Chapter 19
Stochastic Calculus for Jump Processes

In this chapter we present the construction of processes with jumps and independent increments, including the Poisson and compound Poisson processes. We also with stochastic integrals and stochastic calculus with jumps, and with the Girsanov Theorem for jump processes, which will be used for pricing and the determination of risk-neutral probability measures in the next chapter, in relation with market incompleteness.

19.1 The Poisson Process

The most elementary and useful jump process is the *standard Poisson process* $(N_t)_{t\in\mathbb{R}_+}$ which is a *counting process*, *i.e.* $(N_t)_{t\in\mathbb{R}_+}$ has jumps of size $+1$ only, and its paths are constant in between two jumps.

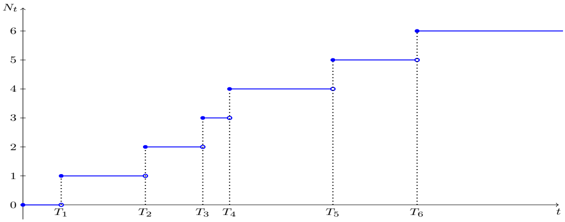

Fig. 19.1: Sample path of a Poisson process $(N_t)_{t\in\mathbb{R}_+}$.

In other words, the value N_t at time t is given by*

$$N_t = \sum_{k\geqslant 1} \mathbb{1}_{[T_k,\infty)}(t), \qquad t \in \mathbb{R}_+, \tag{19.1}$$

where

$$\mathbb{1}_{[T_k,\infty)}(t) = \begin{cases} 1 \text{ if } t \geqslant T_k, \\ \\ 0 \text{ if } 0 \leqslant t < T_k, \end{cases}$$

$k \geqslant 1$, and $(T_k)_{k\geqslant 1}$ is the increasing family of jump times of $(N_t)_{t\in\mathbb{R}_+}$ such that

$$\lim_{k\to\infty} T_k = +\infty.$$

In addition, the Poisson process $(N_t)_{t\in\mathbb{R}_+}$ is assumed to satisfy the following conditions:

1. Independence of increments: for all $0 \leqslant t_0 < t_1 < \cdots < t_n$ and $n \geqslant 1$ the increments

$$N_{t_1} - N_{t_0}, \ldots, \; N_{t_n} - N_{t_{n-1}},$$

   are mutually independent random variables.

2. Stationarity of increments: $N_{t+h} - N_{s+h}$ has the same distribution as $N_t - N_s$ for all $h > 0$ and $0 \leqslant s \leqslant t$.

The meaning of the above stationarity condition is that for all fixed $k \in \mathbb{N}$ we have

$$\mathbb{P}(N_{t+h} - N_{s+h} = k) = \mathbb{P}(N_t - N_s = k),$$

for all $h > 0$, *i.e.*, the value of the probability

$$\mathbb{P}(N_{t+h} - N_{s+h} = k)$$

* The notation N_t is not to be confused with the same notation used for numéraire processes in Chapter 15.

does not depend on $h > 0$, for all fixed $0 \leqslant s \leqslant t$ and $k \in \mathbb{N}$.

Based on the above assumption, given $T > 0$ a time value, a natural question arises:

what is the probability distribution of the random variable N_T?

We already know that N_t takes values in $\mathbb{N}$ and therefore it has a discrete distribution for all $t \in \mathbb{R}_+$.

It is a remarkable fact that the distribution of the increments of $(N_t)_{t\in\mathbb{R}_+}$, can be completely determined from the above conditions, as shown in the following theorem.

As seen in the next result, cf. Theorem 4.1 in Bosq and Nguyen (1996), $N_t - N_s$ has the Poisson distribution with parameter $\lambda(t-s)$.

Theorem 19.1. *Assume that the counting process $(N_t)_{t\in\mathbb{R}_+}$ satisfies the above independence and stationarity Conditions 1 and 2 on page 626. Then for all fixed $0 \leqslant s \leqslant t$ we have*

$$\boxed{\mathbb{P}(N_t - N_s = k) = \mathrm{e}^{-\lambda(t-s)} \frac{(\lambda(t-s))^k}{k!}, \qquad k \geqslant 0,} \tag{19.2}$$

for some constant $\lambda > 0$.

The parameter $\lambda > 0$ is called the intensity of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$ and it is given by

$$\lambda := \lim_{h\to 0} \frac{1}{h}\mathbb{P}(N_h = 1). \tag{19.3}$$

The proof of the above Theorem 19.1 is technical and not included here, cf. *e.g.* Bosq and Nguyen (1996) for details, and we could in fact take this distribution property (19.2) as one of the hypotheses that define the Poisson process.

Precisely, we could restate the definition of the standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity $\lambda > 0$ as being a stochastic process defined by (19.1), which is assumed to have independent increments distributed according to the Poisson distribution, in the sense that for all $0 \leqslant t_0 \leqslant t_1 < \cdots < t_n$,

$$(N_{t_1} - N_{t_0}, \ldots, N_{t_n} - N_{t_{n-1}})$$

is a vector of independent Poisson random variables with respective parameters

$$(\lambda(t_1 - t_0), \ldots, \lambda(t_n - t_{n-1})).$$

In particular, N_t has the Poisson distribution with parameter λt, *i.e.*,

$$\mathbb{P}(N_t = k) = \frac{(\lambda t)^k}{k!} \mathrm{e}^{-\lambda t}, \qquad t > 0.$$

The *expected value* $\mathbb{E}[N_t]$ of N_t can be computed as

$$\begin{aligned}
\mathbb{E}[N_t] &= \sum_{k \geqslant 0} k \mathbb{P}(N_t = k) \\
&= \mathrm{e}^{-\lambda t} \sum_{k \geqslant 0} k \frac{(\lambda t)^k}{k!} \\
&= \mathrm{e}^{-\lambda t} \sum_{k \geqslant 1} \frac{(\lambda t)^k}{(k-1)!} \\
&= \lambda t \, \mathrm{e}^{-\lambda t} \sum_{k \geqslant 0} \frac{(\lambda t)^k}{k!} \\
&= \lambda t, \qquad (19.4)
\end{aligned}$$

cf. Exercise A.1. Similarly, we have

$$\begin{aligned}
\mathbb{E}\left[N_t^2\right] &= \sum_{k \geqslant 0} k^2 \mathbb{P}(N_t = k) \\
&= \mathrm{e}^{-\lambda t} \sum_{k \geqslant 1} k^2 \frac{(\lambda t)^k}{k!} \\
&= \mathrm{e}^{-\lambda t} \sum_{k \geqslant 1} k \frac{(\lambda t)^k}{(k-1)!} \\
&= \mathrm{e}^{-\lambda t} \sum_{k \geqslant 2} \frac{(\lambda t)^k}{(k-2)!} + \mathrm{e}^{-\lambda t} \sum_{k \geqslant 1} \frac{(\lambda t)^k}{(k-1)!} \\
&= (\lambda t)^2 \mathrm{e}^{-\lambda t} \sum_{k \geqslant 0} \frac{(\lambda t)^k}{k!} + \lambda t \, \mathrm{e}^{-\lambda t} \sum_{k \geqslant 0} \frac{(\lambda t)^k}{k!} \\
&= (\lambda t)^2 + \lambda t
\end{aligned}$$

and

$$\operatorname{Var}[N_t] = \mathbb{E}\left[N_t^2\right] - \left(\mathbb{E}[N_t]\right)^2 = \lambda t = \mathbb{E}[N_t].$$

As a consequence, the *dispersion index* of the Poisson process is

$$\frac{\operatorname{Var}[N_t]}{\mathbb{E}[N_t]} = 1, \qquad t \in \mathbb{R}_+. \qquad (19.5)$$

Short Time Behaviour

From (19.3) above we deduce the short time asymptotics*

$$\begin{cases} \mathbb{P}(N_h = 0) = \mathrm{e}^{-h\lambda} = 1 - h\lambda + o(h), & h \to 0, \\ \\ \mathbb{P}(N_h = 1) = h\lambda \mathrm{e}^{-h\lambda} \simeq h\lambda, & h \to 0. \end{cases}$$

By stationarity of the Poisson process we also find more generally that

$$\begin{cases} \mathbb{P}(N_{t+h} - N_t = 0) = \mathrm{e}^{-h\lambda} = 1 - h\lambda + o(h), & h \to 0, \\ \\ \mathbb{P}(N_{t+h} - N_t = 1) = h\lambda \mathrm{e}^{-h\lambda} \simeq h\lambda, & h \to 0, \\ \\ \mathbb{P}(N_{t+h} - N_t = 2) \simeq h^2 \dfrac{\lambda^2}{2} = o(h), & h \to 0, \quad t > 0, \end{cases}$$

for all $t > 0$. This means that within a "short" interval $[t, t+h]$ of length h, the increment $N_{t+h} - N_t$ behaves like a Bernoulli random variable with parameter λh. This fact can be used for the random simulation of Poisson process paths.

More generally, for $k \geqslant 1$ we have

$$\mathbb{P}(N_{t+h} - N_t = k) \simeq h^k \frac{\lambda^k}{k!}, \qquad h \to 0, \qquad t > 0.$$

The intensity of the Poisson process can in fact be made time-dependent (*e.g.* by a time change), in which case we have

$$\mathbb{P}(N_t - N_s = k) = \exp\left(- \int_s^t \lambda(u) du \right) \frac{\left(\int_s^t \lambda(u) du \right)^k}{k!}, \qquad k = 0, 1, 2, \ldots.$$

This is a special case of *Cox processes*. In this case we have in particular

$$\mathbb{P}(N_{t+dt} - N_t = k) = \begin{cases} \mathrm{e}^{-\lambda(t)dt} = 1 - \lambda(t)dt + o(h), & k = 0, \\ \\ \lambda(t) \mathrm{e}^{-\lambda(t)dt} dt \simeq \lambda(t)dt, & k = 1, \\ \\ o(dt), & k \geqslant 2. \end{cases}$$

* The notation $f(h) = o(h^k)$ means $\lim_{h \to 0} f(h)/h^k = 0$, and $f(h) \simeq h^k$ means $\lim_{h \to 0} f(h)/h^k = 1$.

The intensity process $(\lambda(t))_{t\in\mathbb{R}_+}$ can also be made random, as in the case of Cox processes.

Poisson Process Jump Times

In order to determine the distribution of the first jump time T_1 we note that we have the equivalence

$$\{T_1 > t\} \Longleftrightarrow \{N_t = 0\},$$

which implies

$$\mathbb{P}(T_1 > t) = \mathbb{P}(N_t = 0) = \mathrm{e}^{-\lambda t}, \qquad t \in \mathbb{R}_+,$$

i.e., T_1 has an exponential distribution with parameter $\lambda > 0$.

In order to prove the next proposition we note that more generally, we have the equivalence

$$\{T_n > t\} \Longleftrightarrow \{N_t \leqslant n-1\},$$

for all $n \geqslant 1$. This allows us to compute the distribution of T_n with its density. It coincides with the *gamma* distribution with integer parameter $n \geqslant 1$, also known as the Erlang distribution in queueing theory.

Proposition 19.2. *For all $n \geqslant 1$ the probability distribution of T_n has the gamma probability density function*

$$t \longmapsto \lambda^n \mathrm{e}^{-\lambda t} \frac{t^{n-1}}{(n-1)!}$$

on $\mathbb{R}_+$, i.e., for all $t > 0$ the probability $\mathbb{P}(T_n \geqslant t)$ is given by

$$\mathbb{P}(T_n \geqslant t) = \lambda^n \int_t^\infty \mathrm{e}^{-\lambda s} \frac{s^{n-1}}{(n-1)!} ds.$$

Proof. We have

$$\mathbb{P}(T_1 > t) = \mathbb{P}(N_t = 0) = \mathrm{e}^{-\lambda t}, \qquad t \in \mathbb{R}_+,$$

and by induction, assuming that

$$\mathbb{P}(T_{n-1} > t) = \lambda \int_t^\infty \mathrm{e}^{-\lambda s} \frac{(\lambda s)^{n-2}}{(n-2)!} ds, \qquad n \geqslant 2,$$

we obtain

$$\mathbb{P}(T_n > t) = \mathbb{P}(T_n > t \geqslant T_{n-1}) + \mathbb{P}(T_{n-1} > t)$$

$$
\begin{aligned}
&= \mathbb{P}(N_t = n-1) + \mathbb{P}(T_{n-1} > t) \\
&= \mathrm{e}^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} + \lambda \int_t^\infty \mathrm{e}^{-\lambda s} \frac{(\lambda s)^{n-2}}{(n-2)!} ds \\
&= \lambda \int_t^\infty \mathrm{e}^{-\lambda s} \frac{(\lambda s)^{n-1}}{(n-1)!} ds, \qquad t \in \mathbb{R}_+,
\end{aligned}
$$

where we applied an integration by parts to derive the last line. □

In particular, for all $n \in \mathbb{Z}$ and $t \in \mathbb{R}_+$, we have

$$
\mathbb{P}(N_t = n) = p_n(t) = \mathrm{e}^{-\lambda t} \frac{(\lambda t)^n}{n!},
$$

i.e., $p_{n-1} : \mathbb{R}_+ \to \mathbb{R}_+$, $n \geqslant 1$, is the density function of T_n.

