1}{t_k - t_{k-1}} \frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}}
\]

\[\approx \frac{1}{N} \sum_{k=1}^{N} \frac{1}{t_k - t_{k-1}} \log \frac{S_{t_k}}{S_{t_{k-1}}} \]

\[= \frac{1}{T} \sum_{k=1}^{N} \left( \log(S_{t_k}) - \log(S_{t_{k-1}}) \right) \]

\[= \frac{1}{T} \log \frac{S_T}{S_0}.\]

Similarly we can use the squared volatility estimator

\[
\hat{\sigma}_N^2 := \frac{1}{N-1} \sum_{k=1}^{N} \frac{1}{t_k - t_{k-1}} \left( \frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}} \right)^2 - (\hat{\mu}_N)^2
\]
\[
\begin{align*}
&\approx \frac{1}{N} \sum_{k=1}^{N} \frac{1}{t_k - t_{k-1}} \left( \log(S_{t_k}) - \log(S_{t_{k-1}}) \right)^2 - \left( \frac{1}{N} \sum_{k=1}^{N} \frac{1}{t_k - t_{k-1}} \frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}} \right)^2 \\
&= \frac{1}{T} \sum_{k=1}^{N} \left( \frac{\log S_{t_k}}{S_{t_{k-1}}} \right)^2 - \frac{1}{T^2} \left( \frac{\log S_T}{S_0} \right)^2.
\end{align*}
\]

(8.3)

**Realized variance swaps**

Realized variance swaps are forward contracts that allow for the exchange of the estimated volatility (8.3) against a fixed value \( \kappa \sigma \). They are priced using the expected value

\[
\mathbb{E} \left[ \tilde{\sigma}_N^2 \right] = \frac{1}{T} \mathbb{E} \left[ \sum_{k=1}^{N} \left( \frac{\log S_{t_k}}{S_{t_{k-1}}} \right)^2 - \frac{1}{T^2} \left( \frac{\log S_T}{S_0} \right)^2 \right] - \kappa \sigma
\]

of their payoff

\[
\frac{1}{T} \sum_{k=1}^{N} \left( \frac{\log S_{t_k}}{S_{t_{k-1}}} \right)^2 - \frac{1}{T^2} \left( \frac{\log S_T}{S_0} \right)^2 - \kappa \sigma,
\]

where \( \kappa \sigma \) is the volatility level. Note that the above payoff has to be multiplied by the *vega notional*, which is part of the contract, in order to convert it into currency units.

**Heston model**

As an application of the lognormal and gamma approximations, consider the Heston model driven by the stochastic differential equation

\[
dv_t = (a - bv_t)dt + \sigma \sqrt{v_t} dW_t,
\]

we have

\[
\mathbb{E}[v_T] = v_0 e^{-bT} + \frac{a}{b} \left( 1 - e^{-bT} \right),
\]

from which it follows that

\[
\mathbb{E} \left[ R_{0,T}^2 \right] = \mathbb{E} \left[ \int_0^T \sigma_u^2 dt \right]
= v_0 \frac{1 - e^{-bT}}{b} + a \frac{e^{-bT} + bT - 1}{b^2}.
\]

**Independent stochastic volatility**

In the sequel, we assume that the risky asset price process is given by

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\[
\frac{dS_t}{S_t} = rd_t + \sigma_t dB_t,
\]
i.e.

\[
S_t = S_0 \exp \left( rt + \int_0^t \sigma_s dB_s - \frac{1}{2} \int_0^t \sigma_s^2 ds \right), \quad t \in \mathbb{R}_+,
\]
where \((\sigma_t)_{t \in \mathbb{R}_+}\) is a stochastic volatility process which is independent of the Brownian motion \((B_t)_{t \in \mathbb{R}_+}\).

**Lemma 8.1.** Assume that \((\sigma_t)_{t \in \mathbb{R}_+}\) is independent of \((B_t)_{t \in \mathbb{R}_+}\). Then for every \(\lambda > 0\) we have

\[
\mathbb{E} \left[ \exp \left( \lambda \int_0^T \sigma_t^2 dt \right) \right] = e^{-rp_\lambda^+ T} \mathbb{E} \left[ \left( \frac{S_T}{S_0} \right)^{p_\lambda^+} \right], \tag{8.4}
\]

where \(p_\lambda^\pm = 1/2 \pm \sqrt{1/4 + 2\lambda}\).

**Proof.** We have

\[
e^{-rp_\lambda T} \mathbb{E} \left[ \left( \frac{S_T}{S_0} \right)^{p_\lambda} \mid \mathcal{F}_T^\sigma \right] = \mathbb{E} \left[ \exp \left( p_\lambda \int_0^T \sigma_t dB_t - \frac{p_\lambda}{2} \int_0^T \sigma_t^2 dt \right) \mid \mathcal{F}_T^\sigma \right]
= \exp \left( -\frac{p_\lambda}{2} \int_0^T \sigma_t^2 dt \right) \mathbb{E} \left[ \exp \left( p_\lambda \int_0^T \sigma_t dB_t \right) \mid \mathcal{F}_T^\sigma \right]
= \exp \left( -\frac{p_\lambda}{2} \int_0^T \sigma_t^2 dt \right) \exp \left( \frac{p_\lambda^2}{2} \int_0^T \sigma_t^2 dt \right)
= \exp \left( p_\lambda(p_\lambda - 1)/2 \int_0^T \sigma_t^2 dt \right)
= \exp \left( \lambda \int_0^T \sigma_t^2 dt \right),
\]
provided that \(\lambda = p_\lambda(p_\lambda - 1)/2\), and in this case we have

\[
e^{-rp_\lambda T} \mathbb{E} \left[ \left( \frac{S_T}{S_0} \right)^{p_\lambda} \right] = e^{-rp_\lambda T} \mathbb{E} \left[ \left( \frac{S_T}{S_0} \right)^{p_\lambda} \mid \mathcal{F}_T^\sigma \right]
\]

\[
= \mathbb{E} \left[ \exp \left( \lambda \int_0^T \sigma_t^2 dt \right) \right].
\]

It remains to note that the equation \(\lambda = p_\lambda(p_\lambda - 1)/2\), i.e. \(p_\lambda^2 - p_\lambda - 2\lambda = 0\), has for solutions

\[
p_\lambda^\pm = \frac{1}{2} \pm \sqrt{1/4 + 2\lambda},
\]
with \(p_\lambda^- < 0 < p_\lambda^+\) when \(\lambda > 0\). \(\square\)

In addition to the relation

\[\ast\]

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we have the following result.

**Corollary 8.2.** We have

\[
\mathbb{E} \left[ \int_0^T \sigma_t^2 dt \right] = 2rT - 2 \mathbb{E} \left[ \log \frac{S_T}{S_0} \right] = -2 \mathbb{E} \left[ \int_0^T \sigma_t dB_t - \frac{1}{2} \int_0^T \sigma_t^2 dt \right],
\]

Proof. Rewriting (8.4) as

\[
\mathbb{E} \left[ \exp \left( \lambda \int_0^T \sigma_t^2 dt \right) \right] = \mathbb{E} \left[ \exp \left( -rp^\lambda T + p^\lambda \log \frac{S_T}{S_0} \right) \right]
\]

and differentiating this relation with respect to \( \lambda \), we get

\[
\mathbb{E} \left[ \int_0^T \sigma_t^2 dt \exp \left( \lambda \int_0^T \sigma_t^2 dt \right) \right] = -rp^\lambda T \mathbb{E} \left[ \exp \left( -rp^\lambda T + p^\lambda \log \frac{S_T}{S_0} \right) \right] \\
+ p^\lambda \mathbb{E} \left[ \exp \left( -rp^\lambda T + p^\lambda \log \frac{S_T}{S_0} \right) \log \frac{S_T}{S_0} \right] \\
= \pm \frac{rT}{\sqrt{1/4 + 2\lambda}} \mathbb{E} \left[ \exp \left( -rp^\lambda T \right) \left( \frac{S_T}{S_0} \right)^{p^\lambda} \right] \\
\pm \frac{1}{\sqrt{1/4 + 2\lambda}} \mathbb{E} \left[ \exp \left( -rp^\lambda T + p^\lambda \log \frac{S_T}{S_0} \right) \log \frac{S_T}{S_0} \right],
\]

which, when \( \lambda = 0 \), yields

\[
\mathbb{E} \left[ \int_0^T \sigma_t^2 dt \right] = 2rT - 2 \mathbb{E} \left[ \log \frac{S_T}{S_0} \right] = -2 \mathbb{E} \left[ \int_0^T \sigma_t dB_t - \frac{1}{2} \int_0^T \sigma_t^2 dt \right]
\]

for \( p_0^- = 0 \), and

\[
\mathbb{E} \left[ \int_0^T \sigma_t^2 dt \right] = 2e^{-rT} \mathbb{E} \left[ \frac{S_T}{S_0} \log \frac{e^{-rT}S_T}{S_0} \right]
\]

for \( p_0^+ = 1 \). \( \square \)

