Chapter 20
Pricing and Hedging in Jump Models

In this chapter we consider option pricing and hedging in jump-diffusion models. In comparison with the continuous case the situation is further complicated by the existence of multiple risk-neutral probability measures. As a consequence, perfect replicating hedging strategies cannot be computed in general.

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20.1 Market Returns vs Gaussian and Power Tails

The modeling of risky asset by stochastic processes with continuous paths, based on Brownian motions, suffers from several defects. First, the path continuity assumption does not seem reasonable in view of the possibility of sudden price variations (jumps) resulting of market crashes, gaps or opening jumps, see e.g. Chapter 1 of Cont and Tankov (2004). Secondly, the modeling of risky asset prices by Brownian motion relies on the use of the Gaussian distribution which tends to underestimate the probabilities of extreme events.

The R package Quantmod is installed through the command:

```
install.packages("quantmod")
```
The following scripts allow us to fetch DJI and STI index data using Quantmod.

```r
library(quantmod)
getSymbols("^STI", from="1990-01-03", to="2015-02-01", src="yahoo")
stock.rtn<-diff(log(Ad("STI"))); returns <- as.vector(stock.rtn)
m=mean(returns,na.rm=TRUE); s=sd(returns,na.rm=TRUE)
times=index(stock.rtn)
n = sum(is.na(returns))+sum(!is.na(returns))
x=seq(1,n); y=rnorm(n,mean=m,sd=s)
plot(times,returns,pch=19,cex=0.03,xaxs="i",col="blue", ylab="X", xlab="n", main="")
segments(x0 = times, x1 = times, y0 = 0, y1 = returns,col="blue")
points(times,y,pch=19,cex=0.3, col="red", ylab="X", xlab="n", main="")

pnorm(3*s,0,s)-pnorm(-3*s,0,s)
```

The next figures illustrate the mismatch between the distributional properties of market vs Gaussian returns based on the above code.

Fig. 20.1: Market returns vs normalized Gaussian returns.
The following Quantile-Quantile plot is plotting the normalized empirical quantiles against the standard Gaussian quantiles, and is obtained with the `qqnorm` command.

![Empirical Cumulative Distribution](image1.png)

**Fig. 20.2:** Empirical vs Gaussian CDF.

The Kolmogorov-Smirnov test clearly rejects the null (normality) hypothesis.

![Quantile-Quantile Plot](image2.png)

**Fig. 20.3:** Quantile-Quantile plot.
One-sample Kolmogorov-Smirnov test

data: returns
D = 0.075577, p-value < 2.2e-16
alternative hypothesis: two-sided

```r
x <- seq(-0.25, 0.25, length=100)
qx <- dnorm(x,mean=m,sd=s)
stock.dens=density(returns,na.rm=TRUE)
plot(stock.dens, xlab = 'Sample Quantiles', lwd=2, col="red",ylab = '', main = 'Empirical Cumulative Distribution',panel.first = abline(h = 0, col="grey", lwd =0.2))
lines(x, qx, type="l", lty=2, lwd=2, col="blue",xlab="x value",ylab="Density", main="Gaussian cdf")
legend("topleft", legend=c("Empirical cdf", "Gaussian cdf"),col=c("red", "blue"), lty=1:2, cex=0.8)
```

Fig. 20.4: Empirical density vs normalized Gaussian density.

On the other hand, power tail densities can provide a better fit of empirical densities, as shown in Figure 20.5.
The above fitting of empirical density is using a power probability density defined by a rational fraction obtained by the following R script.

```r
install.packages("pracma")
x <- seq(-0.25, 0.25, length=1000)
stock.dens <- density(returns, na.rm=TRUE, from = -0.1, to = 0.1, n = 1000)
library(pracma)
a <- rationalfit(stock.dens$x, stock.dens$y, d1=2, d2=2)
plot(stock.dens$x, stock.dens$y, lwd=2, type = "l", xlab = "x", col="red", ylab = "y", main = "
panel.first = abline(h = 0, col="grey", lwd =0.2))
legend("topleft", legend=c("Empirical density", "power density"),col=c("red", "blue"), lty=1:2, cex=0.8)
```

The output of the `rationalfit` command is

$p1$

```
[1] -0.184717249 -0.001591433  0.001385017
```

$p2$

```
[1]  1.000000e+00 -6.460948e-04  1.314672e-05
```

which yields a rational fraction of the form

\[
x \mapsto \frac{0.001385017 - 0.001591433 \times x - 0.184717249 \times x^2}{1.314672 \times 10^{-5} - 6.460948 \times 10^{-4} \times x + x^2}
\]

\[
\approx -0.184717249 - \frac{0.001591433}{x} + \frac{0.001385017}{x^2},
\]

which approximates the empirical density of DJI returns in the least squares sense.

A solution to this tail problem is to use stochastic processes with jumps, that will account for sudden variations of the asset prices. On the other hand,
such jump models are generally based on the Poisson distribution which has a slower tail decay than the Gaussian distribution. This allows one to assign higher probabilities to extreme events, resulting in a more realistic modeling of asset prices. Stable distributions with parameter $\alpha \in (0, 2)$ provide typical examples of probability laws with power tails, as their probability density functions behave asymptotically as $x \mapsto C_\alpha / |x|^{1+\alpha}$ when $x \to \pm \infty$.

