This chapter is concerned with interest rate modeling, in which the mean reversion property plays an important role. We consider the main short rate models (Vasicek, CIR, CEV, affine models) and the computation of fixed income products, such as bond prices, in such models.

16.1 Short-Term Mean-Reverting Models

Money market accounts with price \((A_t)_{t \in \mathbb{R}^+}\) can be defined from a short rate interest rate process \((r_t)_{t \in \mathbb{R}^+}\) as

\[
\frac{A_{t+dt} - A_t}{A_t} = r_t dt, \quad \frac{dA_t}{A_t} = r_t dt, \quad \frac{dA_t}{dt} = r_t A_t, \quad t \in \mathbb{R}^+,.
\]

with

\[
A_t = A_0 \exp \left( \int_0^t r_s ds \right), \quad t \in \mathbb{R}^+.
\]

Vasicek model

The first model to capture the mean reversion property of interest rates, a property not possessed by geometric Brownian motion, is the Vašiček (1977) model, which is based on the Ornstein-Uhlenbeck process. Here, the short-term interest rate process \((r_t)_{t \in \mathbb{R}^+}\) solves the equation

\[dA_t = \sigma A_t (\kappa (A - \theta) dt - \sqrt{A} dW_t), \quad t \in \mathbb{R}^+,
\]

where \(\sigma > 0\), \(\kappa > 0\), and \(\theta > 0\) are constants, and \(W_t\) is a standard Brownian motion.
\[ dr_t = (a - b r_t) dt + \sigma dB_t, \]  

where \( a, \sigma \in \mathbb{R}, b > 0 \), and \((B_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion, with solution

\[
r_t = r_0 e^{-bt} + \frac{a}{b} (1 - e^{-bt}) + \sigma \int_0^t e^{-(t-s) b} dB_s, \quad t \in \mathbb{R}_+, \tag{16.2}
\]

see Exercise 16.1. The probability distribution of \( r_t \) is Gaussian at all times \( t \), with mean

\[
\mathbb{E}[r_t] = r_0 e^{-bt} + \frac{a}{b} (1 - e^{-bt}),
\]

and variance

\[
\text{Var}[r_t] = \text{Var} \left[ \sigma \int_0^t e^{-(t-s) b} dB_s \right] = \sigma^2 \int_0^t (e^{-(t-s) b})^2 ds = \sigma^2 \int_0^t e^{-2bs} ds = \frac{\sigma^2}{2b} (1 - e^{-2bt}), \quad t \in \mathbb{R}_+,
\]

i.e.

\[
r_t \simeq \mathcal{N} \left( r_0 e^{-bt} + \frac{a}{b} (1 - e^{-bt}), \frac{\sigma^2}{2b} (1 - e^{-2bt}) \right), \quad t > 0.
\]

In particular, the probability density function \( f_t(x) \) of \( r_t \) at time \( t > 0 \) is given by

\[
f_t(x) = \frac{\sqrt{b/\pi}}{\sigma \sqrt{1 - e^{-2bt}}} \exp \left( - \frac{(r_0 e^{-bt} + a(1 - e^{-bt})/b - x)^2}{\sigma^2 (1 - e^{-2bt})/b} \right), \quad x \in \mathbb{R}.
\]

When \( b > 0 \) and in the long run, \( ^* \) i.e. when time \( t \) is large we have

\[
\lim_{t \to \infty} \mathbb{E}[r_t] = \frac{a}{b} \quad \text{and} \quad \lim_{t \to \infty} \text{Var}[r_t] = \frac{\sigma^2}{2b}, \tag{16.3}
\]

and this distribution converges to the Gaussian \( \mathcal{N}(a/b, \sigma^2/(2b)) \) distribution, which is also the invariant (or stationary) distribution of \((r_t)_{t \in \mathbb{R}_+}\), and the process tends to revert to its long term mean \( a/b = \lim_{t \to \infty} \mathbb{E}[r_t] \) which makes the average drift vanish, i.e.:

\[
\lim_{t \to \infty} \mathbb{E}[a - br_t] = a - b \lim_{t \to \infty} \mathbb{E}[r_t] = 0.
\]

\(^*\) “But this long run is a misleading guide to current affairs. In the long run we are all dead.” J. M. Keynes, A Tract on Monetary Reform (1923), Ch. 3, p. 80.
Figure 16.1 presents a random simulation of $t \mapsto r_t$ in the Vasicek model with $r_0 = 3\%$, and shows the mean-reverting property of the process with respect to $a/b = 2.5\%$.

As can be checked from the simulation of Figure 16.1 the value of $r_t$ in the Vasicek model may become negative due to its Gaussian distribution. Although real interest rates can sometimes fall below zero, this can be regarded as a potential drawback of the Vasicek model. The next R code provides a numerical solution of the stochastic differential equation (16.1) using the Euler method.

```
N=10000; t <- 0:(N-1); dt <- 1.0/N; nsim <- 2
a=0.025; b=1; sigma=0.1;
X <- matrix(rnorm(nsim*N,mean=0,sd=sqrt(dt)), nsim, N)
R <- matrix(0,nsim,N);R[,1]=0.03
for (i in 1:nsim){for (j in 2:N){R[i,j]=R[i,j-1]+(a-b*R[i,j-1])*dt+sigma*X[i,j]}}
plot(t,R[1,],xlab = "time",ylab = "",type = "l",ylim = c(R[1,1]-0.2,R[1,1]+0.2),col = 0,axes=FALSE)
axis(2, pos=0);for (i in 1:nsim){lines(t, R[i, ], xlab = "time", type = "l", col = i+3)}
abline(h=a/b,col="blue",lwd=3);abline(h=0)
```

**Example - TNX yield**

We consider the yield of the 10 Year Treasury Note on the Chicago Board Options Exchange (CBOE). Treasury notes usually have a maturity between one and 10 years, whereas treasury bonds have maturities beyond 10 years.

```
library(quantmod)
getSymbols("^TNX",from="2012-01-01",to="2016-01-01",src="yahoo")
rate<Ad(grav.ts1/TNX/grav.ts1)
rate<-rate[!is.na(rate)]
chartSeries(rate,up.col="blue",theme="white")
```

* Eurozone interest rates turned negative in 2014.
The next Figure 16.2 displays the yield of the 10 Year Treasury Note.

Fig. 16.2: CBOE 10 Year Treasury Note yield (TNX).

**Cox-Ingersoll-Ross (CIR) model**

The Cox et al. (1985) (CIR) model brings a solution to the positivity problem encountered with the Vasicek model, by the use the nonlinear stochastic differential equation

\[
\frac{dr_t}{r_t} = \beta (\alpha - r_t) dt + \sigma \sqrt{r_t} dB_t, \tag{16.4}
\]

with \( \alpha > 0, \beta > 0, \sigma > 0 \). The probability distribution of \( r_t \) at time \( t > 0 \) admits the noncentral Chi square probability density function given by

\[
f_t(x) = \frac{2\beta}{\sigma^2 (1 - e^{-\beta t})} \exp \left( -\frac{2\beta (x + r_0 e^{-\beta t})}{\sigma^2 (1 - e^{-\beta t})} \right) \left( \frac{x}{r_0 e^{-\beta t}} \right)^{\alpha \beta / \sigma^2 - 1 / 2} I_{2\alpha \beta / \sigma^2 - 1} \left( \frac{4\beta \sqrt{r_0 x e^{-\beta t}}}{\sigma^2 (1 - e^{-\beta t})} \right), \tag{16.5}
\]

\( x > 0 \), where

\[
I_\lambda(z) := \left( \frac{z}{2} \right) \sum_{k \geq 0} \frac{(z^2 / 4)^k}{k! \Gamma(\lambda + k + 1)}, \quad z \in \mathbb{R},
\]

is the modified Bessel function of the first kind, see Lemma 9 in Feller (1951) and Corollary 24 in Albanese and Lawi (2005). Note that \( f_t(x) \) is not defined at \( x = 0 \) if \( \alpha \beta / \sigma^2 - 1 / 2 < 0 \), i.e. \( \sigma^2 > 2\alpha\beta \), in which case the probability distribution of \( r_t \) admits a point mass at \( x = 0 \). On the other hand, \( r_t \) remains almost surely strictly positive under the Feller condition \( 2\alpha\beta \geq \sigma^2 \), cf. the study of the associated probability density in Lemma 4 of Feller (1951) for \( \alpha, \beta \in \mathbb{R} \).
Figure 16.3 presents a random simulation of $t \mapsto r_t$ in the Cox et al. (1985) (CIR) model in the case $\sigma^2 > 2\alpha\beta$, in which the process is mean reverting with respect to $\alpha = 2.5\%$ and has a nonzero probability of hitting 0.

![Graph of the CIR short rate $t \mapsto r_t$ with $\alpha = 2.5\%$, $\beta = 1$, and $\sigma = 1.3$.](image)

The next R code provides a numerical solution of the stochastic differential equation (16.4) using the Euler method.

```r
N=10000; t <- 0:(N-1); dt <- 1.0/N; nsim <- 2
a=0.025; b=1; sigma=0.1; sd=sqrt(sigma^2/2/b)
X <- matrix(rnorm(nsim*N,mean=0,sd=sqrt(dt)), nsim, N)
for (i in 1:nsim){for (j in 2:N){R[i,j]=max(0,R[i,j-1]+(a-b*R[i,j-1])*dt+sigma*sqrt(R[i,j-1])*X[i,j])}}
plot(t,R[1,],xlab="time",ylab="",type="l",ylim=c(0,R[1,1]+sd/5),col=0,axes=FALSE)
axis(2, pos=0)
for (i in 1:nsim){lines(t, R[i, ], xlab = "time", type = "l", col = i+8)}
abline(h=a/b,col="blue",lwd=3);abline(h=0)
```

In large time $t$, using the asymptotics

$$I_\lambda(z) \simeq_{z \to 0} \frac{1}{\Gamma(\lambda + 1)} \left( \frac{z}{2} \right)^\lambda,$$

the density (16.5) becomes the gamma density

$$f(x) = \lim_{t \to \infty} f_t(x) = \frac{1}{\Gamma(2\alpha\beta/\sigma^2)} \left( \frac{2\beta}{\sigma^2} \right)^{2\alpha\beta/\sigma^2} x^{-1+2\alpha\beta/\sigma^2} e^{-2\beta x/\sigma^2}, \quad x > 0.$$  

with shape parameter $2\alpha\beta/\sigma^2$ and scale parameter $\sigma^2/(2\beta)$, which is also the invariant distribution of $r_t$.