**8.3 Realized Variance Options**

In this section, we consider the realized variance call option with payoff

\[
\left( \int_0^T \sigma_t^2 dt - \kappa^2 \right)^+.
\]
Proposition 8.3. In case $\int_0^t \sigma^2_u dt \geq \kappa^2_\sigma$, the price of the realized variance call option is given by

$$e^{-(T-t)r} \mathbb{E}^* \left( \left( \int_0^T \sigma^2_u du - \kappa^2_\sigma \right)^+ \right| \mathcal{F}_t$$

$$= e^{-(T-t)r} \int_0^t \sigma^2_u du - e^{-(T-t)r} \kappa^2_\sigma + e^{-(T-t)r} \mathbb{E}^* \left[ \int_0^T \sigma^2_u du \right| \mathcal{F}_t].$$

Proof. In case $\int_0^t \sigma^2_u dt \geq \kappa^2_\sigma$, we have

$$e^{-(T-t)r} \mathbb{E}^* \left( \left( \int_0^T \sigma^2_u du - \kappa^2_\sigma \right)^+ \right| \mathcal{F}_t$$

$$= e^{-(T-t)r} \mathbb{E}^* \left( \left( x + \int_{t}^{T} \sigma^2_u du - \kappa^2_\sigma \right)^+ \right| \mathcal{F}_t \biggr| \int_0^t \sigma^2_u du$$

$$= e^{-(T-t)r} \mathbb{E}^* \left( \left( x + \int_{t}^{T} \sigma^2_u du - \kappa^2_\sigma \right) \biggl| \mathcal{F}_t \biggr| \int_0^t \sigma^2_u du$$

$$= e^{-(T-t)r} \int_0^t \sigma^2_u du - e^{-(T-t)r} \kappa^2_\sigma + e^{-(T-t)r} \mathbb{E}^* \left[ \int_0^T \sigma^2_u du \right| \mathcal{F}_t].$$

\[\square\]

Lognormal approximation

In order to estimate the price

$$e^{-(T-t)r} \mathbb{E}^* \left( \left( x + \int_{t}^{T} \sigma^2_u du - \kappa^2_\sigma \right)^+ \right| \mathcal{F}_t \biggr| \int_0^t \sigma^2_u du$$

of the realized variance call option when $R^2_{0,t} = \int_0^t \sigma^2_u dt < \kappa^2_\sigma$, we can approximate $R^2_{t,T} := \int_t^T \sigma^2_u du$ by a lognormal random variable

$$R^2_{t,T} = \int_t^T \sigma^2_u du \sim e^{\tilde{\mu}_{t,T} + \tilde{\sigma}_{t,T} X}$$

with mean $\tilde{\mu}_{t,T}$ and variance $\tilde{\eta}^2_{t,T}$, where $X \sim \mathcal{N}(0,1)$ is a standard normal random variable.

Proposition 8.4. (Lognormal approximation). Under the lognormal approximation, the probability density function $\varphi_{R^2_{t,T}}$ of $R^2_{t,T} := \int_t^T \sigma^2_u du$ can be approximated as
\[
\varphi_{R_{t,T}^2}(x) \approx \frac{1}{x\tilde{\sigma}_{t,T}\sqrt{2(T-t)\pi}} \exp\left(-\frac{(\tilde{\mu}_{t,T} - \log x)^2}{2(T-t)\tilde{\sigma}_{t,T}^2}\right), \quad x > 0, \tag{8.5}
\]

where
\[
\tilde{\mu}_{t,T} := -(T-t)\frac{\tilde{\sigma}_{t,T}^2}{2} + \log \mathbb{E}[R_{t,T}^2],
\]
and
\[
\tilde{\sigma}_{t,T}^2 = \frac{1}{T-t} \log \left(1 + \frac{\text{Var}[R_{t,T}^2]}{(\mathbb{E}[R_{t,T}^2])^2}\right).
\]

**Proof.** The parameters \(\tilde{\mu}_{t,T}\) and \(\tilde{\sigma}_{t,T}^2\) are estimated by matching the first and second moments \(\mathbb{E}[R_{t,T}^2]\) and \(\mathbb{E}[R_{t,T}^4]\) of \(R_{t,T}^4\) to those of the lognormal distribution with mean \(\tilde{\mu}_{t,T}\) and variance \((T-t)\tilde{\sigma}_{t,T}^2\), which yields
\[
\mathbb{E}[R_{t,T}^2] = e^{\tilde{\mu}_{t,T}+(T-t)\tilde{\sigma}_{t,T}^2/2}, \quad \mathbb{E}[R_{t,T}^4] = e^{2(\tilde{\mu}_{t,T}+(T-t)\tilde{\sigma}_{t,T}^2)},
\]
and
\[
\tilde{\mu}_{t,T} = -(T-t)\frac{\tilde{\sigma}_{t,T}^2}{2} + \log \mathbb{E}[R_{t,T}^2] \quad \text{and} \quad \tilde{\sigma}_{t,T}^2 := \frac{1}{T-t} \log \left(\frac{\mathbb{E}[R_{t,T}^4]}{(\mathbb{E}[R_{t,T}^2])^2}\right). \tag{8.6}
\]

By (8.6), the parameters \(\tilde{\mu}_{t,T}\) and \(\tilde{\sigma}_{t,T}^2\) can be estimated from the variance swap price
\[
e^{-(T-t)r} \mathbb{E}^*[R_{t,T}^2 | \mathcal{F}_t] = e^{-(T-t)r} \mathbb{E}^*\left[\int_t^T \sigma_u^2 du \bigg| \mathcal{F}_t\right],
\]
and from the variance power option price
\[
e^{-(T-t)r} \mathbb{E}^*[R_{t,T}^4 | \mathcal{F}_t] = e^{-(T-t)r} \mathbb{E}^*\left[\left(\int_t^T \sigma_u^2 du\right)^2 \bigg| \mathcal{F}_t\right].
\]

By Proposition 8.4, in case \(R_{0,t}^2 = \int_0^t \sigma_u^2 du < \kappa_\sigma^2\) we can estimate the price
\[
e^{-(T-t)r} \mathbb{E}^*\left[\left(x + \int_t^T \sigma_u^2 du - \kappa_\sigma^2\right)^+ \bigg| \mathcal{F}_t\right]_{x=0}^{\int_t^T \sigma_u^2 du}
\]
of the realized variance call option by approximating \(R_{t,T}^2 = \int_t^T \sigma_u^2 du\) by a lognormal random variable, see Friz and Gatheral (2005), Carr and Lee (2007).

**Proposition 8.5.** Under the lognormal approximation (8.5), the price
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\[ V_{C_{t,T}}(c_\sigma) = e^{-(T-t)} E \left[ (x + R_{t,T}^2 - c_\sigma)^+ \right]_{x=R_{0,t}^2} \]

of the realized variance call option can be approximated as

\[ V_{C_{t,T}}(c_\sigma) \approx e^{-(T-t)} E[R_{t,T}^2] \Phi(d_1) - e^{-(T-t)} (c_\sigma - R_{0,t}^2) \Phi(d_2), \quad (8.7) \]

where

\[ d_1 := \frac{\log \left( \frac{E[R_{t,T}^2]}{(c_\sigma - R_{0,t}^2)} \right)}{\tilde{\sigma}_{t,T} \sqrt{T-t}} + \frac{\sqrt{T-t}}{2} \]

\[ = \frac{-\log(c_\sigma - R_{0,t}^2) + \tilde{\mu}_{t,T} + (T-t)\tilde{\sigma}_{t,T}^2}{\tilde{\sigma}_{t,T} \sqrt{T-t}}, \]

and

\[ d_2 := d_1 - \tilde{\sigma}_{t,T} \sqrt{T-t} = \frac{-\log(c_\sigma - R_{0,t}^2) + \tilde{\mu}_{t,T}}{\tilde{\sigma}_{t,T} \sqrt{T-t}}, \]

and \( \Phi \) denotes the standard Gaussian cumulative distribution function.