20.2 Risk-Neutral Probability Measures

Consider an asset price modeled by the equation,

$$dS_t = \mu S_t dt + \sigma S_t dB_t + S_t - dY_t,$$

where $(Y_t)_{t \in \mathbb{R}_+}$ is the compound Poisson process defined in Section 19.2, with jump size distribution $\nu(dx)$ under $\mathbb{P}_\nu$. The equation (20.1) has for solution

$$S_t = S_0 \exp \left( \mu t + \sigma B_t - \frac{\sigma^2}{2} t \right) \prod_{k=1}^{N_t} (1 + Z_k),$$

$t \in \mathbb{R}_+$. An important issue for non-arbitrage pricing is to determine a risk-neutral probability measure (or martingale measure) $\mathbb{P}^*$ under which the discounted process $(e^{-rt}S_t)_{t \in \mathbb{R}_+}$ is a martingale, and this goal can be achieved using the Girsanov Theorem for jump processes, cf. Section 19.6.

We have

$$d(e^{-rt}S_t) = -r e^{-rt}S_t dt + e^{-rt}dS_t$$

$$= (\mu - r) e^{-rt}S_t dt + \sigma e^{-rt}S_t dB_t + e^{-rt}S_t - dY_t$$

$$= (\mu - r + \lambda \mathbb{E}_\nu[Z]) e^{-rt}S_t dt + \sigma e^{-rt}S_t dB_t + e^{-rt}S_t - (dY_t - \lambda \mathbb{E}_\nu[Z] dt),$$

which yields a martingale under $\mathbb{P}$ provided that

$$\mu - r + \lambda \mathbb{E}_\nu[Z] = 0,$$

however that condition may not be satisfied by the market parameters $\mu$, $r$, $\lambda$ and $\mathbb{E}_\nu[Z]$.

In this case, a change of measure might be needed. The discounted asset price process

$$\tilde{S}_t := e^{-rt}S_t, \quad t \in \mathbb{R}_+,$$

satisfies the equation

$$d\tilde{S}_t = (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t dB_t + e^{-rt}\tilde{S}_t - dY_t$$

$$= (\sigma u - \lambda \mathbb{E}_{\tilde{\nu}}[Z])\tilde{S}_t dt + \sigma \tilde{S}_t dB_t + \tilde{S}_t - dY_t.$$
\[
\begin{align*}
\sigma \tilde{S}_t & (dB_t + u dt) + \tilde{S}_t^{-} (dY_t - \tilde{\lambda} \mathbb{E}_{\tilde{\nu}}[Z] dt) \\
& = \sigma \tilde{S}_t (dB_t + u dt + dY_t - \tilde{\lambda} \mathbb{E}_{\tilde{\nu}}[Z] dt),
\end{align*}
\]

where \( u \in \mathbb{R} \). When the drift parameter \( u \), the intensity \( \tilde{\lambda} > 0 \) and a jump size distribution \( \tilde{\nu} \) are chosen to satisfy

\[
\mu - r = \sigma u - \tilde{\lambda} \mathbb{E}_{\tilde{\nu}}[Z],
\]

and \( \sigma u + r - \mu > 0 \), then

\[
\tilde{\lambda} = \frac{\sigma u + r - \mu}{\mathbb{E}_{\tilde{\nu}}[Z]} > 0,
\]

and the Girsanov Theorem 19.18 for jump processes then shows that

\[
dB_t + u dt + dY_t - \tilde{\lambda} \mathbb{E}_{\tilde{\nu}}[Z] dt
\]

is a martingale under the probability measure \( \mathbb{P}_{u, \tilde{\lambda}, \tilde{\nu}} \) defined in (19.31). As a consequence, the discounted price process \( (\tilde{S}_t)_{t \in \mathbb{R}^+} = (e^{-rt} S_t)_{t \in \mathbb{R}^+} \) becomes a martingale under \( \mathbb{P}_{u, \tilde{\lambda}, \tilde{\nu}} \).

In this setting, the non-uniqueness of the risk-neutral probability measure is apparent since additional degrees of freedom are involved in the choices of \( u, \lambda \) and the measure \( \tilde{\nu} \), whereas in the continuous case the choice of \( u = (\mu - r) / \sigma \) in (7.10) was unique.

### 20.3 Pricing in Jump Models

Recall that a market is without arbitrage if and only it admits at least one risk-neutral probability measure.

Consider the probability measure \( \mathbb{P}_{u, \tilde{\lambda}, \tilde{\nu}} \) built in the previous section, under which the discounted asset price

\[
d(e^{-rt} S_t) = e^{-rt} S_t^{-} (dY_t - \tilde{\lambda} \mathbb{E}_{\tilde{\nu}}[Z] dt) + \sigma e^{-rt} S_t d\tilde{B}_t,
\]

is a martingale, and \( \tilde{B}_t = B_t + u dt \) is a standard Brownian motion under \( \mathbb{P}_{u, \tilde{\lambda}, \tilde{\nu}} \).