Other classical mean-reverting models include the Courtadon (1982) model

$$dr_t = \beta(\alpha - r_t)dt + \sigma r_t dB_t,$$
where $\alpha, \beta, \sigma$ are nonnegative, cf. Exercise 16.4, and the exponential Vasicek model
\[
dr_t = r_t(\eta - a \log r_t)dt + \sigma r_t dB_t,
\]
where $a, \eta, \sigma > 0$, cf. Exercises 4.11 and 4.12.

**Constant Elasticity of Variance (CEV) Model**

Constant Elasticity of Variance models are designed to take into account nonconstant volatilities that can vary as a power of the underlying asset price. The Marsh and Rosenfeld (1983) model
\[
dr_t = (\beta r_t^{\gamma-1} + \alpha r_t)dt + \sigma r_t^{\gamma/2} dB_t
\]
(16.7)
where $\alpha \in \mathbb{R}$, $\beta, \sigma > 0$ are constants and $\gamma > 0$ is the variance (or diffusion) elasticity coefficient, covers most of the CEV models. Denoting by $v^2(r) := \sigma^2 r^{\gamma}$ the variance coefficient in (16.7), constant elasticity refers to the constant ratio
\[
\frac{dv(r)/v^2(r)}{dr/r} = 2 \frac{rv'(r)}{v(r)} = 2 \frac{d \log v(r)}{d \log r} = 2 \frac{d \log r^{\gamma/2}}{d \log r} = \gamma
\]
between the relative change $dv(r)/v(r)$ in the variance $v(r)$ and the relative change $dr/r$ in $r$.

For $\gamma = 1$, (16.7) yields the Cox et al. (1985) (CIR) equation
\[
dr_t = (\beta + \alpha r_t)dt + \sigma \sqrt{r_t} dB_t.
\]
For $\beta = 0$ we get the standard CEV model
\[
dr_t = \alpha r_t dt + \sigma r_t^{\gamma/2} dB_t,
\]
and for $\gamma = 2$ and $\beta = 0$ this yields the Dothan (1978) model
\[
dr_t = \alpha r_t dt + \sigma r_t dB_t,
\]
which is a version of geometric Brownian motion used for short-term interest rate modeling.

**Time-dependent affine models**

The class of short rate interest rate models admits a number of generalizations (see the references quoted in the introduction of this chapter), including the class of affine models of the form
\[ dr_t = (\eta(t) + \lambda(t)r_t)dt + \sqrt{\delta(t) + \gamma(t)r_t}dB_t. \]  

(16.8)

Such models are called affine because the associated bonds can be priced using an affine PDE of the type (16.22) below, as will be seen after Proposition 16.2.

The family of affine models also includes:

i) the Ho and Lee (1986) model

\[ dr_t = \theta(t)dt + \sigma dB_t, \]

where \( \theta(t) \) is a deterministic function of time, as an extension of the Merton model \( dr_t = \theta dt + \sigma dB_t \),

ii) the Hull and White (1990) model, cf. Section 16.1,

\[ dr_t = (\theta(t) - \alpha(t)r_t)dt + \sigma(t)dB_t \]

which is itself a time-dependent extension of the Vasicek model.

16.2 Calibration of the Vasicek Model

The Vasicek equation (16.1), i.e.

\[ dr_t = (a - br_t)dt + \sigma dB_t \]

can be discretized according to a discrete-time sequence \((t_k)_{k=0,1,...,n} = (t_0, t_1, \ldots, t_n)\) of \(n+1\) time instants as

\[ r_{t_{k+1}} - r_{t_k} = (a - br_{t_k})\Delta t + \sigma Z_k, \quad k \geq 0, \]

where \( \Delta t := t_{k+1} - t_k \) and \((Z_k)_{k \geq 0}\) is a Gaussian white noise with variance \(\Delta t\), i.e. a sequence of independent, centered and identically distributed \(\mathcal{N}(0,\Delta t)\) Gaussian random variables, which yields

\[ r_{t_{k+1}} = r_{t_k} + (a - br_{t_k})\Delta t + \sigma Z_k = a\Delta t + (1 - b\Delta t)r_{t_k} + \sigma Z_k, \quad k \geq 0. \]

Based on a set \((\tilde{r}_{t_k})_{k=0,1,...,n}\) of market data we consider the quadratic residual

\[ \sum_{k=0}^{n-1} (\tilde{r}_{t_{k+1}} - a\Delta t - (1 - b\Delta t)\tilde{r}_{t_k})^2 \]

(16.9)

which represents the quadratic distance between the observed data sequence \((\tilde{r}_{t_k})_{k=1,2,...,n}\) and its predictions \((a\Delta t + (1 - b\Delta t)\tilde{r}_{t_k})_{k=0,1,...,n-1}\).

In order to minimize the residual (16.9) over \(a\) and \(b\) we use Ordinary Least Square (OLS) regression, and equate the following derivatives to zero.
Namely, we have

\[
\frac{\partial}{\partial a} \sum_{l=0}^{n-1} (\tilde{r}_{t_{l+1}} - a\Delta t - (1 - b\Delta t)\tilde{r}_{t_l})^2
\]

\[
= -2\Delta t \left( -an\Delta t + \sum_{l=0}^{n-1} (\tilde{r}_{t_{l+1}} - (1 - b\Delta t)\tilde{r}_{t_l}) \right)
\]

\[
= 2an(\Delta t)^2 - 2\Delta t \sum_{l=0}^{n-1} \tilde{r}_{t_{l+1}} + 2\Delta t(1 - b\Delta t) \sum_{l=0}^{n-1} \tilde{r}_{t_l}
\]

\[
= 0,
\]

hence

\[
a\Delta t = \frac{\Delta t}{n} \sum_{l=0}^{n-1} (\tilde{r}_{t_{l+1}} - (1 - b\Delta t)\tilde{r}_{t_l}),
\]

and

\[
\frac{\partial}{\partial b} \sum_{k=0}^{n-1} (\tilde{r}_{t_{k+1}} - a\Delta t - (1 - b\Delta t)\tilde{r}_{t_k})^2
\]

\[
= \Delta t \sum_{k=0}^{n-1} \tilde{r}_{t_k} \left( -a\Delta t + \tilde{r}_{t_{k+1}} - (1 - b\Delta t)\tilde{r}_{t_k} \right)
\]

\[
= \Delta t \sum_{k=0}^{n-1} \tilde{r}_{t_k} \left( \tilde{r}_{t_{k+1}} - (1 - b\Delta t)\tilde{r}_{t_k} - \frac{1}{n} \sum_{l=0}^{n-1} \tilde{r}_{t_{l+1}} - (1 - b\Delta t)\tilde{r}_{t_l} \right)
\]

\[
= \Delta t \sum_{k=0}^{n-1} \tilde{r}_{t_k} \tilde{r}_{t_{k+1}} - \frac{\Delta t}{n} \sum_{k,l=0}^{n-1} \tilde{r}_{t_k} \tilde{r}_{t_{l+1}} - \Delta t(1 - b\Delta t) \left( \sum_{k=0}^{n-1} (\tilde{r}_{t_k})^2 - \frac{1}{n} \sum_{k,l=0}^{n-1} \tilde{r}_{t_k} \tilde{r}_{t_l} \right)
\]

\[
= 0.
\]

This leads to estimators for the parameters \(a\) and \(b\), respectively as the empirical mean and covariance of \((\tilde{r}_{t_k})_{k=0,1,...,n}\), i.e.
Short Rates and Bond Pricing

\[
\hat{a}\Delta t = \frac{1}{n} \sum_{k=0}^{n-1} (\tilde{r}_{t_{k+1}} - (1 - \hat{b}\Delta t)\tilde{r}_{t_k}),
\]

and

\[
1 - \hat{b}\Delta t = \frac{1}{n} \sum_{k=0}^{n-1} \tilde{r}_{t_k}\tilde{r}_{t_{k+1}} - \frac{1}{n} \sum_{k,l=0}^{n-1} \tilde{r}_{t_k}\tilde{r}_{t_l}
\]

\[
= \frac{1}{n} \sum_{k=0}^{n-1} \left( \tilde{r}_{t_k} - \frac{1}{n} \sum_{l=0}^{n-1} \tilde{r}_{t_l} \right) \left( \tilde{r}_{t_{k+1}} - \frac{1}{n} \sum_{l=0}^{n-1} \tilde{r}_{t_{l+1}} \right).
\]

This also yields

\[
\sigma^2 \Delta t = \text{Var}[\sigma Z_k]
\]

\[
\simeq \mathbb{E} \left[ (\tilde{r}_{t_{k+1}} - (1 - b\Delta t)\tilde{r}_{t_k} - a\Delta t)^2 \right], \quad k \geq 0,
\]

hence \(\sigma\) can be estimated as

\[
\hat{\sigma}^2 \Delta t = \frac{1}{n} \sum_{k=0}^{n-1} (\tilde{r}_{t_{k+1}} - \tilde{r}_{t_k}(1 - \hat{b}\Delta t) - \hat{a}\Delta t)^2,
\]

as follows from minimizing the residual

\[
\eta \mapsto \sum_{k=0}^{n-1} \left( (\tilde{r}_{t_{k+1}} - \tilde{r}_{t_k}(1 - \hat{b}\Delta t) - \hat{a}\Delta t)^2 - \eta\Delta t \right)^2
\]

over \(\eta > 0\), see also Exercise 16.2.