Proof. The lognormal approximation states that

\[ \varphi_{R_{t,T}^2}(x) \approx \frac{1}{x \tilde{\sigma}_{t,T} \sqrt{2(T-t)\pi}} e^{-(\tilde{\mu}_{t,T} + \log x)^2/(2(T-t)\tilde{\sigma}_{t,T}^2)}, \quad x > 0, \quad (8.8) \]

hence

\[ V_{C_{t,T}}(c_\sigma) = e^{-(T-t)} E \left[ (x + R_{t,T}^2 - c_\sigma)^+ \right]_{x=R_{0,t}^2} \]

\[ = e^{-(T-t)} \int_{c_\sigma}^{\infty} (y - (c_\sigma - R_{0,t}^2))^+ \varphi_{R_{t,T}^2}(y) dy \]

\[ \approx e^{-(T-t)} \left( e^{\tilde{\mu}_{t,T} + (T-t)\tilde{\sigma}_{t,T}^2/2} \Phi(d_1) - (c_\sigma - R_{0,t}^2) \Phi(d_2) \right) \]

\[ = e^{-(T-t)} E[R_{t,T}^2] \Phi(d_1) - e^{-(T-t)} (c_\sigma - R_{0,t}^2) \Phi(d_2). \]

\[ \square \]

Gamma approximation

In case \( R_{0,t}^2 = \int_0^t \sigma_u^2 dt < \kappa_\sigma^2 \), we can also derive an estimate for the price

\[ e^{-(T-t)} E^x \left[ \left( x + \int_t^T \sigma_u^2 du - \kappa_\sigma^2 \right)^+ \right]_{x=\int_0^t \sigma_u^2 du} \]

of the realized variance call option by approximating \( R_{t,T}^2 = \int_t^T \sigma_u^2 du \) by a gamma variable.
Proposition 8.6. (Gamma approximation). Under the gamma approximation the probability density function $\varphi_{R_{t,T}^2}$ of $R_{t,T}^2 := \int_t^T \sigma_u^2 du$ can be approximated as

$$\varphi_{R_{t,T}^2}(x) \approx \frac{e^{-x/\theta_{t,T}} (x/\theta_{t,T})^{-1+\nu_{t,T}}}{\Gamma(\nu_{t,T})}, \quad x > 0,$$

where

$$\theta_{t,T} = \frac{\text{Var}[R_{t,T}^2]}{\mathbb{E}[R_{t,T}^2]} \quad \text{and} \quad \nu_{t,T} = \frac{\mathbb{E}[R_{t,T}^2]}{\theta_{t,T}} = \frac{(\mathbb{E}[R_{t,T}^2])^2}{\text{Var}[R_{t,T}^2]}.$$  \hspace{1cm} (8.11)

Proof. The parameters $\theta_{t,T}$, $\nu_{t,T}$ are estimated by matching the first and second moments of $R_{t,T}^2$ to those of the gamma distribution with scale and shape parameters $\theta_{t,T}$ and $\nu_{t,T}$, which yields

$$\mathbb{E}[R_{t,T}^2] = \nu_{t,T} \theta_{t,T} \quad \text{and} \quad \text{Var}[R_{t,T}^2] = \nu_{t,T} \theta_{t,T}^2,$$

and (8.11). \hfill \Box

Proposition 8.7. Under the gamma approximation (8.10), the price $\mathbb{E}A(\kappa_\sigma, T) = e^{-\theta_{t,T} T} \mathbb{E}\left[\left(x + R_{t,T}^2 - \kappa_\sigma\right)^+\right]_{x=R_{0,t}}$ of the realized variance call option can be approximated as

$$\mathbb{E}A(\kappa_\sigma, T) = e^{-\nu_{t,T} T} \left(\mathbb{E}[R_{t,T}^2] Q\left(1 + \nu_{t,T}, \frac{\kappa_\sigma}{\theta_{t,T}}\right) - \kappa_\sigma Q\left(\nu_{t,T}, \frac{\kappa_\sigma}{\theta_{t,T}}\right)\right),$$

where

$$Q(\lambda, z) := \frac{1}{\Gamma(\lambda)} \int_z^{\infty} t^{\lambda-1} e^{-t} dt, \quad z > 0,$$

is the (normalized) upper incomplete gamma function.

Proof. Using the gamma approximation

$$\varphi_{R_{t,T}^2}(x) \approx \frac{e^{-x/\theta_{t,T}} x^{-1+\nu_{t,T}}}{\Gamma(\nu_{t,T}) (\theta_{t,T})^{\nu_{t,T}}},$$

where $\theta_{t,T}$ and $\nu_{t,T}$ are given by (8.11), we have

$$\mathbb{E}\left[(R_{t,T}^2 - \kappa_\sigma)^+\right] = \int_{\kappa_\sigma}^{\infty} (x - \kappa_\sigma)^+ \varphi_{R_{t,T}^2}(x) dx$$

$$\approx \frac{1}{\Gamma(\nu_{t,T})} \int_{\kappa_\sigma}^{\infty} (x - \kappa_\sigma) x^{-1+\nu_{t,T}} (\theta_{t,T})^{\nu_{t,T}} e^{-x/\theta_{t,T}} dx$$

$$= \frac{1}{\Gamma(\nu_{t,T})} \int_{\kappa_\sigma}^{\infty} (x/\theta_{t,T})^{\nu_{t,T}} e^{-x/\theta_{t,T}} dx - \frac{\kappa_\sigma}{\Gamma(\nu_{t,T})} \int_{\kappa_\sigma}^{\infty} x^{-1+\nu_{t,T}} (\theta_{t,T})^{\nu_{t,T}} e^{-x/\theta_{t,T}} dx$$

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\[
\begin{align*}
\theta_t &= \frac{\theta_t}{\Gamma(\nu_t, T)} \int_0^\infty x^{\nu_t, T} e^{-x} dx - \frac{\kappa}{\Gamma(\nu_t, T)} \int_0^\infty x^{-1+\nu_t, T} e^{-x} dx \\
\Gamma(\nu_t, T) &= \theta_t \nu_t T Q \left( 1 + \nu_t, \frac{\kappa}{\theta_t, T} \right) - \kappa Q \left( \nu_t, \frac{\kappa}{\theta_t, T} \right),
\end{align*}
\]

where

\[
Q(\lambda, z) := \frac{1}{\Gamma(\lambda)} \int_z^\infty t^{\lambda-1} e^{-t} dt, \quad z > 0,
\]

is the (normalized) upper incomplete gamma function, which yields

\[
EA(\kappa, T) = e^{-(T-t)\kappa} \mathbb{E}\left[ (x + R_{1,T} - \kappa) \right] \\
\approx e^{-(T-t)\kappa} \left( \nu_t, T \theta_t, T Q \left( 1 + \nu_t, \frac{\kappa}{\theta_t, T} \right) - \kappa Q \left( \nu_t, \frac{\kappa}{\theta_t, T} \right) \right) \quad (8.14)
\]

\[
= e^{-(T-t)\kappa} \left( \mathbb{E}[R_{1,T}^2] Q \left( 1 + \nu_t, \frac{\kappa}{\theta_t, T} \right) - \kappa Q \left( \nu_t, \frac{\kappa}{\theta_t, T} \right) \right).
\]

\square

Realized variance in the Heston model

Consider now the Heston model driven by the stochastic differential equation

\[
dv_t = (a - bv_t)dt + \sigma \sqrt{v_t} dW_t,
\]

where \(a, b, \sigma > 0\). We have

\[
\mathbb{E}[v_T] = v_0 e^{-bT} + \frac{a}{b} \left( 1 - e^{-bT} \right),
\]

from which it follows

\[
\mathbb{E} \left[ R_{0,T}^2 \right] = \mathbb{E} \left[ \int_0^T \sigma_u^2 du \right] \\
= v_0 \frac{1 - e^{-bT}}{b} + a \frac{e^{-bT} + bT - 1}{b^2}.
\]

We also have

\[
\text{Var}[v_T] = v_0 \frac{\sigma^2}{b} \left( e^{-bT} - e^{-2bT} \right) + \frac{a \sigma^2}{2b^2} \left( 1 - e^{-bT} \right)^2,
\]

see Exercise 4.13, and

\[
\text{Var} \left[ R_{0,T}^2 \right] = v_0 \sigma^2 \frac{1 - 2bT e^{-bT} - e^{-2bT}}{b^3}.
\]
\[ +a \sigma^2 e^{-2bT} + 2bT + 4(bT + 1)e^{-bT} - 5 \],

cf. e.g. Prayoga and Privault (2017). Using the parameters \( \sigma = 1, b = 1.15, a = 0.04 \times b, v_0 = 0.04, T = 1, r = 0 \), we plot the graphs of the lognormal approximation (8.7) and of the gamma approximation (8.12) of realized variance call option prices for \( \kappa \sigma \in [0, 0.2] \). The following graph in Figure 8.2 is obtained with \( t = 0 \) and \( R_{0,0} = 0 \) using this R code, see also § 11 and Figure 11.4 page 153 of Gatheral (2006), and Figure 6 in Friz and Gatheral (2005).

