In this chapter we consider the pricing of fixed income derivatives such as caplets, caps, and swaptions, using change of numéraire and forward swap measures.

### 18.1 Forward Measures and Tenor Structure

The maturity dates are arranged according to a discrete tenor structure

\[ \{0 = T_0 < T_1 < T_2 < \cdots < T_n\} \]

A sample of forward interest rate curve data is given in Table 18.1, which contains the values of \((T_1, T_2, \ldots, T_{23})\) and of \(\{f(t, t + T_i, t + T_i + \delta)\}_{i=1,2,\ldots,23}\), with \(t = 07/05/2003\) and \(\delta = \text{six months}\).

<table>
<thead>
<tr>
<th>Maturity</th>
<th>2D</th>
<th>1W</th>
<th>1M</th>
<th>2M</th>
<th>3M</th>
<th>1Y</th>
<th>2Y</th>
<th>3Y</th>
<th>4Y</th>
<th>5Y</th>
<th>6Y</th>
<th>7Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate (%)</td>
<td>2.55</td>
<td>2.53</td>
<td>2.56</td>
<td>2.52</td>
<td>2.48</td>
<td>2.34</td>
<td>2.49</td>
<td>2.79</td>
<td>3.07</td>
<td>3.31</td>
<td>3.52</td>
<td>3.71</td>
</tr>
<tr>
<td>Maturity</td>
<td>8Y</td>
<td>9Y</td>
<td>10Y</td>
<td>11Y</td>
<td>12Y</td>
<td>13Y</td>
<td>14Y</td>
<td>15Y</td>
<td>20Y</td>
<td>25Y</td>
<td>30Y</td>
<td></td>
</tr>
<tr>
<td>Rate (%)</td>
<td>3.88</td>
<td>4.02</td>
<td>4.14</td>
<td>4.23</td>
<td>4.33</td>
<td>4.40</td>
<td>4.47</td>
<td>4.54</td>
<td>4.74</td>
<td>4.83</td>
<td>4.86</td>
<td></td>
</tr>
</tbody>
</table>

Table 18.1: Forward rates arranged according to a tenor structure.

Recall that by definition of \(P(t, T_i)\) and absence of arbitrage the discounted bond price process
\[ t \mapsto e^{-\int_0^t r_s ds} P(t, T_i), \quad 0 \leq t \leq T_i, \quad i = 1, 2, \ldots, n, \]
is an \( \mathcal{F}_t \)-martingale under the probability measure \( \mathbb{P}^* = \mathbb{P} \), hence it satisfies the Assumption (A) page 494. As a consequence the bond price process can be taken as a numéraire

\[ N^{(i)}_t := P(t, T_i), \quad 0 \leq t \leq T_i, \]
in the definition

\[ \frac{d\hat{\mathbb{P}}_t}{d\mathbb{P}^*} = \frac{1}{P(0, T_i)} e^{-\int_0^{T_i} r_s ds} \tag{18.1} \]
of the forward measure \( \hat{\mathbb{P}}_t \). The following proposition will allow us to price contingent claims using the forward measure \( \hat{\mathbb{P}}_t \), it is a direct consequence of Proposition 15.3, noting that here we have \( P(T_i, T_i) = 1 \).

**Proposition 18.1.** For all sufficiently integrable random variables \( F \) we have

\[ \mathbb{E}^* \left[ F e^{-\int_0^{T_i} r_s ds} \mid \mathcal{F}_t \right] = P(t, T_i) \mathbb{E}_t[F \mid \mathcal{F}_t], \quad 0 \leq t \leq T_i, \quad i = 1, 2, \ldots, n. \tag{18.2} \]

Recall that for all \( T_i, T_j \geq 0 \), the deflated process

\[ t \mapsto \frac{P(t, T_j)}{P(t, T_i)}, \quad 0 \leq t \leq \min(T_i, T_j), \]
is an \( \mathcal{F}_t \)-martingale under \( \hat{\mathbb{P}}_t \), cf. Proposition 15.4.

In the sequel we assume as in (16.22) that the dynamics of the bond price \( P(t, T_i) \) is given by

\[ \frac{dP(t, T_i)}{P(t, T_i)} = r_t dt + \zeta_i(t) dW_t, \tag{18.3} \]
for \( i = 1, 2, \ldots, n \), where \( (W_t)_{t \in \mathbb{R}^+} \) is a standard Brownian motion under \( \mathbb{P}^* \) and \( (r_t)_{t \in \mathbb{R}^+} \) and \( (\zeta_i(t))_{t \in \mathbb{R}^+} \) are adapted processes with respect to the filtration \( (\mathcal{F}_t)_{t \in \mathbb{R}^+} \) generated by \( (W_t)_{t \in \mathbb{R}^+} \), i.e.

\[ P(t, T_i) = P(0, T_i) \exp \left( \int_0^t r_s ds + \int_0^t \zeta_i(s) dW_s - \frac{1}{2} \int_0^t |\zeta_i(s)|^2 ds \right), \quad 0 \leq t \leq T_i, \quad i = 1, 2, \ldots, n. \]

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Proposition 18.2. For all \( i = 1, 2, \ldots, n \), the process
\[
\hat{W}_t^i := W_t - \int_0^t \zeta_i(s) ds, \quad 0 \leq t \leq T_i, \tag{18.4}
\]
is a standard Brownian motion under the forward measure \( \hat{P}_i \).

Proof. The Girsanov Proposition 15.5 applied to the numéraire
\[
N_t^{(i)} := P(t, T_i), \quad 0 \leq t \leq T_i,
\]
as in (15.11), shows that
\[
d\hat{W}_t^i := dW_t - \frac{1}{N_t^{(i)}} dN_t^{(i)} \cdot dW_t \\
= dW_t - \frac{1}{P(t, T_i)} dP(t, T_i) \cdot dW_t \\
= dW_t - \frac{1}{P(t, T_i)} (P(t, T_i) r_t dt + \zeta_i(t) P(t, T_i) dW_t) \cdot dW_t \\
= dW_t - \zeta_i(t) dt,
\]
is a standard Brownian motion under the forward measure \( \hat{P}_i \) for all \( i = 1, 2, \ldots, n \).

We have
\[
d\hat{W}_t^i = dW_t - \zeta_i(t) dt, \quad i = 1, 2, \ldots, n, \tag{18.5}
\]
and
\[
d\hat{W}_t^j = dW_t - \zeta_j(t) dt = d\hat{W}_t^i + (\zeta_i(t) - \zeta_j(t)) dt, \quad i, j = 1, 2, \ldots, n,
\]
which shows that \( (\hat{W}_t^j)_{t \in \mathbb{R}_+} \) has drift \( (\zeta_i(t) - \zeta_j(t))_{t \in \mathbb{R}_+} \) under \( \hat{P}_i \).

Bond price dynamics under the forward measure

In order to apply Proposition 18.1 and to compute the price
\[
\mathbb{E}^* \left[ e^{-\int_t^{T_j} r_s ds C} \bigg| \mathcal{F}_t \right] = P(t, T_j) \mathbb{E}_i[C | \mathcal{F}_t],
\]
of a random claim payoff \( C \), it can be useful to determine the dynamics of the underlying asset processes \( r_t, f(t, T, S) \), and \( P(t, T) \) via their stochastic differential equations written under the forward measure \( \hat{P}_i \).

As a consequence of Proposition 18.2 and (18.3), the dynamics of \( t \mapsto P(t, T_j) \) under \( \hat{P}_i \) is given by
\[
dP(t, T_j)^t \frac{dP(t, T_j)}{P(t, T_j)} = r_t dt + \zeta_i(t)\zeta_j(t) dt + \zeta_j(t)d\widehat{W}^i_t, \quad i, j = 1, 2, \ldots, n, \quad (18.6)
\]

where \((\widehat{W}^i_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion under \(\widehat{P}_i\), and we have

\[
P(t, T_j)
= P(0, T_j) \exp \left( \int_0^t r_s ds + \int_0^t \zeta_j(s) dW_s - \frac{1}{2} \int_0^t |\zeta_j(s)|^2 ds \right) \quad [\text{under } \mathbb{P}^*]
\]

\[
P(t, T_j)
= P(0, T_j) \exp \left( \int_0^t r_s ds + \int_0^t \zeta_j(s) d\widehat{W}_s^j + \frac{1}{2} \int_0^t |\zeta_j(s)|^2 ds \right)
\]

\[
P(t, T_j)
= P(0, T_j) \exp \left( \int_0^t r_s ds + \int_0^t \zeta_j(s) d\widehat{W}_s^i + \int_0^t \zeta_j(s) \zeta_i(s) ds - \frac{1}{2} \int_0^t |\zeta_j(s)|^2 ds \right)
\]

\[
P(t, T_j)
= P(0, T_j) \exp \left( \int_0^t r_s ds + \int_0^t \zeta_j(s) d\widehat{W}_s^i - \frac{1}{2} \int_0^t |\zeta_j(s) - \zeta_i(s)|^2 ds + \frac{1}{2} \int_0^t |\zeta_i(s)|^2 ds \right),
\]

\(t \in [0, T_j], i, j = 1, 2, \ldots, n\). Consequently, the forward price \(P(t, T_j)/P(t, T_i)\) can be written as

\[
\frac{P(t, T_j)}{P(t, T_i)}
= \frac{P(0, T_j)}{P(0, T_i)} \exp \left( \int_0^t (\zeta_j(s) - \zeta_i(s)) d\widehat{W}_s^j + \frac{1}{2} \int_0^t |\zeta_j(s) - \zeta_i(s)|^2 ds \right) \quad [\text{under } \widehat{P}^j]
\]

\[
\frac{P(t, T_j)}{P(t, T_i)}
= \frac{P(0, T_j)}{P(0, T_i)} \exp \left( \int_0^t (\zeta_j(s) - \zeta_i(s)) d\widehat{W}_s^i - \frac{1}{2} \int_0^t |\zeta_i(s) - \zeta_j(s)|^2 ds \right), \quad [\text{under } \widehat{P}^i]
\]

\(t \in [0, \min(T_i, T_j)], i, j = 1, 2, \ldots, n\), which also follows from Proposition 15.6.

**Short rate dynamics under the forward measure**

In case the short rate process \((r_t)_{t \in \mathbb{R}_+}\) is given as the (Markovian) solution to the stochastic differential equation

\[
dr_t = \mu(t, r_t) dt + \sigma(t, r_t) dW_t,
\]

by \((18.5)\) its dynamics will be given under \(\widehat{P}_i\) by

\[
dr_t = \mu(t, r_t) dt + \sigma(t, r_t) (\zeta_i(t) dt + d\widehat{W}_t^i)
\]

\[
= \mu(t, r_t) dt + \sigma(t, r_t) \zeta_i(t) dt + \sigma(t, r_t) d\widehat{W}_t^i. \quad (18.8)
\]

In the case of the Vasicek model, by \((16.23)\) we have
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\[ dr_t = (a - b r_t) dt + \sigma dW_t, \]

and

\[ \zeta_i(t) = -\frac{\sigma}{b} (1 - e^{-b(T_i-t)}), \quad 0 \leq t \leq T_i, \]
hence from (18.8) we have

\[ dr_t = (a - b r_t) dt - \frac{\sigma^2}{b} (1 - e^{-b(T_i-t)}) dt + \sigma d\tilde{W}_t^i \tag{18.9} \]

and we obtain

\[ \frac{dP(t, T_i)}{P(t, T_i)} = r_t dt + \frac{\sigma^2}{b^2} (1 - e^{-b(T_i-t)})^2 dt - \frac{\sigma}{b} (1 - e^{-b(T_i-t)}) d\tilde{W}_t^i \]

from (16.23).