In addition to Proposition 19.2 we could show the following proposition which relies on the *strong Markov property*, see *e.g.* Theorem 6.5.4 of Norris (1998).

Proposition 19.3. *The (random) interjump times*

$$
\tau_k := T_{k+1} - T_k
$$

spent at state $k \in \mathbb{N}$, *with* $T_0 = 0$, *form a sequence of independent identically distributed random variables having the exponential distribution with parameter* $\lambda > 0$, *i.e.*,

$$
\mathbb{P}(\tau_0 > t_0, \ldots, \tau_n > t_n) = \mathrm{e}^{-\lambda(t_0+t_1+\cdots+t_n)}, \qquad t_0, t_1, \ldots, t_n \in \mathbb{R}_+.
$$

As the expectation of the exponentially distributed random variable τ_k with parameter $\lambda > 0$ is given by

$$
\mathbb{E}[\tau_k] = \lambda \int_0^\infty x \,\mathrm{e}^{-\lambda x} dx = \frac{1}{\lambda},
$$

we can check that the higher the intensity λ (*i.e.*, the higher the probability of having a jump within a small interval), the smaller is the time spent in each state $k \in \mathbb{N}$ on average.

In addition, conditionally to $\{N_T = n\}$, the n jump times on $[0, T]$ of the Poisson process $(N_t)_{t \in \mathbb{R}_+}$ are independent uniformly distributed random variables on $[0, T]^n$, cf. *e.g.* § 12.1 of Privault (2018. xvii+372 pp). This fact can be useful for the random simulation of the Poisson process.

As a consequence of Propositions 19.2 and 19.2, random samples of Poisson process jump times can be generated using the following *R* code.

```
lambda = 0.6;n = 20
for (k in 1:n){tau_k <- rexp(n,rate=lambda)/lambda; T_n <- cumsum(tau_k)}
tau_k;T_n
Z<-cumsum(c(0,rep(1,n)))
plot(stepfun(T_n,Z),xlim = c(0,10),ylim = c(0,8),xlab="t",ylab="Nt",pch=1,cex=0.8,col="blue",lw
     =2,main="")
```

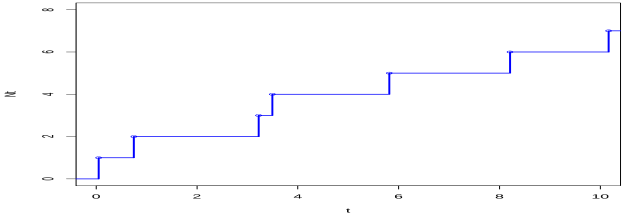

Fig. 19.2: Sample path of a Poisson process $(N_t)_{t\in\mathbb{R}_+}$.

Compensated Poisson Martingale

From (19.4) above we deduce that

$$\mathbb{E}[N_t - \lambda t] = 0, \tag{19.6}$$

i.e., the compensated Poisson process $(N_t - \lambda t)_{t\in\mathbb{R}_+}$ has *centered increments*. Since in addition $(N_t - \lambda t)_{t\in\mathbb{R}_+}$ also has independent increments, we get the following proposition, cf. e.g. Example 2 page 228. We let

$$\mathcal{F}_t := \sigma\big(N_s \ : \ s \in [0,t]\big), \qquad t \in \mathbb{R}_+,$$

denote the *filtration* generated by the Poisson process $(N_t)_{t\in\mathbb{R}_+}$.

Proposition 19.4. *The compensated Poisson process*

$$(N_t - \lambda t)_{t\in\mathbb{R}_+}$$

is a martingale *with respect* $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$.

Extensions of the Poisson process include Poisson processes with time-dependent intensity, and with random time-dependent intensity (Cox processes). Poisson processes belong to the family of *renewal processes* which are counting processes of the form

$$N_t = \sum_{n\geqslant 1} \mathbb{1}_{[T_n,\infty)}(t), \qquad t \in \mathbb{R}_+,$$

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for which $\tau_k := T_{k+1} - T_k$, $k \geqslant 0$, is a sequence of independent identically distributed random variables.

19.2 Compound Poisson Process

The Poisson process itself appears to be too limited to develop realistic price models as its jumps are of constant size. Therefore there is some interest in considering jump processes that can have random jump sizes.

Let $(Z_k)_{k\geqslant 1}$ denote an *i.i.d.* sequence of square-integrable random variables distributed as the common random variable Z with probability distribution $\nu(dy)$ on $\mathbb{R}$, independent of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$. We have

$$\mathbb{P}(Z \in [a,b]) = \nu([a,b]) = \int_a^b \nu(dy), \quad -\infty < a \leqslant b < \infty, \quad k \geqslant 1.$$

Definition 19.5. *The process $(Y_t)_{t\in\mathbb{R}_+}$ given by the random sum*

$$Y_t := Z_1 + Z_2 + \cdots + Z_{N_t} = \sum_{k=1}^{N_t} Z_k, \qquad t \in \mathbb{R}_+, \tag{19.7}$$

*is called a compound Poisson process.**

Letting Y_{t^-} denote the left limit

$$Y_{t^-} := \lim_{s\nearrow t} Y_s, \qquad t > 0,$$

we note that the jump size

$$\Delta Y_t := Y_t - Y_{t^-}, \qquad t \in \mathbb{R}_+,$$

of $(Y_t)_{t\in\mathbb{R}_+}$ at time t is given by the relation

$$\Delta Y_t = Z_{N_t}\Delta N_t, \qquad t \in \mathbb{R}_+, \tag{19.8}$$

where

$$\Delta N_t := N_t - N_{t^-} \in \{0,1\}, \qquad t \in \mathbb{R}_+,$$

denotes the jump size of the standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$, and N_{t^-} is the left limit

$$N_{t^-} := \lim_{s\nearrow t} N_s, \qquad t > 0,$$

* We use the convention $\sum_{k=1}^{n} Z_k = 0$ if $n = 0$, so that $Y_0 = 0$.

For a typical example of a compound Poisson process we can assume that jump sizes are Gaussian distributed with mean δ and variance η^2, in which case $\nu(dy)$ is given by

$$\nu(dy) = \frac{1}{\sqrt{2\pi\eta^2}} \mathrm{e}^{-(y-\delta)^2/(2\eta^2)} dy.$$

The next Figure 19.3 represents a sample path of a compound Poisson process, with here $Z_1 = 0.9$, $Z_2 = -0.7$, $Z_3 = 1.4$, $Z_4 = 0.6$, $Z_5 = -2.5$, $Z_6 = 1.5$, $Z_7 = -0.5$, with the relation

$$Y_{T_k} = Y_{T_k^-} + Z_k, \qquad k \geqslant 1.$$

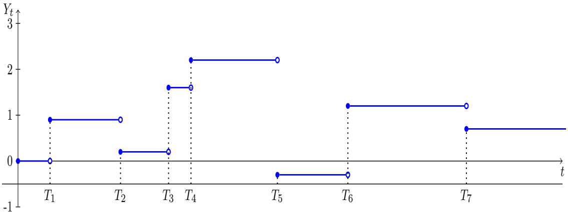

Fig. 19.3: Sample path of a compound Poisson process $(Y_t)_{t\in\mathbb{R}_+}$.

```
n<-100
x<-cumsum(rexp(50,rate=0.5))
y<-cumsum(c(0,rexp(50,rate=0.5)))
plot(stepfun(x,y),xlim = c(0,10),do.points = F,main="L=0.5")
y<-cumsum(c(0,rnorm(50,0,1)))
plot(stepfun(x,y),xlim = c(0,10),do.points = F,main="L=0.5")
```

Given that $\{N_T = n\}$, the n jump sizes of $(Y_t)_{t\in\mathbb{R}_+}$ on $[0, T]$ are independent random variables which are distributed on $\mathbb{R}$ according to $\nu(dx)$. Based on this fact, the next proposition allows us to compute the moment generating function (MGF) of the increment $Y_T - Y_t$.

Proposition 19.6. *For any $t \in [0, T]$ we have*

$$\mathbb{E}\left[\exp\left(\alpha(Y_T - Y_t)\right)\right] = \exp\left(\lambda(T-t)\int_{-\infty}^{\infty}(\mathrm{e}^{\alpha y} - 1)\nu(dy)\right), \quad \alpha \in \mathbb{R}. \tag{19.9}$$

Proof. Since N_t has a Poisson distribution with parameter $t > 0$ and is independent of $(Z_k)_{k\geqslant 1}$, for all $\alpha \in \mathbb{R}$ we have, by conditioning on the value of $N_T - N_t = n$,

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$$
\begin{aligned}
\mathbb{E}\left[\exp\left(\alpha(Y_T - Y_t)\right)\right] &= \mathbb{E}\left[\exp\left(\alpha \sum_{k=N_t+1}^{N_T} Z_k\right)\right] \\
&= \mathbb{E}\left[\exp\left(\alpha \sum_{k=N_t+1}^{N_T} Z_k\right)\right] = \mathbb{E}\left[\exp\left(\alpha \sum_{k=1}^{N_T-N_t} Z_k\right)\right] \\
&= \sum_{n\geqslant 0} \mathbb{E}\left[\exp\left(\alpha \sum_{k=1}^{n} Z_k\right) \middle| N_T - N_t = n\right] \mathbb{P}(N_T - N_t = n) \\
&= \mathrm{e}^{-\lambda(T-t)} \sum_{n\geqslant 0} \frac{\lambda^n}{n!}(T-t)^n \, \mathbb{E}\left[\exp\left(\alpha \sum_{k=1}^{n} Z_k\right)\right] \\
&= \mathrm{e}^{-\lambda(T-t)} \sum_{n\geqslant 0} \frac{\lambda^n}{n!}(T-t)^n \prod_{k=1}^{n} \mathbb{E}\left[\exp\left(\alpha Z_k\right)\right] \\
&= \mathrm{e}^{-\lambda(T-t)} \sum_{n\geqslant 0} \frac{\lambda^n}{n!}(T-t)^n \left(\mathbb{E}\left[\exp\left(\alpha Z\right)\right]\right)^n \\
&= \exp\left(\lambda(T-t)\left(\mathbb{E}\left[\exp\left(\alpha Z\right)\right] - 1\right)\right) \\
&= \exp\left(\lambda(T-t)\int_{-\infty}^{\infty} \mathrm{e}^{\alpha y}\nu(dy) - \lambda(T-t)\int_{-\infty}^{\infty} \nu(dy)\right) \\
&= \exp\left(\lambda(T-t)\int_{-\infty}^{\infty} (\mathrm{e}^{\alpha y} - 1)\nu(dy)\right),
\end{aligned}
$$

since the probability distribution $\nu(dy)$ of Z satisfies

$$
\mathbb{E}\left[\exp\left(\alpha Z\right)\right] = \int_{-\infty}^{\infty} \mathrm{e}^{\alpha y}\nu(dy) \quad \text{and} \quad \int_{-\infty}^{\infty} \nu(dy) = 1.
$$

□

From the moment generating function (19.9) we can compute the expectation of Y_t for fixed t as the product of the mean number of jump times $\mathbb{E}[N_t] = \lambda t$ and the mean jump size $\mathbb{E}[Z]$, *i.e.*,

$$
\mathbb{E}[Y_t] = \frac{\partial}{\partial \alpha} \mathbb{E}[\mathrm{e}^{\alpha Y_t}]_{|\alpha=0} = \lambda t \int_{-\infty}^{\infty} y \nu(dy) = \mathbb{E}[N_t]\,\mathbb{E}[Z] = \lambda t\,\mathbb{E}[Z]. \tag{19.10}
$$

Note that the above identity requires to exchange the differentiation and expectation operators, which is possible when the moment generating function (19.9) takes finite values for all α in a certain neighborhood $(-\varepsilon, \varepsilon)$ of 0.