![Graph of variance call prices](image)

**Fig. 8.2:** One-year variance call prices.

Figure 8.2 also includes the lognormal volatility approximation obtained from

\[
\mathbb{E}[R_{t,T}] = e^{\tilde{\mu}_{t,T} + (T-t)\tilde{\sigma}_{t,T}^2(y)/2}, \quad \text{and} \quad \mathbb{E}[R_{t,T}^2] = e^{2(\tilde{\mu}_{t,T} + (T-t)\tilde{\sigma}_{t,T}^2)},
\]

i.e.

\[
\tilde{\mu}_{t,T} = -(T-t)\frac{\tilde{\sigma}_{t,T}^2}{2} + \log \mathbb{E}[R_{t,T}] \quad \text{and} \quad \tilde{\sigma}_{t,T}^2 := \frac{1}{T-t} \log \left( \frac{\mathbb{E}[R_{t,T}^2]}{\left( \mathbb{E}[R_{t,T}] \right)^2} \right),
\]

with

\[
VC_{t,T}(\kappa \sigma) \approx e^{-(T-t)r} \mathbb{E}[R_{t,T}^2]\Phi(d_1) - e^{-(T-t)r}(\kappa \sigma - R_{0,t}^2)\Phi(d_2),
\]

where

\[
d_1 := \frac{\log \left( \left( \mathbb{E}[R_{t,T}] \right)^2 / (\kappa \sigma - R_{0,t}^2) \right)}{2\tilde{\sigma}_{t,T} \sqrt{T-t}} + 2\tilde{\sigma}_{t,T} \sqrt{T-t}.
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\[
= -\log(\kappa \sigma - R_{0,t}^2) + 2\bar{\mu}_tT + 4(T - t)\delta_{t,T}^2, \\
\]

and

\[
d_2 := d_1 - 2\bar{\sigma}_{t,T}\sqrt{T - t} = -\log(\kappa \sigma - R_{0,t}^2) + 2\bar{\mu}_tT, \\
\]

see § 8.4 in Friz and Gatheral (2005), where \(\mathbb{E}[R_{t,T}]\) is obtained as

\[
\mathbb{E}[R_{t,T}] = \frac{1}{2\sqrt{\pi}} \int_0^\infty \left(1 - \mathbb{E}[e^{-\lambda R_{t,T}^2}]\right) \frac{d\lambda}{\lambda^3}, \\
\]

see § 3.1 therein, with

\[
\mathbb{E}[e^{-\lambda R_{0,T}^2}] = \exp\left(-\frac{2v_0\lambda(1 - e^{-bT})}{b + b + e^{-bT}(b - b)} - \frac{a}{\sigma^2}(b - b)T - \frac{2a}{\sigma^2}\log\frac{b + b + e^{-bT}(b - b)}{2b}\right), \\
\]

where \(b := \sqrt{b^2 + 2\lambda\sigma^2}\).

The gamma variance approximation appears to be more accurate than the lognormal approximations for large values of \(\kappa \sigma\), which can be consistent with the fact that the long run distribution of the CIR-Heston process has the gamma density

\[
f(x) = \frac{1}{\Gamma(2a/\sigma^2)} \left(\frac{2b}{\sigma^2}\right)^{2a/\sigma^2} x^{-1+2a/\sigma^2} e^{-2bx/\sigma^2}, \quad x > 0, \\
\]

with shape parameter \(2a/\sigma^2\) and scale parameter \(\sigma^2/(2b)\), which is also the invariant distribution of \(v_t\).

8.4 PDE Method

In the sequel we will assume that \((B_{t}^{(1)})_{t \in \mathbb{R}^+}\) is a standard Brownian motion under \(\mathbb{P}^*\), i.e. the discounted price process \((e^{-rt}S_t)_{t \in \mathbb{R}^+}\) is a martingale under \(\mathbb{P}^*\). For simplicity of exposition we will make the assumption that \((B_{t}^{(2)})_{t \in \mathbb{R}^+}\) is also a standard Brownian motion under \(\mathbb{P}^*\).

**Proposition 8.8.** Consider a vanilla option with payoff \(h(S_T)\) priced as

\[
V_t = f(t, v_t, S_t) = e^{-(T-t)r} \mathbb{E}^*[h(S_T) \mid \mathcal{F}_t], \quad 0 \leq t \leq T. \\
\]

The function \(f(t, y, x)\) satisfies the Heston PDE...
\[
\frac{\partial f}{\partial t}(t,v,x) + r x \frac{\partial f}{\partial x}(t,v,x) + \frac{1}{2} v x^2 \frac{\partial^2 f}{\partial x^2}(t,v,x) + \mu(t,v) \frac{\partial f}{\partial v}(t,v,x) + \frac{1}{2} \beta^2(t,v) \frac{\partial^2 f}{\partial v^2}(t,v,x) + \rho \beta(t,v) x \sqrt{v} \frac{\partial^2 f}{\partial v \partial x}(t,v,x) = r f(t,v,x),
\]

under the terminal condition \(f(T,v,x) = h(x)\).

**Proof.** By Itô calculus with respect to the correlated Brownian motions \((B_t^{(1)})_{t \in \mathbb{R}^+}\) and \((B_t^{(2)})_{t \in \mathbb{R}^+}\), the portfolio value \(f(t,v_t,S_t)\) can be differentiated as follows:

\[
df(t,v_t,S_t) = \frac{\partial f}{\partial t}(t,v_t,S_t)dt + r v_t \frac{\partial f}{\partial x}(t,v_t,S_t)dt + \sqrt{v_t} S_t \frac{\partial f}{\partial v}(t,v_t,S_t) dB_t^{(1)} + \frac{1}{2} v_t S_t^2 \frac{\partial^2 f}{\partial x^2}(t,v_t,S_t)dt + \mu(t,v_t) \frac{\partial f}{\partial v}(t,v_t,S_t)dt \\
+ \beta(t,v_t) \frac{\partial f}{\partial v}(t,v_t,S_t) dB_t^{(2)} + \frac{1}{2} \beta^2(t,v_t) \frac{\partial^2 f}{\partial v^2}(t,v_t,S_t)dt \\
+ \beta(t,v_t) \sqrt{v_t} S_t \frac{\partial^2 f}{\partial v \partial x}(t,v_t,S_t) dB_t^{(1)} dB_t^{(2)}.
\]

Assuming that \((B_t^{(2)})_{t \in \mathbb{R}^+}\) is also a standard Brownian motion under the risk-neutral probability measure\(^*\) \(\mathbb{P}^*\) and knowing that the discounted portfolio price process \((e^{-rt} f(t,v_t,S_t))_{t \in \mathbb{R}^+}\) is also a martingale under \(\mathbb{P}^*\), from the relation

\[
d(e^{-rt} f(t,v_t,S_t)) = -r e^{-rt} f(t,v_t,S_t) dt + e^{-rt} df(t,v_t,S_t),
\]

we obtain

\(^*\) When this condition is not satisfied we need to introduce a drift that yields a market price of volatility.
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\[- rf(t, v_t, S_t) dt + \frac{\partial f}{\partial t}(t, v_t, S_t) dt + r S_t \frac{\partial f}{\partial x}(t, v_t, S_t) dt + \frac{1}{2} v_t S_t^2 \frac{\partial^2 f}{\partial x^2}(t, v_t, S_t) dt \]

\[+ \mu(t, v_t) \frac{\partial f}{\partial v}(t, v_t, S_t) dt + \frac{1}{2} \beta^2(t, v_t) \frac{\partial^2 f}{\partial v^2}(t, v_t, S_t) dt \]

\[+ \rho \beta(t, v_t) S_t \sqrt{v_t} \frac{\partial^2 f}{\partial v \partial x}(t, v_t, S_t) dt \]

\[= 0 , \]

and the pricing PDE (8.15).