18.2 Bond Options

The next proposition can be obtained as an application of the Margrabe formula (15.28) of Proposition 15.12 by taking \( X_t = P(t, T_j), \ N_t^{(i)} = P(t, T_i) \), and \( \tilde{X}_t = X_t / N_t^{(i)} = P(t, T_j) / P(t, T_i) \). In the Vasicek model, this formula has been first obtained in Jamshidian (1989).

We work with a standard Brownian motion \((W_t)_{t \in \mathbb{R}^+}\) under \(P^*\), generating the filtration \((\mathcal{F}_t)_{t \in \mathbb{R}^+}\), and an \((\mathcal{F}_t)_{t \in \mathbb{R}^+}\)-adapted short rate process \((r_t)_{t \in \mathbb{R}^+}\).

**Proposition 18.3.** Let \( 0 \leq T_i \leq T_j \) and assume as in (16.22) that the dynamics of the bond prices \( P(t, T_i) \), \( P(t, T_j) \) under \(P^*\) are given by

\[ \frac{dP(t, T_i)}{P(t, T_i)} = r_t dt + \zeta_i(t) dW_t, \quad \frac{dP(t, T_j)}{P(t, T_j)} = r_t dt + \zeta_j(t) dW_t, \]

where \((\zeta_i(t))_{t \in \mathbb{R}^+}\) and \((\zeta_j(t))_{t \in \mathbb{R}^+}\) are deterministic volatility functions. Then the price of a bond call option on \( P(T_i, T_j) \) with payoff

\[ C := (P(T_i, T_j) - \kappa)^+ \]

can be written as

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where \( v^2(t, T_i) := \int_t^{T_i} |\zeta_i(s) - \zeta_j(s)|^2 ds \) and

\[
\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy, \quad x \in \mathbb{R},
\]

is the Gaussian cumulative distribution function.

**Proof.** First, we note that using \( N_t^{(i)} := P(t, T_i) \) as a numéraire the price of a bond call option on \( P(T_i, T_j) \) with payoff \( F = (P(T_i, T_j) - \kappa)^+ \) can be written from Proposition 15.3 using the forward measure \( \mathbb{P}_t \), or directly by (15.5), as

\[
\mathbb{E}^* \left[ e^{-\int_t^{T_i} r_s ds} (P(T_i, T_j) - \kappa)^+ \left| \mathcal{F}_t \right. \right] = P(t, T_i) \mathbb{E}_t \left[ (P(T_i, T_j) - \kappa)^+ \left| \mathcal{F}_t \right. \right].
\]

Next, by (18.7) or by solving (15.13) in Proposition 15.6 we can write \( P(T_i, T_j) \) as the geometric Brownian motion

\[
P(T_i, T_j) = \frac{P(T_i, T_j)}{P(t, T_i)} = \frac{P(t, T_j)}{P(t, T_i)} \exp \left( \int_t^{T_i} (\zeta_j(s) - \zeta_i(s)) d\tilde{W}_s^i - \frac{1}{2} \int_t^{T_i} |\zeta_i(s) - \zeta_j(s)|^2 ds \right),
\]

under the forward measure \( \mathbb{P}_t \), and rewrite (18.11) as

\[
\mathbb{E}^* \left[ e^{-\int_t^{T_i} r_s ds} (P(T_i, T_j) - \kappa)^+ \left| \mathcal{F}_t \right. \right] = P(t, T_i) \mathbb{E}_t \left[ \left( \frac{P(t, T_j)}{P(t, T_i)} e^{\int_t^{T_i} (\zeta_j(s) - \zeta_i(s)) d\tilde{W}_s^i - \frac{1}{2} \int_t^{T_i} |\zeta_i(s) - \zeta_j(s)|^2 ds - \kappa} \right)^+ \left| \mathcal{F}_t \right. \right].
\]

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Since \((\zeta_i(s))_{s \in [0, T_i]} \) and \((\zeta_j(s))_{s \in [0, T_j]} \) in (18.3) are deterministic volatility functions, \(P(T_i, T_j)\) is a lognormal random variable given \(\mathcal{F}_t\) under \(\hat{\mathcal{P}}_i\) and we can use Lemma 7.8 to price the bond option by the zero-rate Black-Scholes formula

\[
Bl\left(P(t, T_j), \kappa P(t, T_i), v(t, T_i) / \sqrt{T_i - t}, 0, T_i - t\right)
\]

with underlying asset price \(P(t, T_j)\), strike price \(\kappa P(t, T_i)\), volatility parameter

\[
v(t, T_i) / \sqrt{T_i - t} = \sqrt{\int_t^{T_i} |\zeta_i(s) - \zeta_j(s)|^2 ds / T_i - t},
\]

time to maturity \(T_i - t\), and zero interest rate, which yields (18.10).

Note that from Corollary 15.14 the decomposition (18.10) gives the self-financing portfolio in the assets \(P(t, T_i)\) and \(P(t, T_j)\) for the claim with payoff \((P(T_i, T_j) - \kappa)^+\).

In the Vasicek case the above bond option price could also be computed from the joint distribution of \((r_T, \int_t^T r_s ds)\), which is Gaussian, or from the dynamics (18.6)-(18.9) of \(P(t, T)\) and \(r_t\) under \(\hat{\mathcal{P}}_i\), cf. § 7.3 of Privault (2012), and Kim (2002) for the CIR and other short rate models with correlated Brownian motions.

### 18.3 Caplet Pricing

A caplet is an option contract that offers protection against the fluctuations of a variable (or floating) rate with respect to a fixed rate \(\kappa\). The payoff of a caplet on the yield (or spot forward rate) \(f(T_i, T_i, T_i+1)\) with strike price \(\kappa\) can be written as

\[
(f(T_i, T_i, T_i+1) - \kappa)^+,
\]

priced at time \(t \in [0, T_i]\) from Proposition 15.3 using the forward measure \(\hat{\mathcal{P}}_i\) as

\[
\mathbb{E}^* \left[ e^{-\int_t^{T_i+1} r_s ds} (f(T_i, T_i, T_i+1) - \kappa)^+ | \mathcal{F}_t \right] = P(t, T_i+1) \mathbb{E}_{i+1} \left[ (f(T_i, T_i, T_i+1) - \kappa)^+ | \mathcal{F}_i \right],
\]

by taking \(N_t^{i+1} = P(t, T_i+1)\) as a numéraire. Next, we consider the caplet with payoff \((L(T_i, T_i, T_i+1) - \kappa)^+\) on the LIBOR rate

\[
L(t, T_i, T_i+1) = \frac{1}{T_i+1 - T_i} \left( \frac{P(t, T_i)}{P(t, T_i+1)} - 1 \right), \quad 0 \leq t \leq T_i < T_i+1,
\]
which is a deflated process using the forward numéraire process \((P(t, T_{i+1}))_{t \in [0,T_{i+1}]}\), and therefore a martingale under the probability measure \(\hat{\mathbb{P}}_{i+1}\) defined in (18.1), from Proposition 15.4. The caplet on \(L(T_i, T_i, T_{i+1})\) can be priced at time \(t \in [0,T_i]\) as

\[
\mathbb{E}^\ast \left[ e^{-\int_t^{T_{i+1}} r_s ds} (L(T_i, T_i, T_{i+1}) - \kappa)^+ \right| \mathcal{F}_t \right] = \mathbb{E}^\ast \left[ e^{-\int_t^{T_{i+1}} r_s ds} \left( \frac{1}{T_{i+1} - T_i} \left( \frac{P(t, T_i)}{P(t, T_{i+1})} - 1 \right) - \kappa \right)^+ \right| \mathcal{F}_t \right].
\] (18.13)

The next formula (18.15) is known as the Black caplet formula. It allows us to price and hedge a caplet using a portfolio based on the bonds \(P(t, T_i)\) and \(P(t, T_{i+1})\), cf. (18.16) below, when \(L(t, T_i, T_{i+1})\) is modeled in the BGM model of Section 17.5.

**Proposition 18.4.** Assume that \(L(t, T_i, T_{i+1})\) is modeled in the BGM model as

\[
\frac{dL(t, T_i, T_{i+1})}{L(t, T_i, T_{i+1})} = \gamma_i(t) d\hat{W}^{i+1}_{t},
\] (18.14)

\(0 \leq t \leq T_i, \ i = 1, 2, \ldots, n - 1\), where \(t \mapsto \gamma_i(t)\) is a deterministic volatility function of time, \(i = 1, 2, \ldots, n - 1\). The caplet on \(L(T_i, T_i, T_{i+1})\) is priced at time \(t \in [0,T_i]\) as

\[
\mathbb{E}^\ast \left[ e^{-\int_t^{T_{i+1}} r_s ds} (L(T_i, T_i, T_{i+1}) - \kappa)^+ \right| \mathcal{F}_t \right] = P(t, T_{i+1}) L(t, T_i, T_{i+1}) \Phi(d_+(t, T_i)) - \kappa P(t, T_{i+1}) \Phi(d_-(t, T_i)),
\] (18.15)

\(0 \leq t \leq T_{i+1}\), where

\[
d_+(t, T_i) = \frac{\log(L(t, T_i, T_{i+1})/\kappa) + (T_i - t)\sigma_i^2(t, T_i)/2}{\sigma_i(t, T_i)\sqrt{T_i - t}},
\]

and

\[
d_-(t, T_i) = \frac{\log(L(t, T_i, T_{i+1})/\kappa) - (T_i - t)\sigma_i^2(t, T_i)/2}{\sigma_i(t, T_i)\sqrt{T_i - t}},
\]

and

\[
|\sigma_i(t, T_i)|^2 = \frac{1}{T_i - t} \int_t^{T_i} |\gamma_i|^2(s)ds.
\]

**Proof.** Taking \(P(t, T_{i+1})\) as a numéraire, the forward price

\[
\hat{X}_t := \frac{P(t, T_i)}{P(t, T_{i+1})} = 1 + (T_{i+1} - T_i) L(T_i, T_i, T_{i+1})
\]

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and the forward LIBOR rate process \((L(t, T_i, T_{i+1})_{t \in [0, T_i]}\) are martingales under \(\hat{P}_{i+1}\) by Proposition 15.4. More precisely, by (18.14) we have

\[
L(T_i, T_i, T_{i+1}) = L(t, T_i, T_{i+1}) \exp \left( \int_t^{T_i} \gamma_i(s) d\hat{W}_s + \frac{1}{2} \int_t^{T_i} |\gamma_i(s)|^2 ds \right),
\]