Relation (19.10) states that the mean value of Y_t is the mean jump size $\mathbb{E}[Z]$ times the mean number of jumps $\mathbb{E}[N_t]$. It can also be directly using series summations as

$$
\begin{aligned}
\mathbb{E}[Y_t] &= \mathbb{E}\left[\sum_{k=1}^{N_t} Z_k\right] \\
&= \sum_{n\geqslant 1} \mathbb{E}\left[\sum_{k=1}^{n} Z_k \Big| N_t = n\right] \mathbb{P}(N_t = n) \\
&= \mathrm{e}^{-\lambda t} \sum_{n\geqslant 1} \frac{\lambda^n t^n}{n!} \mathbb{E}\left[\sum_{k=1}^{n} Z_k \Big| N_t = n\right] \\
&= \mathrm{e}^{-\lambda t} \sum_{n\geqslant 1} \frac{\lambda^n t^n}{n!} \mathbb{E}\left[\sum_{k=1}^{n} Z_k\right] \\
&= \lambda t \, \mathrm{e}^{-\lambda t} \, \mathbb{E}[Z] \sum_{n\geqslant 1}^{\frac{(\lambda t)^{n-1}}{(n-1)!}} \\
&= \lambda t \, \mathbb{E}[Z] \\
&= \mathbb{E}[N_t] \, \mathbb{E}[Z].
\end{aligned}
$$

Regarding the variance, we have

$$
\begin{aligned}
\mathbb{E}[Y_t^2] &= \frac{\partial^2}{\partial \alpha^2} \mathbb{E}[\,\mathrm{e}^{\alpha Y_t}]_{|\alpha=0} \\
&= \lambda t \int_{-\infty}^{\infty} y^2 \nu(dy) + (\lambda(T-t))^2 \left(\int_{-\infty}^{\infty} y \nu(dy)\right)^2 \\
&= \lambda t \, \mathbb{E}[Z^2] + (\lambda t \, \mathbb{E}[Z])^2,
\end{aligned}
$$

which yields

$$
\boxed{\mathrm{Var}\,[Y_t] = \lambda t \int_{-\infty}^{\infty} y^2 \nu(dy) = \lambda t \, \mathbb{E}[|Z|^2] = \mathbb{E}[N_t] \, \mathbb{E}[|Z|^2].}
$$

As a consequence, the *dispersion index* of the compound Poisson process

$$
\frac{\mathrm{Var}\,[Y_t]}{\mathbb{E}[Y_t]} = \frac{\mathbb{E}[|Z|^2]}{\mathbb{E}[Z]}, \qquad t \in \mathbb{R}_+.
$$

is the dispersion index of the random jump size Z. By a multivariate version of Theorem 22.17, the above identity can be used to show the next proposition.

Proposition 19.7. *The compound Poisson process*

$$
Y_t = \sum_{k=1}^{N_t} Z_k, \qquad t \in \mathbb{R}_+,
$$

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has independent increments, i.e. for any finite sequence of times $t_0 < t_1 < \cdots < t_n$*, the increments*

$$Y_{t_1} - Y_{t_0}, \ Y_{t_2} - Y_{t_1}, \ldots, \ Y_{t_n} - Y_{t_{n-1}}$$

are mutually independent random variables.

Proof. This result relies on the fact that the result of Proposition 19.6 can be extended to sequences $0 \leqslant t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$, as

$$\begin{aligned}
\mathbb{E}\left[\prod_{k=1}^n \mathrm{e}^{i\alpha_k (Y_{t_k} - Y_{t_{k-1}})}\right] &= \mathbb{E}\left[\exp\left(i\sum_{k=1}^n \alpha_k (Y_{t_k} - Y_{t_{k-1}})\right)\right] \\
&= \exp\left(\lambda \sum_{k=1}^n (t_k - t_{k-1}) \int_{-\infty}^{\infty} (\mathrm{e}^{i\alpha_k y} - 1)\nu(dy)\right) \\
&= \prod_{k=1}^n \exp\left(\lambda (t_k - t_{k-1}) \int_{-\infty}^{\infty} (\mathrm{e}^{i\alpha_k y} - 1)\nu(dy)\right) \\
&= \prod_{k=1}^n \mathbb{E}\left[\mathrm{e}^{i\alpha_k (Y_{t_k} - Y_{t_{k-1}})}\right].
\end{aligned}$$

□

Since the compensated compound Poisson process also has independent and centered increments by (19.6) we have the following counterpart of Proposition 19.4, cf. also Example 2 page 228.

Proposition 19.8. *The compensated compound Poisson process*

$$M_t := Y_t - \lambda t \, \mathbb{E}[Z], \qquad t \in \mathbb{R}_+,$$

is a martingale.

By construction, compound Poisson processes only have a *finite* number of jumps on any interval. They belong to the family of *Lévy processes* which may have an infinite number of jumps on any finite time interval, see *e.g.* § 4.4.1 of Cont and Tankov (2004).

19.3 Stochastic Integrals with Jumps

Based on the relation

$$\Delta Y_t = Z_{N_t} \Delta N_t,$$

we can define the stochastic integral of a stochastic process $(\phi_t)_{t\in\mathbb{R}_+}$ with respect to $(Y_t)_{t\in\mathbb{R}_+}$ by

$$\int_0^T \phi_t dY_t = \int_0^T \phi_t Z_{N_t} dN_t := \sum_{k=1}^{N_T} \phi_{T_k} Z_k. \tag{19.11}$$

As a consequence of Proposition 19.6 we can derive the following version of the Lévy-Khintchine formula:

$$\mathbb{E}\left[\exp\left(\int_0^T f(t)dY_t\right)\right] = \exp\left(\lambda \int_0^T \int_{-\infty}^{\infty} (\mathrm{e}^{yf(t)} - 1)\nu(dy)dt\right)$$

for $f : [0, T] \longrightarrow \mathbb{R}$ a bounded deterministic function of time.

Note that the expression (19.11) of $\int_0^T \phi_t dY_t$ has a natural financial interpretation as the value at time T of a portfolio containing a (possibly fractional) quantity ϕ_t of a risky asset at time t, whose price evolves according to random returns Z_k, generating profits/losses $\phi_{T_k} Z_k$ at random times T_k.

In particular the compound Poisson process $(Y_t)_{t\in\mathbb{R}_+}$ in (19.5) admits the stochastic integral representation

$$Y_t = Y_0 + \sum_{k=1}^{N_t} Z_k = Y_0 + \int_0^t Z_{N_s} dN_s.$$

The next result is also called the smoothing formula, cf. Theorem 9.2.1 in Brémaud (1999).

Proposition 19.9. *Let $(\phi_t)_{t\in\mathbb{R}_+}$ be a stochastic process* adapted *to the filtration generated by $(Y_t)_{t\in\mathbb{R}_+}$ and such that*

$$\mathbb{E}\left[\int_0^T |\phi_t| dt\right] < \infty, \qquad T > 0.$$

The expected value of the compound Poisson compensated stochastic integral can be expressed as

$$\mathbb{E}\left[\int_0^T \phi_{t^-} dY_t\right] = \mathbb{E}\left[\int_0^T \phi_{t^-} Z_t dN_t\right] = \lambda\,\mathbb{E}[Z]\,\mathbb{E}\left[\int_0^T \phi_t dt\right], \tag{19.12}$$

where ϕ_{t^-} denotes the left limit

$$\phi_{t^-} := \lim_{s \nearrow t} \phi_s, \qquad t > 0.$$

Proof. By Proposition 19.8 the compensated compound Poisson process $(Y_t - \lambda t\,\mathbb{E}[Z])_{t\in\mathbb{R}_+}$ is a *martingale*, and as a consequence the indefinite stochastic integral

$$t \longmapsto \int_0^t \phi_{s^-} d(Y_s - \lambda \mathbb{E}[Z]s) = \int_0^t \phi_{s^-}(Z_{N_s} dN_s - \lambda \mathbb{E}[Z]ds)$$

is also a martingale, by an argument similar to that in the proof of Proposition 7.1 because the adaptedness of $(\phi_t)_{t\in\mathbb{R}_+}$ to the filtration generated by $(Y_t)_{t\in\mathbb{R}_+}$, makes $(\phi_{t^-})_{t>0}$ *predictable*, *i.e.* adapted with respect to the filtration

$$\mathcal{F}_{t^-} := \sigma\big(Y_s \ : \ s \in [0,t)\ \big), \qquad t > 0.$$

It remains to use the fact that the expectation of a martingale remains constant over time, which shows that

$$\begin{aligned}
0 &= \mathbb{E}\left[\int_0^T \phi_{t^-}(dY_t - \lambda \mathbb{E}[Z]dt)\right] \\
&= \mathbb{E}\left[\int_0^T \phi_{t^-} dY_t\right] - \lambda \mathbb{E}[Z]\, \mathbb{E}\left[\int_0^T \phi_{t^-} dt\right] \\
&= \mathbb{E}\left[\int_0^T \phi_{t^-} dY_t\right] - \lambda \mathbb{E}[Z]\, \mathbb{E}\left[\int_0^T \phi_t dt\right].
\end{aligned}$$

□

For example, taking $\phi_t = Y_t := N_t$ we have

$$\int_0^T N_{t^-} dN_t = \sum_{k=1}^{N_T} (k-1) = \frac{1}{2} N_T(N_T - 1),$$

hence

$$\mathbb{E}\left[\int_0^T N_{t^-} dN_t\right] = \frac{1}{2}\left(\mathbb{E}\left[N_T^2\right] - \mathbb{E}[N_T]\right) = \frac{(\lambda T)^2}{2}.$$

On the other hand, we check that

$$\begin{aligned}
\lambda\, \mathbb{E}\left[\int_0^T N_{t^-} dt\right] &= \lambda\, \mathbb{E}\left[\int_0^T N_t dt\right] \\
&= \lambda \int_0^T \mathbb{E}\left[N_t\right] dt \\
&= \lambda^2 \int_0^T t dt \\
&= \frac{(\lambda T)^2}{2},
\end{aligned}$$

as in (19.12).

Note however that while the identity in expectations (19.12) holds for the left limit ϕ_{t^-}, it need not hold for ϕ_t itself. Indeed, taking $\phi_t = Y_t := N_t$ we have

$$\int_0^T N_t dN_t = \sum_{k=1}^{N_T} k = \frac{1}{2} N_T(N_T+1),$$

hence

$$\begin{aligned}\mathbb{E}\left[\int_0^T N_t dN_t\right] &= \frac{1}{2}\left(\mathbb{E}\left[N_T^2\right] + \mathbb{E}[N_T]\right) \\ &= \frac{1}{2}\left((\lambda T)^2 + 2\lambda T\right) \\ &= \frac{(\lambda T)^2}{2} + \lambda T \\ &\neq \lambda\,\mathbb{E}\left[\int_0^T N_t dt\right].\end{aligned}$$

Under similar conditions, the compound Poisson compensated stochastic integral can be shown to satisfy the Itô isometry (19.13) in the next proposition.

Proposition 19.10. *Let $(\phi_t)_{t\in\mathbb{R}_+}$ be a stochastic process* adapted *to the filtration generated by $(Y_t)_{t\in\mathbb{R}_+}$, and such that*

$$\mathbb{E}\left[\int_0^T |\phi_t|^2 dt\right] < \infty, \qquad T > 0.$$

The expected value of the squared compound Poisson compensated stochastic integral can be computed as

$$\boxed{\mathbb{E}\left[\left(\int_0^T \phi_{t^-}(dY_t - \lambda\,\mathbb{E}[Z]dt)\right)^2\right] = \lambda\,\mathbb{E}[|Z|^2]\,\mathbb{E}\left[\int_0^T |\phi_t|^2 dt\right],} \tag{19.13}$$

Note that in (19.13), the generic jump size Z is squared but λ is not.