\[\Box\]

Heston model

In the Heston model with \( \mu(t, v) = -\lambda(v - m) \) and \( \beta(t, v) = \eta \sqrt{v} \), from (8.15) we find the PDE

\[
\frac{\partial f}{\partial t}(t, v, x) + rx \frac{\partial f}{\partial x}(t, v, x) + \frac{1}{2} v x^2 \frac{\partial^2 f}{\partial x^2}(t, v, x) - \lambda(v - m) \frac{\partial f}{\partial v}(t, v, x) + \frac{1}{2} \eta^2 v \frac{\partial^2 f}{\partial v^2}(t, v, x) + \rho \eta x v \frac{\partial^2 f}{\partial v \partial x}(t, v, x) = rf(t, v, x).
\]

The solution of this PDE has been expressed in Heston (1993) as a complex integral by inversion of a characteristic function.

Using the change of variable \( y = \log x \) with \( g(t, v, y) = f(t, v, e^y) \), the PDE (8.17) is transformed into

\[
\frac{\partial g}{\partial t}(t, v, y) + r \frac{\partial g}{\partial y}(t, v, y) + \frac{1}{2} v \frac{\partial^2 g}{\partial y^2}(t, v, y) - \frac{1}{2} v \frac{\partial g}{\partial y}(t, v, x)
\]

\[- \lambda(v - m) \frac{\partial g}{\partial v}(t, v, y) + \frac{1}{2} \eta^2 v \frac{\partial^2 g}{\partial v^2}(t, v, y) + \rho \eta x v \frac{\partial^2 g}{\partial v \partial y}(t, v, y) = rg(t, v, y).
\]

Using the Fourier transform

\[\widehat{g}(t, v, z) := \int_{-\infty}^{\infty} e^{-iyz} g(t, v, y) dy\]

and the relation

\[iz \widehat{g}(t, v, z) = \int_{-\infty}^{\infty} e^{-iyz} \frac{\partial g}{\partial y}(t, v, y) dy,\]

we find, using the rule \( i^2 = -1 \), that \( \widehat{g}(t, v, z) \) satisfies the equation

\[
\frac{\partial \widehat{g}}{\partial t}(t, v, z) + irz \widehat{g}(t, v, z) - \frac{1}{2} vz^2 \widehat{g}(t, v, z) - iz \frac{1}{2} v \widehat{g}(t, v, z)
\]
\[-\lambda(v-m)\frac{\partial \tilde{g}}{\partial v}(t,v,z) + v\eta^2 \frac{\partial^2 \tilde{g}}{\partial v^2}(t,v,z) + ip\eta z v \frac{\partial \tilde{g}}{\partial v}(t,v,z) = r\tilde{g}(t,v,z),\]

which is an affine PDE with respect to the variable \(v\) with \(z\) a constant parameter. This equation can be solved in closed form, and the final solution \(g(t,v,y)\) can then be obtained by the Fourier inversion

\[g(t,v,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{izy} \tilde{g}(t,v,z) dz,\]  

(8.18)


**Delta hedging in the Heston model**

Consider a portfolio of the form

\[V_t = \eta_t e^{rt} + \xi_t S_t\]

based on the riskless asset \(A_t = e^{rt}\) and on the risky asset \(S_t\). When this portfolio is self-financing we have

\[dV_t = df(t, v_t, S_t) = r\eta_t e^{rt} dt + \xi_t dS_t \]
\[= r\eta_t e^{rt} dt + \xi_t (rS_t dt + S_t \sqrt{v_t} dB_t^{(1)}) \]
\[= rV_t dt + \xi_t S_t \sqrt{v_t} dB_t^{(1)} \]
\[= rf(t, v_t, S_t) dt + \xi_t S_t \sqrt{v_t} dB_t^{(1)}.\]  

(8.19)

However, trying to match (8.16) to (8.19) yields

\[\sqrt{v_t} S_t \frac{\partial f}{\partial x}(t, v_t, S_t) dB_t^{(1)} + \beta(t, v_t) \frac{\partial f}{\partial v}(t, v_t, S_t) dB_t^{(2)} = \xi_t S_t \sqrt{v_t} dB_t^{(1)},\]  

(8.20)

which admits no solution unless \(\beta(t, v) = 0\), i.e. when volatility is deterministic. A solution to that problem is to consider instead a portfolio

\[V_t = f(t, v_t, S_t) = \eta_t e^{rt} + \xi_t S_t + \zeta_t P(t, v_t, S_t)\]

that includes an additional asset with price \(P(t, v_t, S_t)\), which can be an option depending on the volatility \(v_t\).

**Proposition 8.9.** The self-financing portfolio allocation \((\xi_t, \zeta_t)_{t\in[0,T]}\) in the assets \((e^{rt}, S_t, P(t, v_t, S_t))_{t\in[0,T]}\) with portfolio price

\[V_t = f(t, v_t, S_t) = \eta_t e^{rt} + \xi_t S_t + \zeta_t P(t, v_t, S_t)\]
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is given by

\[ \zeta_t = \frac{\partial f(t, v_t, S_t)}{\partial v} (t, v_t, S_t), \]  

(8.21)

and

\[ \xi_t = \frac{\partial f}{\partial x}(t, v_t, S_t) - \frac{\partial f}{\partial v}(t, v_t, S_t) \frac{\partial P}{\partial v}(t, v_t, S_t). \]  

(8.22)

Proof. Here, (8.19) is replaced with

\[ dV_t = df(t, v_t, S_t) \]
\[ = r \eta_t e^{rt} dt + \xi_t dS_t + \zeta_t dP(t, v_t, S_t) \]
\[ = r \eta_t e^{rt} dt + \zeta_t (rS_t dt + S_t \sqrt{v_t} dB_t^{(1)}) + r \zeta_t S_t \frac{\partial P}{\partial x}(t, v_t, S_t) dt \]
\[ + \zeta_t \mu(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dt + \zeta_t \frac{\partial P}{\partial t}(t, v_t, S_t) dt + \frac{1}{2} \zeta_t S_t^2 v_t \frac{\partial^2 P}{\partial x^2}(t, v_t, S_t) dt \]
\[ + \frac{1}{2} \zeta_t \beta^2(t, v_t) \frac{\partial^2 P}{\partial v^2}(t, v_t, S_t) dt + \rho \zeta_t \beta(t, v_t) S_t \sqrt{v_t} \frac{\partial^2 P}{\partial x \partial v}(t, v_t, S_t) dt \]
\[ + \zeta_t S_t \sqrt{v_t} \frac{\partial P}{\partial v}(t, v_t, S_t) dB_t^{(1)} + \zeta_t \beta(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dB_t^{(2)} \]
\[ = (V_t - \zeta_t P(t, v_t, S_t)) r dt + \xi_t S_t \sqrt{v_t} dB_t^{(1)} + r \zeta_t S_t \frac{\partial P}{\partial x}(t, v_t, S_t) dt \]
\[ + \zeta_t \mu(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dt + \zeta_t \frac{\partial P}{\partial t}(t, v_t, S_t) dt + \frac{1}{2} \zeta_t S_t^2 v_t \frac{\partial^2 P}{\partial x^2}(t, v_t, S_t) dt \]
\[ + \frac{1}{2} \zeta_t \beta^2(t, v_t) \frac{\partial^2 P}{\partial v^2}(t, v_t, S_t) dt + \rho \zeta_t \beta(t, v_t) S_t \sqrt{v_t} \frac{\partial^2 P}{\partial x \partial v}(t, v_t, S_t) dt \]  