**Time series modeling**

Defining \(\tilde{r}_{t_k} := r_{t_k} - a/b, k \in \mathbb{N}\), we have

\[
\tilde{r}_{t_{k+1}} = \tilde{r}_{t_k} - \frac{a}{b}
\]

\[
= r_{t_k} - \frac{a}{b} + (a - b r_{t_k})\Delta t + \sigma Z_k
\]

\[
= r_{t_k} - \frac{a}{b} - b \left( r_{t_k} - \frac{a}{b} \right) \Delta t + \sigma Z_k
\]
In other words, the sequence \((\tilde{r}_k)_{k \geq 0}\) is modeled according to an autoregressive AR(1) time series with parameter \(\alpha = 1 - b\Delta t\), in which the current state \(X_n\) of the system is expressed as the linear combination

\[
X_n := \sigma Z_n + \alpha X_{n-1}, \quad n \geq 1, \tag{16.10}
\]

which can be solved recursively as the causal series

\[
X_n = \sigma Z_n + \alpha (\sigma Z_{n-1} + \alpha X_{n-2}) = \cdots = \sigma \sum_{k \geq 0} \alpha^k Z_{n-k},
\]

which converges when \(|\alpha| < 1\), i.e. \(|1 - b\Delta t| < 1\), in which case the time series \((X_n)_{n \geq 0}\) is weakly stationary, with

\[
\mathbb{E}[X_n] = \sigma \sum_{k \geq 0} \alpha^k \mathbb{E}[Z_{n-k}]
\]

\[
= \sigma \mathbb{E}[Z_0] \sum_{k \geq 0} \alpha^k
\]

\[
= \frac{\sigma}{1 - \alpha} \mathbb{E}[Z_0]
\]

\[
= 0, \quad n \geq 0.
\]

The variance of \(X_n\) is given by

\[
\text{Var}[X_n] = \sigma^2 \text{Var}\left[ \sum_{k \geq 0} \alpha^k Z_{n-k} \right]
\]

\[
= \sigma^2 \Delta t \sum_{k \geq 0} \alpha^{2k}
\]

\[
= \sigma^2 \Delta t \sum_{k \geq 0} (1 - b\Delta t)^{2k}
\]

\[
= \frac{\sigma^2 \Delta t}{1 - (1 - b\Delta t)^2}
\]

\[
= \frac{\sigma^2 \Delta t}{2b\Delta t - b^2(\Delta t)^2}
\]

\[
\approx \frac{\sigma^2}{2b}, \quad [\Delta t \approx 0],
\]

which is the expected variance (16.3) of the Vasicek process in the stationary regime.
Example - TNX yield calibration

The next code is estimating the parameters of the Vasicek model using the 10 Year Treasury Note yield data of Figure 16.2.

```r
ratek <- as.vector(rate)
ratekplus1 <- c(ratek[-1], 0)
b <- (sum(ratek*ratekplus1) - sum(ratek)*sum(ratekplus1)/n)/(sum(ratek*ratek) - sum(ratek)*sum(ratek)/n)
a <- sum(ratekplus1)/n - b*sum(ratek)/n
sigma <- sqrt(sum((ratekplus1-b*ratek-a)^2)/n)
```

Parameter estimation can also be implemented using the linear regression command

\[ \text{lm}(\text{c(diff(ratek))} \sim \text{ratek}[1:length(ratek)-1]) \]

in R, which estimates the values of \( a\Delta t \) and \(-b\Delta t\) in the regression

\[ r_{tk+1} - r_{tk} = (a - br_{tk})\Delta t + \sigma Z_k, \quad k \in \mathbb{N}, \]

Coefficients:

<table>
<thead>
<tr>
<th>(Intercept)</th>
<th>ratek[1:length(ratek) - 1]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.017110</td>
<td>-0.007648</td>
</tr>
</tbody>
</table>

The code below is generating Vasicek random samples according to the AR(1) time series (16.10), see Figure 16.4.

```r
x11()
for (i in 1:100) {ar.sim<-arima.sim(model=list(ar=c(b)), n.start=100, n)
y=a/b+sigma*ar.sim
y=y+ratek[1]-y[1]
time <- as.POSIXct(time(rate), format = "%Y-%m-%d")
yield <- xts(x = y, order.by = time)
chartSeries(yield, up.col="blue", theme="white", yrange=c(0, max(ratek)))
Sys.sleep(0.5)}
```
16.3 Zero-Coupon and Coupon Bonds

A zero-coupon bond is a contract priced \( P(t, T) \) at time \( t < T \) to deliver the face value (or par value) \( P(T, T) = 1 \) at time \( T \). In addition to its value at maturity, a bond may yield a periodic coupon payment at regular time intervals until the maturity date.

The computation of the arbitrage price \( P_0(t, T) \) of a zero-coupon bond based on an underlying short-term interest rate process \( (r_t)_{t \in \mathbb{R}^+} \) is a basic and important issue in interest rate modeling.

**Constant short rate**

In case the short-term interest rate is a constant \( r_t = r \), \( t \in \mathbb{R}^+ \), a standard arbitrage argument shows that the price \( P(t, T) \) of the bond is given by

\[
P(t, T) = e^{-r(T-t)}, \quad 0 \leq t \leq T.
\]
Indeed, if \( P(t, T) > e^{-r(T-t)} \) we could issue a bond at the price \( P(t, T) \) and invest this amount at the compounded risk free rate \( r \), which would yield \( P(t, T) e^{r(T-t)} > 1 \) at time \( T \).

On the other hand, if \( P(t, T) < e^{-r(T-t)} \) we could borrow \( P(t, T) \) at the rate \( r \) to buy a bond priced \( P(t, T) \). At maturity time \( T \) we would receive $1 and refund only \( P(t, T) e^{r(T-t)} < 1 \).

The price \( P(t, T) = e^{-r(T-t)} \) of the bond is the value of \( P(t, T) \) that makes the potential profit \( P(t, T) e^{r(T-t)} - 1 \) vanish for both traders.

**Deterministic short rates**

Similarly to the above, when the short-term interest rate process \((r(t))_{t \in \mathbb{R}_+}\) is a deterministic function of time, a similar argument shows that

\[
P(t, T) = e^{-\int_t^T r(s) \, ds}, \quad 0 \leq t \leq T.
\]

(16.11)

**Stochastic short rates**

In case \((r_t)_{t \in \mathbb{R}_+}\) is an \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\)-adapted random process the formula (16.11) is no longer valid as it relies on future information, and we replace it with the averaged discounted payoff

\[
P(t, T) = \mathbb{E}^* \left[ e^{-\int_t^T r_s \, ds} \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T,
\]

(16.12)

under a risk-neutral probability measure \( \mathbb{P}^* \). It is natural to write \( P(t, T) \) as a conditional expectation under a martingale measure, as the use of conditional expectation helps to “filter out” the (random/unknown) future information past time \( t \) contained in \( \int_t^T r_s \, ds \). The expression (16.12) makes sense as the “best possible estimate” of the future quantity \( e^{-\int_t^T r_s \, ds} \) in mean square sense, given information known up to time \( t \).

**Coupon bonds**

Pricing bonds with nonzero coupon is not difficult since in general the amount and periodicity of coupons are deterministic.* In the case of the succession of coupon payments \( c_1, c_2, \ldots, c_n \) at times \( T_1, T_2, \ldots, T_n \in (t, T] \), another application of the above absence of arbitrage argument shows that the price \( P_c(t, T) \) of the coupon bond with discounted (deterministic) coupon payments is given by the linear combination of zero-coupon bond prices

* However, coupon default cannot be excluded.
\[ P_c(t, T) := \mathbb{E}^* \left[ \sum_{k=1}^{n} c_k e^{-\int_t^{T_k} r_s ds} \mid \mathcal{F}_t \right] + \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right] \]
\[ = \sum_{k=1}^{n} c_k \mathbb{E}^* \left[ e^{-\int_t^{T_k} r_s ds} \mid \mathcal{F}_t \right] + P_0(t, T) \]
\[ = P_0(t, T) + \sum_{k=1}^{n} c_k P_0(t, T_k), \quad 0 \leq t \leq T_1, \quad (16.13) \]

which represents the present value at time \( t \) of future $1 receipts at times \( T_1, T_2, \ldots, T_n \), in addition to a terminal $1 payment.

In the case of a constant coupon rate \( c \) paid at regular time intervals \( \tau = T_{k+1} - T_0, \) \( k = 0, 1, \ldots, n \), with \( T_0 = t \) and constant deterministic short rate \( r \), we find

\[ P_c(t, T) = e^{-rn\tau} + c \sum_{k=1}^{n} e^{-(T_k-t)r} \]
\[ = e^{-rn\tau} + c \sum_{k=1}^{n} e^{-k\tau r} \]
\[ = e^{-rn\tau} + c \frac{e^{-r\tau} - e^{-r(n+1)\tau}}{1 - e^{-r\tau}}. \]

In terms of the discrete-time interest rate \( \tilde{r} := e^{r\tau} - 1 \), we have

\[ P_c(t, T) = \frac{1}{(1 + \tilde{r})^n} + \frac{c}{\tilde{r}} \left( 1 - \frac{1}{(1 + \tilde{r})^n} \right). \]

In the case of a continuous-time coupon rate \( c > 0 \), the above discrete-time calculation \( (16.13) \) can be reinterpreted as follows:

\[ P_c(t, T) = P_0(t, T) + c \int_t^T P_0(t, u) du \quad (16.14) \]
\[ = e^{-r(T-t)} + c \int_0^{T-t} e^{-ru} du \]
\[ = e^{-r(T-t)} + c \frac{1 - e^{-r(T-t)}}{r}, \]
\[ = \frac{c}{r} + \frac{r-c}{r} e^{-r(T-t)}, \quad 0 \leq t \leq T, \quad (16.15) \]

which satisfies the differential equation

\[ dP_c(t, T) = -cdt + rP_c(t, T)dt, \quad 0 \leq t \leq T, \]
see also Figure 16.8 below.

In the sequel, we will mostly consider zero-coupon bonds priced as \( P(t,T) = P_0(t,T), 0 \leq t \leq T, \) in the setting of stochastic short rates.

**Martingale property of discounted bond prices**

The following proposition shows that Assumption (A) of Chapter 15 is satisfied, in other words, the bond price process \( t \mapsto -\to P(t,T) \) can be used as a numéraire.

**Proposition 16.1.** The discounted bond price process

\[
t \mapsto -\to P(t,T) := e^{-\int_0^t r_s ds} P(t,T) \]

is a martingale under \( \mathbb{P}^* \).