Then the arbitrage price of a claim with payoff \( C \) is given by

\[
e^{-(T-t)r} \mathbb{E}_{u, \tilde{\lambda}, \tilde{\nu}}[C \mid \mathcal{F}_t]
\]

under \( \mathbb{P}_{u, \tilde{\lambda}, \tilde{\nu}} \).
Clearly the price (20.4) of $C$ is no longer unique in the presence of jumps due to the infinity of choices satisfying the martingale condition (20.3), and such a market is not complete, except if either $\tilde{\lambda} = \lambda = 0$, or $(\sigma = 0$ and $\tilde{\nu} = \nu = \delta_1$).

Various techniques can be used for the selection of a risk-neutral probability measure, such as the determination of a minimal entropy risk-neutral probability measure $\mathbb{P}_{u,\tilde{\lambda},\tilde{\nu}}^*$ that minimizes the Kullback-Leibler relative entropy

$$ Q \mapsto I(Q, \mathbb{P}) := \mathbb{E} \left[ \frac{dQ}{d\mathbb{P}} \log \frac{dQ}{d\mathbb{P}} \right] $$

among the probability measures $Q$ equivalent to $\mathbb{P}$.

**Pricing Vanilla Options**

The price of a vanilla option with payoff of the form $\phi(S_T)$ on the underlying asset $S_T$ can be written from (20.4) as

$$ e^{-(T-t)r} \mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[\phi(S_T) \mid \mathcal{F}_t], \quad (20.5) $$

where the expectation can be computed as

$$ \mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[\phi(S_T) \mid \mathcal{F}_t] $$

$$ = \mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}} \left[ \phi \left( S_0 \exp \left( \mu T + \sigma B_T - \frac{\sigma^2}{2} T \right) \prod_{k=1}^{N_T} (1 + Z_k) \right) \mid \mathcal{F}_t \right] $$

$$ = \mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}} \left[ \phi \left( S_t \exp \left( (T-t)\mu + (B_T - B_t)\sigma - \frac{\sigma^2}{2} (T-t) \right) \prod_{k=N_t+1}^{N_T} (1 + Z_k) \right) \mid \mathcal{F}_t \right] $$

$$ = \mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}} \left[ \phi \left( x \exp \left( (T-t)\mu + (B_T - B_t)\sigma - \frac{\sigma^2}{2} (T-t) \right) \prod_{k=N_t+1}^{N_T} (1 + Z_k) \right) \right]_{x=S_t} $$

$$ = \sum_{n \geq 0} \mathbb{P}_{u,\tilde{\lambda},\tilde{\nu}}(N_T - N_t = n) \mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}} \left[ \phi \left( x e^{(T-t)\mu+(B_T-B_t)\sigma-(T-t)\sigma^2/2} \prod_{k=N_t+1}^{N_T} (1 + Z_k) \right) \mid N_T - N_t = n \right]_{x=S_t} $$

$$ = \frac{e^{-(T-t)\tilde{\lambda}}}{n!} \sum_{n \geq 0} \frac{(T-t)\tilde{\lambda}^n}{n!} $$

$$ \times \mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}} \left[ \phi \left( x e^{(T-t)\mu+(B_T-B_t)\sigma-(T-t)\sigma^2/2} \prod_{k=1}^{n} (1 + Z_k) \right) \right]_{x=S_t} $$

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Recall that by the Markov property of the function $f$ for the function $S_t$, the terminal condition $S_T$ and $S_t$ is a martingale under $P_{u,\tilde{\nu}}$ with the terminal condition $S_T$, i.e.

$$
E_{u,\tilde{\nu}}[\phi(S_T) \mid F_t]
$$

hence the price of the vanilla option with payoff $\phi(S_T)$ is given by

$$
e^{-(T-t)r} E_{u,\tilde{\nu}}[\phi(S_T) \mid F_t]
$$

$$=rac{1}{\sqrt{2(T-t)\pi}} e^{-(r+\tilde{\lambda})(T-t)} \sum_{n \geq 0} \frac{(\tilde{\lambda}(T-t))^n}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}
$$

$$
\phi \left( S_t e^{(T-t)u + \sigma x - (T-t)\sigma^2/2} \prod_{k=1}^{n} (1 + z_k) \right) e^{-(x+(T-t)u)/(2(T-t))} \tilde{\nu}(dz_1) \cdots \tilde{\nu}(dz_n) dx.
$$

### 20.4 Black-Scholes PDE with Jumps

Recall that by the Markov property of $(S_t)_{t \in \mathbb{R}}$, the price (20.5) at time $t$ of the option with payoff $\phi(S_T)$ can be written as a function $f(t, S_t)$ of $t$ and $S_t$, i.e.

$$f(t, S_t) = e^{-(T-t)r} E_{u,\tilde{\nu}}[\phi(S_T) \mid F_t], \quad (20.6)$$

with the terminal condition $f(T, x) = \phi(x)$. In addition, the process

$$t \mapsto e^{(T-t)r} f(t, S_t)
$$

is a martingale under $P_{u,\tilde{\nu}}$ by the same argument as in (7.1).