(8.23)
\[ + \zeta_t S_t \sqrt{v_t} \frac{\partial P}{\partial v}(t, v_t, S_t) dB_t^{(1)} + \zeta_t \beta(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dB_t^{(2)} \]
\[ = r f(t, v_t, S_t) dt + \xi_t S_t \sqrt{v_t} dB_t^{(1)} + r \zeta_t S_t \frac{\partial P}{\partial x}(t, v_t, S_t) dt \]
\[ + \zeta_t \mu(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dt + \zeta_t \frac{\partial P}{\partial t}(t, v_t, S_t) dt + \frac{1}{2} \zeta_t S_t^2 v_t \frac{\partial^2 P}{\partial x^2}(t, v_t, S_t) dt \]
\[ + \frac{1}{2} \zeta_t \beta^2(t, v_t) \frac{\partial^2 P}{\partial v^2}(t, v_t, S_t) dt + \rho \zeta_t \beta(t, v_t) S_t \sqrt{v_t} \frac{\partial^2 P}{\partial x \partial v}(t, v_t, S_t) dt \]
\[ + \zeta_t S_t \sqrt{v_t} \frac{\partial P}{\partial v}(t, v_t, S_t) dB_t^{(1)} + \zeta_t \beta(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dB_t^{(2)} , \]

and by matching (8.23) to (8.16), the equation (8.20) now becomes

\[ \sqrt{v_t} S_t \frac{\partial f}{\partial x}(t, v_t, S_t) dB_t^{(1)} + \beta(t, v_t) \frac{\partial f}{\partial v}(t, v_t, S_t) dB_t^{(2)} \]
which show that

\[
\beta(t, v_t) \frac{\partial f}{\partial v}(t, v_t, S_t) = \zeta_t \beta(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t),
\]

This leads to the equations

\[
\begin{align*}
\sqrt{v_t S_t} \frac{\partial f}{\partial x}(t, v_t, S_t) &= \xi_t \sqrt{v_t} + \zeta_t \sqrt{v_t} \frac{\partial P}{\partial x}(t, v_t, S_t), \\
\beta(t, v_t) \frac{\partial f}{\partial v}(t, v_t, S_t) &= \zeta_t \beta(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t),
\end{align*}
\]

which show that

\[
\zeta_t = \frac{\partial f(t, v_t, S_t)}{\partial v} = \frac{\partial f(t, v_t, S_t)}{\partial P} \frac{\partial P(t, v_t, S_t)}{\partial v}.
\]

and

\[
\xi_t = \frac{1}{S_t \sqrt{v_t}} \left( \sqrt{v_t S_t} \frac{\partial f}{\partial x}(t, v_t, S_t) - \zeta_t \sqrt{v_t} \frac{\partial P}{\partial x}(t, v_t, S_t) \right)
= \frac{\partial f}{\partial x}(t, v_t, S_t) - \zeta_t \frac{\partial P}{\partial x}(t, v_t, S_t)
= \frac{\partial f}{\partial x}(t, v_t, S_t) - \frac{\partial f}{\partial v}(t, v_t, S_t) \frac{\partial P(t, v_t, S_t)}{\partial v(t, v_t, S_t)}.
\]

\[\square\]

We note in addition that identifying the “\(dt\)” terms when equating (8.23) to (8.16) would now lead to the more complicated PDE

\[
(f(t, v_t, S_t) - \zeta_t P(t, v_t, S_t)) r + r \zeta_t S_t \frac{\partial P}{\partial x}(t, v_t, S_t) + \zeta_t \mu(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t)
+ \zeta_t \beta(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) + \frac{1}{2} \zeta_t \beta^2(t, v_t) \frac{\partial^2 P}{\partial x^2}(t, v_t, S_t) + \frac{1}{2} \zeta_t \beta^2(t, v_t) \frac{\partial^2 P}{\partial v^2}(t, v_t, S_t)
+ \rho \zeta_t \beta(t, v_t) S_t \sqrt{v_t} \frac{\partial^2 P}{\partial x \partial v}(t, v_t, S_t)
= \frac{\partial f}{\partial t}(t, v_t, S_t) + r S_t \frac{\partial f}{\partial x}(t, v_t, S_t) + \frac{1}{2} v_t S_t^2 \frac{\partial^2 f}{\partial x^2}(t, v_t, S_t) + \mu(t, v_t) \frac{\partial f}{\partial v}(t, v_t, S_t)
+ \frac{1}{2} \beta^2(t, v_t) \frac{\partial^2 f}{\partial v^2}(t, v_t, S_t) + \rho \beta(t, v_t) S_t \sqrt{v_t} \frac{\partial^2 f}{\partial v \partial x}(t, v_t, S_t),
\]

which can be rewritten using (8.21) as

\[
\frac{\partial f}{\partial v}(t, v, x) \left( -r P(t, v, x) + \frac{\partial P}{\partial t}(t, v, x) + r x \frac{\partial P}{\partial x}(t, v, x) + \mu(t, v) \frac{\partial P}{\partial v}(t, v, x) \right)
\]

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+ \frac{\partial f}{\partial v}(t,v,x) \left( \frac{x^2 v}{2} \frac{\partial^2 P}{\partial x^2}(t,v,x) + \frac{1}{2} \beta^2(t,v) \frac{\partial^2 P}{\partial v^2}(t,v,x) + \rho \beta(t,v) x \sqrt{v} \frac{\partial^2 P}{\partial x \partial v}(t,v,x) \right) \\
= \frac{\partial P}{\partial v}(t,v,x) \left( -rf(t,v,x) + \frac{\partial f}{\partial t}(t,v,x) + r \frac{\partial f}{\partial x}(t,v,x) + \frac{v x^2}{2} \frac{\partial^2 f}{\partial x^2}(t,v,x) \right) \\
+ \frac{\partial f}{\partial v}(t,v,x) \left( \mu(t,v) \frac{\partial f}{\partial v}(t,v,x) + \frac{1}{2} \beta^2(t,v) \frac{\partial^2 f}{\partial v^2}(t,v,x) + \rho \beta(t,v) x \sqrt{v} \frac{\partial^2 f}{\partial v \partial x}(t,v,x) \right).

Therefore, letting

\lambda(t,v,x) \tag{8.24} \\
:= \frac{1}{\frac{\partial P}{\partial v}(t,v,x)} \left( -rf(t,v,x) + \frac{\partial f}{\partial t}(t,v,x) + r \frac{\partial f}{\partial x}(t,v,x) + \frac{v x^2}{2} \frac{\partial^2 f}{\partial x^2}(t,v,x) \right) \\
+ \frac{1}{\frac{\partial f}{\partial v}(t,v,x)} \left( \mu(t,v) \frac{\partial f}{\partial v}(t,v,x) + \frac{1}{2} \beta^2(t,v) \frac{\partial^2 f}{\partial v^2}(t,v,x) + \rho \beta(t,v) x \sqrt{v} \frac{\partial^2 f}{\partial v \partial x}(t,v,x) \right) \tag{8.25}

defines a function \(\lambda(t,v,x)\) that depends only on the parameters \((t,v,x)\) and not on \(P\), without requiring \((B^2_t)_{t \in \mathbb{R}^+}\) to be a standard Brownian motion under \(\mathbb{P}^*\). The function \(\lambda(t,v,x)\) is linked to the market price of volatility risk, cf. Chapter 1 of Gatheral (2006) § 2.4.1 in Fouque et al. (2011) and Fouque et al. (2000) for details.

Combining (8.24)-(8.26) allows us to rewrite the pricing PDE as

\[
\frac{\partial f}{\partial t}(t,v,x) + r x \frac{\partial f}{\partial x}(t,v,x) + \frac{v x^2}{2} \frac{\partial^2 f}{\partial x^2}(t,v,x) + \frac{1}{2} \beta^2(t,v) \frac{\partial^2 f}{\partial v^2}(t,v,x) + \rho \beta(t,v) x \sqrt{v} \frac{\partial^2 f}{\partial v \partial x}(t,v,x) = rf(t,v,x) + \lambda(t,v,x) \frac{\partial f}{\partial v}(t,v,x),
\]

and (8.15) corresponds to the choice \(\lambda(t,v,x) = -\mu(t,v)\), which corresponding to a vanishing market price of volatility risk.