**Proof.** By (16.12) we have

\[
-\to P(t,T) = e^{-\int_0^t r_s ds} P(t,T) \\
= e^{-\int_0^t r_s ds} \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} \ \bigg| \mathcal{F}_t \right] \\
= \mathbb{E}^* \left[ e^{-\int_0^t r_s ds} e^{-\int_t^T r_s ds} \bigg| \mathcal{F}_t \right] \\
= \mathbb{E}^* \left[ e^{-\int_0^T r_s ds} \bigg| \mathcal{F}_t \right],
\]

and this suffices to conclude since by the “tower property” (22.38) of conditional expectations, any process \( (X_t)_{t \in \mathbb{R}_+} \) of the form \( t \mapsto X_t := \mathbb{E}^*[F \mid \mathcal{F}_t], \)
\( F \in L^1(\Omega) \), is a martingale, see also Relation (7.1). \( \square \)

**16.4 Bond Pricing PDE**

We assume from now on that the underlying short rate process is solution to the stochastic differential equation

\[
dr_t = \mu(t,r_t) dt + \sigma(t,r_t) dB_t \quad (16.16)
\]

where \( (B_t)_{t \in \mathbb{R}_+} \) is a standard Brownian motion under \( \mathbb{P}^* \). Note that specifying the dynamics of \( (r_t)_{t \in \mathbb{R}_+} \) under the historical probability measure \( \mathbb{P} \) will also lead to a notion of market price of risk (MPoR) for the modeling of short rates.

As all solutions of stochastic differential equations such as (16.16) have the Markov property, cf. e.g. Theorem V-32 of Protter (2004), the arbitrage price \( P(t,T) \) can be rewritten as a function \( F(t,r_t) \) of \( r_t \), i.e.
Proof. By Itô’s formula, we have

\[
\begin{align*}
  d\left( e^{-\int_0^t r_s ds} P(t, T) \right) &= -r_t e^{-\int_0^t r_s ds} P(t, T) dt + e^{-\int_0^t r_s ds} dP(t, T) \\
  &= -r_t e^{-\int_0^t r_s ds} F(t, r_t) dt + e^{-\int_0^t r_s ds} dF(t, r_t) \\
  &= -r_t e^{-\int_0^t r_s ds} F(t, r_t) dt + e^{-\int_0^t r_s ds} \left( \frac{\partial F}{\partial t}(t, r_t) + \frac{\partial F}{\partial x}(t, r_t) \right) dt \\
  &\quad + \frac{1}{2} e^{-\int_0^t r_s ds} \left( \frac{\partial^2 F}{\partial x^2}(t, r_t) \right)^2 + e^{-\int_0^t r_s ds} \left( \frac{\partial F}{\partial t}(t, r_t) \right) dt \\
  &= -r_t e^{-\int_0^t r_s ds} F(t, r_t) dt + e^{-\int_0^t r_s ds} \left( \frac{\partial F}{\partial t}(t, r_t) \right) dr_t \\
  &\quad + \frac{1}{2} e^{-\int_0^t r_s ds} \left( \frac{\partial^2 F}{\partial x^2}(t, r_t) \right)^2 + e^{-\int_0^t r_s ds} \left( \frac{\partial F}{\partial t}(t, r_t) \right) dt \\
  &= e^{-\int_0^t r_s ds} \left( -r_t F(t, r_t) + \mu(t, r_t) \frac{\partial F}{\partial x}(t, r_t) + \frac{1}{2} \sigma^2(t, r_t) \frac{\partial^2 F}{\partial x^2}(t, r_t) + \frac{\partial F}{\partial t}(t, r_t) \right) dt.
\end{align*}
\]

(16.20)
Given that \( t \mapsto e^{-\int_0^t r_s ds} P(t, T) \) is a martingale, the above expression (16.20) should only contain terms in \( dB_t \) (cf. Corollary II-6-1, page 72 of Protter (2004)), and all terms in \( dt \) should vanish inside (16.20). This leads to the identities

\[
\begin{aligned}
\begin{cases}
    r_t F(t, r_t) \\
    \frac{\partial F}{\partial t}(t, r_t) + \frac{\partial^2 F}{\partial x^2}(t, r_t)
\end{cases}
\end{aligned}
\]

which recover (16.18). Condition (16.19) is due to the fact that \( P(T, T) = 1 \).

In the case of an interest rate process modeled by (16.8), we have

\[
\mu(t, x) = \eta(t) + \lambda(t) x, \quad \text{and} \quad \sigma(t, x) = \sqrt{\delta(t) + \gamma(t) x},
\]

hence (16.18) yields the affine PDE

\[
x F(t, x) = \frac{\partial F}{\partial t}(t, x) + (\eta(t) + \lambda(t) x) \frac{\partial F}{\partial x}(t, x) + \frac{1}{2} (\delta(t) + \gamma(t) x) \frac{\partial^2 F}{\partial x^2}(t, x)
\]

(16.22)

with time-dependent coefficients, \( t \in \mathbb{R}_+, x \in \mathbb{R} \). By (16.21a), the proof of Proposition 16.2 also shows that

\[
\frac{dP(t, T)}{P(t, T)} = \frac{1}{P(t, T)} \left( r_t P(t, T) dt + e^{-\int_0^t r_s ds} \sigma(t, r_t) \frac{\partial F}{\partial x}(t, r_t) dB_t \right)
\]

(16.23)

**Vasicek case**

In the Vasicek case

\[
dr_t = (a - br_t) dt + \sigma dW_t,
\]

the bond price takes the form
\[ F(t, r_t) = P(t, T) = e^{A(T-t) + r_tC(T-t)}, \]

where \( A(\cdot) \) and \( C(\cdot) \) are functions of time, cf. (16.31) below, and (16.23) yields

\[
\frac{dP(t, T)}{P(t, T)} = r_t dt - \frac{\sigma}{b} (1 - e^{-(T-t)b}) dW_t, \tag{16.24}
\]

since \( F(t, x) = e^{A(T-t) + xC(T-t)} \).

Note that more generally, all affine short rate models as defined in Relation (16.8), including the Vasicek model, will yield a bond pricing formula of the form

\[ P(t, T) = e^{A(T-t) + r_tC(T-t)}, \]

cf. e.g. § 3.2.4. of Brigo and Mercurio (2006).

**Probabilistic solution of the Vasicek PDE**

Next, we solve the PDE (16.18), written with \( \mu(t, x) = a - bx \) and \( \sigma(t, x) = \sigma \) in the Vašíček (1977) model

\[ dr_t = (a - br_t) dt + \sigma dB_t \tag{16.25} \]

as

\[
\left\{ \begin{aligned}
xF(t, x) &= \frac{\partial F}{\partial t} (t, x) + (a - bx) \frac{\partial F}{\partial x} (t, x) + \frac{\sigma^2}{2} \frac{\partial^2 F}{\partial x^2} (t, x), \\
F(T, x) &= 1.
\end{aligned} \right. \tag{16.26}
\]

For this, Proposition 16.3 relies on a direct computation of the conditional expectation

\[ F(t, r_t) = P(t, T) = \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right]. \tag{16.27} \]

See also Exercise 16.6 for a bond pricing formula in the Cox et al. (1985) (CIR) model.

**Proposition 16.3.** The zero-coupon bond price in the Vasicek model (16.25) can be expressed as

\[ P(t, T) = e^{A(T-t) + r_tC(T-t)}, \quad 0 \leq t \leq T, \tag{16.28} \]

where \( A(x) \) and \( C(x) \) are functions of time to maturity given by

\[ C(x) := -\frac{1}{b} (1 - e^{-bx}), \tag{16.29} \]

and
Short Rates and Bond Pricing

\[ A(x) := \frac{4ab - 3a^2}{4b^3} + \frac{\sigma^2 - 2ab}{2b^2} x + \frac{\sigma^2 - ab}{b^3} e^{-bx} - \frac{\sigma^2}{4b^3} e^{-2bx} \quad (16.30) \]

\[
= - \left( \frac{a}{b} - \frac{\sigma^2}{2b^2} \right) (x + C(x)) - \frac{\sigma^2}{4b^2} C^2(x), \quad x \geq 0.
\]

**Proof.** Recall that in the Vasicek model, the short rate \((r_t)_{t \in \mathbb{R}^+}\) solution of (16.25) has the expression

\[
r_t = g(t) + \int_0^t h(t, s) dB_s = r_0 e^{-bt} + \frac{a}{b} (1 - e^{-bt}) + \sigma \int_0^t e^{-(t-s)b} dB_s,
\]

where \(g\) and \(h\) are the deterministic functions

\[
g(t) := r_0 e^{-bt} + \frac{a}{b} (1 - e^{-bt}), \quad t \in \mathbb{R}^+,
\]

and

\[
h(t, s) := \sigma e^{-(t-s)b}, \quad 0 \leq s \leq t.
\]

Using the fact that Wiener integrals are Gaussian random variables and the Gaussian moment generating function, we have

\[
P(t, T) = \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right]
\]

\[
= \mathbb{E}^* \left[ e^{-\int_t^T (g(s) + \int_0^s h(s, u) dB_u) ds} \mid \mathcal{F}_t \right]
\]

\[
= \exp \left( -\int_t^T g(s) ds \right) \mathbb{E}^* \left[ e^{-\int_t^T \int_0^s h(s, u) dB_u ds} \mid \mathcal{F}_t \right]
\]

\[
= \exp \left( -\int_t^T g(s) ds - \int_0^t \int_0^T h(u, t) h(s, u) dB_u ds \right) \mathbb{E}^* \left[ e^{-\int_t^T \int_0^T h(s, u) dB_u ds} \mid \mathcal{F}_t \right]
\]

\[
= \exp \left( -\int_t^T g(s) ds - \int_0^t \int_t^T h(s, u) dB_u ds \right) \mathbb{E}^* \left[ e^{-\int_t^T \int_0^T h(s, u) dB_u ds} \mid \mathcal{F}_t \right]
\]

\[
= \exp \left( -\int_t^T g(s) ds - \int_0^t \int_t^T h(s, u) dB_u ds \right) \mathbb{E}^* \left[ e^{-\int_t^T h(s, u) dB_u ds} \right]
\]

\[
= \exp \left( -\int_t^T g(s) ds - \int_0^t \int_t^T h(s, u) dB_u ds + \frac{1}{2} \int_t^T \left( \int_u^T h(s, u) ds \right)^2 du \right)
\]

\[
\times \exp \left( \frac{\sigma^2}{2} \int_t^T \left( \int_u^T e^{-(s-u)b} ds \right)^2 du \right)
\]

\[
= \exp \left( -\int_t^T \left( r_0 e^{-bs} + \frac{a}{b} (1 - e^{-bs}) \right) ds - \frac{\sigma}{b} \int_0^t e^{-(T-u)b} dB_u \right)
\]
\[ \times \exp \left( \frac{\sigma^2}{2} \int_t^T e^{2bu} \left( \frac{e^{-bu} - e^{-bT}}{b} \right)^2 \, du \right) \]

\[ = \exp \left( -\frac{r_t}{b} (1 - e^{-(T-t)b}) + \frac{1}{b} (1 - e^{-(T-t)b}) \left( r_0 e^{-bt} + \frac{a}{b} (1 - e^{-bt}) \right) \right) \times \exp \left( -\int_t^T \left( r_0 e^{-bs} + \frac{a}{b} (1 - e^{-bs}) \right) \, ds + \frac{\sigma^2}{2} \int_t^T e^{2bu} \left( \frac{e^{-bu} - e^{-bT}}{b} \right)^2 \, du \right) \]

\[ = e^{A(T-t)+r_tC(T-t)}, \quad (16.31) \]

where \( A(x) \) and \( C(x) \) are the functions given by (16.29) and (16.30).