In this section we derive a Partial Integro-Differential Equation (PIDE) for the function $(t, x) \mapsto f(t, x)$. We have

$$dS_t = rS_t dt + \sigma S_t d\tilde{B}_t + S_{t-} (dY_t - \tilde{\lambda} E_{\tilde{\nu}}[Z] dt), \quad (20.7)$$

where $\tilde{B}_t = B_t + ut$ is a standard Brownian motion under $P_{u,\tilde{\nu}}$. Next, by the Itô formula with jumps (19.19), we have

$$df(t, S_t)
$$

$$=rac{\partial f}{\partial t}(t, S_t) dt + rS_t \frac{\partial f}{\partial x}(t, S_t) dt + \sigma S_t \frac{\partial f}{\partial x}(t, S_t) d\tilde{B}_t + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 f}{\partial x^2}(t, S_t) dt
$$

$$- \tilde{\lambda} E_{\tilde{\nu}}[Z] S_t \frac{\partial f}{\partial x}(t, S_t) dt + (f(t, S_{t-} (1 + Z_{N_t})) - f(t, S_{t-})) dN_t
$$

$$= \sigma S_t \frac{\partial f}{\partial x}(t, S_t) d\tilde{B}_t + (f(t, S_{t-} (1 + Z_{N_t})) - f(t, S_{t-})) dN_t
$$
we conclude to the vanishing of the terms (20.8)-(20.9) above, the discounted portfolio price process

\[ -\lambda \mathbb{E}_\nu[(f(t, x(1 + Z)) - f(t, x))]_{x=S_t} dt \]
\[ + \left( \frac{\partial f}{\partial t}(t, S_t) + r S_t \frac{\partial f}{\partial x}(t, S_t) + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 f}{\partial x^2}(t, S_t) \right) dt \]
\[ + \left( \lambda \mathbb{E}_\nu[(f(t, x(1 + Z)) - f(t, x))]_{x=S_t} - \lambda \mathbb{E}_\nu[Z] S_t \frac{\partial f}{\partial x}(t, S_t) \right) dt. \]

Based on the discounted portfolio price differential
\[ d(e^{-rt} f(t, S_t)) \]
\[ = e^{-rt} \sigma S_t \frac{\partial f}{\partial x}(t, S_t) d\hat{B}_t \]
\[ + e^{-rt} \left( f(t, S_t-(1 + Z_{N_t})) - f(t, S_t-) \right) dN_t - \lambda \mathbb{E}_\nu[(f(t, x(1 + Z)) - f(t, x))]_{x=S_t} dt \]
\[ + e^{-rt} \left( -rf(t, S_t) + \frac{\partial f}{\partial t}(t, S_t) + r S_t \frac{\partial f}{\partial x}(t, S_t) + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 f}{\partial x^2}(t, S_t) \right) dt \]
\[ + e^{-rt} \left( \lambda \mathbb{E}_\nu[(f(t, x(1 + Z)) - f(t, x))]_{x=S_t} - \lambda \mathbb{E}_\nu[Z] S_t \frac{\partial f}{\partial x}(t, S_t) \right) dt, \]
\[ (20.8) \]
\[ (20.9) \]

obtained from the Itô Table 19.1 with jumps, and the facts that

- the Brownian motion \((\hat{B}_t)_{t \in \mathbb{R}^+}\) is a martingale under \(\mathbb{P}_{u,\lambda,\nu}\),
- by the smoothing formula Proposition 19.9 the process given by the differential
\[ (f(t, S_t-(1 + Z_{N_t})) - f(t, S_t-)) dN_t - \lambda \mathbb{E}_\nu[(f(t, x(1 + Z)) - f(t, x))]_{x=S_t} dt, \]

is a martingale under \(\mathbb{P}_{u,\lambda,\nu}\),
- the discounted portfolio price process \(t \mapsto e^{-rt} f(t, S_t)\), is also a martingale under the risk-neutral probability measure \(\mathbb{P}_{u,\lambda,\nu}\),

we conclude to the vanishing of the terms (20.8)-(20.9) above, i.e.
\[ -rf(t, S_t) + \frac{\partial f}{\partial t}(t, S_t) + r S_t \frac{\partial f}{\partial x}(t, S_t) + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 f}{\partial x^2}(t, S_t) \]
\[ + \lambda \mathbb{E}_\nu[(f(t, x(1 + Z)) - f(t, x))]_{x=S_t} - \lambda \mathbb{E}_\nu[Z] S_t \frac{\partial f}{\partial x}(t, S_t) = 0, \]
or
\[ \frac{\partial f}{\partial t}(t, x) + r x \frac{\partial f}{\partial x}(t, x) + \frac{\sigma^2}{2} x^2 \frac{\partial^2 f}{\partial x^2}(t, x) \]
\[ + \lambda \int_{-\infty}^{\infty} (f(t, x(1 + y)) - f(t, x)) \nu(dy) - \lambda x \frac{\partial f}{\partial x}(t, x) \int_{-\infty}^{\infty} y \nu(dy) = rf(t, x), \]

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which leads to the Partial \textit{Integro-Differential} Equation (PIDE)

\[
rf(t, x) = \frac{\partial f}{\partial t}(t, x) + rx \frac{\partial f}{\partial x}(t, x) + \frac{\sigma^2}{2} x^2 \frac{\partial^2 f}{\partial x^2}(t, x)
+ \lambda \int_{-\infty}^{\infty} \left(f(t, x(1 + y)) - f(t, x) - yx \frac{\partial f}{\partial x}(t, x)\right) \tilde{\nu}(dy),
\]
\hspace{1cm} (20.10)

under the terminal condition \(f(T, x) = \phi(x)\).