### 8.5 Perturbation Analysis

We refer to Chapter 4 of Fouque et al. (2011) for the contents of this section. Consider the time-rescaled model

\[ \]
\[
\begin{align*}
\begin{cases}
\diff S_t &= r S_t \diff t + S_t \sqrt{v_t/\varepsilon} \diff B_t^{(1)} \\
\diff v_t &= \mu(v_t) \diff t + \beta(v_t) \diff B_t^{(2)}.
\end{cases}
\tag{8.27}
\end{align*}
\]

We note that \(v^\varepsilon_t := v_{t/\varepsilon}\) satisfies the SDE
\[
\diff v^\varepsilon_t = \frac{1}{\varepsilon} \mu(v^\varepsilon_t) \diff t + \frac{1}{\sqrt{\varepsilon}} \beta(v^\varepsilon_t) \diff B_t^{(2)}
\]
with
\[
(dB_t^{(2)})^2 \approx \frac{dt}{\varepsilon} \approx \frac{1}{\varepsilon} (dB_t^{(2)})^2 \approx \left( \frac{1}{\sqrt{\varepsilon}} dB_t^{(2)} \right)^2,
\]
hence the SDE for \(v^\varepsilon_t\) can be rewritten as the slow-fast system
\[
\diff v^\varepsilon_t = \frac{1}{\varepsilon} \mu(v^\varepsilon_t) \diff t + \frac{1}{\sqrt{\varepsilon}} \beta(v^\varepsilon_t) \diff B_t^{(2)}.
\]

In other words, \(\varepsilon \to 0\) corresponds to fast mean-reversion and (8.27) can be rewritten as
\[
\begin{align*}
\begin{cases}
\diff S_t &= r S_t \diff t + \sqrt{v^\varepsilon_t} S_t \diff B_t^{(1)} \\
\diff v^\varepsilon_t &= \frac{1}{\varepsilon} \mu(v^\varepsilon_t) \diff t + \frac{1}{\sqrt{\varepsilon}} \beta(v^\varepsilon_t) \diff B_t^{(2)},
\end{cases}
\quad \varepsilon > 0.
\end{align*}
\]

The perturbed PDE
\[
\begin{align*}
\frac{\partial f_\varepsilon}{\partial t}(t, v, x) + r x \frac{\partial f_\varepsilon}{\partial x}(t, v, x) + \frac{v x^2}{2} \frac{\partial^2 f_\varepsilon}{\partial x^2}(t, v, x) + \frac{1}{\varepsilon} \mu(v) \frac{\partial f_\varepsilon}{\partial v}(t, v, x) \\
+ \frac{1}{2\varepsilon} \beta^2(v) \frac{\partial^2 f_\varepsilon}{\partial v^2}(t, v, x) + \frac{\rho}{\sqrt{\varepsilon}} \beta(v) x \sqrt{v} \frac{\partial^2 f_\varepsilon}{\partial v \partial x}(t, v, x) &= rf_\varepsilon(t, v, x)
\end{align*}
\]
with terminal condition \(f_\varepsilon(T, v, x) = (x - K)^+\) rewrites as
\[
\begin{align*}
\frac{1}{\varepsilon} \mathcal{L}_0 f_\varepsilon(t, v, x) + \frac{1}{\sqrt{\varepsilon}} \mathcal{L}_1 f_\varepsilon(t, v, x) + \mathcal{L}_2 f_\varepsilon(t, v, x) &= rf_\varepsilon(t, v, x),
\tag{8.28}
\end{align*}
\]
where
The solution

\[ L_0 f_\varepsilon(t, v, x) := \frac{1}{2} \beta^2(v) \frac{\partial^2 f_\varepsilon}{\partial v^2}(t, v, x) + \mu(v) \frac{\partial f_\varepsilon}{\partial v}(t, v, x), \]

\[ L_1 f_\varepsilon(t, v, x) := \rho x \beta(v) \sqrt{\varepsilon} \frac{\partial^2 f_\varepsilon}{\partial v \partial x}(t, v, x), \]

\[ L_2 f_\varepsilon(t, v, x) := \frac{\partial f_\varepsilon}{\partial t}(t, v, x) + rx \frac{\partial f_\varepsilon}{\partial x}(t, v, x) + \frac{vx^2}{2} \frac{\partial^2 f_\varepsilon}{\partial x^2}(t, v, x). \]

Note that

- \( L_0 \) is the infinitesimal generator of the process \((v^1_s)_{s \in \mathbb{R}^+}\), see (8.32) below, and

- \( L_2 \) is the Black-Scholes operator, i.e. \( L_2 f = rf \) is the Black-Scholes PDE.

The solution \( f_\varepsilon(t, v, x) \) will be expanded as

\[ f_\varepsilon(t, v, x) = f^{(0)}(t, v, x) + \sqrt{\varepsilon} f^{(1)}(t, v, x) + \varepsilon f^{(2)}(t, v, x) + \cdots \quad (8.29) \]

with \( f(T, v, x) = (x - K)^+ \), \( f^{(1)}(T, v, x) = 0 \), and \( f^{(2)}(T, v, x) = 0 \). Since \( L_0 \) contains only differentials with respect to \( v \), we will choose \( f^{(0)}(t, v, x) \) of the form

\[ f^{(0)}(t, v, x) = f^{(0)}(t, x), \]

cf. § 4.2.1 of Fouque et al. (2011) for details, with

\[ L_0 f^{(0)}(t, x) = L_1 f^{(0)}(t, x) = 0. \quad (8.30) \]

**Proposition 8.10.** *(Fouque et al. (2011), § 3.2).* The first order term \( f_0(t, v) \) in (8.29) satisfies the Black-Scholes PDE

\[ rf^{(0)}(t, x) = \frac{\partial f^{(0)}}{\partial t}(t, x) + rx \frac{\partial f^{(0)}}{\partial x}(t, x) + \frac{\eta^2}{2} \int_0^\infty v\phi(v)dv \frac{\partial^2 f^{(0)}}{\partial x^2}(t, x) \]

with the terminal condition \( f^{(0)}(T, x) = (x - K)^+ \).

**Proof.** By identifying the terms of order \( 1/\sqrt{\varepsilon} \) when plugging (8.29) in (8.28) we have

\[ L_0 f^{(1)}(t, v, x) + L_1 f^{(0)}(t, x) = 0, \]

hence \( L_0 f^{(1)}(t, v, x) = 0 \). Similarly, by identifying the terms that do not depend on \( \varepsilon \) in (8.28) and taking \( f^{(1)}(t, v, x) = f^{(1)}(t, x) \), we have \( L_1 f^{(1)} = 0 \) and

\[ L_0 f^{(2)}(t, v, x) + L_2 f^{(0)}(t, x) = 0. \quad (8.31) \]

Using the Itô formula, we have
By differentiation with respect to $h$ hence by (8.31) we find
\[ w = \mathbb{E} \left[ f^{(2)}(t, v^1_{s}, x) \right] = f^{(2)}(t, v^1_{0}, x) + \mathbb{E} \left[ \int_{0}^{s} \frac{\partial f^{(2)}}{\partial x} (t, v^1_{r}, x) dB^{(2)}_{r} \right] + \mathbb{E} \left[ \int_{0}^{s} \left( \mu(v^1_{r}) \frac{\partial f^{(2)}}{\partial v} (t, v^1_{r}, x) + \frac{1}{2} \beta^2(v^1_{r}) \frac{\partial^2 f^{(2)}}{\partial v^2}(t, v^1_{r}, x) \right) d\tau \right] \]

\[ = \mathbb{E} \left[ f^{(2)}(t, v^1_{0}, x) \right] + \int_{0}^{s} \mathbb{E} \left[ \mathcal{L}_0 f^{(2)}(t, v^1_{r}, x) \right] d\tau. \] (8.32)

When the process $(v^1_{t})_{t \in \mathbb{R}^+}$ is started under its stationary (or invariant) probability distribution with density function $\phi(v)$ we have

\[ \mathbb{E} \left[ f^{(2)}(t, v^1_{\tau}, x) \right] = \int_{0}^{\infty} f^{(2)}(t, v, x) \phi(v) dv, \quad \tau \in \mathbb{R}^+, \]

hence (8.32) rewrites as

\[ \int_{0}^{\infty} f^{(2)}(t, v, x) \phi(v) dv = \int_{0}^{\infty} f^{(2)}(t, v, x) \phi(v) dv + \int_{0}^{s} \int_{0}^{\infty} \mathcal{L}_0 f^{(2)}(t, v, x) \phi(v) dv d\tau. \]