\(0 \leq t \leq T_i\), i.e. \(t \mapsto L(t, T_i, T_{i+1})\) is a geometric Brownian motion with volatility \(\gamma_i(t)\) under \(\hat{P}_{i+1}\). Hence by (18.12), since \(N_{T_{i+1}}^{(t+1)} = 1\), we have

\[
\mathbb{E}^* \left[ e^{-\int_t^{T_{i+1}} r_s ds} (L(T_i, T_i, T_{i+1}) - \kappa)^+ \bigg| \mathcal{F}_t \right] = P(t, T_{i+1}) \mathbb{E}^* \left[ (L(T_i, T_i, T_{i+1}) - \kappa)^+ \bigg| \mathcal{F}_t \right] = P(t, T_{i+1}) (L(t, T_i, T_{i+1}) \Phi(d_+(t, T_i)) - \kappa \Phi(d_-(t, T_i))) = P(t, T_{i+1}) \text{Bl}(L(t, T_i, T_{i+1}), \kappa, \sigma_i(t, T_i), 0, T_i - t),
\]

t \in [0, T_i]\), where

\[
\text{Bl}(x, \kappa, \sigma, r, \tau) = x \Phi(d_+(t, T_i)) - \kappa e^{-r\tau} \Phi(d_-(t, T_i))
\]
is the Black-Scholes function, with

\[
|\sigma_i(t, T_i)|^2 = \frac{1}{T_i - t} \int_t^{T_i} |\gamma_i|^2(s) ds.
\]

In general we may also write (18.15) as

\[
(T_{i+1} - T_i) \mathbb{E}^* \left[ e^{-\int_t^{T_{i+1}} r_s ds} (L(T_i, T_i, T_{i+1}) - \kappa)^+ \bigg| \mathcal{F}_t \right] = P(t, T_{i+1}) \left( \frac{P(t, T_i)}{P(t, T_{i+1})} - 1 \right) \Phi(d_+(t, T_i)) - \kappa(T_{i+1} - T_i) P(t, T_{i+1}) \Phi(d_-(t, T_i)) = (P(t, T_i) - P(t, T_{i+1})) \Phi(d_+(t, T_i)) - \kappa(T_{i+1} - T_i) P(t, T_{i+1}) \Phi(d_-(t, T_i)),
\]

and by Corollary 15.14 this gives the self-financing portfolio strategy

\[
(\Phi(d_+(t, T_i)), -\Phi(d_+(t, T_i)) - \kappa(T_{i+1} - T_i) \Phi(d_-(t, T_i))) \quad (18.16)
\]
in the bonds \((P(t, T_i), P(t, T_{i+1}))\) with maturities \(T_i\) and \(T_{i+1}\), cf. Corollary 15.15 and Privault and Teng (2012).

The formula (18.15) is also known as the Black (1976) formula Black (1976) when applied to options on underlying futures or forward contracts on commodities, which are modeled according to (18.14). In this case, the bond price \(P(t, T_{i+1})\) can be simply modeled as \(P(t, T_{i+1}) = e^{-r(T_{i+1} - t)}\) and (18.15)
becomes
\[ e^{-r(T_{i+1} - t)} L(t, T_i, T_{i+1}) \Phi(d_+(t, T_i)) - \kappa e^{-r(T_{i+1} - t)} \Phi(d_-(t, T_i)), \]
where \( L(t, T_i, T_{i+1}) \) is the underlying future price.

Floorlets

Similarly, a floorlet on \( L(T_i, T_i, T_{i+1}) \) with strike price \( \kappa \) is a contract with payoff \( \kappa - L(T_i, T_i, T_{i+1}) \), priced at time \( t \in [0, T_i] \) as

\[
E^* \left[ e^{-\int_t^{T_{i+1}} r_s ds} (\kappa - L(T_i, T_i, T_{i+1}))^+ \middle| F_t \right] = P(t, T_{i+1}) E_{i+1}^* \left[ (\kappa - L(T_i, T_i, T_{i+1}))^+ \middle| F_t \right] = P(t, T_{i+1}) (\kappa \Phi(-d_-(T_i - t)) - L(t, T_i, T_{i+1}) \Phi(-d_+(T_i - t))), \quad (18.17)
\]

0 \leq t \leq T_{i+1}. Floorlets are analog to put options and can be similarly priced by the call/put parity in the Black-Scholes formula.

Cap Pricing

More generally, one can consider caps that are relative to a given tenor structure \( \{T_1, T_2, \ldots, T_n\} \), with discounted payoff

\[
\sum_{k=1}^{n-1} (T_{k+1} - T_k) e^{-\int_t^{T_{k+1}} r_s ds} (L(T_k, T_k, T_{k+1}) - \kappa)^+. \]

Pricing formulas for caps are easily deduced from analog formulas for caplets, since the payoff of a cap can be decomposed into a sum of caplet payoffs. Thus, the cap price at time \( t \in [0, T_1] \) is given by

\[
E^* \left[ \sum_{k=1}^{n-1} (T_{k+1} - T_k) e^{-\int_t^{T_{k+1}} r_s ds} (L(T_k, T_k, T_{k+1}) - \kappa)^+ \middle| F_t \right] = \sum_{k=1}^{n-1} (T_{k+1} - T_k) E^* \left[ e^{-\int_t^{T_{k+1}} r_s ds} (L(T_k, T_k, T_{k+1}) - \kappa)^+ \middle| F_t \right] = \sum_{k=1}^{n-1} (T_{k+1} - T_k) P(t, T_{k+1}) \hat{E}_{k+1}^* \left[ (L(T_k, T_k, T_{k+1}) - \kappa)^+ \middle| F_t \right]. \quad (18.18)
\]

In the BGM model (18.14) the cap with payoff
Pricing of Interest Rate Derivatives

\[ \sum_{k=1}^{n-1} (T_{k+1} - T_k) (L(T_k, T_k, T_{k+1}) - \kappa)^+ \]

can be priced at time \( t \in [0, T_1] \) by the Black formula

\[ \sum_{k=1}^{n-1} (T_{k+1} - T_k) P(t, T_{k+1}) Bl(L(t, T_k, T_{k+1}), \kappa, \sigma_k(t, T_k), 0, T_k - t), \]

where

\[ |\sigma_k(t, T_k)|^2 \leq 1 \]

and

\[ T_k - t \]

with in particular, when \( j = i + 1 \),

\[ P(t, T_i, T_{i+1}) = (T_{i+1} - T_i) P(t, T_{i+1}), \quad 0 \leq t \leq T_i. \]

1 \( \leq i < n \). The annuity numéraire can be also used to price a bond ladder. It satisfies the following martingale property, which can be proved by linearity and the fact that \( t \mapsto e^{-\int_0^t r_s \, ds} P(t, T_k) \) is a martingale for all \( k = 1, 2, \ldots, n \), under Assumption (A).

Remark 18.5. The discounted annuity numéraire

\[ t \mapsto e^{-\int_0^t r_s \, ds} P(t, T_i, T_j) = e^{-\int_0^t r_s \, ds} \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}), \quad 0 \leq t \leq T_i, \]

is a martingale under \( P^* \).

The forward swap measure \( \widehat{P}_{i,j} \) is defined, according to Definition 15.1, by

\[ \frac{d\widehat{P}_{i,j}}{dP^*} := e^{-\int_0^{T_i} r_s \, ds} \frac{N_t^{i,j}}{N_0^{i,j}} = e^{-\int_0^{T_i} r_s \, ds} \frac{P(T_i, T_i, T_j)}{P(0, T_i, T_j)}, \quad 1 \leq i < j \leq n. \]

Remark 18.6. We have

\[ \widehat{P}_{i,j} \]

18.4 Forward Swap Measures

In this section we introduce the forward measures to be used for the pricing of swaptions, and we study their properties. We start with the definition of the annuity numéraire

\[ N_t^{i,j} := P(t, T_i, T_j) = \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}), \quad 0 \leq t \leq T_i, \quad (18.19) \]

with in particular, when \( j = i + 1 \),

\[ P(t, T_i, T_{i+1}) = (T_{i+1} - T_i) P(t, T_{i+1}), \quad 0 \leq t \leq T_i. \]
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\[ E^* \left[ \frac{d\tilde{P}_{i,j}}{dP^*} \bigg| \mathcal{F}_t \right] = \frac{1}{P(0, T_i, T_j)} E^* \left[ e^{-\int_0^T r_s ds} P(T_i, T_i, T_j) \bigg| \mathcal{F}_t \right] = \frac{1}{P(0, T_i, T_j)} E^* \left[ e^{-\int_0^T r_s ds} \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(T_i, T_{k+1}) \bigg| \mathcal{F}_t \right] = \frac{1}{P(0, T_i, T_j)} \sum_{k=i}^{j-1} (T_{k+1} - T_k) E^* \left[ e^{-\int_0^T r_s ds} P(T_i, T_{k+1}) \bigg| \mathcal{F}_t \right] = \frac{1}{P(0, T_i, T_j)} e^{-\int_0^T r_s ds} \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) = e^{-\int_0^T r_s ds} \frac{P(t, T_i, T_j)}{P(0, T_i, T_j)}, \]

\[ 0 \leq t \leq T_i, \text{ by Remark 18.5, and} \]

\[ \frac{d\tilde{P}_{i,j}}{dP^*} \bigg| \mathcal{F}_t = e^{-\int_0^T r_s ds} \frac{P(T_i, T_i, T_j)}{P(t, T_i, T_j)}, \quad 0 \leq t \leq T_{i+1}, \quad (18.21) \]

by Relation (15.3) in Lemma 15.2.

We also know that the deflated process

\[ t \mapsto v^{i,j}_k(t) := \frac{P(t, T_k)}{P(t, T_i, T_j)}, \quad i, j, k = 1, 2, \ldots, n, \]

is an \( \mathcal{F}_t \)-martingale under \( \tilde{P}_{i,j} \) by Proposition 15.4. It follows that the LIBOR swap rate

\[ S(t, T_i, T_j) = \frac{P(t, T_i) - P(t, T_j)}{P(t, T_i, T_j)} = v^{i,j}_i(t) - v^{i,j}_j(t), \quad 0 \leq t \leq T_i, \]

defined in Proposition 17.8 is also a martingale under the forward swap measure \( \tilde{P}_{i,j} \).