A major technical difficulty when solving the PIDE (20.10) numerically is that the operator

\[
\frac{\partial f}{\partial x}(t, x) \mapsto \int_{-\infty}^{\infty} \left(f(t, x(1 + y)) - f(t, x) - yx \frac{\partial f}{\partial x}(t, x)\right) \tilde{\nu}(dy)
\]

is \textit{nonlocal}, therefore adding significant difficulties to the application of standard discretization schemes, cf. \textit{e.g.} Section 21.2.

In addition we have found that the change \(df(t, S_t)\) in the portfolio price (20.6) is given by

\[
df(t, S_t) = \sigma S_t \frac{\partial f}{\partial x}(t, S_t) dB_t + rf(t, S_t) dt
+ (f(t, S_t(1 + Z)) - f(t, S_t)) dN_t - \tilde{\lambda} \mathbb{E}_\phi[(f(t, x(1 + Z)) - f(t, x))]_{x=S_t} dt.
\]

In the case of Poisson jumps with fixed size \(a\), \textit{i.e.} when \(Y_t = a N_t\) and \(\nu(dx) = \delta_a(dx)\), the PIDE (20.10) reads

\[
rf(t, x) = \frac{\partial f}{\partial t}(t, x) + rx \frac{\partial f}{\partial x}(t, x) + \frac{\sigma^2}{2} x^2 \frac{\partial^2 f}{\partial x^2}(t, x)
+ \tilde{\lambda} \left(f(t, x(1 + a)) - f(t, x) - ax \frac{\partial f}{\partial x}(t, x)\right),
\]

and we have

\[
df(t, S_t) = \sigma S_t \frac{\partial f}{\partial x}(t, S_t) dB_t + rf(t, S_t) dt
+ (f(t, S_t(1 + a)) - f(t, S_t)) dN_t - \tilde{\lambda}(f(t, S_t(1 + a)) - f(t, S_t)) dt.
\]

\subsection*{20.5 Exponential Models}

Instead of modeling the asset price \((S_t)_{t \in \mathbb{R}_+}\) through a stochastic exponential (20.2) solution of the stochastic differential equation with jumps of the form
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(20.1), we may consider an exponential price process of the form

\[
S_t := S_0 e^{\mu t + \sigma B_t + Y_t}
\]

\[
= S_0 \exp \left( \mu t + \sigma B_t + \sum_{k=1}^{N_t} Z_k \right)
\]

\[
= S_0 e^{\mu t + \sigma B_t} \prod_{k=1}^{N_t} e^{Z_k}
\]

\[
= S_0 e^{\mu t + \sigma B_t} \prod_{0 \leq s \leq t} e^{\Delta Y_s}, \quad t \in \mathbb{R}^+,
\]

from Relation (19.8), i.e. \( \Delta Y_t = Z_{N_t} \Delta N_t \). The process \((S_t)_{t \in \mathbb{R}^+}\) is equivalently given by the log-return dynamics

\[
d\log S_t = \mu dt + \sigma d B_t + d Y_t, \quad t \in \mathbb{R}^+.
\]

In this exponential model we also have

\[
S_t = S_0 e^{(\mu \sigma^2 / 2)t + \sigma B_t - \sigma^2 t / 2 + Y_t}
\]

and the process \( S_t \) satisfies the stochastic differential equation

\[
dS_t = \left( \mu + \frac{\sigma^2}{2} \right) S_t dt + \sigma S_t d B_t + S_t \left( e^{\Delta Y_t} - 1 \right) d N_t
\]

\[
= \left( \mu + \frac{\sigma^2}{2} \right) S_t dt + \sigma S_t d B_t + S_t \left( e^{Z_{N_t}} - 1 \right) d N_t,
\]

hence the process \( S_t \) has jumps of size \( S_{T_k} (e^{Z_k} - 1) \), \( k \geq 1 \), and (20.3) reads

\[
\mu + \frac{\sigma^2}{2} - r = \sigma u - \tilde{\lambda} \mathbb{E}_{\tilde{\nu}}[e^Z - 1].
\]

Under this condition we can choose a risk-neutral probability measure \( \mathbb{P}_{u, \tilde{\lambda}, \tilde{\nu}} \) under which \( (e^{-rt} S_t)_{t \in \mathbb{R}^+} \) is a martingale, and the expected value

\[
e^{-(T-t)r} \mathbb{E}_{u, \tilde{\lambda}, \tilde{\nu}}[\phi(S_T) \mid \mathcal{F}_t]
\]

represents a (non-unique) arbitrage price at time \( t \in [0, T] \) for the contingent claim with payoff \( \phi(S_T) \).