\[ \square \]

**Analytical solution of the Vasicek PDE**

In order to solve the PDE (16.26) analytically, we may look for a solution of the form

\[ F(t, x) = e^{A(T-t)+xC(T-t)}, \quad (16.32) \]

where \( A(\cdot) \) and \( C(\cdot) \) are functions to be determined under the conditions \( A(0) = 0 \) and \( C(0) = 0 \). Substituting (16.32) into the PDE (16.18) with the Vasicek coefficients \( \mu(t,x) = (a - bx) \) and \( \sigma(t,x) = \sigma \) shows that

\[ xe^{A(T-t)+xC(T-t)} = -(A'(T-t) + xC'(T-t)) e^{A(T-t)+xC(T-t)} \]

\[ + (a - bx)C(T-t) e^{A(T-t)+xC(T-t)} \]

\[ + \frac{1}{2} \sigma^2 C^2(T-t) e^{A(T-t)+xC(T-t)}, \]

i.e.

\[ x = -A'(T-t) - xC'(T-t) + (a - bx)C(T-t) + \frac{1}{2} \sigma^2 C^2(T-t). \]

By identification of terms for \( x = 0 \) and \( x \neq 0 \), this yields the system of Riccati and linear differential equations

\[ \begin{cases} 
A'(s) = aC(s) + \frac{\sigma^2}{2} C^2(s) \\
C'(s) = -1 - bC(s),
\end{cases} \]

which can be solved to recover the above value of \( P(t, T) = F(t, r_t) \) via

\[ C(s) = -\frac{1}{b} (1 - e^{-bs}) \]

and
\[ A(t) = A(0) + \int_0^t A'(s)\,ds \]
\[ = \int_0^t \left( aC(s) + \frac{\sigma^2}{2}C^2(s) \right)\,ds \]
\[ = \int_0^t \left( \frac{a}{b} (1 - e^{-bs}) + \frac{\sigma^2}{2b^2} (1 - e^{-bs})^2 \right)\,ds \]
\[ = \frac{a}{b} \int_0^t (1 - e^{-bs})\,ds + \frac{\sigma^2}{2b^2} \int_0^t (1 - e^{-bs})^2\,ds \]
\[ = \frac{4ab - 3\sigma^2}{4b^3} + \frac{\sigma^2 - 2ab}{2b^2} t + \frac{\sigma^2 - ab}{b^3} e^{-bt} - \frac{\sigma^2}{4b^3} e^{-2bt}, \quad t \in \mathbb{R}_+. \]

**Vasicek bond price simulations**

In this section we consider again the Vasicek model, in which the short rate \((r_t)_{t \in \mathbb{R}_+}\) is solution to (16.1). Figure 16.6 presents a random simulation of the zero-coupon bond price (16.28) in the Vasicek model with \(\sigma = 10\%\), \(r_0 = 5\%\), \(b = 0.5\), and \(a = 0.025\). The graph of the corresponding deterministic zero-coupon bond price with \(r = r_0 = 5\%\) is also shown in Figure 16.6.

![Graph of zero-coupon bond price](https://www.ntu.edu.sg/home/nprivault/indext.html)

**Fig. 16.6:** Graphs of \(t \mapsto F(t,r_t) = P(t,T)\) vs \(t \mapsto e^{-r_0(T-t)}\).

Figure 16.7 presents a random simulation of the coupon bond price (16.14) in the Vasicek model with \(\sigma = 2\%\), \(r_0 = 3.5\%\), \(b = 0.5\), \(a = 0.025\), and coupon rate \(c = 5\%\). The graph of the corresponding deterministic coupon bond price (16.15) with \(r = r_0 = 3.5\%\) is also shown in Figure 16.7.
Figure 16.7: Graph of $t \mapsto P_c(t, T)$ for a bond with a 5% coupon rate.

Figure 16.8 presents market price data for a coupon bond with coupon rate $c = 6.25\%$.

**Zero-coupon bond price and yield data**

The following zero-coupon bond price data was downloaded at EMMA from the Municipal Securities Rulemaking Board.

**ORANGE CNTY CALIF PENSION OBLIG CAP APPREC-TAXABLE-REF-SER A (CA)**
CUSIP: 68428LBB9
Dated Date: 06/12/1996 (June 12, 1996)
Maturity Date: 09/01/2016 (September 1st, 2016)
Interest Rate: 0.0%
Principal Amount at Issuance: $26,056,000
Initial Offering Price: 19.465
library(quantmod)
bondprice <- read.table("bond_data_R.txt",col.names =
c("Date","HighPrice","LowPrice","HighYield","LowYield","Count","Amount"))
head(bondprice)
time <- as.POSIXct(bondprice$Date, format = "%Y-%m-%d")
price <- xts(x = bondprice$HighPrice, order.by = time)
yield <- xts(x = bondprice$HighYield, order.by = time)
chartSeries(price,up.col="blue",theme="white")
chartSeries(yield,up.col="blue",theme="white")

<table>
<thead>
<tr>
<th>Date</th>
<th>HighPrice</th>
<th>LowPrice</th>
<th>HighYield</th>
<th>LowYield</th>
<th>Count</th>
<th>Amount</th>
</tr>
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<td>98.982</td>
<td>1.666</td>
<td>1.501</td>
<td>2</td>
<td>20000</td>
</tr>
<tr>
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<td>99.183</td>
<td>1.250</td>
<td>1.250</td>
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<td>10000</td>
</tr>
<tr>
<td>2015-12-21</td>
<td>97.952</td>
<td>97.952</td>
<td>3.014</td>
<td>3.014</td>
<td>1</td>
<td>10000</td>
</tr>
<tr>
<td>2015-12-17</td>
<td>99.141</td>
<td>98.550</td>
<td>2.123</td>
<td>1.251</td>
<td>5</td>
<td>610000</td>
</tr>
<tr>
<td>2015-12-07</td>
<td>98.770</td>
<td>98.770</td>
<td>1.714</td>
<td>1.714</td>
<td>2</td>
<td>10000</td>
</tr>
<tr>
<td>2015-12-04</td>
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<td>98.118</td>
<td>2.628</td>
<td>2.280</td>
<td>2</td>
<td>10000</td>
</tr>
</tbody>
</table>

The next Figure 16.10 plots the bond yield \( y(t,T) \) defined as

\[
y(t,T) = - \frac{\log P(t,T)}{T - t}, \quad \text{or} \quad P(t,T) = e^{-(T-t)y(t,T)}, \quad 0 \leq t \leq T.
\]
Bond pricing in the Dothan model

In the Dothan (1978) model, the short-term interest rate process \((r_t)_{t \in \mathbb{R}_+}\) is modeled according to a geometric Brownian motion

\[
dr_t = \mu r_t dt + \sigma r_t dB_t,
\]

(16.33)

where the volatility \(\sigma > 0\) and the drift \(\mu \in \mathbb{R}\) are constant parameters and \((B_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion. In this model the short-term interest rate \(r_t\) remains always positive, while the proportional volatility term \(\sigma r_t\) accounts for the sensitivity of the volatility of interest rate changes to the level of the rate \(r_t\).

On the other hand, the Dothan model is the only lognormal short rate model that allows for an analytical formula for the zero-coupon bond price

\[
P(t, T) = \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T.
\]

(16.35)

For convenience of notation we let \(p = 1 - 2\mu/\sigma^2\) and rewrite (16.33) as

\[
dr_t = (1-p)\frac{\sigma^2}{2} r_t dt + \sigma r_t dB_t,
\]

with solution

\[
r_t = r_0 e^{\sigma B_t - p\sigma^2 t/2}, \quad t \in \mathbb{R}_+.
\]

(16.34)

By the Markov property of \((r_t)_{t \in \mathbb{R}_+}\), the bond price \(P(t, T)\) is a function \(F(t, r_t)\) of \(r_t\) and time \(t \in [0, T]\):

\[
P(t, T) = F(t, r_t) = \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} \mid r_t \right], \quad 0 \leq t \leq T.
\]

(16.35)
By computation of the conditional expectation (16.35) using (13.8) we easily obtain the following result, cf. Proposition 1.2 of Pintoux and Privault (2011), where the function \( \theta(v, t) \) is defined in (13.4).

**Proposition 16.4.** The zero-coupon bond price \( P(t, T) = F(t, r_t) \) is given for all \( p \in \mathbb{R} \) by

\[
F(t, x) = e^{-\sigma^2 p^2 (T-t)/8} \int_0^\infty \int_0^\infty e^{-ux} \exp \left( -2 \frac{(1 + z^2)}{\sigma^2 u} \right) \theta \left( \frac{4z}{\sigma^2 u}, \frac{(T-t)\sigma^2}{4} \right) \frac{du}{u} \frac{dz}{z^{p+1}},
\]

\( x > 0 \).