The following pricing formula is then stated for a given integrable claim with payoff of the form \( P(T_i, T_i, T_j)F \), using the forward swap measure \( \tilde{P}_{i,j} \):

\[ E^* \left[ e^{-\int_0^T r_s ds} P(T_i, T_i, T_j)F \bigg| \mathcal{F}_t \right] = P(t, T_i, T_j) E^* \left[ \frac{d\tilde{P}_{i,j}}{dP^*} \bigg| \mathcal{F}_t \right] \mathcal{F}_t \]

\[ = P(t, T_i, T_j) E_{i,j} \big[ F \big| \mathcal{F}_t \big], \quad \text{(18.22)} \]

after applying (18.20) and (18.21) on the last line, or Proposition 15.3.
18.5 Swaption Pricing on the LIBOR

A swaption on the forward rate $f(T_i, T_k, T_{k+1})$ is a contract meant for protection against a risk based on an interest rate swap, and has the payoff

$$\left( \sum_{k=i}^{j-1} (T_{k+1} - T_k) e^{-\int_{T_i}^{T_{k+1}} r_s ds} ( f(T_i, T_k, T_{k+1}) - \kappa ) \right)^+$$

at time $T_i$, i.e.

$$e^{-\int_{T_i}^{T_{i+1}} r_s ds} \left( \sum_{k=i}^{j-1} (T_{k+1} - T_k) e^{-\int_{T_i}^{T_{k+1}} r_s ds} ( f(T_i, T_k, T_{k+1}) - \kappa ) \right)^+$$

$$= \left( \sum_{k=i}^{j-1} (T_{k+1} - T_k) e^{-\int_{T_i}^{T_{k+1}} r_s ds} ( f(T_i, T_k, T_{k+1}) - \kappa ) \right)^+$$

after discounting from time $T_i$ to time $t \in [0, T_i]$. This swaption can be priced at time $t \in [0, T_i]$ under the risk-neutral probability measure $\mathbb{P}^*$ as

$$\mathbb{E}^* \left[ \left( \sum_{k=i}^{j-1} (T_{k+1} - T_k) e^{-\int_{T_i}^{T_{k+1}} r_s ds} ( f(T_i, T_k, T_{k+1}) - \kappa ) \right)^+ \bigg| \mathcal{F}_t \right]$$

$$= \mathbb{E}^* \left[ e^{-\int_{T_i}^{T_{i+1}} r_s ds} \left( \sum_{k=i}^{j-1} (T_{k+1} - T_k) e^{-\int_{T_i}^{T_{k+1}} r_s ds} ( f(T_i, T_k, T_{k+1}) - \kappa ) \right)^+ \bigg| \mathcal{F}_t \right]$$

$$= \mathbb{E}^* \left[ e^{-\int_{T_i}^{T_{i+1}} r_s ds} \mathbb{E}^* \left[ \left( \sum_{k=i}^{j-1} (T_{k+1} - T_k) e^{-\int_{T_i}^{T_{k+1}} r_s ds} ( f(T_i, T_k, T_{k+1}) - \kappa ) \right)^+ \bigg| \mathcal{F}_{T_i} \right] \bigg| \mathcal{F}_t \right].$$

When $j = i + 1$, the swaption price (18.25) coincides with the price at time $t$ of a caplet on $[T_i, T_{i+1}]$ since

$$\mathbb{E}^* \left[ e^{-\int_{T_i}^{T_{i+1}} r_s ds} ((T_{i+1} - T_i) P(T_i, T_{i+1}) ( f(T_i, T_i, T_{i+1}) - \kappa ))^+ \bigg| \mathcal{F}_t \right]$$

$$= (T_{i+1} - T_i) \mathbb{E}^* \left[ e^{-\int_{T_i}^{T_{i+1}} r_s ds} P(T_i, T_{i+1}) ( f(T_i, T_i, T_{i+1}) - \kappa )^+ \bigg| \mathcal{F}_t \right]$$

$$= (T_{i+1} - T_i) \mathbb{E}^* \left[ e^{-\int_{T_i}^{T_{i+1}} r_s ds} \mathbb{E}^* \left[ e^{-\int_{T_i}^{T_{i+1}} r_s ds} \bigg| \mathcal{F}_{T_i} \right] ( f(T_i, T_i, T_{i+1}) - \kappa )^+ \bigg| \mathcal{F}_t \right]$$

$$= (T_{i+1} - T_i) \mathbb{E}^* \left[ e^{-\int_{T_i}^{T_{i+1}} r_s ds} e^{-\int_{T_i}^{T_{i+1}} r_s ds} ( f(T_i, T_i, T_{i+1}) - \kappa )^+ \bigg| \mathcal{F}_{T_i} \right] \bigg| \mathcal{F}_t \right]$$

$$= (T_{i+1} - T_i) \mathbb{E}^* \left[ e^{-\int_{T_i}^{T_{i+1}} r_s ds} ( f(T_i, T_i, T_{i+1}) - \kappa )^+ \bigg| \mathcal{F}_t \right].$$

(18.24)
In the sequel and in practice the price (18.23) of the swaption will be evaluated as
\[
\mathbb{E}^* \left[ e^{-\int_t^{T_i} r_s ds} \left( \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(T_i, T_{k+1}) (f(T_i, T_k, T_{k+1}) - \kappa) \right) \bigg| \mathcal{F}_t \right],
\]
(18.25)
t \in [0, T_i], \ i.e. the discount factor \( e^{-\int_t^{T_i} r_s ds} \) is replaced with its conditional expectation given \( \mathcal{F}_t \), which is the bond price \( P(T_i, T_{k+1}) \).

**Proposition 18.7.** The above term (18.25) can be upper bounded by the cap price (18.18) as
\[
\mathbb{E}^* \left[ e^{-\int_t^{T_i} r_s ds} \left( \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(T_i, T_{k+1}) (f(T_i, T_k, T_{k+1}) - \kappa) \right) \bigg| \mathcal{F}_t \right] 
\leq \mathbb{E}^* \left[ \sum_{k=i}^{j-1} (T_{k+1} - T_k) e^{-\int_t^{T_{k+1}} r_s ds} (f(T_i, T_k, T_{k+1}) - \kappa)^+ \bigg| \mathcal{F}_t \right],
\]

\( 0 \leq t \leq T_i \).

**Proof.** Using the inequality
\[
(x_1 + x_2 + \cdots + x_m)^+ \leq x_1^+ + x_2^+ + \cdots + x_m^+, \quad x_1, x_2, \ldots, x_m \in \mathbb{R},
\]
we have
\[
\mathbb{E}^* \left[ e^{-\int_t^{T_i} r_s ds} \left( \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(T_i, T_{k+1}) (f(T_i, T_k, T_{k+1}) - \kappa) \right) \bigg| \mathcal{F}_t \right] 
\leq \mathbb{E}^* \left[ \sum_{k=i}^{j-1} (T_{k+1} - T_k) e^{-\int_t^{T_{k+1}} r_s ds} P(T_i, T_{k+1}) (f(T_i, T_k, T_{k+1}) - \kappa)^+ \bigg| \mathcal{F}_t \right]
\]
\[
= \sum_{k=i}^{j-1} (T_{k+1} - T_k) \mathbb{E}^* \left[ e^{-\int_t^{T_{k+1}} r_s ds} P(T_i, T_{k+1}) (f(T_i, T_k, T_{k+1}) - \kappa)^+ \bigg| \mathcal{F}_t \right]
\]
\[
= \sum_{k=i}^{j-1} (T_{k+1} - T_k) \mathbb{E}^* \left[ e^{-\int_t^{T_{k+1}} r_s ds} \mathbb{E}^* \left[ e^{-\int_t^{T_{k+1}} r_s ds} \bigg| \mathcal{F}_{T_i} \right] (f(T_i, T_k, T_{k+1}) - \kappa)^+ \bigg| \mathcal{F}_t \right]
\]
\[
= \sum_{k=i}^{j-1} (T_{k+1} - T_k) \mathbb{E}^* \left[ e^{-\int_t^{T_{k+1}} r_s ds} (f(T_i, T_k, T_{k+1}) - \kappa)^+ \bigg| \mathcal{F}_{T_i} \bigg| \mathcal{F}_t \right]
\]
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\[ \sum_{k=0}^{j-1} (T_{k+1} - T_k) E^* \left[ e^{-\int_t^{T_{k+1}} r_s \, ds} \left( f(T_i, T_k, T_{k+1}) - \kappa \right)^+ \bigg| F_t \right] \]

\[ = E^* \left[ \sum_{k=0}^{j-1} (T_{k+1} - T_k) e^{-\int_t^{T_{k+1}} r_s \, ds} \left( f(T_i, T_k, T_{k+1}) - \kappa \right)^+ \bigg| F_t \right], \]

for \( 0 \leq t \leq T_i \).

In case the forward rate \( f(t, T, S) \) is replaced with the LIBOR rate \( L(t, T, S) \) defined in Proposition 17.8, the payoff of the swaption can be rewritten as in the following lemma which is a direct consequence of the definition of the swap rate \( S(T_i, T_i, T_j) \).

**Lemma 18.8.** The swaption payoff (18.25) can be rewritten as

\[ \left( \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(T_i, T_{k+1}) (L(T_i, T_k, T_{k+1}) - \kappa) \right)^+ \]

\[ = (P(T_i, T_i) - P(T_i, T_j) - \kappa P(T_i, T_j))^+ \quad (18.26) \]

\[ = P(T_i, T_i, T_j) (S(T_i, T_i, T_j) - \kappa)^+. \quad (18.27) \]

**Proof.** The relation

\[ \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) (L(t, T_k, T_{k+1}) - S(t, T_i, T_j)) = 0 \]

that defines the forward swap rate \( S(t, T_i, T_j) \) shows that

\[ \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) L(t, T_k, T_{k+1}) \]

\[ = S(t, T_i, T_j) \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) \]

\[ = P(t, T_i, T_j) S(t, T_i, T_j) \]

\[ = P(t, T_i) - P(t, T_j) \]

as in the proof of Corollary 17.8, hence by the definition (18.19) of \( P(t, T_i, T_j) \) we have

\[ \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) (L(t, T_k, T_{k+1}) - \kappa) \]

\[ = P(t, T_i) - P(t, T_j) - \kappa P(t, T_i, T_j) \]

\[ = P(t, T_i, T_j) (S(t, T_i, T_j) - \kappa), \]
and for $t = T_i$ we get
\[
\left( \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(T_i, T_{k+1}) (L(T_i, T_k, T_{k+1}) - \kappa) \right)^+ 
= P(T_i, T_i, T_j) (S(T_i, T_i, T_j) - \kappa)^+.
\]

\[\square\]

The next proposition simply states that a swaption on the LIBOR rate can be priced as a European call option on the swap rate $S(T_i, T_i, T_j)$ under the forward swap measure $\hat{P}_{i,j}$.

**Proposition 18.9.** The price (18.25) of the European swaption with payoff
\[
\left( \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(T_i, T_{k+1}) (L(T_i, T_k, T_{k+1}) - \kappa) \right)^+ \tag{18.28}
\]
on the LIBOR market can be written under the forward swap measure $\hat{P}_{i,j}$ as
\[
P(t, T_i, T_j) \hat{E}_{i,j} [(S(T_i, T_i, T_j) - \kappa)^+ | \mathcal{F}_t], \quad 0 \leq t \leq T_i.
\]

**Proof.** As a consequence of (18.22) and Lemma 18.8, we find
\[
\lim_{i \to \infty} e^{-\int_t^{T_i} r_s ds} \left( \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(T_i, T_{k+1}) (L(T_i, T_k, T_{k+1}) - \kappa) \right)^+ | \mathcal{F}_t 
\]
\[
= \lim_{i \to \infty} e^{-\int_t^{T_i} r_s ds} (P(T_i, T_i) - P(T_i, T_j) - \kappa P(T_i, T_i, T_j))^+ | \mathcal{F}_t \tag{18.29}
\]
\[
= P(t, T_i, T_j) \hat{E}_{i,j} \left[ \left( S(T_i, T_i, T_j) - \kappa \right)^+ | \mathcal{F}_t \right] \tag{18.30}
\]

In the next Proposition 18.10, we price a swaption with payoff (18.28) or equivalently (18.27), by modeling the swap rate $(S(t, T_i, T_j))_{0 \leq t \leq T_i}$ using standard Brownian motion $(\hat{W}^i_{t,j})_{0 \leq t \leq T_i}$ under the forward measure $\hat{P}_{i,j}$, see Exercise 18 for swaption pricing without the Black-Scholes formula.