Proof. From the stochastic Fubini-type theorem we have

$$\left(\int_0^T \phi_{t^-}(dY_t - \lambda\,\mathbb{E}[Z]dt)\right)^2 \tag{19.14}$$

$$= 2\int_0^T \phi_{t^-}\int_0^{t^-} \phi_{s^-}(dY_s - \lambda\,\mathbb{E}[Z]ds)(dY_t - \lambda\,\mathbb{E}[Z]dt) \tag{19.15}$$

$$+\int_0^T |\phi_{t^-}|^2 |Z_{N_t}|^2 dN_t, \tag{19.16}$$

where integration over the diagonal $\{s = t\}$ has been excluded in (19.15) as the inner integral has an upper limit t^- rather than t. Next, taking expectation on both sides of (19.14)-(19.16) we find

$$\mathbb{E}\left[\left(\int_0^T \phi_{t^-}(dY_t - \lambda\,\mathbb{E}[Z]dt)\right)^2\right] = \mathbb{E}\left[\int_0^T |\phi_{t^-}|^2 |Z_{N_t}|^2 dN_t\right]$$
$$= \lambda\,\mathbb{E}[|Z|^2]\,\mathbb{E}\left[\int_0^T |\phi_t|^2 dt\right],$$

where we used the vanishing of the expectation of the double stochastic integral:

$$\mathbb{E}\left[\int_0^T \phi_{t^-} \int_0^{t^-} \phi_{s^-}(dY_s - \lambda\,\mathbb{E}[Z]ds)(dY_t - \lambda\,\mathbb{E}[Z]dt)\right] = 0,$$

and the martingale property of the compensated compound Poisson process

$$t \longmapsto \left(\sum_{k=1}^{N_t} |Z_k|^2\right) - \lambda t\,\mathbb{E}[Z^2], \qquad t \in \mathbb{R}_+,$$

as in the proof of Proposition 19.9. The isometry relation (19.13) can also be proved using simple predictable processes, similarly to the proof of Proposition 4.16. □

Next, take $(B_t)_{t\in\mathbb{R}_+}$ a standard Brownian motion independent of $(Y_t)_{t\in\mathbb{R}_+}$ and $(X_t)_{t\in\mathbb{R}_+}$ a jump-diffusion process of the form

$$X_t := \int_0^t u_s dB_s + \int_0^t v_s ds + Y_t, \qquad t \in \mathbb{R}_+,$$

where $(u_t)_{t\in\mathbb{R}_+}$ is a stochastic process which is adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ generated by $(B_t)_{t\in\mathbb{R}_+}$ and $(Y_t)_{t\in\mathbb{R}_+}$, and such that

$$\mathbb{E}\left[\int_0^T |\phi_t|^2 |u_t|^2 dt\right] < \infty \quad \text{and} \quad \mathbb{E}\left[\int_0^T |\phi_t v_t| dt\right] < \infty, \qquad T > 0.$$

We define the stochastic integral of $(\phi_t)_{t\in\mathbb{R}_+}$ with respect to $(X_t)_{t\in\mathbb{R}_+}$ by

$$\int_0^T \phi_t dX_t := \int_0^T \phi_t u_t dB_t + \int_0^T \phi_t v_t dt + \int_0^T \phi_t dY_t$$
$$:= \int_0^T \phi_t u_t dB_t + \int_0^T \phi_t v_t dt + \sum_{k=1}^{N_T} \phi_{T_k} Z_k, \qquad T > 0.$$

For the mixed continuous-jump martingale

$$X_t := \int_0^t u_s dB_s + Y_t - \lambda t\,\mathbb{E}[Z], \qquad t \in \mathbb{R}_+,$$

we then have the isometry:

$$\mathbb{E}\left[\left(\int_0^T \phi_{t^-} dX_t\right)^2\right] = \mathbb{E}\left[\int_0^T |\phi_t|^2 |u_t|^2 dt\right] + \lambda\, \mathbb{E}[|Z|^2]\, \mathbb{E}\left[\int_0^T |\phi_t|^2 dt\right]. \tag{19.17}$$

provided that $(\phi_s)_{s\in\mathbb{R}_+}$ is adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ generated by $(B_t)_{t\in\mathbb{R}_+}$ and $(Y_t)_{t\in\mathbb{R}_+}$. The isometry formula (19.17) will be used in Section 20.6 for mean-variance hedging in jump-diffusion models.

More generally, when $(X_t)_{t\in\mathbb{R}_+}$ contains an additional drift term,

$$X_t = X_0 + \int_0^t u_s dB_s + \int_0^t v_s ds + \int_0^t \eta_s dY_s, \qquad t \in \mathbb{R}_+,$$

the stochastic integral of $(\phi_t)_{t\in\mathbb{R}_+}$ with respect to $(X_t)_{t\in\mathbb{R}_+}$ is given by

$$\begin{aligned}\int_0^T \phi_s dX_s &:= \int_0^T \phi_s u_s dB_s + \int_0^T \phi_s v_s ds + \int_0^T \eta_s \phi_s dY_s \\ &= \int_0^T \phi_s u_s dB_s + \int_0^T \phi_s v_s ds + \sum_{k=1}^{N_T} \phi_{T_k} \eta_{T_k} Z_k, \qquad T > 0.\end{aligned}$$

19.4 Itô Formula with Jumps

The next proposition gives the simplest instance of the Itô formula with jumps, in the case of a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity λ.

Proposition 19.11. *Itô formula for the standard Poisson process. We have*

$$f(N_t) = f(0) + \int_0^t (f(N_s) - f(N_{s^-}))dN_s, \qquad t \in \mathbb{R}_+,$$

where N_{s^-} denotes the left limit $N_{s^-} = \lim_{h\searrow 0} N_{s-h}$.

Proof. We note that

$$N_s = N_{s^-} + 1 \;\text{ if }\; dN_s = 1 \;\text{ and }\; k = N_{T_k} = 1 + N_{T_k^-}, \qquad k \geqslant 1.$$

Hence we have the *telescoping sum*

$$\begin{aligned} f(N_t) &= f(0) + \sum_{k=1}^{N_t} (f(k) - f(k-1)) \\ &= f(0) + \sum_{k=1}^{N_t} (f(N_{T_k}) - f(N_{T_k^-})) \end{aligned}$$

$$
\begin{aligned}
&= f(0) + \sum_{k=1}^{N_t} (f(1 + N_{T_k^-}) - f(N_{T_k^-})) \\
&= f(0) + \int_0^t (f(1 + N_{s^-}) - f(N_{s^-})) dN_s \\
&= f(0) + \int_0^t (f(N_s) - f(N_s - 1)) dN_s \\
&= f(0) + \int_0^t (f(N_s) - f(N_{s^-})) dN_s,
\end{aligned}
$$

where N_{s^-} denotes the left limit $N_{s^-} = \lim_{h \searrow 0} N_{s-h}$. □

The next result deals with the compound Poisson process $(Y_t)_{t \in \mathbb{R}_+}$ via a similar argument.

Proposition 19.12. *Itô formula for the compound Poisson process. We have the* pathwise Itô formula

$$
f(Y_t) = f(0) + \int_0^t (f(Y_s) - f(Y_{s^-})) dN_s, \qquad t \in \mathbb{R}_+. \tag{19.18}
$$

Proof. We have

$$
\begin{aligned}
f(Y_t) &= f(0) + \sum_{k=1}^{N_t} (f(Y_{T_k}) - f(Y_{T_k^-})) \\
&= f(0) + \sum_{k=1}^{N_t} (f(Y_{T_k^-} + Z_k) - f(Y_{T_k^-})) \\
&= f(0) + \int_0^t (f(Y_{s^-} + Z_{N_s}) - f(Y_{s^-})) dN_s \\
&= f(0) + \int_0^t (f(Y_s) - f(Y_{s^-})) dN_s, \qquad t \in \mathbb{R}_+.
\end{aligned}
$$

□

The formula (19.18) can be decomposed using a compensated Poisson stochastic integral as

$$
f(Y_t) = f(0) + \int_0^t (f(Y_s) - f(Y_{s^-}))(dN_s - \lambda ds) + \lambda \int_0^t (f(Y_s) - f(Y_{s^-})) ds.
$$

More generally we have the following result.

Proposition 19.13. *For an Itô process of the form*

$$
X_t = X_0 + \int_0^t u_s dB_s + \int_0^t v_s ds + \int_0^t \eta_s dY_s, \qquad t \in \mathbb{R}_+,
$$

we have the Itô formula

$$f(X_t) = f(X_0) + \int_0^t v_s f'(X_s) ds + \int_0^t u_s f'(X_s) dB_s + \frac{1}{2} \int_0^t f''(X_s) |u_s|^2 ds$$
$$+ \int_0^t (f(X_s) - f(X_{s^-})) dN_s, \qquad t \in \mathbb{R}_+. \tag{19.19}$$

Proof. By combining the Itô formula for Brownian motion with the above argument we find

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t u_s f'(X_s) dB_s + \frac{1}{2} \int_0^t f''(X_s) |u_s|^2 ds + \int_0^t v_s f'(X_s) ds \\ &\quad + \sum_{k=1}^{N_T} \left(f\left(X_{T_k^-} + \eta_{T_k} Z_k\right) - f\left(X_{T_k^-}\right) \right) \\ &= f(X_0) + \int_0^t u_s f'(X_s) dB_s + \frac{1}{2} \int_0^t f''(X_s) |u_s|^2 ds + \int_0^t v_s f'(X_s) ds \\ &\quad + \int_0^t (f(X_{s^-} + \eta_s Z_{N_s}) - f(X_{s^-})) dN_s, \quad t \in \mathbb{R}_+, \end{aligned}$$

which yields (19.19). □

The integral Itô formula (19.19) can be rewritten in differential notation as

$$df(X_t) = v_t f'(X_t) dt + u_t f'(X_t) dB_t + \frac{|u_t|^2}{2} f''(X_t) dt + (f(X_t) - f(X_{t^-})) dN_t, \tag{19.20}$$

$t \in \mathbb{R}_+$.

Jump processes with infinite activity

Given an Itô process of the form

$$X_t := X_0 + \int_0^t u_s dB_s + \int_0^t \eta_s dY_t, \qquad t \in \mathbb{R}_+,$$

the Itô formula with jumps (19.19) can be rewritten as

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t v_s f'(X_{s^-}) ds + \int_0^t u_s f'(X_{s^-}) dB_s \\ &\quad + \frac{1}{2} \int_0^t f''(X_s) |u_s|^2 ds + \int_0^t \eta_s f'(X_{s^-}) dY_s \\ &\quad + \int_0^t \left(f(X_s) - f(X_{s^-}) - \Delta X_s f'(X_{s^-}) \right) dN_s, \qquad t \in \mathbb{R}_+, \end{aligned}$$

This version: August 17, 2019
http://www.ntu.edu.sg/home/nprivault/indext.html

where we used the relation $dX_s = \eta_s \Delta Y_s$, which implies

$$\int_0^t \eta_s f'(X_{s^-}) dY_s = \int_0^t \Delta X_s f'(X_{s^-}) dN_s, \qquad t \geqslant 0.$$

The above Poisson stochastic integral can be written as

$$\begin{aligned}
&\int_0^t \big(f(X_s) - f(X_{s^-}) - \Delta X_s f'(X_{s^-})\big) dN_s \\
&= \int_0^t \big(f(X_{s^-} + \eta_s \Delta Y_s) - f(X_{s^-}) - \eta_s \Delta Y_s f'(X_{s^-})\big) dN_s \\
&= \int_0^t \big(f(X_{s^-} + \eta_s Z_{N_s}) - f(X_{s^-}) - \eta_s Z_{N_s} f'(X_{s^-})\big) dN_s \\
&= \int_0^t \big(f(X_{s^-} + \eta_s Z_{1+N_{s^-}}) - f(X_{s^-}) - \eta_s Z_{N_s} f'(X_{s^-})\big) dN_s,
\end{aligned}$$

and when $\eta(s)$ is a deterministic function of time, it can be compensated into the martingale

$$\begin{aligned}
&\int_0^t \big(f(X_s) - f(X_{s^-}) - \Delta X_s f'(X_{s^-})\big)\, dN_s \\
&\qquad - \int_0^t \mathbb{E}\big[f(x + \eta(s)Z) - f(x) - \eta(s) Z f'(x)\big]_{x = X_{s^-}} ds \\
&\quad = \int_0^t \big(f(X_s) - f(X_{s^-}) - \eta(s) Z f'(X_{s^-})\big)\, dN_s \\
&\qquad - \lambda \int_0^t \int_{-\infty}^{\infty} \big(f(X_{s^-} + \eta(s) y) - f(X_{s^-}) - \eta(s) y f'(X_{s^-})\big)\, \nu(dy) ds.
\end{aligned}$$

This above formulation is at the basis of the extension of Itô's formula to Lévy processes with an infinite number of jumps on any interval, using the bound

$$|f(x+y) - f(x) - y f'(x)| \leqslant C y^2, \qquad y \in [-1, 1],$$

that follows from Taylor's theorem for f a $\mathcal{C}^2(\mathbb{R})$ function, cf. *e.g.* Theorem 4.4.7 in Applebaum (2004) in the setting of Poisson random measures. Such processes, also called "infinite activity Lévy processes" are also useful in financial modeling, Cont and Tankov (2004) and include the gamma process, stable processes, variance gamma processes, inverse Gaussian processes, etc, as in the following illustrations.