**Proof.** By Proposition 13.1, cf. Proposition 2 in Yor (1992), the probability distribution of the time integral \( \int_0^{T-t} e^{\sigma B_s - p \sigma^2 s/2} ds \) is given by

\[
P \left( \int_0^{T-t} e^{\sigma B_s - p \sigma^2 s/2} ds \in dy \right)
\]

\[
= \int_{-\infty}^\infty P \left( \int_0^t e^{\sigma B_s - p \sigma^2 s/2} ds \in dy, B_t - p \sigma t/2 \in dz \right)
\]

\[
= \frac{\sigma}{2} \int_{-\infty}^\infty e^{-p \sigma s/2 - p^2 \sigma^2 t/8} \exp \left( -2 \frac{1 + e^{\sigma^2 z}}{\sigma^2} \right) \theta \left( \frac{4 e^{\sigma^2 z/2}}{\sigma^2}, \frac{\sigma^2 t}{4} \right) \frac{dy}{y} \frac{dz}{z^{p+1}},
\]

\( y > 0 \), where the exchange of integrals is justified by the Fubini theorem and the nonnegativity of integrands. Hence, by (13.8) and (16.34) we find

\[
F(t, r_t) = P(t, T)
\]

\[
= \mathbb{E}^* \left[ \exp \left( -\int_t^T r_s ds \right) \mid \mathcal{F}_t \right]
\]

\[
= \mathbb{E}^* \left[ \exp \left( -r_t \int_t^T e^{\sigma(B_s - B_t) - \sigma^2 p(s-t)/2} ds \right) \mid \mathcal{F}_t \right]
\]

\[
= \mathbb{E}^* \left[ \exp \left( -x \int_t^T e^{\sigma(B_s - B_t) - \sigma^2 p(s-t)/2} ds \right) \right]_{x = r_t}
\]

\[
= \mathbb{E}^* \left[ \exp \left( -x \int_0^{T-t} e^{\sigma B_s - \sigma^2 p s/2} ds \right) \right]_{x = r_t}
\]

\[
= \int_0^\infty e^{-r_t y} P \left( \int_0^{T-t} e^{\sigma B_s - p \sigma^2 s/2} ds \in dy \right)
\]

\[
= e^{-(T-t)p^2 \sigma^2/8} \int_0^\infty e^{-r_t y} \int_0^\infty \exp \left( -2 \frac{1 + z^2}{\sigma^2 y} \right) \frac{dz}{z^{p+1}}.
\]
The zero-coupon bond price \( P(t, T) = F(t, r_t) \) in the Dothan model can also be written for all \( p \in \mathbb{R} \) as

\[
F(t, x) = \frac{(2x)^{p/2}}{2\pi^{p/2} \sigma^p} \int_0^\infty u e^{-(p^2+u^2)\sigma^2 t/8} \sinh(\pi u) \left| \Gamma \left( -\frac{p}{2} + i\frac{u}{2} \right) \right|^2 K_{iu} \left( \frac{\sqrt{8x}}{\sigma} \right) du \\
+ \frac{(2x)^{p/2}}{\sigma^p} \sum_{k \geq 0} \frac{2(p-2k)^+}{k!(p-k)!} e^{\sigma^2 k (k-p) t/2} K_{p-2k} \left( \frac{\sqrt{8x}}{\sigma} \right), \quad x > 0, \ t > 0,
\]

cf. Corollary 2.2 of Pintoux and Privault (2010), see also Privault and Uy (2013) for numerical computations. Zero-coupon bond prices in the Dothan model can also be computed by the conditional expression

\[
\mathbb{E} \left[ \exp \left( -\int_0^T r_t dt \right) \right] = \int_0^\infty \mathbb{E} \left[ \exp \left( -\int_0^T r_t dt \right) \bigg| r_T = z \right] d\mathbb{P}(r_T \leq z), \tag{16.37}
\]

where \( r_T \) has the lognormal distribution

\[
d\mathbb{P}(r_T \leq z) = \mathbb{P}(r_0 e^{\sigma B_T - \sigma^2 T/2} \leq z) = \frac{1}{z\sqrt{2\pi\sigma^2 T}} e^{- \left( \frac{\sigma^2 T}{2} + \log(z/r_0)^2 \right)/\left(2\sigma^2 T\right)}.
\]

In Proposition 16.5 we note that the conditional Laplace transform

\[
\mathbb{E} \left[ \exp \left( -\int_0^T r_t dt \right) \bigg| r_T = z \right]
\]

cf. (16.41) above, can be computed by a closed-form integral expression based on the modified Bessel function of the second kind

\[
K_\zeta(z) := \frac{z^\zeta}{2^{\zeta+1}} \int_0^\infty \exp \left(-u - \frac{z^2}{4u} \right) \frac{du}{u^{\zeta+1}}, \quad \zeta \in \mathbb{R}, \ z \in \mathbb{C}, \tag{16.38}
\]

cf. e.g. Watson (1995) page 183, provided that the real part \( \mathcal{R}(z^2) \) of \( z^2 \in \mathbb{C} \) is positive.

**Proposition 16.5.** Privault and Yu (2016), Proposition 4.1. Taking \( r_0 = 1 \), for all \( \lambda, z > 0 \) we have

\[
\mathbb{E} \left[ \exp \left( -\lambda \int_0^T r_s ds \right) \bigg| r_T = z \right] = \frac{4e^{-\sigma^2 T/8}}{\pi^{3/2} \sigma^2 p(z)} \sqrt{\frac{\lambda}{T}} \times \int_0^\infty e^{2(\pi^2-\xi^2)/(\sigma^2 T)} \sin \left( \frac{4\pi \xi}{\sigma^2 T} \right) \sinh \left( \frac{K_1 \left( \sqrt{8\lambda}\sqrt{1+2\sqrt{z}\cosh \xi + z/\sigma} \right)}{\sqrt{1+2\sqrt{z}\cosh \xi + z}} \right) d\xi.
\]

Note however that the numerical evaluation of (16.39) fails for small values of \( T > 0 \), and for this reason the integral can be estimated by a gamma
approximation, cf. (16.40) below. Under the gamma approximation we can approximate the conditional bond price on the Dothan short rate \( r_t \) as

\[
E \left[ \exp \left( -\lambda \int_0^T r_t \, dt \right) \mid r_T = z \right] \simeq (1 + \lambda \theta(z))^{-\nu(z)},
\]

where the parameters \( \nu(z) \) and \( \theta(z) \) are determined by conditional moment fitting to a gamma distribution, as

\[
\theta(z) := \frac{\text{Var}[\Lambda_T \mid S_T = z]}{E[\Lambda_T \mid S_T = z]}, \quad \nu(z) := \frac{(E[\Lambda_T \mid S_T = z])^2}{\text{Var}[\Lambda_T \mid S_T = z]} = \frac{E[\Lambda_T \mid S_T = z]}{\theta},
\]

cf. Privault and Yu (2016), which yields

\[
E \left[ \exp \left( -\lambda \int_0^T r_s \, ds \right) \right] \simeq \int_0^\infty (1 + \lambda \theta(z))^{-\nu(z)} \, dP(r_T \leq z). \tag{16.40}
\]

Note that \( \theta(z) \) is known in physics as the Fano factor or dispersion index which measures the dispersion of the probability distribution of \( \Lambda_T \) given that \( S_T = z \). Figures 16.11 shows that the stratified gamma approximation (16.40) matches the Monte Carlo estimate, while the use of the integral expressions (16.37) and (16.39) leads to numerical instabilities.

Fig. 16.11: Approximation of Dothan bond prices \( t \mapsto F(t, x) \) with \( \sigma = 0.3 \) and \( T = 10 \).

Related computations for yield options in the Cox et al. (1985) (CIR) model can also be found in Prayoga and Privault (2017).

**Path integrals in option pricing**

In physics, the Feynman path integral
\[ \psi(y, t) := \int_{x(0)=x, \ x(t)=y} Dx(\cdot) \exp \left( \frac{i}{\hbar} S(x(\cdot)) \right) \]

where \( \hbar \) is the Planck constant and \( S(x(\cdot)) \) is the action given by the Lagrangian
\[ L(x(s), \dot{x}(s), s) := \frac{1}{2} m (\dot{x}(s))^2 - V(x(s)), \]
as
\[ S(x(\cdot)) = \int_0^t L(x(s), \dot{x}(s), s) ds = \int_0^t \left( \frac{1}{2} m (\dot{x}(s))^2 - V(x(s)) \right) ds \]
\[ \simeq \sum_{i=1}^N \left( \frac{(x(t_i) - x(t_{i-1}))^2}{2(t_i - t_{i-1})^2} - V(x(t_{i-1})) \right) \Delta t_i, \]
solves the Schrödinger equation
\[ ih \frac{\partial \psi}{\partial t}(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}(x, t) + V(x(t)) \psi(x, t). \]

After the Wick rotation \( t \mapsto -it \), the function
\[ \phi(y, t) := \int_{x(0)=x, \ x(t)=y} Dx(\cdot) \exp \left( -\frac{1}{\hbar} S(x(\cdot)) \right) \]
where \( S(x(\cdot)) \) is the action
\[ S(x(\cdot)) := \int_0^t L(x(s), \dot{x}(s), s) ds = \int_0^t \left( \frac{1}{2} m (\dot{x}(s))^2 + V(x(s)) \right) ds \]
\[ \simeq \sum_{i=1}^N \left( \frac{(x(t_i) - x(t_{i-1}))^2}{2(t_i - t_{i-1})^2} + V(x(t_{i-1})) \right) \Delta t_i, \]
solves the heat equation
\[ \hbar \frac{\partial \phi}{\partial t}(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2 \phi}{\partial x^2}(x, t) + V(x(t)) \phi(x, t). \]

Given the action
\[ S(x(\cdot)) = \int_0^t \left( \frac{1}{2} m (\dot{x}(s))^2 + V(x(s)) \right) ds \]
\[ \simeq \sum_{i=1}^N \left( \frac{(x(t_i) - x(t_{i-1}))^2}{2(t_i - t_{i-1})^2} + V(x(t_{i-1})) \right) \Delta t_i, \]
we can rewrite the Euclidean path integral as

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\[
\phi(y, t) = \int_{x(0)=x, \ t=0}^{x(t)=y} D(x(\cdot)) \exp \left( -\frac{1}{\hbar} S(x(\cdot)) \right)
\]

\[
= \int_{x(0)=x, \ t=0}^{x(t)=y} D(x(\cdot)) \exp \left( -\frac{1}{2\hbar} \sum_{i=1}^{N} \frac{(x(t_i) - x(t_{i-1}))^2}{2\Delta t_i} - \frac{1}{\hbar} \sum_{i=1}^{N} V(x(t_{i-1})) \right)
\]

\[
= \mathbb{E}^\ast \left[ \exp \left( -\frac{1}{\hbar} \int_0^t V(B_s) \, ds \right) \mid B_0 = x, B_t = y \right].
\]

This type of path integral computation

\[
\phi(y, t) = \mathbb{E}^\ast \left[ \exp \left( -\int_0^t V(B_s) \, ds \right) \mid B_0 = x, B_t = y \right]. \tag{16.41}
\]

is particularly useful for bond pricing, as (16.41) can be interpreted as the price of a bond with short-term interest rate process \((r_t)_{t\in\mathbb{R}^+} := (V(B_t))_{t\in\mathbb{R}^+}\) conditionally to the value of the endpoint \(B_t = y\), cf. (16.39) below. It can also be useful for exotic option pricing, cf. Chapter 13, and for risk management, see e.g. Kakushadze (2015). The path integral (16.41) can be estimated either by closed-form expressions using Partial Differential Equations (PDEs) or probability densities, by approximations such as (conditional) Moment matching, or by Monte Carlo estimation, from the paths of a Brownian bridge as shown in Figure 16.12.