**Proposition 18.10.** Assume that the LIBOR swap rate (17.17) is modeled as a geometric Brownian motion under $\hat{P}_{i,j}$, i.e.
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\[ dS(t, T_i, T_j) = S(t, T_i, T_j)\tilde{\sigma}_{i,j}(t)d\tilde{W}^{i,j}_t, \]

where \((\tilde{\sigma}_{i,j}(t))_{t \in \mathbb{R}_+}\) is a deterministic volatility function of time. Then the swaption with payoff

\[ (P(T, T_i) - P(T, T_j) - \kappa P(T_i, T_j))^+ = P(T, T_i, T_j) (S(T_i, T_i, T_j) - \kappa)^+ \]

can be priced using the Black-Scholes formula as

\[
\mathbb{E}^* \left[ e^{-\int_t^{T_i} r_s ds} P(T_i, T_i, T_j) (S(T_i, T_i, T_j) - \kappa)^+ \bigg| \mathcal{F}_t \right]
\]

\[
= (P(t, T_i) - P(t, T_j)) \Phi_+ (d_+(T_i - t))
\]

\[ -\kappa \Phi_- (d_- (T_i - t)) \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}), \]

where

\[
d_+(T_i - t) = \frac{\log(S(t, T_i, T_j)/\kappa) + \sigma^2_{i,j}(t, T_i) / 2}{\sigma_{i,j}(t, T_i) \sqrt{T_i - t}},
\]

and

\[
d_- (T_i - t) = \frac{\log(S(t, T_i, T_j)/\kappa) - \sigma^2_{i,j}(t, T_i) / 2}{\sigma_{i,j}(t, T_i) \sqrt{T_i - t}},
\]

and

\[
|\sigma_{i,j}(t, T_i)|^2 = \frac{1}{T_i - t} \int_t^{T_i} |\tilde{\sigma}(s)|^2 ds, \quad 0 \leq t \leq T.
\]

**Proof.** Since \(S(t, T_i, T_j)\) is a geometric Brownian motion with volatility function \((\tilde{\sigma}(t))_{t \in \mathbb{R}_+}\) under \(\hat{\mathbb{P}}_{i,j}\), by (18.26)-(18.27) and (18.29)-(18.30) we have

\[
\mathbb{E}^* \left[ e^{-\int_t^{T_i} r_s ds} P(T_i, T_i, T_j) (S(T_i, T_i, T_j) - \kappa)^+ \bigg| \mathcal{F}_t \right]
\]

\[
= \mathbb{E}^* \left[ e^{-\int_t^{T_i} r_s ds} (P(T_i) - P(T_j) - \kappa P(T_i, T_j))^+ \bigg| \mathcal{F}_t \right]
\]

\[
= P(t, T_i, T_j) \mathbb{E}_{i,j} [(S(T_i, T_i, T_j) - \kappa)^+] \bigg| \mathcal{F}_t \bigg]
\]

\[
= P(t, T_i, T_j) \text{Bl}(S(t, T_i, T_j), \kappa, \sigma_{i,j}(t, T_i), 0, T_i - t)
\]

\[
= P(t, T_i, T_j) (S(t, T_i, T_j) \Phi_+ (t, S(t, T_i, T_j)) - \kappa \Phi_- (t, S(t, T_i, T_j)))
\]

\[
= (P(t, T_i) - P(t, T_j)) \Phi_+ (t, S(t, T_i, T_j)) - \kappa P(t, T_i, T_j) \Phi_- (t, S(t, T_i, T_j))
\]

\[
= (P(t, T_i) - P(t, T_j)) \Phi_+ (t, S(t, T_i, T_j))
\]

\[ -\kappa \Phi_- (t, S(t, T_i, T_j)) \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}). \]
In addition, the hedging strategy

\[
\Phi_+ (t, S(t, T_i, T_j)), -\kappa \Phi_- (t, S(t, T_i, T_j))(T_{i+1} - T_i), \ldots \\
\ldots, -\kappa \Phi_- (t, S(t, T_i, T_j))(T_{j-1} - T_{j-2}), -\Phi_+ (t, S(t, T_i, T_j))
\]

based on the assets \((P(t, T_i), \ldots, P(t, T_j))\) is self-financing by Corollary 15.15, cf. also Privault and Teng (2012).

Swaption prices can also be computed by an approximation formula, from the exact dynamics of the swap rate \(S(t, T_i, T_j)\) under \(\mathbb{P}_{i,j}\), based on the bond price dynamics of the form (18.3), cf. Schoenmakers (2005), page 17.

Swaption volatilities can be estimated from swaption prices as implied volatilities from the Black pricing formula:

\[
\text{Fig. 18.1: Implied swaption volatilities.}
\]

Implied swaption volatilities can then be used to calibrate the BGM model, cf. Schoenmakers (2005), Privault and Wei (2009), § 11.4 of Privault (2012).

Bermudan swaption pricing in Quantlib

The Bermudan swaption on the tenor structure \(\{T_i, \ldots, T_j\}\) is priced as the supremum

\[
\sup_{\tau \in \{T_i, \ldots, T_{j-1}\}} \mathbb{E}^* \left[ e^{-\int_{\tau}^T r_s ds} \left( \sum_{k=\tau}^{j-1} (T_{k+1} - T_k)P(\tau, T_{k+1})(L(\tau, T_k, T_{k+1}) - \kappa) \right)^+ \bigg| \mathcal{F}_t \right] \\
= \sup_{\tau \in \{T_i, \ldots, T_{j-1}\}} \mathbb{E}^* \left[ e^{-\int_{\tau}^T r_s ds} P(\tau, \tau) - P(\tau, T_j) - \kappa P(\tau, T_j) \right]^+ \bigg| \mathcal{F}_t \\
= \sup_{\tau \in \{T_i, \ldots, T_{j-1}\}} \mathbb{E}^* \left[ e^{-\int_{\tau}^T r_s ds} P(\tau, T_j) (S(\tau, \tau, T_j) - \kappa)^+ \bigg| \mathcal{F}_t \right],
\]

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where the supremum is over all stopping times $\tau$ taking values in $\{T_i, \ldots, T_j\}$.

Bermudan swaptions can be priced using this code* in (R)quantlib, with the following output:

**Summary of pricing results for Bermudan Swaption**

Price (in bp) of Bermudan swaption is 24.92137
Stike is NULL (ATM strike is 0.05 )
Model used is: Hull-White using analytic formulas
Calibrated model parameters are:
a = 0.04641
sigma = 0.005869

This modified code† can be used in particular the pricing of ordinary swaptions, with the output:

**Summary of pricing results for Bermudan Swaption**

Price (in bp) of Bermudan swaption is 22.45436
Stike is NULL (ATM strike is 0.05 )
Model used is: Hull-White using analytic formulas
Calibrated model parameters are:
a = 0.07107
sigma = 0.006018

**Exercises**

Exercise 18.1 Consider a floorlet on a three-month LIBOR rate in nine month’s time, with a notional principal amount of $1 million. The term structure is flat at 4.95% per year with discrete compounding, and the volatility of the forward LIBOR rate in nine months is 10%. The annual floor rate with simple compounding is 4.5% and the floorplet price is quoted in basis points (one basis point = 0.01%).

a) What are the key assumptions on the LIBOR rate in nine month in order to apply Black’s formula to price this floorlet?
b) Compute the price of this floorlet using Black’s formula as an application of Proposition 18.4 and (18.17), using the functions $\Phi(d_+)$ and $\Phi(d_-)$.

Exercise 18.2 Consider a European swaption giving its holder the right to enter into a 3-year annual pay swap in four years, where a fixed rate of 5%
is paid and the LIBOR rate is received. Assume that the yield curve is flat at 5% per annum with annual compounding and the volatility of the swap rate is 20%. The notional principal is $10 millions and the swaption price is quoted in basis points.

a) What are the key assumptions in order to apply Black’s formula to value this swaption?
b) Compute the price of this swaption using Black’s formula as an application of Proposition 18.10, using the functions $\Phi(d_+)$ and $\Phi(d_-)$.

Exercise 18.3 Consider two bonds with maturities $T_1$ and $T_2$, $T_1 < T_2$, which follow the stochastic differential equations

$$dP(t, T_1) = r_t P(t, T_1) dt + \zeta_1(t) P(t, T_1) dW_t$$

and

$$dP(t, T_2) = r_t P(t, T_2) dt + \zeta_2(t) P(t, T_2) dW_t.$$

a) Using Itô calculus, show that the forward process $P(t, T_2)/P(t, T_1)$ is a driftless geometric Brownian motion driven by $d\hat{W}_t := dW_t - \zeta_1(t) dt$ under the $T_1$-forward measure $P^*$.
b) Compute the price $E^* \left[ e^{-\int_0^{T_1} r_s ds} (K - P(T_1, T_2))^+ \right]$ of a bond put option using change of numéraire and the Black-Scholes formula.

Hint: Given $X$ a centered Gaussian random variable with mean $m$ and variance $v^2$ given $\mathcal{F}_t$, we have:

$$E \left[ (\kappa - e^X)^+ \mid \mathcal{F}_t \right] = \kappa \Phi \left( \frac{v}{2} + \frac{1}{v} (m + v^2/2 - \log \kappa) \right) - e^{m+v^2/2} \Phi \left( -\frac{v}{2} + \frac{1}{v} (m + v^2/2 - \log \kappa) \right).$$

Exercise 18.4 Given two bonds with maturities $T$, $S$ and prices $P(t, T)$, $P(t, S)$, consider the LIBOR rate

$$L(t, T, S) := \frac{P(t, T) - P(t, S)}{(S - T) P(t, S)}$$

at time $t \in [0, T]$, modeled as

$$dL(t, T, S) = \mu_t L(t, T, S) dt + \sigma L(t, T, S) dW_t, \quad 0 \leq t \leq T, \quad (18.31)$$

where $(W_t)_{t \in [0, T]}$ is a standard Brownian motion under the risk-neutral probability measure $P^*$, $\sigma > 0$ is a constant, and $(\mu_t)_{t \in [0, T]}$ is an adapted process. Let

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F_t = \mathbb{E}^*[e^{-\int^T_0 r_s ds} (\kappa - L(T, T, S))^+] \mid F_t]

denote the price at time t of a floorlet option with strike price \( \kappa \), maturity \( T \), and payment date \( S \).

a) Rewrite the value of \( F_t \) using the forward measure \( \hat{\mathbb{P}}_S \) with maturity \( S \).

b) What is the dynamics of \( L(t, T, S) \) under the forward measure \( \hat{\mathbb{P}}_S \)?

c) Write down the value of \( F_t \) using the Black-Scholes formula.

Hint. Given \( X \) a centered Gaussian random variable with variance \( v^2 \) we have

\[
\mathbb{E}^*[ (\kappa - e^{m + X})^+] = \kappa \Phi(- (m - \log \kappa) / v) - e^{m + v^2 / 2} \Phi(- v - (m - \log \kappa) / v),
\]

where \( \Phi \) denotes the Gaussian cumulative distribution function.