1. Gamma process.

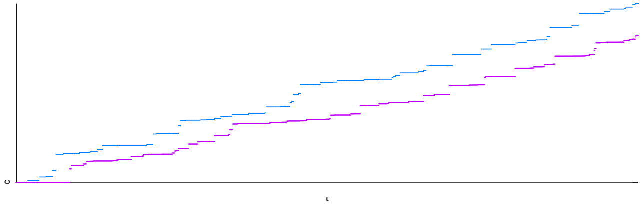

Fig. 19.4: Sample trajectories of a gamma process.

The next R code can be used to generate the gamma process paths of Figure 19.4.

```
N=2000; t <- 0:N; dt <- 1.0/N; nsim <- 6; alpha=20.0
X = matrix(0, nsim, N)
for (i in 1:nsim){X[i,1]=0;X[i,]=rgamma(N,alpha*dt);}
X <- cbind(rep(0, nsim), t(apply(X, 1, cumsum)))
plot(t, X[1, ], xlab = "time", type = "l", ylim = c(0, 2*N*alpha*dt), col = 0)
for (i in 1:nsim){points(t, X[i, ], xlab = "time", type = "p", pch=20, cex =0.02, col = i)}
```

2. Variance gamma process.

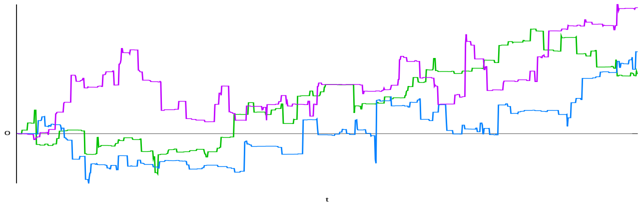

Fig. 19.5: Sample trajectories of a variance gamma process.

3. Inverse Gaussian process.

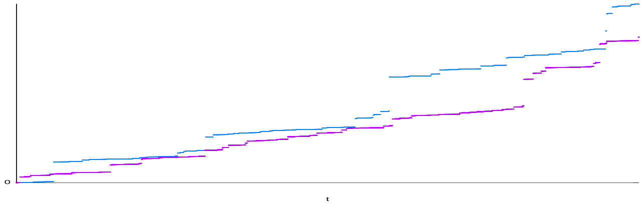

Fig. 19.6: Sample trajectories of an inverse Gaussian process.

4. Negative Inverse Gaussian process.

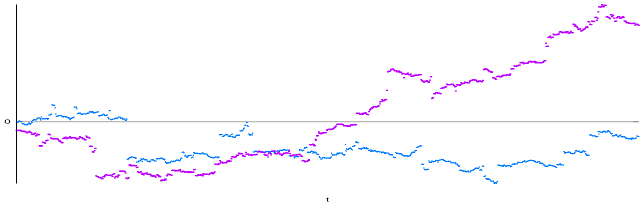

Fig. 19.7: Sample trajectories of a negative inverse Gaussian process.

5. Stable process.

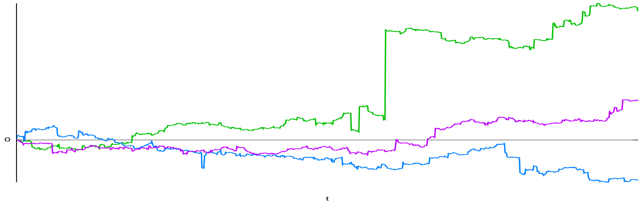

Fig. 19.8: Sample trajectories of a stable process.

The above sample paths of a stable process can be compared to the USD/CNY exchange rate over the year 2015, according to the date retrieved from the following code.

```
library(quantmod)
getSymbols("CNY=X",from="2015-01-01",to="2015-12-6",src="yahoo")
rate=Ad(`CNY=X`)
chartSeries(rate,up.col="blue",theme="white")
```

The adjusted close price Ad() is the closing price after adjustments for applicable splits and dividend distributions.

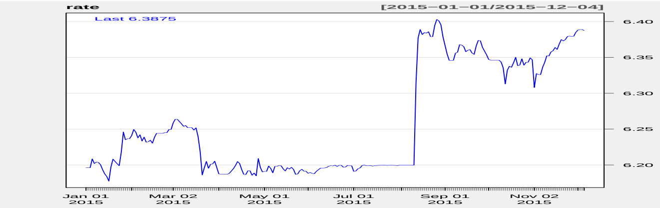

Fig. 19.9: USD/CNY Exchange rate data.

Itô multiplication table with jumps

For a stochastic process $(X_t)_{t\in\mathbb{R}_+}$ is given by

$$X_t = \int_0^t u_s dB_s + \int_0^t v_s ds + \int_0^t \eta_s dN_s, \qquad t \in \mathbb{R}_+,$$

the Itô formula with jumps reads

$$\begin{aligned}
f(X_t) &= f(0) + \int_0^t u_s f'(X_s) dB_s + \frac{1}{2}\int_0^t |u_s|^2 f''(X_s) dB_s \\
&\quad + \int_0^t v_s f'(X_s) ds + \int_0^t (f(X_{s^-} + \eta_s) - f(X_{s^-})) dN_s \\
&= f(0) + \int_0^t u_s f'(X_s) dB_s + \frac{1}{2}\int_0^t |u_s|^2 f''(X_s) dB_s \\
&\quad + \int_0^t v_s f'(X_s) ds + \int_0^t (f(X_s) - f(X_{s^-})) dN_s.
\end{aligned}$$

Given two Itô processes $(X_t)_{t\in\mathbb{R}_+}$ and $(Y_t)_{t\in\mathbb{R}_+}$ written in differential notation as

$$dX_t = u_t dB_t + v_t dt + \eta_t dN_t, \qquad t \in \mathbb{R}_+,$$

and

$$dY_t = a_t dB_t + b_t dt + c_t dN_t, \qquad t \in \mathbb{R}_+,$$

the Itô formula for jump processes can also be written as

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t \cdot dY_t$$

where the product $dX_t \cdot dY_t$ is computed according to the following extension of the Itô multiplication Table 4.1. The relation $dB_t \cdot dN_t = 0$ is due to the fact that $(N_t)_{t\in\mathbb{R}_+}$ has finite variation on any finite interval.

$\cdot$	dt	dB_t	dN_t
dt	0	0	0
dB_t	0	dt	0
dN_t	0	0	dN_t

Table 19.1: Itô multiplication table with jumps.

In other words, we have

$$\begin{aligned} dX_t \cdot dY_t &= (v_t dt + u_t dB_t + \eta_t dN_t)(b_t dt + a_t dB_t + c_t dN_t) \\ &= b_t v_t (dt)^2 + b_t u_t dt \cdot dB_t + b_t \eta_t dt \cdot dN_t + c_t v_t dt \cdot dN_t \\ &\quad + a_t v_t dt dB_t + a_t u_t (dB_t)^2 + a_t \eta_t dB_t \cdot dN_t \\ &\quad + c_t u_t dN_t \cdot dB_t + c_t u_t (dB_t)^2 + c_t \eta_t dN_t \cdot dN_t \\ &= a_t u_t dt + c_t \eta_t dN_t, \end{aligned}$$

since

$$dN_t \cdot dN_t = (dN_t)^2 = dN_t,$$

as $\Delta N_t \in \{0,1\}$. In particular, we have

$$(dX_t)^2 = (v_t dt + u_t dB_t + \eta_t dN_t)^2 = u_t^2 dt + \eta_t^2 dN_t.$$

19.5 Stochastic Differential Equations with Jumps

In the continuous asset price model, the returns of the riskless asset $(A_t)_{t\in\mathbb{R}_+}$ and risky asset process $(S_t)_{t\in\mathbb{R}_+}$ are modeled as

$$\frac{dA_t}{A_t} = r dt \quad \text{and} \quad \frac{dS_t}{S_t} = \mu dt + \sigma dB_t.$$

In this section we are interested in using jump processes in order to model an asset price process $(S_t)_{t\in\mathbb{R}_+}$.

In the case of discontinuous asset prices, let us start with the simplest example of a constant market return η written as

$$\eta := \frac{S_t - S_{t^-}}{S_{t^-}}, \tag{19.21}$$

assuming the presence of a jump at time t, *i.e.*, $dN_t = 1$. Using the relation $dS_t := S_t - S_{t^-}$, (19.21) rewrites as

$$\eta dN_t = \frac{S_t - S_{t^-}}{S_{t^-}} = \frac{dS_t}{S_{t^-}}, \tag{19.22}$$

or

$$dS_t = \eta S_{t^-} dN_t, \tag{19.23}$$

which is a stochastic differential equation with respect to the standard Poisson process, with constant volatility $\eta \in \mathbb{R}$. Note that the left limit S_{t^-} in (19.23) occurs naturally from the definition (19.22) of market returns when dividing by the previous index value S_{t^-}. The use of the left limit S_{t^-} turns out to be necessary when computing pathwise solutions by solving for S_t from S_{t^-}.

i) Constant market returns. In the presence of a jump at time t, the equation (19.21) also reads

$$S_t = (1+\eta)S_{t^-}, \qquad dN_t = 1,$$

which can be applied by induction at the successive jump times $T_1, T_2, \ldots, T_{N_t}$ until time t, to derive the solution

$$S_t = S_0(1+\eta)^{N_t}, \qquad t \in \mathbb{R}_+,$$

of (19.23).

ii) Time-dependent market returns. Next, consider the case where η_t is time-dependent, *i.e.*,

$$dS_t = \eta_t S_{t^-} dN_t. \tag{19.24}$$

At each jump time T_k, Relation (19.24) reads

$$dS_{T_k} = S_{T_k} - S_{T_k^-} = \eta_{T_k} S_{T_k^-},$$

i.e.,

$$S_{T_k} = (1+\eta_{T_k})S_{T_k^-},$$

and repeating this argument for all $k = 1, 2, \ldots, N_t$ yields the product solution

$$\begin{aligned} S_t &= S_0 \prod_{k=1}^{N_t} (1+\eta_{T_k}) \\ &= S_0 \prod_{\substack{\Delta N_s = 1 \\ 0 \leqslant s \leqslant t}} (1+\eta_s) \\ &= S_0 \prod_{0 \leqslant s \leqslant t} (1+\eta_s \Delta N_s), \quad t \in \mathbb{R}_+. \end{aligned}$$

By a similar argument, we obtain the following proposition.

Proposition 19.14. *The stochastic differential equation with jumps*

$$dS_t = \mu_t S_t dt + \eta_t S_{t^-}(dN_t - \lambda dt), \qquad (19.25)$$

admits the solution

$$\boxed{S_t = S_0 \exp\left(\int_0^t \mu_s ds - \lambda \int_0^t \eta_s ds\right) \prod_{k=1}^{N_t}(1+\eta_{T_k}), \qquad t \in \mathbb{R}_+.}$$

Note that the equations

$$dS_t = \mu_t S_t dt + \eta_t S_{t^-}(dN_t - \lambda dt)$$

and

$$dS_t = \mu_t S_t dt + \eta_t S_{t^-}(dN_t - \lambda dt)$$

are equivalent because $S_{t^-}dt = S_t dt$ as the set $(T_k)_{k \geqslant 1}$ has zero measure of length.

A random simulation of the numerical solution of the above equation (19.25) is given in Figure 19.10 for $\eta = 1.29$ and constant $\mu = \mu_t$, $t \in \mathbb{R}_+$.

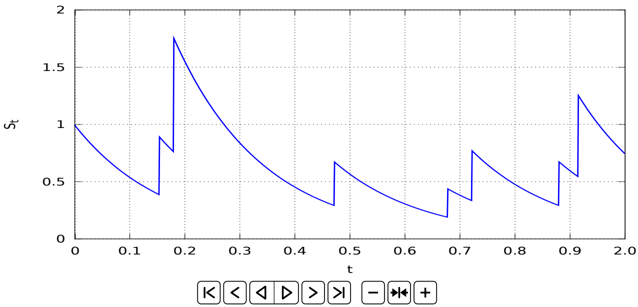

Fig. 19.10: Geometric Poisson process.*

The above simulation can be compared to the real sales ranking data of Figure 19.11.

* The animation works in Acrobat Reader on the entire pdf file.