Fig. 16.12: Brownian bridge.

**Exercises**

Exercise 16.1 Show that the solution of the equation

\[
dr_t = (a - br_t) \, dt + \sigma dB_t,
\]

where \(a, \sigma \in \mathbb{R}, \ b > 0\), is

\[
r_t = r_0 e^{-bt} + \frac{a}{b} (1 - e^{-bt}) + \sigma \int_0^t e^{-(t-s)b} dB_s, \quad t \in \mathbb{R}^+.
\]

\[\hat{\circ} \]

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Exercise 16.2 Consider the Chan-Karolyi-Longstaff-Sanders (CKLS) interest rate model (Chan et al. (1992)) parametrized as
\[ dr_t = (a - br_t) dt + \sigma r_t^\gamma dB_t, \]
and time-discretized as
\[ r_{t_k+1} = r_{t_k} + (a - br_{t_k}) \Delta t + \sigma r_{t_k}^\gamma Z_k \]
where \( \Delta t := t_{k+1} - t_k \) and \((Z_k)_{k\geq 0}\) is an i.i.d sequence of \( \mathcal{N}(0,\Delta t) \) random variables. Assuming that \( a, b, \gamma > 0 \) are known, find an unbiased estimator \( \hat{\sigma}^2 \) for the variance coefficient \( \sigma^2 \), based on a market data set \((\tilde{r}_{tk})_{k=0,1,\ldots,n}\).

Exercise 16.3 Let \((B_t)_{t\in \mathbb{R}^+}\) denote a standard Brownian motion started at 0 under the risk-neutral probability measure \( \mathbb{P}^* \). We consider a short-term interest rate process \((r_t)_{t\in \mathbb{R}^+}\) in a Ho-Lee model with constant deterministic volatility, defined by
\[ dr_t = adt + \sigma dB_t, \]
where \( a \in \mathbb{R} \) and \( \sigma > 0 \). Let \( P(t,T) \) will denote the arbitrage price of a zero-coupon bond in this model:
\[ P(t,T) = \mathbb{E}^* \left[ \exp \left( -\int_t^T r_s ds \right) \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T. \] (16.44)
a) State the bond pricing PDE satisfied by the function \( F(t,x) \) defined via
\[ F(t,x) := \mathbb{E}^* \left[ \exp \left( -\int_t^T r_s ds \right) \mid r_t = x \right], \quad 0 \leq t \leq T. \]
b) Compute the arbitrage price \( F(t,r_t) = P(t,T) \) from its expression (16.44) as a conditional expectation.

Hint. One may use the integration by parts argument
\[ \int_t^T B_s ds = TB_T - tB_t - \int_t^T s dB_s = (T-t)B_t + T(B_T - B_t) - \int_t^T s dB_s \]
and the Gaussian moment generating function \( \mathbb{E}[e^{\lambda X}] = e^{\lambda^2 \eta^2/2} \) for \( X \approx \mathcal{N}(0,\eta^2) \).
c) Check that the function $F(t, x)$ computed in Question (b) does satisfy the PDE derived in Question (a).

Exercise 16.4 Consider the Couradon (1982) model

$$dr_t = \beta(\alpha - r_t)dt + \sigma r_t dB_t, \quad (16.45)$$

where $\alpha, \beta, \sigma$ are nonnegative, which is a particular case of the Chan-Karolyi-Longstaff-Sanders (CKLS) model (Chan et al. (1992)) with $\gamma = 1$. Show that the solution of (16.45) is given by

$$r_t = \alpha \beta \int_0^t \frac{S_t}{S_u} du + r_0 S_t, \quad t \in \mathbb{R}_+, \quad (16.46)$$

where $(S_t)_{t \in \mathbb{R}_+}$ is the geometric Brownian motion solution of $dS_t = -\beta S_t dt + \sigma S_t dB_t$ with $S_0 = 1$.

Exercise 16.5 Consider the Marsh and Rosenfeld (1983) interest rate model

$$dr_t = (\beta r_t^{-(1-\gamma)} + \alpha r_t)dt + \sigma r_t^{\gamma/2} dB_t.$$  

Given that the discounted bond price process is a martingale, derive the bond pricing PDE satisfied by the function $F(t, x)$ such that

$$F(t, r_t) = P(t, T) = \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right] = \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} \mid r_t \right].$$

Exercise 16.6 Consider the Cox et al. (1985) (CIR) process $(r_t)_{t \in \mathbb{R}_+}$ solution of

$$dr_t = -ar_t dt + \sigma \sqrt{r_t} dB_t,$$

where $a, \sigma > 0$ are constants $(B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion started at 0.

a) Write down the bond pricing PDE for the function $F(t, x)$ given by

$$F(t, x) := \mathbb{E}^* \left[ \exp \left( - \int_t^T r_s ds \right) \mid r_t = x \right], \quad 0 \leq t \leq T.$$  

*Hint:* Use Itô calculus and the fact that the discounted bond price is a martingale.

b) Show that the PDE of Question (a) admits a solution of the form $F(t, x) = e^{A(T-t) + xC(T-t)}$ where the functions $A(s)$ and $C(s)$ satisfy ordinary differential equations to be also written down together with the values of $A(0)$ and $C(0)$.
Exercise 16.7 Convertible bonds. Consider an underlying asset price process $(S_t)_{t \in \mathbb{R}_+}$ given by
\[ dS_t = rS_t dt + \sigma S_t dB^{(1)}_t, \]
and a short-term interest rate process $(r_t)_{t \in \mathbb{R}_+}$ given by
\[ dr_t = \gamma(t, r_t) dt + \eta(t, r_t) dB^{(2)}_t, \]
where $(B^{(1)}_t)_{t \in \mathbb{R}_+}$ and $(B^{(2)}_t)_{t \in \mathbb{R}_+}$ are two correlated Brownian motions under the risk-neutral probability measure $\mathbb{P}^*$, with $dB^{(1)}_t \cdot dB^{(2)}_t = \rho dt$. A convertible bond is a corporate bond that can be exchanged into a quantity $\alpha > 0$ of the underlying company’s stock $S_\tau$ at a future time $\tau$, whichever has a higher value, where $\alpha$ is a conversion rate.

a) Find the payoff of the convertible bond at time $\tau$.
b) Rewrite the convertible bond payoff at time $\tau$ as the linear combination of $P(\tau, T)$ and a call option payoff on $S_\tau$, whose strike price is to be determined.
c) Write down the corporate bond price at time $t \in [0, \tau]$ as a function $C(t, S_t, r_t)$ of the underlying asset price and interest rate, using a discounted conditional expectation, and show that the discounted corporate bond price
\[ e^{-\int_0^t r_s ds} C(t, S_t, r_t), \quad t \in [0, \tau], \]
is a martingale.
d) Write down $d \left( e^{-\int_0^t r_s ds} C(t, S_t, r_t) \right)$ using the Itô formula and derive the pricing PDE satisfied by the function $C(t, x, y)$ together with its terminal condition.
e) Taking the bond price $P(t, T)$ as a numéraire, price the convertible bond as a European option with strike price $K = 1$ on an underlying asset priced $Z_t := S_t / P(t, T)$, $t \in [0, \tau]$ under the forward measure $\tilde{\mathbb{E}}_T$.
f) Assuming the bond price dynamics
\[ dP(t, T) = r_t P(t, T) dt + \sigma(t) P(t, T) dB_t, \]
determine the dynamics of the process $(Z_t)_{t \in \mathbb{R}_+}$ under the forward measure $\tilde{\mathbb{E}}_T$.
g) Assuming that $(Z_t)_{t \in \mathbb{R}_+}$ can be modeled as a geometric Brownian motion, price the corporate bond option using the Black-Scholes formula.

Exercise 16.8 Bond duration. Compute the duration
\[ D_c(0, n) := - \frac{1 + r}{P_c(0, n)} \frac{\partial}{\partial r} P_c(0, n), \]
of a discrete-time coupon bond priced as
\[ P_c(0, n) = \frac{1}{(1 + r)^n} + c \sum_{k=1}^{n} \frac{1}{(1 + r)^k} \]
\[ = \frac{1}{(1 + r)^n} + \frac{c}{r} \left( 1 - \frac{1}{(1 + r)^n} \right), \]
where \( r > 0 \), and \( c \geq 0 \) denotes the coupon rate. What happens when \( n \) becomes large?