Exercise 18.5 Jamshidian’s trick. Consider a European swaption with payoff

\[
(P(T_i, T_i) - P(T_i, T_j) - \kappa P(T_i, T_i, T_j))^+ = \left(1 - \kappa \sum_{k=i}^{j-1} c_{k+1} P(T_i, T_{k+1}) \right)^+,
\]

where \( c_j \) contains the final coupon payment. We assume that the bond prices are functions \( P(T_i, T_{k+1}) = F_{k+1}(T_i, r_{T_i}) \) of \( r_{T_i} \) that are increasing in the variable \( r_{T_i} \), for all \( k = i, i+1, \ldots, j-1 \), where \( (r_t)_{t\in\mathbb{R}^+} \) denotes the short rate process. Show, by choosing \( \gamma_\kappa \) such that

\[
\kappa \sum_{k=i}^{j-1} c_{k+1} F_{k+1}(T_i, \gamma_\kappa) = 1,
\]

that the swaption can be priced as a sum of weighted bond put options under the forward measure \( \hat{\mathbb{P}}_i \) with numéraire \( N_t^{(i)} := P(t, T_i) \).

Exercise 18.6 Path freezing. Consider \( n \) bonds with prices \( (P(t, T_i))_{i=1,\ldots,n} \) and the bond option with payoff

\[
\left(\sum_{i=2}^n c_i P(T_0, T_i) - \kappa P(T_0, T_1) \right)^+ = P(T_0, T_1) (X_{T_0} - \kappa)^+,
\]

where \( N_t := P(t, T_1) \) is taken as numéraire and

\[
X_t := \frac{1}{P(t, T_1)} \sum_{i=2}^n c_i P(t, T_i) = \sum_{i=2}^n c_i \tilde{P}(t, T_i), \quad 0 \leq t \leq T_1.
\]
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with \( \hat{P}(t, T_i) := P(t, T_i) / P(t, T_1), i = 2, 3, \ldots, n. \)

a) Assuming the dynamics \( d\hat{P}(t, T_i) = \sigma_i(t)\hat{P}(t, T_i)d\hat{W}_t \) under the forward measure \( \hat{P}_1 \), write down the dynamics of \( X_t \) as \( dX_t = \sigma_t X_t d\hat{W}_t \), where \( \sigma_t \) is to be computed explicitly.

b) Approximating \( (\hat{P}(t, T_i))_{t \in [0, T_i]} \) by \( \hat{P}(0, T_i) \) and \( (P(t, T_2, T_n))_{t \in [0, T_2]} \) by \( P(0, T_2, T_n) \), find a deterministic approximation \( \tilde{\sigma}(t) \) of \( \sigma_t \), and deduce an expression of the option price

\[
\mathbb{E}^* \left[ e^{-\int_0^{T_1} r_s ds} \left( \sum_{i=2}^{n} c_i P(T_0, T_i) - \kappa P(T_0, T_1) \right)^+ \right] = P(0, T_1) \mathbb{E}[ (X_{T_0} - \kappa)^+ ]
\]

using the Black-Scholes formula.

**Hint:** Given \( X \) a centered Gaussian random variable with variance \( v^2 \), we have:

\[
\mathbb{E} \left[ (xe^{X-v^2/2} - \kappa)^+ \right] = x\Phi(v/2 + (\log(\kappa/x))/v) - \kappa\Phi(-v/2 + (\log(\kappa/x))/v).
\]

**Exercise 18.7** We work in the short rate model

\( dr_t = \sigma dB_t \),

where \( (B_t)_{t \in \mathbb{R}_+} \) is a standard Brownian motion under \( \mathbb{P}^* \), and \( \hat{P}_2 \) is the forward measure defined by

\[
\frac{d\hat{P}_2}{d\mathbb{P}^*} = \frac{1}{P(0, T_2)} e^{-\int_0^{T_2} r_s ds}.
\]

a) State the expressions of \( \zeta_1(t) \) and \( \zeta_2(t) \) in

\[
\frac{dP(t, T_i)}{P(t, T_1)} = r_i dt + \zeta_i(t) dB_t, \quad i = 1, 2,
\]

and the dynamics of the \( P(t, T_1)/P(t, T_2) \) under \( \hat{P}_2 \), where \( P(t, T_1) \) and \( P(t, T_2) \) are bond prices with maturities \( T_1 \) and \( T_2 \).

**Hint.** Use Exercise 16.3 and the relation (16.22).

b) State the expression of the forward rate \( f(t, T_1, T_2) \).

c) Compute the dynamics of \( f(t, T_1, T_2) \) under the forward measure \( \hat{P}_2 \) with

\[
\frac{d\hat{P}_2}{d\mathbb{P}^*} = \frac{1}{P(0, T_2)} e^{-\int_0^{T_2} r_s ds}.
\]

d) Compute the price

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\[(T_2 - T_1) \mathbb{E}^* \left[ e^{-\int_t^{T_2} r_s ds} (f(T_1, T_1, T_2) - \kappa)^+ \mid \mathcal{F}_t \right] \]

of a cap at time \( t \in [0, T_1] \), using the expectation under the forward measure \( \hat{\mathbb{P}}_2 \).

e) Compute the dynamics of the swap rate process

\[ S(t, T_1, T_2) = \frac{P(t, T_1) - P(t, T_2)}{(T_2 - T_1)P(t, T_2)}, \quad t \in [0, T_1], \]

under \( \hat{\mathbb{P}}_2 \).

f) Using (18.24), compute the swaption price

\[(T_2 - T_1) \mathbb{E}^* \left[ e^{-\int_t^{T_1} r_s ds} P(T_1, T_2) (S(T_1, T_1, T_2) - \kappa)^+ \mid \mathcal{F}_t \right] \]

on the swap rate \( S(T_1, T_1, T_2) \) using the expectation under the forward swap measure \( \hat{\mathbb{P}}_{1,2} \).

Exercise 18.8 Consider three zero-coupon bonds \( P(t, T_1) \), \( P(t, T_2) \) and \( P(t, T_3) \) with maturities \( T_1 = \delta \), \( T_2 = 2\delta \) and \( T_3 = 3\delta \) respectively, and the forward LIBOR \( L(t, T_1, T_2) \) and \( L(t, T_2, T_3) \) defined by

\[ L(t, T_i, T_{i+1}) = \frac{1}{\delta} \left( \frac{L(t, T_i)}{L(t, T_{i+1})} - 1 \right), \quad i = 1, 2. \]

Assume that \( L(t, T_1, T_2) \) and \( L(t, T_2, T_3) \) are modeled in the BGM model by

\[
\frac{dL(t, T_1, T_2)}{L(t, T_1, T_2)} = e^{-at}d\hat{W}_t^2, \quad 0 \leq t \leq T_1, \tag{18.32}
\]

and \( L(t, T_2, T_3) = b, 0 \leq t \leq T_2 \), for some constants \( a, b > 0 \), where \( \hat{W}_t^2 \) is a standard Brownian motion under the forward rate measure \( \hat{\mathbb{P}}_2 \) defined by

\[
\frac{d\hat{\mathbb{P}}_2}{d\mathbb{P}^*} = e^{-\int_0^{T_2} r_s ds} \frac{P(0, T_2)}{P(0, T_2)}.
\]

a) Compute \( L(t, T_1, T_2), 0 \leq t \leq T_2 \) by solving Equation (18.32).

b) Show that the price at time \( t \) of the caplet with strike price \( \kappa \) can be written as

\[ \mathbb{E}^* \left[ e^{-\int_t^{T_2} r_s ds} (L(T_1, T_1, T_2) - \kappa)^+ \mid \mathcal{F}_t \right] = P(t, T_2) \hat{\mathbb{E}}_2 \left[ (L(T_1, T_1, T_2) - \kappa)^+ \mid \mathcal{F}_t \right], \]

where \( \hat{\mathbb{E}}_2 \) denotes the expectation under the forward measure \( \hat{\mathbb{P}}_2 \).
c) Using the hint below, compute the price at time $t$ of the caplet with strike price $\kappa$ on $L(T_1, T_1, T_2)$.

d) Compute
\[
\frac{P(t, T_1)}{P(t, T_1, T_3)}, \quad 0 \leq t \leq T_1, \quad \text{and} \quad \frac{P(t, T_3)}{P(t, T_1, T_3)}, \quad 0 \leq t \leq T_2,
\]
in terms of $b$ and $L(t, T_1, T_2)$, where $P(t, T_1, T_3)$ is the annuity numéraire
\[
P(t, T_1, T_3) = \delta P(t, T_2) + \delta P(t, T_3), \quad 0 \leq t \leq T_2.
\]
e) Compute the dynamics of the swap rate
\[
t \mapsto S(t, T_1, T_3) = \frac{P(t, T_1) - P(t, T_3)}{P(t, T_1, T_3)}, \quad 0 \leq t \leq T_1,
\]
i.e. show that we have
\[
dS(t, T_1, T_3) = \sigma_{1,3}(t)S(t, T_1, T_3)d\widehat{W}_t^2,
\]
where $\sigma_{1,3}(t)$ is a stochastic process to be determined.

f) Using the Black-Scholes formula, compute an approximation of the swap-rate price
\[
IE^* \left[ e^{\int_0^T r_s ds} P(T_1, T_1, T_3)(S(T_1, T_1, T_3) - \kappa)^+ \mid \mathcal{F}_t \right] = P(t, T_1, T_3)E_2 \left[ (S(T_1, T_1, T_3) - \kappa)^+ \mid \mathcal{F}_t \right],
\]
at time $t \in [0, T_1]$. You will need to approximate $\sigma_{1,3}(s), s \geq t$, by “freezing” all random terms at time $t$.

Hint. Given $X$ a centered Gaussian random variable with variance $v^2$ we have
\[
IE^* \left[ (e^{m+X} - \kappa)^+ \right] = e^{m+v^2/2}\Phi(v + (m - \log \kappa)/v) - \kappa\Phi((m - \log \kappa)/v),
\]
where $\Phi$ denotes the Gaussian cumulative distribution function.