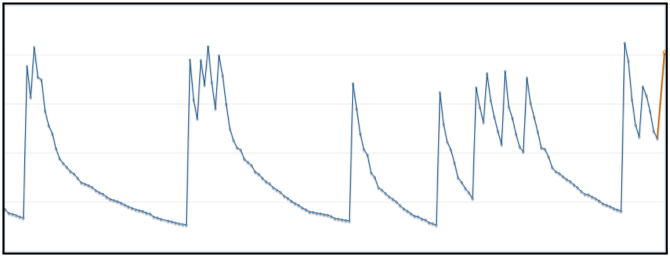

Fig. 19.11: Ranking data.

Next, consider the equation

$$dS_t = \mu_t S_t dt + \eta_t S_{t^-} (dY_t - \lambda \mathbb{E}[Z] dt)$$

driven by the compensated compound Poisson process $(Y_t - \lambda \mathbb{E}[Z] t)_{t \in \mathbb{R}_+}$, also written as

$$dS_t = \mu_t S_t dt + \eta_t S_{t^-} (Z_{N_t} dN_t - \lambda \mathbb{E}[Z] dt),$$

with solution

$$S_t = S_0 \exp\left(\int_0^t \mu_s ds - \lambda \mathbb{E}[Z] \int_0^t \eta_s ds \right) \prod_{k=1}^{N_t} (1 + \eta_{T_k} Z_k) \qquad t \in \mathbb{R}_+. \tag{19.26}$$

A random simulation of the geometric compound Poisson process (19.26) is given in Figure 19.12.

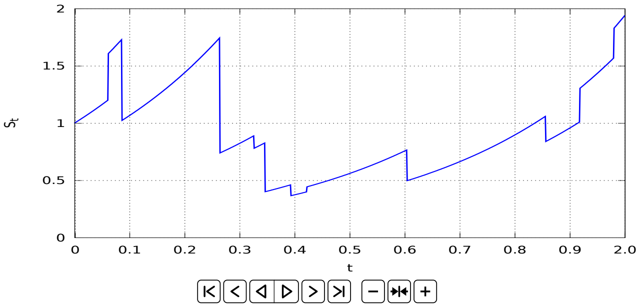

Fig. 19.12: Geometric compound Poisson process.*

In the case of a jump-diffusion stochastic differential equation of the form

$$dS_t = \mu_t S_t dt + \eta_t S_{t^-}(dY_t - \lambda \mathbb{E}[Z]dt) + \sigma_t S_t dB_t,$$

we get

$$S_t = S_0 \exp\left(\int_0^t \mu_s ds - \lambda \mathbb{E}[Z] \int_0^t \eta_s ds + \int_0^t \sigma_s dB_s - \frac{1}{2}\int_0^t |\sigma_s|^2 ds\right) \times \prod_{k=1}^{N_t}(1+\eta_{T_k} Z_k), \qquad t \in \mathbb{R}_+.$$

A random simulation of the geometric Brownian motion with compound Poisson jumps is given in Figure 19.13.

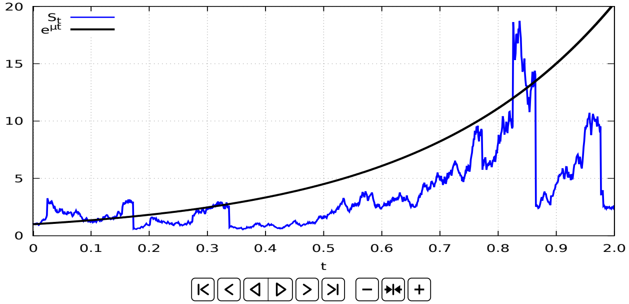

Fig. 19.13: Geometric Brownian motion with compound Poisson jumps.*

By rewriting S_t as

$$S_t = S_0 \exp\left(\int_0^t \mu_s ds + \int_0^t \eta_s(dY_s - \lambda \mathbb{E}[Z]ds) + \int_0^t \sigma_s dB_s - \frac{1}{2}\int_0^t |\sigma_s|^2 ds\right) \times \prod_{k=1}^{N_t}\left((1+\eta_{T_k} Z_k)\,\mathrm{e}^{-\eta_{T_k} Z_k}\right),$$

$t \in \mathbb{R}_+$, one can extend this jump model to processes with an infinite number of jumps on any finite time interval, cf. Cont and Tankov (2004). The next Figure 19.14 shows a number of downward and upward jumps occuring in the SMRT historical share price data, with a typical geometric Brownian behavior in between jumps.

* The animation works in Acrobat Reader on the entire pdf file.

* The animation works in Acrobat Reader on the entire pdf file.

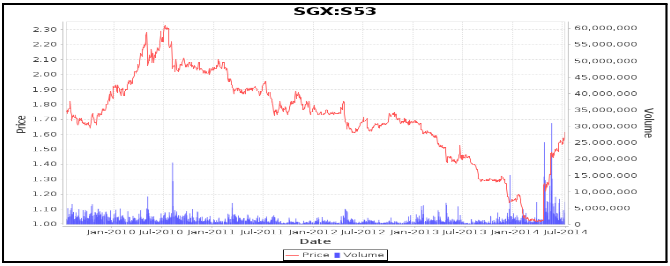

Fig. 19.14: SMRT Share price.

19.6 Girsanov Theorem for Jump Processes

Recall that in its simplest form, cf. Section 7.2, the Girsanov Theorem for Brownian motion states the following.

Under the probability measure $\widetilde{\mathbb{P}}_{-\mu}$ defined by

$$d\widetilde{\mathbb{P}}_{-\mu} := \mathrm{e}^{-\mu B_T - \mu^2 T/2} d\mathbb{P},$$

the random variable $B_T + \mu T$ has the centered Gaussian distribution $\mathcal{N}(0, T)$.

This fact follows from the calculation

$$\begin{aligned}\widetilde{\mathbb{E}}_{-\mu}[f(B_T + \mu T)] &= \mathbb{E}[f(B_T + \mu T)\,\mathrm{e}^{-\mu B_T - \mu^2 T/2}] \\ &= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} f(x + \mu T)\,\mathrm{e}^{-\mu x - \mu^2 T/2}\,\mathrm{e}^{-x^2/(2T)} dx \\ &= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} f(x + \mu T)\,\mathrm{e}^{-(x+\mu T)^2/(2T)} dx \\ &= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} f(y)\,\mathrm{e}^{-y^2/(2T)} dy \\ &= \mathbb{E}[f(B_T)], \end{aligned} \tag{19.27}$$

for any bounded measurable function f on $\mathbb{R}$, which shows that $B_T + \mu T$ is a centered Gaussian random variable under $\widetilde{\mathbb{P}}_{-\mu}$.

More generally, the Girsanov Theorem states that $(B_t + \mu t)_{t\in[0,T]}$ is a standard Brownian motion under $\widetilde{\mathbb{P}}_{-\mu}$.

When Brownian motion is replaced with a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$, a spatial shift of the type

$$B_t \longmapsto B_t + \mu t$$

can no longer be used because $N_t + \mu t$ cannot be a Poisson process, whatever the change of probability applied, since by construction, the paths of the standard Poisson process has jumps of unit size and remain constant between jump times.

The correct way to extend the Girsanov Theorem to the Poisson case is to replace the space shift with a shift in the intensity of the Poisson process as in the following statement. Assume that the random variable N_T has the Poisson distribution $\mathcal{P}(\lambda T)$ with parameter λT under $\mathbb{P}_\lambda$.

Under the probability measure $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$ defined by

$$d\widetilde{\mathbb{P}}_{\tilde{\lambda}} := \mathrm{e}^{-(\tilde{\lambda}-\lambda)T}\left(\frac{\tilde{\lambda}}{\lambda}\right)^{N_T} d\mathbb{P}_\lambda,$$

the random variable N_T has a Poisson distribution with intensity $\tilde{\lambda}T$.

This fact follows from the relation

$$\widetilde{\mathbb{P}}_{\tilde{\lambda}}(N_T = k) = \mathrm{e}^{-(\tilde{\lambda}-\lambda)T}\left(\frac{\tilde{\lambda}}{\lambda}\right)^{k}\mathbb{P}_\lambda(N_T = k) = \mathrm{e}^{-\tilde{\lambda}T}\frac{\tilde{\lambda}^k}{k!}, \quad k \in \mathbb{N}.$$

Assume now that $(N_t)_{t\in\mathbb{R}_+}$ is a standard Poisson process with intensity λ under a probability measure $\mathbb{P}_\lambda$. In order to extend (19.27) to the Poisson case we can replace the space shift with a *time contraction* (or dilation)

$$N_t \longmapsto N_{t/(1+c)} \quad \text{or} \quad N_t \longmapsto N_{(1+c)t},$$

by a factor $1+c$, where

$$c := -1 + \frac{\tilde{\lambda}}{\lambda} > -1,$$

or $\tilde{\lambda} = (1+c)\lambda$. We note that

$$\begin{aligned}
\mathbb{P}_\lambda\big(N_{(1+c)T} = k\big) &= \frac{(\lambda(1+c)T)^k}{k!}\,\mathrm{e}^{-\lambda(1+c)T} \\
&= (1+c)^k\,\mathrm{e}^{-\lambda cT}\mathbb{P}_\lambda(N_T = k) \\
&= \widetilde{\mathbb{P}}_{\tilde{\lambda}}(N_T = k), \quad k \in \mathbb{N},
\end{aligned}$$

and by analogy with (19.27) we have

$$\begin{aligned}\mathbb{E}_\lambda\left[f(N_{(1+c)T})\right] &= \sum_{k\geqslant 0} f(k)\mathbb{P}_\lambda\big(N_{(1+c)T}=k\big) \\ &= \mathrm{e}^{-\lambda cT}\sum_{k\geqslant 0} f(k)(1+c)^k\mathbb{P}_\lambda(N_T=k) \\ &= \mathrm{e}^{-\lambda cT}\,\mathbb{E}\left[f(N_T)(1+c)^{N_T}\right] \\ &= \mathrm{e}^{-\lambda cT}\int_\Omega (1+c)^{N_T} f(N_T)d\mathbb{P}_\lambda \\ &= \int_\Omega f(N_T)d\widetilde{\mathbb{P}}_{\tilde\lambda} \\ &= \widetilde{\mathbb{E}}_{\tilde\lambda}[f(N_T)],\end{aligned} \tag{19.28}$$

for any bounded function f on $\mathbb{N}$. In other words, taking $f(x) := \mathbb{1}_{\{x\leqslant n\}}$ we have

$$\mathbb{P}_\lambda\big(N_{(1+c)T}\leqslant n\big) = \widetilde{\mathbb{P}}_{\tilde\lambda}(N_T\leqslant n), \qquad n\in\mathbb{N},$$

or

$$\widetilde{\mathbb{P}}_{\tilde\lambda}(N_{T/(1+c)}\leqslant n) = \mathbb{P}_\lambda\big(N_T\leqslant n\big), \qquad n\in\mathbb{N}.$$

As a consequence, we have the following proposition.

Proposition 19.15. *Let $\lambda, \tilde\lambda > 0$, and set*

$$c := -1 + \frac{\tilde\lambda}{\lambda} > -1.$$

The process $(N_{t/(1+c)})_{t\in\mathbb{R}_+}$ is a standard Poisson process with intensity λ under $\widetilde{\mathbb{P}}_{\tilde\lambda}$. In particular, the compensated Poisson process

$$N_{t/(1+c)} - \lambda t, \qquad t\in\mathbb{R}_+,$$

is a martingale under $\widetilde{\mathbb{P}}_{\tilde\lambda}$.

Proof. As in (19.28) we have

$$\mathbb{E}[f(N_T)] = \widetilde{\mathbb{E}}_{\tilde\lambda}\big[f(N_{T/(1+c)})\big],$$

i.e., under $\widetilde{\mathbb{P}}_{\tilde\lambda}$ the distribution of $N_{T/(1+c)}$ is that of a standard Poisson random variable with parameter λT. As a consequence, $\big(N_{t/(1+c)}\big)_{t\in\mathbb{R}_+}$ is a standard Poisson process with intensity λ under $\widetilde{\mathbb{P}}_{\tilde\lambda}$, and since $\big(N_{t/(1+c)}\big)_{t\in\mathbb{R}_+}$ has independent increments, the compensated process $(N_{t/(1+c)} - \lambda t)_{t\in\mathbb{R}_+}$ is a martingale under $\widetilde{\mathbb{P}}_{\tilde\lambda}$ by (7.2). $\square$

Similarly, since

$$(N_t - (1+c)\lambda t)_{t\in\mathbb{R}_+} = (N_t - \tilde\lambda t)_{t\in\mathbb{R}_+}$$

has independent increments, the compensated Poisson process

$$N_t - (1+c)\lambda t = N_t - \tilde{\lambda} t$$

is a martingale under $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$. We also have

$$N_{t/(1+c)} = \sum_{n\geqslant 1} \mathbb{1}_{[T_n,\infty)}(t/(1+c)) = \sum_{n\geqslant 1} \mathbb{1}_{[(1+c)T_n,\infty)}(t), \quad t \in \mathbb{R}_+,$$

which shows that the jump times $((1+c)T_n)_{n\geqslant 1}$ of $(N_{t/(1+c)})_{t\in[0,T]}$ are distributed under $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$ as the jump times of a Poisson process with intensity λ.