By differentiation with respect to $s > 0$ this yields

\[ \int_{0}^{\infty} \mathcal{L}_0 f^{(2)}(t, v, x) \phi(v) dv = 0, \]

hence by (8.31) we find

\[ \int_{0}^{\infty} \mathcal{L}_2 f^{(0)}(t, x) \phi(v) dv = 0, \]

cf. § 3.2 of Fouque et al. (2011), i.e. we find

\[ \frac{\partial f^{(0)}}{\partial t} (t, x) + r x \frac{\partial f^{(0)}}{\partial x} (t, x) + \frac{\eta^2}{2} \int_{0}^{\infty} v \phi(v) dv \frac{\partial^2 f^{(0)}}{\partial x^2}(t, x) = r f^{(0)}(t, x), \]

with the terminal condition $f^{(0)}(T, x) = (x - K)^+$. \hfill \Box

As a consequence of Proposition 8.10, the first order term $f^{(0)}(t, x)$ in the expansion (8.28) is the Black-Scholes function

\[ f^{(0)}(t, x) = \text{Bl} \left( S_t, K, r, T - t, \sqrt{\int_{0}^{\infty} v \phi(v) dv} \right), \]

with the averaged squared volatility

\[ \int_{0}^{\infty} v \phi(v) dv = \mathbb{E} \left[ v^1_{\tau} \right], \quad \tau \in \mathbb{R}^+, \] (8.33)

under the stationary distribution of the process with infinitesimal generator $\mathcal{L}_0$, i.e. the stationary distribution of the solution to

\[ dv^1_t = \mu(v^1_t) dt + \beta(v^1_t) dB^{(2)}_t. \]
Perturbation analysis in the Heston model

We have

\[
\begin{align*}
    dS_t &= rS_t dt + S_t \sqrt{v_t} dB_t^{(1)} \\
    dv_t^\varepsilon &= -\frac{\lambda}{\varepsilon} (v_t^\varepsilon - m) dt + \eta \sqrt{\frac{v_t^\varepsilon}{\varepsilon}} dB_t^{(2)},
\end{align*}
\]

under the modified short mean-reversion time scale, and the SDE can be rewritten as

\[
dv_t^\varepsilon = -\frac{\lambda}{\varepsilon} (v_t^\varepsilon - m) dt + \eta \sqrt{\frac{v_t^\varepsilon}{\varepsilon}} dB_t^{(2)}.
\]

In other words, \( \varepsilon \to 0 \) corresponds to fast mean reversion, in which \( v_t^\varepsilon \) becomes close to its mean \((8.33)\).

Recall, cf. (16.6), that the CIR process \( (v^1_t)_{t \in \mathbb{R}_+} \) has a gamma invariant (or stationary) distribution with shape parameter \( 2\lambda m/\eta^2 \), scale parameter \( \eta^2/(2\lambda) \), and probability density function \( \phi \) given by

\[
\phi(v) = \frac{1}{\Gamma(2\lambda m/\eta^2)(\eta^2/(2\lambda))^{2\lambda m/\eta^2}} v^{-1+2\lambda m/\eta^2} e^{-2v\lambda/\eta^2} \cdot 1_{[0, \infty)}(v), \quad v \in \mathbb{R},
\]

and mean

\[
\int_0^\infty v \phi(v) dv = m.
\]

Hence the first expansion term \( f^{(0)}(t, x) \) in (8.28) reads

\[
f^{(0)}(t, x) = B_l(S_t, K, r, T - t, \sqrt{m}),
\]

with the averaged squared volatility

\[
\int_0^\infty v \phi(v) dv = m = \mathbb{E}[v_\tau^1], \quad \tau \in \mathbb{R}_+,
\]

under the stationary distribution of the process with infinitesimal generator \( \mathcal{L}_0 \), i.e. the stationary distribution of the solution to

\[
dv^1_t = \mu(v^1_t) dt + \beta(v^1_t) dB_t^{(2)}.
\]

In Figure 8.3, cf. Privault and She (2016), related approximations of put option prices are plotted against the value of \( v \) with correlation \( \rho = -0.5 \) and \( \varepsilon = 0.01 \) in the \( \alpha \)-hypergeometric stochastic volatility model of Fonseca and Martini (2016), based on the series expansion of Han et al. (2013), and compared to a Monte Carlo curve requiring 300,000 samples and 30,000 time steps.
Fig. 8.3: Option price approximations plotted against $v$ with $\rho = -0.5$.

Exercises

Exercise 8.1 (Gatheral (2006), Chapter 11). Compute the expected total realized variance on the time interval $[0, T]$ in the Heston model, with

$$dv_t = -\lambda(v_t - m)dt + \eta\sqrt{v_t}dB_t, \quad t \in [0, T].$$

Exercise 8.2 Compute the variance swap rate

$$VST := \frac{1}{T} \mathbb{E} \left[ \lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{S_{kT/n} - S_{(k-1)T/n}}{S_{(k-1)T/n}} \right)^2 \right] = \frac{1}{T} \mathbb{E} \left[ \int_0^T \frac{1}{S_t^2} (dS_t)^2 \right]$$

on the index whose level $S_t$ is given in the following two models.

a) Heston model. Here, $(S_t)_{t \in \mathbb{R}_+}$ is given by the system of stochastic differential equations

$$\begin{cases} 
  dS_t = (r - \alpha v_t)S_t dt + S_t \sqrt{\beta + v_t} dB_t^{(1)} \ \\
  dv_t = -\lambda(v_t - m)dt + \gamma \sqrt{v_t} dB_t^{(2)},
\end{cases}$$

where $(B_t^{(1)})_{t \in \mathbb{R}_+}$ and $(B_t^{(2)})_{t \in \mathbb{R}_+}$ are standard Brownian motions with correlation $\rho \in [-1, 1]$ and $\alpha \geq 0$, $\beta \geq 0$, $\lambda > 0$, $m > 0$, $r > 0$, $\gamma > 0$.

b) SABR model with $\beta = 1$. The index level $S_t$ is given by the system of stochastic differential equations
Stochastic Volatility

\[
\begin{align*}
\{ &dS_t = \sigma_t S_t dB_t^{(1)} \\
&d\sigma_t = \alpha \sigma_t dB_t^{(2)},
\end{align*}
\]

where \( \alpha > 0 \) and \((B_t^{(1)})_{t\in\mathbb{R}_+}\) and \((B_t^{(2)})_{t\in\mathbb{R}_+}\) are standard Brownian motions with correlation \( \rho \in [-1, 1] \).

Exercise 8.3 (Carr and Lee (2008)) Consider an underlying asset price \((S_t)_{t\in\mathbb{R}_+}\) given by \[dS_t = r S_t dt + \sigma_t S_t dB_t,\] where \((B_t)_{t\in\mathbb{R}_+}\) is a standard Brownian motion and \((\sigma_t)_{t\in\mathbb{R}_+}\) is an (adapted) stochastic volatility process. The riskless asset is priced \(A_t := e^{rt}, t \in [0, T]\). We consider a realized variance swap with strike \(\kappa = 0\) and payoff \[\int_0^T \sigma_t^2 dt.\]

a) Show that the payoff \[\int_0^T \sigma_t^2 dt\] of the realized variance swap satisfies

\[\int_0^T \sigma_t^2 dt = 2 \int_0^T \frac{dS_t}{S_t} - 2 \log \frac{S_T}{S_0}.\]  \hfill (8.34)

b) Show that the price \(V_t := e^{-(T-t)r} \mathbb{E}^* \left[ \int_0^T \sigma_t^2 dt \mid \mathcal{F}_t \right]\) of the variance swap at time \(t \in [0, T]\) satisfies

\[V_t = L_t + 2(T-t)r e^{-(T-t)r} + 2 e^{-(T-t)r} \int_0^t \frac{dS_u}{S_u},\]  \hfill (8.35)

where

\[L_t := -2 e^{-(T-t)r} \mathbb{E}^* \left[ \log \frac{S_T}{S_0} \mid \mathcal{F}_t \right]\]

is the price at time \(t\) of the log-contract (see Neuberger (1994), Demeterfi et al. (1999)) with payoff \(-2 \log(S_T/S_0)\), see also Exercises 6.9 and 7.9.

c) Show that the portfolio made at time \(t \in [0, T]\) of:

- one log-contract priced \(L_t\),
- \(2 e^{-(T-t)r}/S_t\) in shares priced \(S_t\),
- \(2 e^{-rT} \left( \int_0^t \frac{dS_u}{S_u} + (T-t)r - 1 \right)\) in the riskless asset \(A_t = e^{rt}\),

hedges the realized variance swap.

d) Show that the above portfolio is self-financing.