Exercise 18.9 Bond option hedging. Consider a portfolio allocation $(\xi^T_t, \xi^S_t)_{t \in [0, T]}$ made of two bonds with maturities $T, S$, and value
\[
V_t = \xi^T_t P(t, T) + \xi^S_t P(t, S), \quad 0 \leq t \leq T,
\]
at time $t$. We assume that the portfolio is self-financing, i.e.
\[
dV_t = \xi^T_t dP(t, T) + \xi^S_t dP(t, S), \quad 0 \leq t \leq T, \quad (18.33)
\]
and that it hedges the claim payoff \((P(T, S) - \kappa)^+\), so that

\[
V_t = \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} (P(T, S) - \kappa)^+ \mid \mathcal{F}_t \right] = P(t, T) \mathbb{E}_T \left[ (P(T, S) - \kappa)^+ \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T.
\]

a) Show that we have

\[
\mathbb{E}^* \left[ e^{-\int_t^T r_s ds} (P(T, S) - K)^+ \mid \mathcal{F}_t \right] = P(0, T) \mathbb{E}_T \left[ (P(T, S) - K)^+ \right] + \int_0^t \xi_t^T dP(s, T) + \int_0^t \xi_t^S dP(s, S).
\]

b) Show that under the self-financing condition (18.33), the deflated portfolio price \(\tilde{V}_t = e^{-\int_0^t r_s ds} V_t\) satisfies

\[
d\tilde{V}_t = \xi_t^T d\tilde{P}(t, T) + \xi_t^S d\tilde{P}(t, S),
\]

where

\[
\tilde{P}(t, T) := e^{-\int_0^t r_s ds} P(t, T), \quad t \in [0, T],
\]

and

\[
\tilde{P}(t, S) := e^{-\int_0^t r_s ds} P(t, S), \quad t \in [0, S],
\]

denote the discounted bond prices.

c) From now on we work in the framework of Proposition 18.3, and we let the function \(C(x, v)\) be defined by

\[
C(X_t, v(t, T)) := \mathbb{E}_T \left[ (P(T, S) - K)^+ \mid \mathcal{F}_t \right],
\]

where \(X_t\) is the forward price \(X_t := P(t, S)/P(t, T), \ t \in [0, T]\), and

\[
v^2(t, T) := \int_t^T |\sigma_s^S - \sigma_s^T|^2 ds.
\]

Show that

\[
\mathbb{E}_T \left[ (P(T, S) - K)^+ \mid \mathcal{F}_t \right] = \mathbb{E}_T \left[ (P(T, S) - K)^+ \right] + \int_0^t \frac{\partial C}{\partial x}(X_u, v(u, T)) dX_u, \quad t \geq 0.
\]

\textit{Hint:} Use the martingale property and the Itô formula.

d) Show that the discounted portfolio price \(\tilde{V}_t = V_t / P(t, T)\) satisfies

\[
d\tilde{V}_t = \frac{\partial C}{\partial x}(X_t, v(t, T)) dX_t
\]

\[
= \frac{P(t, S)}{P(t, T)} \frac{\partial C}{\partial x}(X_t, v(t, T))(\sigma_t^S - \sigma_t^T) d\tilde{B}_t^T.
\]
e) Show that
\[ dV_t = P(t,S) \frac{\partial C}{\partial x}(X_t, v(t,T))(\sigma_t^S - \sigma_t^T)dB_t + \tilde{V}_t dP(t,T). \]

f) Show that
\[ d\tilde{V}_t = \tilde{P}(t,S) \frac{\partial C}{\partial x}(X_t, v(t,T))(\sigma_t^S - \sigma_t^T)dB_t + \tilde{V}_t d\tilde{P}(t,T). \]

g) Compute the hedging strategy \( (\xi_t^T, \xi_t^S)_{t \in [0,T]} \) of the bond option.

h) Show that
\[ \frac{\partial C}{\partial x}(x,v) = \Phi \left( \frac{\log(x/K) + \tau v^2/2}{\sqrt{\tau v}} \right), \]
and compute the hedging strategy \( (\xi_t^T, \xi_t^S)_{t \in [0,T]} \) in terms of the normal cumulative distribution function \( \Phi \).

Exercise 18.10 Consider a LIBOR rate \( L(t,T,S), t \in [0,T], \) modeled as
\[ dL(t,T,S) = \mu_t L(t,T,S) dt + \sigma_t L(t,T,S) dW_t, \]
where \( (W_t)_{t \in [0,T]} \) is a standard Brownian motion under the risk-neutral probability measure \( \mathbb{P}^* \), \( (\mu_t)_{t \in [0,T]} \) is an adapted process, and \( \sigma(t) > 0 \) is a deterministic volatility function of time \( t \).

a) What is the dynamics of \( L(t,T,S) \) under the forward measure \( \tilde{\mathbb{P}} \) with numéraire \( N_t := P(t,S) \)?

b) Rewrite the price
\[ \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} \phi(L(T,T,S)) | \mathcal{F}_t \right] \] (18.34)
at time \( t \in [0,T] \) of an option with payoff function \( \phi \) using the forward measure \( \tilde{\mathbb{P}} \).

c) Write down the above option price (18.34) using an integral.

Exercise 18.11 Given \( n \) bonds with maturities \( T_1, T_2, \ldots, T_n \), consider the annuity numéraire
\[ P(t,T_i,T_j) = \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t,T_{k+1}) \]
and the swap rate
\[ S(t,T_i,T_j) = \frac{P(t,T_i) - P(t,T_j)}{P(t,T_i,T_j)} \]
at time \( t \in [0,T_i] \), modeled as

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\[ dS(t, T_i, T_j) = \mu_t S(t, T_i, T_j) dt + \sigma S(t, T_i, T_j) dW_t, \quad 0 \leq t \leq T_i, \quad (18.35) \]

where \((W_t)_{t \in [0,T_i]}\) is a standard Brownian motion under the risk-neutral probability measure \(\mathbb{P}^\ast\), \((\mu_t)_{t \in [0,T]}\) is an adapted process and \(\sigma > 0\) is a constant. Let

\[
\mathbb{E}^* \left[ e^{-\int_{T_i}^{T_j} r_s ds} P(T_i, T_i, T_j) \phi(S(T_i, T_i, T_j)) \mid \mathcal{F}_t \right] \quad (18.36)
\]

at time \(t \in [0, T_i]\) of an option with payoff function \(\phi\).

a) Rewrite the option price \(18.36\) at time \(t \in [0, T_i]\) using the forward swap measure \(\mathbb{P}_{i,j}\) defined from the annuity numéraire \(P(t, T_i, T_j)\).

b) What is the dynamics of \(S(t, T_i, T_j)\) under the forward swap measure \(\mathbb{P}_{i,j}\)?

c) Write down the above option price \(18.34\) using a Gaussian integral.

d) Apply the above to the computation at time \(t \in [0, T_i]\) of the put swaption price

\[
\mathbb{E}^* \left[ e^{-\int_{T_i}^{T_j} r_s ds} P(T_i, T_i, T_j) (\kappa - S(T_i, T_i, T_j))^+ \mid \mathcal{F}_t \right]
\]

with strike price \(\kappa\), using the Black-Scholes formula.

Hint. Given \(X\) a centered Gaussian random variable with variance \(v^2\) we have

\[
\mathbb{E}[(\kappa - e^{m+X})^+] = \kappa \Phi(-(m - \log \kappa)/v) - e^{m+v^2/2} \Phi(-v - (m - \log \kappa)/v),
\]

where \(\Phi\) denotes the Gaussian cumulative distribution function.

Exercise 18.12 Consider a bond market with two bonds with maturities \(T_1, T_2\), whose prices \(P(t, T_1), P(t, T_2)\) at time \(t\) are given by

\[
\frac{dP(t, T_1)}{P(t, T_1)} = r_t dt + \zeta_1(t) dB_t, \quad \frac{dP(t, T_2)}{P(t, T_2)} = r_t dt + \zeta_2(t) dB_t,
\]

where \((r_t)_{t \in \mathbb{R}_+}\) is a short-term interest rate process, \((B_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion generating a filtration \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\), and \(\zeta_1(t), \zeta_2(t)\) are volatility processes. The LIBOR rate \(L(t, T_1, T_2)\) is defined by

\[
L(t, T_1, T_2) = \frac{P(t, T_1) - P(t, T_2)}{P(t, T_2)}.
\]

Recall that a caplet on the LIBOR market can be priced at time \(t \in [0, T_1]\) as

\[
\mathbb{E} \left[ e^{-\int_{T_1}^{T_2} r_s ds} (L(T_1, T_1, T_2) - \kappa)^+ \mid \mathcal{F}_t \right] 
= P(t, T_2) \mathbb{E} \left[ (L(T_1, T_1, T_2) - \kappa)^+ \mid \mathcal{F}_t \right], \quad (18.37)
\]
under the forward measure \( \hat{\mathbb{P}} \) defined by
\[
\frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*} = e^{-\int_0^{T_1} r_s ds} \frac{P(T_1, T_2)}{P(0, T_2)},
\]
under which
\[
\hat{B}_t := B_t - \int_0^t \zeta_2(s) ds, \quad t \in \mathbb{R}+, \tag{18.38}
\]
is a standard Brownian motion.

In the sequel we let \( L_t = L(t, T_1, T_2) \) for simplicity of notation.

a) Using Itô calculus, show that the LIBOR rate satisfies
\[
dL_t = L_t \sigma(t) d\hat{B}_t, \quad 0 \leq t \leq T_1, \tag{18.39}
\]
where the LIBOR rate volatility is given by
\[
\sigma(t) = \frac{P(t, T_1)(\zeta_1(t) - \zeta_2(t))}{P(t, T_1) - P(t, T_2)}.
\]

b) Solve the equation (18.39) on the interval \([t, T_1]\), and compute \( L_{T_1} \) from the initial condition \( L_t \).

c) Assuming that \( \sigma(t) \) in (18.39) is a deterministic volatility function of time \( t \), show that the price
\[
P(t, T_2) \mathbb{E}[L_{T_1} - \kappa^+] \mid \mathcal{F}_t
\]
of the caplet can be written as \( P(t, T_2)C(L_t, v(t, T_1)) \), where \( v^2(t, T_1) = \int_t^{T_1} |\sigma(s)|^2 ds \), and \( C(t, v(t, T_1)) \) is a function of \( L_t \) and \( v(t, T_1) \).

d) Consider a portfolio allocation \((\xi_t^{(1)}, \xi_t^{(2)})_{t \in [0, T_1]}\) made of bonds with maturities \( T_1, T_2 \) and value
\[
V_t = \xi_t^{(1)} P(t, T_1) + \xi_t^{(2)} P(t, T_2),
\]
at time \( t \in [0, T_1] \). We assume that the portfolio is self-financing, i.e.
\[
dV_t = \xi_t^{(1)} dP(t, T_1) + \xi_t^{(2)} dP(t, T_2), \quad 0 \leq t \leq T_1, \tag{18.40}
\]
and that it hedges the claim payoff \((L_{T_1} - \kappa)^+\), so that
\[
V_t = \mathbb{E}\left[ e^{-\int_0^{T_1} r_s ds} (P(T_1, T_2)(L_{T_1} - \kappa))^+ \mid \mathcal{F}_t \right]
= P(t, T_2) \mathbb{E}[L_{T_1} - \kappa]^+ \mid \mathcal{F}_t,
\]
\( 0 \leq t \leq T_1 \). Show that we have
e) Show that under the self-financing condition (18.40), the discounted portfolio price \( \tilde{V}_t = e^{-\int_0^t r_s ds} V_t \) satisfies

\[
d\tilde{V}_t = \xi_t^{(1)} d\tilde{P}(t, T_1) + \xi_t^{(2)} d\tilde{P}(t, T_2),
\]

where \( \tilde{P}(t, T_1) := e^{-\int_0^t r_s ds} P(t, T_1) \) and \( \tilde{P}(t, T_2) := e^{-\int_0^t r_s ds} P(t, T_2) \) denote the discounted bond prices.

f) Show that

\[
\tilde{E}\left[(L_{T_1} - \kappa)^+ \mid \mathcal{F}_t\right] = \tilde{E}\left[(L_{T_1} - \kappa)^+\right] + \int_0^t \frac{\partial C}{\partial x}(L_u, v(u, T_1)) dL_u,
\]

and that the discounted portfolio price \( \tilde{V}_t = V_t / P(t, T_2) \) satisfies

\[
d\tilde{V}_t = \frac{\partial C}{\partial x}(L_t, v(t, T_1)) dL_t = L_t \frac{\partial C}{\partial x}(L_t, v(t, T_1)) \sigma(t) d\tilde{B}_t.
\]

*Hint:* use the martingale property and the Itô formula.

g) Show that

\[
dV_t = (P(t, T_1) - P(t, T_2)) \frac{\partial C}{\partial x}(L_t, v(t, T_1)) \sigma(t) dB_t + \tilde{V}_t dP(t, T_2).
\]

h) Show that

\[
d\tilde{V}_t = \frac{\partial C}{\partial x}(L_t, v(t, T_1)) d(\tilde{P}(t, T_1) - \tilde{P}(t, T_2))
\]

\[
+ \left( \tilde{V}_t - L_t \frac{\partial C}{\partial x}(L_t, v(t, T_1)) \right) d\tilde{P}(t, T_2),
\]

and deduce the values of the hedging portfolio allocation \( (\xi_t^{(1)}, \xi_t^{(2)})_{t \in \mathbb{R}^+} \).