When $\mu \neq r$, the discounted price process $(\widetilde{S}_t)_{t\in\mathbb{R}_+} = (\mathrm{e}^{-rt}S_t)_{t\in\mathbb{R}_+}$ written as

$$\frac{d\widetilde{S}_t}{\widetilde{S}_{t^-}} = (\mu - r)dt + \sigma(dN_t - \lambda dt) \tag{19.29}$$

is not a martingale under $\mathbb{P}_\lambda$. However, we can rewrite (19.29) as

$$\frac{d\widetilde{S}_t}{\widetilde{S}_{t^-}} = \sigma\left(dN_t - \left(\lambda - \frac{\mu - r}{\sigma}\right)dt\right)$$

and letting

$$\tilde{\lambda} := \lambda - \frac{\mu - r}{\sigma} = (1+c)\lambda$$

with

$$c := -\frac{\mu - r}{\sigma\lambda},$$

we have

$$\frac{d\widetilde{S}_t}{\widetilde{S}_{t^-}} = \sigma\left(dN_t - \tilde{\lambda}dt\right)$$

hence the discounted price process $(\widetilde{S}_t)_{t\in\mathbb{R}_+}$ is martingale under the probability measure $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$ defined as

$$d\widetilde{\mathbb{P}}_{\tilde{\lambda}} := \mathrm{e}^{-\lambda cT}(1+c)^{N_T}d\mathbb{P}_\lambda = \mathrm{e}^{(\mu-r)/\sigma}\left(1 - \frac{\mu - r}{\sigma\lambda}\right)^{N_T} d\mathbb{P}_\lambda.$$

We note that if

$$\mu - r < \sigma\lambda$$

then the risk-neutral probability measure $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$ exists and is unique, therefore by Theorems 5.7 and 5.11 the market is without arbitrage and complete.

Girsanov Theorem for compound Poisson processes

In the case of compound Poisson processes, the Girsanov Theorem can be extended to variations in jump sizes in addition to time variations, and we have the following more general result.

Theorem 19.16. *Let $(Y_t)_{t\geqslant 0}$ be a compound Poisson process with intensity $\lambda > 0$ and jump size distribution $\nu(dx)$. Consider another intensity parameter $\tilde{\lambda} > 0$ and jump size distribution $\tilde{\nu}(dx)$, and let*

$$\psi(x) := \frac{\tilde{\lambda}}{\lambda}\frac{\tilde{\nu}(dx)}{\nu(dx)} - 1, \qquad x \in \mathbb{R}. \tag{19.30}$$

Then,

under the probability measure

$$d\widetilde{\mathbb{P}}_{\tilde{\lambda},\tilde{\nu}} := \mathrm{e}^{-(\tilde{\lambda}-\lambda)T} \prod_{k=1}^{N_T}(1+\psi(Z_k))d\widetilde{\mathbb{P}}_{\lambda,\nu},$$

the process

$$Y_t := \sum_{k=1}^{N_t} Z_k, \qquad t \in \mathbb{R}_+,$$

is a compound Poisson process with

- *modified intensity $\tilde{\lambda} > 0$, and*
- *modified jump size distribution $\tilde{\nu}(dx)$.*

Proof. For any bounded measurable function f on $\mathbb{R}$, we extend (19.28) to the following change of variable

$$\begin{aligned}
\mathbb{E}_{\tilde{\lambda},\tilde{\nu}}[f(Y_T)] &= \mathrm{e}^{-(\tilde{\lambda}-\lambda)T}\,\mathbb{E}_{\lambda,\nu}\left[f(Y_T)\prod_{i=1}^{N_T}(1+\psi(Z_i))\right] \\
&= \mathrm{e}^{-(\tilde{\lambda}-\lambda)T}\sum_{k\geqslant 0}\mathbb{E}_{\lambda,\nu}\left[f\left(\sum_{i=1}^{k} Z_i\right)\prod_{i=1}^{k}(1+\psi(Z_i))\;\middle|\; N_T = k\right]\mathbb{P}_\lambda(N_T = k) \\
&= \mathrm{e}^{-\tilde{\lambda}T}\sum_{k\geqslant 0}\frac{(\lambda T)^k}{k!}\,\mathbb{E}_{\lambda,\nu}\left[f\left(\sum_{i=1}^{k} Z_i\right)\prod_{i=1}^{k}(1+\psi(Z_i))\right] \\
&= \mathrm{e}^{-\tilde{\lambda}T}\sum_{k\geqslant 0}\frac{(\lambda T)^k}{k!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(z_1+\cdots+z_k)\prod_{i=1}^{k}(1+\psi(z_i))\nu(dz_1)\cdots\nu(dz_k)
\end{aligned}$$

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http://www.ntu.edu.sg/home/nprivault/indext.html

$$= \mathrm{e}^{-\tilde{\lambda}T} \sum_{k \geqslant 0} \frac{(\tilde{\lambda}T)^k}{k!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(z_1 + \cdots + z_k) \left(\prod_{i=1}^{k} \frac{\tilde{\nu}(dz_i)}{\nu(dz_i)} \right) \nu(dz_1) \cdots \nu(dz_k)$$
$$= \mathrm{e}^{-\tilde{\lambda}T} \sum_{k \geqslant 0} \frac{(\tilde{\lambda}T)^k}{k!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(z_1 + \cdots + z_k) \tilde{\nu}(dz_1) \cdots \tilde{\nu}(dz_k).$$

This shows that under $\mathbb{P}_{\tilde{\lambda},\tilde{\nu}}$, Y_T has the distribution of a compound Poisson process with intensity $\tilde{\lambda}$ and jump size distribution $\tilde{\nu}$. We refer to Proposition 9.6 of Cont and Tankov (2004) for the independence of increments of $(Y_t)_{t \in \mathbb{R}_+}$ under $\widetilde{\mathbb{P}}_{\tilde{\lambda},\tilde{\nu}}$. $\square$

Example. In case $\nu \simeq \mathcal{N}(\alpha, \sigma^2)$ and $\tilde{\nu} \simeq \mathcal{N}(\beta, \eta^2)$ we have

$$\nu(dx) = \frac{dx}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\alpha)^2 \right), \quad \tilde{\nu}(dx) = \frac{dx}{\sqrt{2\pi\eta^2}} \exp\left(-\frac{1}{2\eta^2}(x-\beta)^2 \right),$$

$x \in \mathbb{R}$, hence

$$\frac{\tilde{\nu}(dx)}{\nu(dx)} = \frac{\eta}{\sigma} \exp\left(\frac{1}{2\eta^2}(x-\beta)^2 - \frac{1}{2\sigma^2}(x-\alpha)^2 \right), \qquad x \in \mathbb{R},$$

and $\psi(x)$ in (19.30) is given by

$$1 + \psi(x) = \frac{\tilde{\lambda}}{\lambda} \frac{\tilde{\nu}(dx)}{\nu(dx)} = \frac{\tilde{\lambda}\eta}{\lambda\sigma} \exp\left(\frac{1}{2\eta^2}(x-\beta)^2 - \frac{1}{2\sigma^2}(x-\alpha)^2 \right), \quad x \in \mathbb{R}.$$

Note that the compound Poisson process with intensity $\tilde{\lambda} > 0$ and jump size distribution $\tilde{\nu}$ can be built as

$$X_t := \sum_{k=1}^{N_{\tilde{\lambda}t/\lambda}} h(Z_k),$$

provided that $\tilde{\nu}$ is the image measure of ν by the function $h : \mathbb{R} \to \mathbb{R}$, *i.e.*,

$$\mathbb{P}(h(Z_k) \in A) = \mathbb{P}(Z_k \in h^{-1}(A)) = \nu(h^{-1}(A)) = \tilde{\nu}(A),$$

for all (measurable) subsets A of $\mathbb{R}$. As a consequence of Theorem 19.16 we have the following proposition.

Proposition 19.17. *The compensated process*

$$Y_t - \tilde{\lambda}t\, \mathbb{E}_{\tilde{\nu}}[Z]$$

is a martingale under the probability measure $\widetilde{\mathbb{P}}_{\tilde{\lambda},\tilde{\nu}}$ *defined by*

$$d\widetilde{\mathbb{P}}_{\tilde{\lambda},\tilde{\nu}} = \mathrm{e}^{-(\tilde{\lambda}-\lambda)T} \prod_{k=1}^{N_T} (1+\psi(Z_k)) d\widetilde{\mathbb{P}}_{\lambda,\nu}.$$

Finally, the Girsanov Theorem can be extended to the linear combination of a standard Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ and a compound Poisson process $(Y_t)_{t\in\mathbb{R}_+}$ independent of $(B_t)_{t\in\mathbb{R}_+}$, as in the following result which is a particular case of Theorem 33.2 of Sato (1999).

Theorem 19.18. *Let $(Y_t)_{t\geqslant 0}$ be a compound Poisson process with intensity $\lambda > 0$ and jump size distribution $\nu(dx)$. Consider another jump size distribution $\tilde{\nu}(dx)$ and intensity parameter $\tilde{\lambda} > 0$, and let*

$$\psi(x) := \frac{\tilde{\lambda}}{\lambda} \frac{d\tilde{\nu}}{d\nu}(x) - 1, \qquad x \in \mathbb{R},$$

and let $(u_t)_{t\in\mathbb{R}_+}$ be a bounded adapted process. Then the process

$$\left(B_t + \int_0^t u_s ds + Y_t - \tilde{\lambda}\, \mathbb{E}_{\tilde{\nu}}[Z] t \right)_{t\in\mathbb{R}_+}$$

is a martingale under the probability measure

$$d\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}} = \exp\left(-(\tilde{\lambda}-\lambda)T - \int_0^T u_s dB_s - \frac{1}{2}\int_0^T |u_s|^2 ds \right) \prod_{k=1}^{N_T} (1+\psi(Z_k)) d\widetilde{\mathbb{P}}_{\lambda,\nu}. \tag{19.31}$$

As a consequence of Theorem 19.18, if

$$B_t + \int_0^t v_s ds + Y_t \tag{19.32}$$

is not a martingale under $\widetilde{\mathbb{P}}_{\lambda,\nu}$, it will become a martingale under $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$ provided that $u, \tilde{\lambda}$ and $\tilde{\nu}$ are chosen in such a way that

$$v_s = u_s - \tilde{\lambda}\, \mathbb{E}_{\tilde{\nu}}[Z], \qquad s \in \mathbb{R}, \tag{19.33}$$

in which case (19.32) can be rewritten into the martingale decomposition

$$dB_t + u_t dt + dY_t - \tilde{\lambda}\, \mathbb{E}_{\tilde{\nu}}[Z] dt,$$

in which both $\left(B_t + \int_0^t u_s ds \right)_{t\in\mathbb{R}_+}$ and $\left(Y_t - \tilde{\lambda} t\, \mathbb{E}_{\tilde{\nu}}[Z] \right)_{t\in\mathbb{R}_+}$ are martingales under $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$

When $\tilde{\lambda} = \lambda = 0$, Theorem 19.18 coincides with the usual Girsanov Theorem for Brownian motion, in which case (19.33) admits only one solution given by $u = v$ and there is uniqueness of $\widetilde{\mathbb{P}}_{u,0,0}$. Note that uniqueness occurs also when $u = 0$ in the absence of Brownian motion with Poisson jumps of fixed size a (*i.e.*, $\tilde{\nu}(dx) = \nu(dx) = \delta_a(dx)$) since in this case (19.33) also admits only one solution $\tilde{\lambda} = v$ and there is uniqueness of $\widetilde{\mathbb{P}}_{0,\tilde{\lambda},\delta_a}$. These remarks will be of importance for arbitrage pricing in jump models in Chapter 20.