This arbitrage price can be expressed as

\[
e^{-(T-t)r} \mathbb{E}_{u, \tilde{\lambda}, \tilde{\nu}}[\phi(S_T) \mid \mathcal{F}_t] = e^{-(T-t)r} \mathbb{E}_{u, \tilde{\lambda}, \tilde{\nu}}[\phi(S_0 e^{\mu T + \sigma B_T + Y_T}) \mid \mathcal{F}_t]
\]

\[
= e^{-(T-t)r} \mathbb{E}_{u, \tilde{\lambda}, \tilde{\nu}}[\phi(S_t e^{(T-t)(\mu + (B_T - B_t)\sigma + Y_T - Y_t)} \mid \mathcal{F}_t]
\]

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where

\[ \text{Gaussian} \]

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and we have

\[ e^{-ut} \text{as in (20.3), hence by the Girsanov Theorem 19.18 for jump processes, } B_t + ut + Y_t - \tilde{\lambda} \mathbb{E}_\tilde{\nu}[e^{\tilde{Z}} - 1]t \text{ is a martingale and } B_t + ut \text{ is a standard Brownian motion under } \mathbb{P}_{u,\tilde{\lambda},\tilde{\nu}}. \]

For simplicity we choose \( u = 0 \), which yields

\[ \mu = r - \sigma^2/2 - \tilde{\lambda}(e^{\delta + \eta^2/2} - 1), \]

and we have

\[ e^{-t} \mathbb{E}_{\tilde{\lambda},\tilde{\nu}}[\phi(S_T) | \mathcal{F}_t] \]

\[ = e^{-t} \mathbb{E}_{\tilde{\lambda},\tilde{\nu}}[\phi(S_T)|\mathcal{F}_t] \]

\[ \times \mathbb{E}_{\tilde{\lambda},\tilde{\nu}}[\phi(x e^{(T-t)\mu + (B_T-B_t)\sigma + \sum_{k=1}^{n} Z_k})]_{x=S_t} \]

\[ = e^{-t} \mathbb{E}_{\tilde{\lambda},\tilde{\nu}}[\phi(x e^{(T-t)\mu + (B_T-B_t)\sigma + \sum_{k=1}^{n} Z_k})]_{x=S_t} \]

\[ = e^{-t} \mathbb{E}_{\tilde{\lambda},\tilde{\nu}}[\phi(x e^{(T-t)\mu + (B_T-B_t)\sigma + \sum_{k=1}^{n} Z_k})]_{x=S_t} \]

\[ = e^{-t} \mathbb{E}_{\tilde{\lambda},\tilde{\nu}}[\phi(x e^{(T-t)\mu + (B_T-B_t)\sigma + \sum_{k=1}^{n} Z_k})]_{x=S_t} \]

\[ \text{where} \]

\[ X_n := (B_T - B_t) \sigma + \sum_{k=1}^{n} (Z_k - \delta) \sim \mathcal{N}(0, (T-t)\sigma^2 + \eta^2), \quad n \geq 0, \]
is a centered Gaussian random variable with variance
\[ v_n^2 := (T - t)\sigma^2 + \sum_{k=1}^{n} \text{Var} Z_k = (T - t)\sigma^2 + n\eta^2. \]

Hence when \( \phi(x) = (x - \kappa)^+ \) is the payoff function of a European call option, using the relation
\[ B_t(x, \kappa, v_n^2 / \tau, r, \tau) = e^{-r\tau} \mathbb{E} \left[ (x e^{X_n - v_n^2 / 2 + r\tau} - K)^+ \right] \]
we get
\[
e^{-(T-t)r-(T-t)\lambda} \mathbb{E}_{\lambda, \nu \mid F_t} \left[ (S_T - \kappa)^+ \right] = \sum_{n \geq 0} \frac{(T-t)\lambda^n}{n!} e^{n\delta + n\eta^2 / 2 + \lambda(e^\delta + \eta^2 / 2 - 1)(T-t) + n\delta + X_n - \kappa} \bigg|_{x=S_t} \]
\[ = \sum_{n \geq 0} \frac{(T-t)\tilde{\lambda}^n}{n!} \mathbb{E} \left[ (x e^{r - \sigma^2 / 2 - \lambda(e^\delta + \eta^2 / 2 - 1)}(T-t) + n\delta + X_n - \kappa)^+ \right] \bigg|_{x=S_t} \]
\[ = \sum_{n \geq 0} \frac{(T-t)\tilde{\lambda}^n}{n!} \mathbb{E} \left[ (x e^{n\delta + n\eta^2 / 2 - \lambda(e^\delta + \eta^2 / 2 - 1)}(T-t) + X_n - v^2 n/2 + (T-t)r - \kappa)^+ \right] \bigg|_{x=S_t} \]
\[ = e^{-(T-t)\tilde{\lambda}} \sum_{n \geq 0} \frac{(T-t)\tilde{\lambda}^n}{n!} \times B_t(S_{t} e^{n\delta + n\eta^2 / 2 - \lambda(e^\delta + \eta^2 / 2 - 1)}(T-t), \kappa, \sigma^2 + n\eta^2 / (T-t), r, T-t). \]

We may also write
\[
e^{-(T-t)r-(T-t)\lambda} \mathbb{E}_{\lambda, \nu \mid F_t} \left[ (S_T - \kappa)^+ \right] = \sum_{n \geq 0} \frac{(T-t)\lambda^n}{n!} e^{n\delta + n\eta^2 / 2 - \lambda(e^\delta + \eta^2 / 2 - 1)(T-t)} \times B_t(S_{t}, \kappa e^{-n\delta - n\eta^2 / 2 + \lambda(e^\delta + \eta^2 / 2 - 1)(T-t) + \sigma^2 + n\eta^2 / (T-t), r, T-t)} \]
\[ = e^{-\lambda e^{\delta + \eta^2 / 2}} \sum_{n \geq 0} \frac{\tilde{\lambda} e^{\delta + n\eta^2 / 2}(T-t)^n}{n!} \times B_t(S_{t}, \sigma^2 + n\eta^2 / (T-t), r + n\delta + \eta^2 / T-t - \tilde{\lambda}(e^\delta + \eta^2 / 2 - 1), T-t) \].