Exercise 16.9 Let \( (r_t)_{t \in \mathbb{R}_+} \) denote a short-term interest rate process. For any \( T > 0 \), let \( P(t, T) \) denote the price at time \( t \in [0, T] \) of a zero-coupon bond defined by the stochastic differential equation
\[ \frac{dP(t, T)}{P(t, T)} = r_t dt + \sigma_t^T dB_t, \quad 0 \leq t \leq T, \quad (16.47) \]
under the terminal condition \( P(T, T) = 1 \), where \( (\sigma_t^T)_{t \in [0, T]} \) is an adapted process. We define the forward measure \( \mathbb{P}_T \) by
\[ \mathbb{E}^* \left[ \frac{d\mathbb{P}_T}{d\mathbb{P}^*} \middle| F_t \right] = \frac{P(t, T)}{P(0, T)} e^{-\int_0^t r_s ds}, \quad 0 \leq t \leq T. \]
Recall that
\[ B^T_t := B_t - \int_0^t \sigma_s^T ds, \quad 0 \leq t \leq T, \]
is a standard Brownian motion under \( \mathbb{P}_T \).

a) Solve the stochastic differential equation (16.47).

b) Derive the stochastic differential equation satisfied by the discounted bond price process
\[ t \mapsto e^{-\int_0^t r_s ds} P(t, T), \quad 0 \leq t \leq T, \]
and show that it is a martingale.

c) Show that
\[ \mathbb{E}^* \left[ e^{-\int_0^t r_s ds} \middle| F_t \right] = e^{-\int_0^t r_s ds} P(t, T), \quad 0 \leq t \leq T. \]
d) Show that
\[ P(t, T) = \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} \middle| F_t \right], \quad 0 \leq t \leq T. \]
e) Compute \( P(t, S)/P(t, T), 0 \leq t \leq T, \) show that it is a martingale under \( \mathbb{P}_T \) and that

\[ \therefore \]
\[ P(T, S) = \frac{P(t, S)}{P(t, T)} \exp \left( \int_t^T (\sigma_s^S - \sigma_s^T) dB_s^T - \frac{1}{2} \int_t^T (\sigma_s^S - \sigma_s^T)^2 ds \right). \]

f) Assuming that \((\sigma_t^T)_{t \in [0, T]}\) and \((\sigma_t^S)_{t \in [0, S]}\) are deterministic functions of time, compute the price
\[
\mathbb{E}^* \left[ e^{-\int_t^T r_s ds} (P(T, S) - \kappa)^+ \mid \mathcal{F}_t \right] = P(t, T) \mathbb{E}_T \left[ (P(T, S) - \kappa)^+ \mid \mathcal{F}_t \right]
\]
of a bond option with strike price \(\kappa\).

Recall that if \(X\) is a centered Gaussian random variable with mean \(m_t\) and variance \(v_t^2\) given \(\mathcal{F}_t\), we have
\[
\mathbb{E} \left[ (e^X - K)^+ \mid \mathcal{F}_t \right] = e^{m_t + v_t^2/2} \Phi \left( \frac{v_t}{2} + \frac{1}{v_t} \left( m_t + v_t^2/2 - \log K \right) \right)
- K \Phi \left( -\frac{v_t}{2} + \frac{1}{v_t} \left( m_t + v_t^2/2 - \log K \right) \right)
\]
where \(\Phi(x), x \in \mathbb{R},\) denotes the Gaussian cumulative distribution function.

Exercise 16.10 (Exercise 4.13 continued). Write down the bond pricing PDE for the function
\[ F(t, x) = \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} \mid r_t = x \right] \]
and show that in case \(\alpha = 0\) the corresponding bond price \(P(t, T)\) equals
\[ P(t, T) = e^{-r_t B(T-t)}, \quad 0 \leq t \leq T, \]
where
\[ B(x) := \frac{2(e^{\gamma x} - 1)}{2\gamma + (\beta + \gamma)(e^{\gamma x} - 1)}, \]
with \(\gamma = \sqrt{\beta^2 + 2\sigma^2}\).

Exercise 16.11 Consider a zero-coupon bond with prices \(P(1, 2) = 91.74\%\) and \(P(0, 2) = 83.40\%\) at times \(t = 0\) and \(t = 1\).

a) Compute the corresponding yields \(y_{0,1}, y_{0,2}\) and \(y_{1,2}\) at times \(t = 0\) and \(t = 1\).

b) Assume that \$0.1 coupons are paid at times \(t = 1\) and \(t = 2\). Price the corresponding coupon bond at times \(t = 0\) and \(t = 1\) using the yields \(y_0\) and \(y_1\).
Exercise 16.12 Consider the Vasicek process \( (r_t)_{t \in \mathbb{R}_+} \) solution of the equation
\[
    dr_t = (a - br_t)dt + \sigma dB_t.
\] (16.48)

a) Consider the discretization
\[
    r_{tk+1} := r_{tk} + (a - br_{tk})\Delta t \pm \sigma \sqrt{\Delta t}, \quad k = 0, 1, 2, \ldots
\]
of the equation (16.48) with \( p(r_{t_0}) = p(r_{t_1}) = 1/2 \) and
\[
    \mathbb{E}[\Delta r_{t_1}] = (a - br_{t_0})\Delta t + \sigma p(r_{t_0})\sqrt{\Delta t} - \sigma q(r_{t_0})\sqrt{\Delta t} = (a - br_{t_0})\Delta t
\]
and
\[
    \mathbb{E}[\Delta r_{t_2}] = (a - br_{t_1})\Delta t + \sigma p(r_{t_0})\sqrt{\Delta t} - \sigma q(r_{t_0})\sqrt{\Delta t} = (a - br_{t_1})\Delta t.
\]
Does this discretization lead to a binomial tree?

b) Using the Girsanov Theorem, find a probability measure \( Q \) under which the process \( (r_t/\sigma)_{t \in [0,T]} \) with
\[
    \frac{dr_t}{\sigma} = \frac{a - br_t}{\sigma} dt + dB_t
\]
is a standard Brownian motion.

Hint: By the Girsanov Theorem, the process \( X_t = X_0 + \int_0^t u_s ds + B_t \) is a martingale under the probability measure \( Q \) with density
\[
    \frac{dQ}{dP} = \exp \left( - \int_0^T u_t dB_t - \frac{1}{2} \int_0^T (u_t)^2 dt \right)
\]
with respect to \( P \).

c) Using the approximation
\[
    \exp \left( \frac{1}{\sigma^2} \int_0^T (a - br_t)dr_t - \frac{1}{2\sigma^2} \int_0^T (a - br_t)^2 dt \right)
\]
\[
    \approx 2^{T/\Delta T} \prod_{0 < t < T} \left( \frac{1}{2} \pm \frac{a - br_t}{2\sigma} \sqrt{\Delta t} \right),
\]
show that the Vasicek process can be discretized along the binomial tree.
Exercise 16.13  Black-Derman-Toy model (Black et al. (1990)). Consider a one-step interest rate model in which the short-term interest rate $r_0$ on $[0,1]$ can turn into two possible values $r_1^u = r_0 e^{\mu \Delta t + \sigma \sqrt{\Delta t}}$ and $r_1^d = r_0 e^{\mu \Delta t - \sigma \sqrt{\Delta t}}$ on $[1,2]$ with equal probabilities $1/2$ at time $\Delta t = 1$ year and $\sigma = 22\%$ per year, and a zero-coupon bonds with prices $P(0, 1)$ and $P(0, 2)$ at time $t = 0$.

a) Write down the value of $P(1, 2)$ using $r_1^u$ and $r_1^d$.
b) Write down the value of $P(0, 2)$ using $r_1^u$, $r_1^d$ and $r_0$.
c) Estimate the value of $r_0$ from the market price $P(0, 1) = 91.74$.
d) Estimate the values of $r_1^u$ and $r_1^d$ from the market price $P(0, 2) = 83.40$.

Exercise 16.14  Consider a yield curve $(f(t, t, T))_{0 \leq t \leq T}$ and a bond paying coupons $c_1, c_2, \ldots, c_n$ at times $T_1, T_2, \ldots, T_n$ until maturity $T_n$, and priced as

$$P(t, T_n) = \sum_{k=1}^{n} c_k e^{-(T_k-t)f(t,t,T_k)}, \quad 0 \leq t \leq T_1,$$

where $c_n$ is inclusive of the last coupon payment and the nominal $\$1$ value of the bond. Let $\hat{f}(t, t, T_n)$ denote the compounded yield to maturity defined by equating

$$P(t, T_n) = \sum_{k=1}^{n} c_k e^{-(T_k-t)f(t,t,T_n)}, \quad 0 \leq t \leq T_1,$$  \hspace{1cm} (16.49)
i.e. \( \tilde{f}(t, t, T_n) \) solves the equation

\[
F(t, \tilde{f}(t, t, T_n)) = P(t, T_n), \quad 0 \leq t \leq T_1,
\]

with

\[
F(t, x) := \sum_{k=1}^{n} c_k e^{-(T_k - t)x}, \quad 0 \leq t \leq T_1.
\]

The bond duration \( D(t, T_n) \) is the relative sensitivity of \( P(t, T_n) \) with respect to \( \tilde{f}(t, t, T_n) \), defined as

\[
D(t, T_n) := -\frac{1}{P(t, T_n)} \frac{\partial F}{\partial x}(t, \tilde{f}(t, t, T_n)), \quad 0 \leq t \leq T_1.
\]

The bond convexity \( C(t, T_n) \) is defined as

\[
C(t, T_n) := \frac{1}{P(t, T_n)} \frac{\partial^2 F}{\partial x^2}(t, \tilde{f}(t, t, T_n)), \quad 0 \leq t \leq T_1.
\]

a) Compute the bond duration in case \( n = 1 \).

b) Show that the bond duration \( D(t, T_n) \) can be interpreted as an average of times to maturity weighted by the respective discounted bond payoffs.

c) Show that the bond convexity \( C(t, T_n) \) satisfies

\[
C(t, T_n) = (D(t, T_n))^2 + (S(t, T_n))^2,
\]

where \( S(t, T_n) \) measures the dispersion of the duration of the bond payoffs around the portfolio duration \( D(t, T_n) \).

d) Consider now the zero-coupon yield defined as

\[
P(t, t + \alpha(T_n - t)) = \exp \left( -\alpha (T_n - t) f_\alpha(t, t, T_n) \right),
\]

where \( \alpha \in (0, 1) \), i.e.

\[
f_\alpha(t, t, T_n) := -\frac{1}{\alpha(T_n - t)} \log P(t, t + \alpha(T_n - t)), \quad 0 \leq t \leq T_n.
\]

Compute the bond duration associated to the yield \( f_\alpha(t, t, T_n) \) in affine bond pricing models of the form

\[
P(t, T) = e^{A(T-t)+r_tB(T-t)}, \quad 0 \leq t \leq T.
\]

e) (Wu (2000)) Compute the bond duration associated to the yield \( f_\alpha(t, t, T_n) \) in the Vasicek model, in which

\[
B(T-t) := \frac{1 - e^{-(T-t)b}}{b}, \quad 0 \leq t \leq T.
\]