When $\mu \neq r$, the discounted price process $(\widetilde{S}_t)_{t\in\mathbb{R}_+} = (\mathrm{e}^{-rt}S_t)_{t\in\mathbb{R}_+}$ defined by

$$\frac{d\widetilde{S}_t}{\widetilde{S}_{t^-}} = (\mu - r)dt + \sigma dB_t + \eta(dY_t - \lambda t\,\mathbb{E}_\nu[Z])$$

is not martingale under $\mathbb{P}_{\lambda,\nu}$, however we can rewrite the equation as

$$\frac{d\widetilde{S}_t}{\widetilde{S}_{t^-}} = +\sigma(udt + dB_t) + \eta\left(dY_t - \left(\frac{u\sigma}{\eta} + \lambda\,\mathbb{E}_{\tilde{\nu}}[Z] - \frac{\mu - r}{\eta}\right)dt\right)$$

and choosing u, $\tilde{\nu}$, and $\tilde{\lambda}$ such that

$$\tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[Z] = \frac{u\sigma}{\eta} + \lambda\,\mathbb{E}_{\tilde{\nu}}[Z] - \frac{\mu - r}{\eta}, \tag{19.34}$$

we have

$$\frac{d\widetilde{S}_t}{\widetilde{S}_{t^-}} = \sigma(udt + dB_t) + \eta\left(dY_t - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[Z]dt\right)$$

hence the discounted price process $(\widetilde{S}_t)_{t\in\mathbb{R}_+}$ is martingale under the probability measure $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$, and the market is without arbitrage by Theorem 5.7 and the existence of a risk-neutral probability measure $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$. However, the market is not complete due to the non uniqueness of solutions $(u, \tilde{\nu}, \tilde{\lambda})$ to (19.34), and Theorem 5.11 does not apply in this situation.

Exercises

Exercise 19.1 Consider a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity $\lambda > 0$, started at $N_0 = 0$.

a) Solve the stochastic differential equation

$$dS_t = \eta S_{t^-}dN_t - \eta\lambda S_t dt = \eta S_{t^-}(dN_t - \lambda dt).$$

b) Using the first Poisson jump time T_1, solve the stochastic differential equation

$$dS_t = -\lambda\eta S_t dt + dN_t, \qquad t \in (0, T_2).$$

Exercise 19.2 Consider a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity $\lambda > 0$.

a) Solve the stochastic differential equation $dX_t = \alpha X_t dt + \sigma dN_t$ over the time intervals $[0, T_1)$, $[T_1, T_2)$, $[T_2, T_3)$, $[T_3, T_4)$, where $X_0 = 1$.
b) Write a differential equation for $f(t) := \mathbb{E}[X_t]$, and solve it for $t \in \mathbb{R}_+$.

Exercise 19.3 Consider a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity $\lambda > 0$.

a) Solve the stochastic differential equation $dX_t = \sigma X_{t^-} dN_t$ for $(X_t)_{t\in\mathbb{R}_+}$, where $\sigma > 0$ and $X_0 = 1$.
b) Show that the solution $(S_t)_{t\in\mathbb{R}_+}$ of the stochastic differential equation

$$dS_t = rdt + \sigma S_{t^-} dN_t,$$

is given by $S_t = S_0 X_t + r X_t \int_0^t X_s^{-1} ds$.
c) Compute $\mathbb{E}[X_t]$ and $\mathbb{E}[X_t / X_s]$, $0 \leqslant s \leqslant t$.
d) Compute $\mathbb{E}[S_t]$, $t \in \mathbb{R}_+$.

Exercise 19.4 Let $(N_t)_{t\in\mathbb{R}_+}$ be a standard Poisson process with intensity $\lambda > 0$, started at $N_0 = 0$.

a) Is the process $t \mapsto N_t - 2\lambda t$ a *sub*martingale, a martingale, or a *super*martingale?
b) Let $r > 0$. Solve the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_{t^-}(dN_t - \lambda dt).$$

c) Is the process $t \mapsto S_t$ of Question (b) a *sub*martingale, a martingale, or a *super*martingale?
d) Compute the price at time 0 of the European call option with strike price $K = S_0 \, \mathrm{e}^{(r-\lambda\sigma)T}$, where $\sigma > 0$.

Exercise 19.5 Affine stochastic differential equation with jumps. Consider a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity $\lambda > 0$.

a) Solve the stochastic differential equation $dX_t = a dN_t + \sigma X_{t^-} dN_t$, where $\sigma > 0$, and $a \in \mathbb{R}$.
b) Compute $\mathbb{E}[X_t]$ for $t \in \mathbb{R}_+$.

Exercise 19.6 Consider the compound Poisson process $Y_t := \sum_{k=1}^{N_t} Z_k$, where $(N_t)_{t\in\mathbb{R}_+}$ is a standard Poisson process with intensity $\lambda > 0$, and $(Z_k)_{k\geqslant 1}$ is an *i.i.d.* sequence of $\mathcal{N}(0,1)$ Gaussian random variables. Solve the stochastic differential equation

$$dS_t = rS_t dt + \eta S_{t^-} dY_t,$$

where $\eta, r \in \mathbb{R}$.

Exercise 19.7 Show, by direct computation or using the moment generating function (19.9), that the variance of the compound Poisson process Y_t with intensity $\lambda > 0$ satisfies

$$\mathrm{Var}\,[Y_t] = \lambda t\, \mathbb{E}[|Z|^2] = \lambda t \int_{-\infty}^{\infty} x^2 \nu(dx).$$

Exercise 19.8 Consider an exponential compound Poisson process of the form

$$S_t = S_0\, \mathrm{e}^{\mu t + \sigma B_t + Y_t}, \qquad t \in \mathbb{R}_+,$$

where $(Y_t)_{t\in\mathbb{R}_+}$ is a compound Poisson process of the form (19.7).

a) Derive the stochastic differential equation with jumps satisfied by $(S_t)_{t\in\mathbb{R}_+}$.
b) Let $r > 0$. Find a family $(\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}})$ of probability measures under which the discounted asset price $\mathrm{e}^{-rt} S_t$ is a martingale.

Exercise 19.9 Consider $(N_t)_{t\in\mathbb{R}_+}$ a standard Poisson process with intensity $\lambda > 0$ under a probability measure $\mathbb{P}$. Let $(S_t)_{t\in\mathbb{R}_+}$ be defined by the stochastic differential equation

$$dS_t = \mu S_t dt + Z_{N_t} S_{t^-} dN_t, \tag{19.35}$$

where $(Z_k)_{k\geqslant 1}$ is an *i.i.d.* sequence of random variables of the form

$$Z_k = \mathrm{e}^{X_k} - 1, \quad \text{where} \quad X_k \simeq \mathcal{N}(0, \sigma^2), \quad k \geqslant 1.$$

a) Solve the equation (19.35).
b) We assume that μ and the risk-free interest rate $r > 0$ are chosen such that the discounted process $(\mathrm{e}^{-rt} S_t)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}$. What relation does this impose on μ and r?
c) Under the relation of Question (b), compute the price at time t of a European call option on S_T with strike price κ and maturity T, using a series expansion of Black-Scholes functions.

Exercise 19.10 Consider a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity $\lambda > 0$ under a probability measure $\mathbb{P}$. Let $(S_t)_{t\in\mathbb{R}_+}$ be the mean-reverting process defined by the stochastic differential equation

$$dS_t = -\alpha S_t dt + \sigma(dN_t - \beta dt), \tag{19.36}$$

where $S_0 > 0$ and $\alpha, \beta > 0$.

a) Solve the equation (19.36) for S_t.
b) Compute $f(t) := \mathbb{E}[S_t]$ for all $t \in \mathbb{R}_+$.
c) Under which condition on α, β, σ and λ does the process S_t become a *sub*martingale?
d) Propose a method for the calculation of expectations of the form $\mathbb{E}[\phi(S_T)]$ where ϕ is a payoff function.

Exercise 19.11 Let $(N_t)_{t\in[0,T]}$ be a standard Poisson process started at $N_0 = 0$, with intensity $\lambda > 0$ under the probability measure $\mathbb{P}_\lambda$, and consider the compound Poisson process $(Y_t)_{t\in[0,T]}$ with *i.i.d.* jump sizes $(Z_k)_{k\geqslant 1}$ of distribution $\nu(dx)$.

a) Under the probability measure $\mathbb{P}_\lambda$, the process $t \mapsto Y_t - \lambda t(t + \mathbb{E}[Z])$ is a:

submartingale	*martingale*	*supermartingale*

b) Consider the process $(S_t)_{t\in[0,T]}$ given by

$$dS_t = \mu S_t dt + \sigma S_{t^-} dY_t.$$

Find $\tilde{\lambda}$ such that the discounted process $(\widetilde{S}_t)_{t\in[0,T]} := (e^{-rt} S_t)_{t\in[0,T]}$ is a martingale under the probability measure $\mathbb{P}_{\tilde{\lambda}}$ defined by its density

$$\frac{d\mathbb{P}_{\tilde{\lambda}}}{d\mathbb{P}_\lambda} := \mathrm{e}^{-(\tilde{\lambda}-\lambda)T} \left(\frac{\tilde{\lambda}}{\lambda}\right)^{N_T}.$$

with respect to $\mathbb{P}_\lambda$.

c) Price the forward contract with payoff $S_T - \kappa$.

Exercise 19.12 Consider $(Y_t)_{t\in\mathbb{R}_+}$ a compound Poisson process written as

$$Y_t = \sum_{k=1}^{N_t} Z_k, \qquad t \in \mathbb{R}_+,$$

where $(N_t)_{t\in\mathbb{R}_+}$ a standard Poisson process with intensity $\lambda > 0$ and $(Z_k)_{k\geqslant 1}$ is an *i.i.d* family of random variables with probability distribution $\nu(dx)$ on $\mathbb{R}$, under a probability measure $\mathbb{P}$. Let $(S_t)_{t\in\mathbb{R}_+}$ be defined by the stochastic differential equation

$$dS_t = \mu S_t dt + S_{t^-} dY_t. \qquad (19.37)$$

a) Solve the equation (19.37).

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b) We assume that μ, $\nu(dx)$ and the risk-free interest rate $r > 0$ are chosen such that the discounted process $(\mathrm{e}^{-rt} S_t)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}$. What relation does this impose on μ, $\nu(dx)$ and r?

c) Under the relation of Question (b), compute the price at time t of a European call option on S_T with strike price κ and maturity T, using a series expansion of integrals.

Exercise 19.13 Consider a standard Poisson process $(N_t)_{t\in[0,T]}$ with intensity $\lambda > 0$ and a standard Brownian motion $(B_t)_{t\in[0,T]}$ independent of $(N_t)_{t\in[0,T]}$ under the probability measure $\mathbb{P}_\lambda$. Let also $(Y_t)_{t\in[0,T]}$ be a compound Poisson process with *i.i.d.* jump sizes $(Z_k)_{k\geqslant 1}$ of distribution $\nu(dx)$ under $\mathbb{P}_\lambda$, and consider the jump process $(S_t)_{t\in[0,T]}$ solution of

$$dS_t = rS_t dt + \sigma S_t dB_t + \eta S_{t^-}(dY_t - \tilde{\lambda} t\,\mathbb{E}[Z_1]).$$

with $r, \sigma, \eta, \lambda, \tilde{\lambda} > 0$.

a) Assume that $\tilde{\lambda} = \lambda$. Under the probability measure $\mathbb{P}_\lambda$, the discounted price process $(\mathrm{e}^{-rt} S_t)_{t\in[0,T]}$ is a:

submartingale |	*martingale* |	*supermartingale* |

b) Assume $\tilde{\lambda} > \lambda$. Under the probability measure $\mathbb{P}_\lambda$, the discounted price process $(\mathrm{e}^{-rt} S_t)_{t\in[0,T]}$ is a:

submartingale |	*martingale* |	*supermartingale* |

c) Assume $\tilde{\lambda} < \lambda$. Under the probability measure $\mathbb{P}_\lambda$, the discounted price process $(\mathrm{e}^{-rt} S_t)_{t\in[0,T]}$ is a:

submartingale |	*martingale* |	*supermartingale* |

d) Consider the probability measure $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$ defined by its density

$$\frac{d\widetilde{\mathbb{P}}_{\tilde{\lambda}}}{d\mathbb{P}_\lambda} := \mathrm{e}^{-(\tilde{\lambda}-\lambda)T}\left(\frac{\tilde{\lambda}}{\lambda}\right)^{N_T}.$$

with respect to $\mathbb{P}_\lambda$. Under the probability measure $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$, the discounted price process $(\mathrm{e}^{-rt} S_t)_{t\in[0,T]}$ is a:

submartingale |	*martingale* |	*supermartingale* |