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20.6 Self-Financing Hedging with Jumps

Consider a portfolio valued

\[ V_t := \eta_t A_t + \xi_t S_t = \eta_t e^{rt} + \xi_t S_t \]

at time \( t \in \mathbb{R}_+ \), and satisfying the self-financing condition (5.3), i.e.

\[ dV_t = \eta_t dA_t + \xi_t dS_t = r\eta_t e^{rt} dt + \xi_t dS_t. \]

Assuming that the portfolio price takes the form \( V_t = f(t, S_t) \) for all times \( t \in [0, T] \), by (20.7) we have

\[
\begin{align*}
   dV_t &= df(t, S_t) \\
         &= r\eta_t e^{rt} dt + \xi_t dS_t \\
         &= r\eta_t e^{rt} dt + \xi_t (rS_t dt + \sigma S_t d\hat{B}_t + S_t^{-} (dY_t - \tilde{\lambda} \mathbb{E}_v[Z] dt)) \\
         &= rV_t dt + \sigma \xi_t S_t d\hat{B}_t + \xi_t S_t^{-} (dY_t - \tilde{\lambda} \mathbb{E}_v[Z] dt) \\
         &= r f(t, S_t) dt + \sigma \xi_t S_t d\hat{B}_t + \xi_t S_t^{-} (dY_t - \tilde{\lambda} \mathbb{E}_v[Z] dt),
\end{align*}
\]

(20.12)

has to match

\[
\begin{align*}
   df(t, S_t) &= rf(t, S_t) dt + \sigma S_t \frac{\partial f}{\partial x}(t, S_t) d\hat{B}_t \\
               &+ (f(t, S_t^{-} (1 + Z_{N_t})) - f(t, S_t^{-})) dN_t - \tilde{\lambda} \mathbb{E}_v[(f(t, x(1 + Z)) - f(t, x))]_{x=S_t} dt,
\end{align*}
\]

(20.13)

which is obtained from (20.11).

In such a situation we say that the claim payoff \( C \) can be exactly replicated.

Exact replication is possible in essentially only two situations:

(i) **Continuous market**, \( \lambda = \tilde{\lambda} = 0 \). In this case we find the usual Black-Scholes Delta:

\[ \xi_t = \frac{\partial f}{\partial x}(t, S_t). \]

(20.14)

(ii) **Poisson jump market**, \( \sigma = 0 \) and \( Y_t = a N_t \), \( \nu(dx) = \delta_a(dx) \). In this case we find

\[ \xi_t = \frac{1}{a S_t^{-}} (f(t, S_t^{-} (1 + a)) - f(t, S_t^{-})). \]

(20.15)

Note that in the limit \( a \to 0 \) this expression recovers the Black-Scholes Delta formula (20.14).

When Conditions (i) or (ii) above are not satisfied, exact replication is not possible and this results into an hedging error given from (20.12) and (20.13).
by

\[ V_T - \phi(S_T) = V_T - f(T, S_T) \]
\[ = V_0 + \int_0^T dV_t - f(0, S_0) - \int_0^T df(t, S_t) \]
\[ = V_0 - f(0, S_0) + \sigma \int_0^T S_t \left( \xi_t - \frac{\partial f}{\partial x}(t, S_t) \right) d\widehat{B}_t \]
\[ + \int_0^T \xi_t S_{t^-} (Z_{N_t}dN_t - \lambda \mathbb{E}_\nu[Z])dt \]
\[ - \int_0^T (f(t, S_{t^-} (1 + Z_{N_t})) - f(t, S_{t^-}))dN_t \]
\[ + \lambda \int_0^T \mathbb{E}_\nu[(f(t, x(1 + Z)) - f(t, x))]_{x=S_t} dt. \]

Assuming for simplicity that \( Y_t = aN_t \), i.e. \( \nu(dx) = \delta_a(dx) \), we get

\[ V_T - f(T, S_T) = V_0 - f(0, S_0) + \sigma \int_0^T S_{t^-} \left( \xi_t - \frac{\partial f}{\partial x}(t, S_{t^-}) \right) d\widehat{B}_t \]
\[ - \int_0^T (f(t, S_{t^-} (1 + a)) - f(t, S_{t^-}) - a\xi_t S_{t^-})(dN_t - \lambda dt), \]