Exercise 18.13 Consider a bond market with tenor structure \( \{T_i, \ldots, T_j\} \) and \( j - i + 1 \) bonds with maturities \( T_i, \ldots, T_j \), whose prices \( P(t, T_i), \ldots P(t, T_j) \) at time \( t \) are given by

\[
\frac{dP(t, T_k)}{P(t, T_k)} = r_t dt + \zeta_k(t) dB_t, \quad k = i, \ldots, j,
\]
where \((r_t)_{t \in \mathbb{R}^+}\) is a short-term interest rate process and \((B_t)_{t \in \mathbb{R}^+}\) denotes a standard Brownian motion generating a filtration \((\mathcal{F}_t)_{t \in \mathbb{R}^+}\), and \(\zeta_i(t), \ldots, \zeta_j(t)\) are volatility processes.

The swap rate \(S(t, T_i, T_j)\) is defined by
\[
S(t, T_i, T_j) = \frac{P(t, T_i) - P(t, T_j)}{P(t, T_i, T_j)},
\]
where
\[
P(t, T_i, T_j) = \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1})
\]
is the annuity numéraire. Recall that a swaption on the LIBOR market can be priced at time \(t \in [0, T_i]\) as
\[
\mathbb{E}^* \left[ e^{-\int_t^{T_i} r_s ds} \left( \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(T_i, T_{k+1}) (S(T_i, T_k, T_{k+1}) - \kappa) \right)^+ \bigg| \mathcal{F}_t \right]
= P(t, T_i, T_j) \mathbb{E}_{i,j} \left[ (S(T_i, T_j) - \kappa)^+ \bigg| \mathcal{F}_t \right],
\]
under the forward swap measure \(\mathbb{P}_{i,j}\) defined by
\[
\frac{d\mathbb{P}_{i,j}}{d\mathbb{P}^*} = e^{-\int_t^{T_i} r_s ds} \frac{P(T_i, T_j)}{P(0, T_i, T_j)}, \quad 1 \leq i < j \leq n,
\]
under which
\[
\hat{B}_t^{i,j} := B_t - \sum_{k=i}^{j-1} (T_{k+1} - T_k) \frac{P(t, T_{k+1})}{P(t, T_i, T_j)} \zeta_{k+1}(t) dt
\]
is a standard Brownian motion. Recall that the swap rate can be modeled as
\[
dS(t, T_i, T_j) = S(t, T_i, T_j) \sigma_{i,j}(t) d\hat{B}_t^{i,j}, \quad 0 \leq t \leq T_i,
\]
where the swap rate volatilities are given by
\[
\sigma_{i,j}(t) = \sum_{l=i}^{j-1} (T_{l+1} - T_l) \frac{P(t, T_{l+1})}{P(t, T_i, T_j)} \left( \zeta_l(t) - \zeta_{l+1}(t) \right)
+ \frac{P(t, T_j)}{P(t, T_i) - P(t, T_j)} \left( \zeta_i(t) - \zeta_j(t) \right)
\]
\(1 \leq i, j \leq n\), cf. e.g. Proposition 10.8 of Privault (2009). In the sequel we denote \(S_t = S(t, T_i, T_j)\) for simplicity of notation.
Solving the equation (18.43) on the interval \([t, T_i]\), and compute \(S(T_i, T_i, T_j)\) from the initial condition \(S(t, T_i, T_j)\).

b) Assuming that \(\sigma_{i,j}(t)\) is a deterministic volatility function of time \(t\) for \(1 \leq i, j \leq n\), show that the price (18.30) of the swaption can be written as

\[
P(t, T_i, T_j) C(S_t, v(t, T_i)),
\]

where

\[
v^2(t, T_i) = \int_t^{T_i} |\sigma_{i,j}(s)|^2 ds,
\]

and \(C(x, v)\) is a function to be specified using the Black-Scholes formula Bl\((x, K, \sigma, r, \tau)\), with

\[
E[(x e^{m+X} - K)^+] = \Phi(v + (m + \log(x/K))/v) - K \Phi((m + \log(x/K))/v),
\]

where \(X\) is a centered Gaussian random variable with mean \(m = r\tau - v^2/2\) and variance \(v^2\).

c) Consider a portfolio allocation \((\xi_{t}^{(i)}, \ldots, \xi_{t}^{(j)})_{t \in [0, T_i]}\) made of bonds with maturities \(T_i, \ldots, T_j\) and value

\[
V_t = \sum_{k=i}^{j} \xi_{t}^{(k)} P(t, T_k),
\]

at time \(t \in [0, T_i]\). We assume that the portfolio is self-financing, i.e.

\[
dV_t = \sum_{k=i}^{j} \xi_{t}^{(k)} dP(t, T_k), \quad 0 \leq t \leq T_i, \quad (18.45)
\]

and that it hedges the claim payoff \((S(T_i, T_i, T_j) - \kappa)^+\), so that

\[
V_t = E^t \left[ e^{-\int_t^{T_i} r_s ds} \left( \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(T_i, T_k) (L(T_i, T_k, T_{k+1}) - \kappa) \right)^+ \left| F_t \right. \right]
\]

\[
\quad = P(t, T_i, T_j) \mathbb{E}_{i,j} \left[ (S(T_i, T_i, T_j) - \kappa)^+ \left| F_t \right. \right],
\]

\(0 \leq t \leq T_i\). Show that

\[
E^* \left[ e^{-\int_t^{T_i} r_s ds} \left( \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(T_i, T_k) (L(T_i, T_k, T_{k+1}) - \kappa) \right)^+ \left| F_t \right. \right]
\]

\[
\quad = P(0, T_i, T_j) \mathbb{E}_{i,j} \left[ (S(T_i, T_i, T_j) - \kappa)^+ \right] + \sum_{k=i}^{j} \int_0^t \xi_{s}^{(k)} dP(s, T_i),
\]

\(0 \leq t \leq T_i\).
d) Show that under the self-financing condition (18.45), the discounted portfolio price \( \tilde{V}_t = e^{-\int_0^t r_s ds} V_t \) satisfies

\[
d\tilde{V}_t = \sum_{k=i}^j \xi_{tk}^{(k)} d\tilde{P}(t, T_k),
\]

where \( \tilde{P}(t, T_k) = e^{-\int_0^t r_s ds} P(t, T_k), \) \( k = i, i + 1 \ldots, j, \) denote the discounted bond prices.

e) Show that

\[
\mathbb{E}_{i,j} [(S(T_i, T_i, T_j) - \kappa)^+] \bigg| \mathcal{F}_t
\]

\[
= \mathbb{E}_{i,j} [(S(T_i, T_i, T_j) - \kappa)^+] + \int_0^t \frac{\partial C}{\partial x}(S_u, v(u, T_i)) dS_u.
\]

*Hint:* use the martingale property and the Itô formula.

f) Show that the discounted portfolio price \( \tilde{V}_t = V_t / P(t, T_i, T_j) \) satisfies

\[
d\tilde{V}_t = \frac{\partial C}{\partial x}(S_t, v(t, T_i)) dS_t = S_t \frac{\partial C}{\partial x}(S_t, v(t, T_i)) \sigma_t^{i,j} dB + \tilde{V}_t dP(t, T_i, T_j).
\]

g) Show that

\[
dV_t = \frac{P(t, T_i) - P(t, T_j)}{P(t, T_j) - P(t, T_i)} \frac{\partial C}{\partial x}(S_t, v(t, T_i)) \sigma_t^{i,j} d\tilde{B}_t + \tilde{V}_t dP(t, T_i, T_j).
\]

h) Show that

\[
dV_t = S_t \xi_t(t) \frac{\partial C}{\partial x}(S_t, v(t, T_i)) \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) dB_t
\]

\[
+(\tilde{V}_t - S_t \frac{\partial C}{\partial x}(S_t, v(t, T_i))) \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) \zeta_{k+1}(t) dB_t
\]

\[
+ \frac{\partial C}{\partial x}(S_t, v(t, T_i)) P(t, T_j) (\zeta_i(t) - \zeta_j(t)) dB_t.
\]

i) Show that

\[
d\tilde{V}_t = \frac{\partial C}{\partial x}(S_t, v(t, T_i)) d(\tilde{P}(t, T_i) - \tilde{P}(t, T_j))
\]

\[
+(\tilde{V}_t - S_t \frac{\partial C}{\partial x}(S_t, v(t, T_i))) d\tilde{P}(t, T_i, T_j).
\]

j) Show that

\[
\frac{\partial C}{\partial x}(x, v(t, T_i)) = \Phi \left( \frac{\log(x/K)}{v(t, T_i)} + \frac{v(t, T_i)}{2} \right).
\]
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k) Show that we have
\[
d\tilde{V}_t = \Phi \left( \frac{\log(S_t/K)}{v(t, T_i)} + \frac{v(t, T_i)}{2} \right) d(\tilde{P}(t, T_i) - \tilde{P}(t, T_j)) \\
- \kappa \Phi \left( \frac{\log(S_t/K)}{v(t, T_i)} - \frac{v(t, T_i)}{2} \right) d\tilde{P}(t, T_i, T_j).
\]

l) Show that the hedging strategy is given by
\[
\xi_t^{(i)} = \Phi \left( \frac{\log(S_t/K)}{v(t, T_i)} + \frac{v(t, T_i)}{2} \right),
\]
\[
\xi_t^{(j)} = -\Phi \left( \frac{\log(S_t/K)}{v(t, T_i)} + \frac{v(t, T_i)}{2} \right) - \kappa(T_j - T_{j-1}) \Phi \left( \frac{\log(S_t/K)}{v(t, T_i)} - \frac{v(t, T_i)}{2} \right),
\]
and
\[
\xi_t^{(k)} = -\kappa(T_{k+1} - T_k) \Phi \left( \frac{\log(S_t/K)}{v(t, T_i)} - \frac{v(t, T_i)}{2} \right), \quad i \leq k \leq j - 2.
\]