hence the mean square hedging error is given by

\[ \mathbb{E}_{u,\lambda} \left[ (V_T - f(T, S_T))^2 \right] \]
\[ = (V_0 - f(0, S_0))^2 + \sigma^2 \mathbb{E}_{u,\lambda} \left[ \int_0^T S_{t^-} \left( \xi_t - \frac{\partial f}{\partial x}(t, S_{t^-}) \right) d\widehat{B}_t \right]^2 \]
\[ + \mathbb{E}_{u,\lambda} \left[ \left( \int_0^T (f(t, S_{t^-} (1 + a)) - f(t, S_{t^-}) - a\xi_t S_{t^-})(dN_t - \lambda dt) \right)^2 \right] \]
\[ = (V_0 - f(0, S_0))^2 + \sigma^2 \mathbb{E}_{u,\lambda} \left[ \int_0^T S_{t^-}^2 \left( \xi_t - \frac{\partial f}{\partial x}(t, S_{t^-}) \right)^2 dt \right] \]
\[ + \lambda \mathbb{E}_{u,\lambda} \left[ \int_0^T ((f(t, S_{t^-} (1 + a)) - f(t, S_{t^-}) - a\xi_t S_{t^-})^2 dt \right], \]

where we applied the Itô isometry (19.17). Clearly, the initial portfolio value \( V_0 \) that minimizes the above quantity is

\[ V_0 = f(0, S_0) = e^{-rT} \mathbb{E}_{u,\lambda,\nu}[\phi(S_T)]. \]

When hedging only the risk generated by the Brownian part we let

\[ \xi_t = \frac{\partial f}{\partial x}(t, S_{t^-}) \]
as in the Black-Scholes model, and in this case the hedging error due to the presence of jumps becomes
\[
\mathbb{E}_{u,\lambda} \left[ (V_T - f(T, S_T))^2 \right] = \tilde{\lambda} \mathbb{E}_{u,\lambda} \left[ \int_0^T \left( (f(t, S_t - (1 + a)) - f(t, S_t - a\xi_t S_t -)) \right)^2 dt \right].
\]

Next, let us find the optimal strategy \((\xi_t)_{t \in \mathbb{R}_+}\) that minimizes the remaining hedging error
\[
\mathbb{E}_{u,\lambda} \left[ \int_0^T \left( \sigma^2 S_t^2 \left( \xi_t - \frac{\partial f}{\partial x}(t, S_t -) \right)^2 + \tilde{\lambda}((f(t, S_t - (1 + a)) - f(t, S_t - a\xi_t S_t -))^2 \right) dt \right].
\]

For all \(t \in [0, T]\), the almost-sure minimum of
\[
\xi_t \mapsto \sigma^2 S_t^2 \left( \xi t - \frac{\partial f}{\partial x}(t, S_t -) \right)^2 + \tilde{\lambda}((f(t, S_t - (1 + a)) - f(t, S_t - a\xi_t S_t -))^2
\]
is given by differentiation with respect to \(\xi_t\), as the solution of
\[
2\sigma^2 S_t^2 \left( \xi_t - \frac{\partial f}{\partial x}(t, S_t -) \right) - 2a\tilde{\lambda}S_t - ((f(t, S_t - (1 + a)) - f(t, S_t - a\xi_t S_t -)) = 0,
\]
i.e.
\[
\xi_t = \frac{\sigma^2}{\sigma^2 + a^2 \tilde{\lambda}} \frac{\partial f}{\partial x}(t, S_t -) + \frac{a^2 \tilde{\lambda}}{\sigma^2 + a^2 \tilde{\lambda}} \times \frac{f(t, S_t - (1 + a)) - f(t, S_t -)}{aS_t -},
\]

\(t \in (0, T]\). We note that the optimal strategy \((20.16)\) is a weighted average of the Brownian and jump hedging strategies \((20.14)\) and \((20.15)\) according to the respective variance parameters \(\sigma^2\) and \(a^2 \tilde{\lambda}\) of the continuous and jump components.

Clearly, if \(a \tilde{\lambda} = 0\) we get
\[
\xi_t = \frac{\partial f}{\partial x}(t, S_t -), \quad t \in [0, T],
\]
which is the Black-Scholes perfect replication strategy, and when \(\sigma = 0\) we recover
\[
\xi_t = \frac{f(t, (1 + a)S_t -) - f(t, S_t -)}{aS_t -}, \quad t \in [0, T].
\]
which is \((20.15)\).
Note that the fact that perfect replication is not possible in a jump-diffusion model can be interpreted as a more realistic feature of the model, as perfect replication is not possible in the real world.

See Jeanblanc and Privault (2002) for an example of a complete market model with jumps, in which continuous and jump noise are mutually excluding each other over time.

In the following table we summarizes the properties of geometric Brownian motion vs jump-diffusion models in terms of asset price and market behaviors.

<table>
<thead>
<tr>
<th>Properties</th>
<th>Geometric Brownian motion</th>
<th>Jump-diffusion model</th>
<th>Real world</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discontinuous asset prices</td>
<td>✗</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Fat tailed market returns</td>
<td>✗</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Complete market</td>
<td>✓</td>
<td>✗</td>
<td>✗</td>
</tr>
<tr>
<td>Unique prices and risk-neutral measure</td>
<td>✓</td>
<td>✗</td>
<td>✗</td>
</tr>
</tbody>
</table>

Table 20.1: Market models and properties.

Exercises

Exercise 20.1 Consider a standard Poisson process \( (N_t)_{t \in \mathbb{R}_+} \) with intensity \( \lambda > 0 \) under a probability measure \( P \). Let \( (S_t)_{t \in \mathbb{R}_+} \) be defined by the stochastic differential equation

\[
dS_t = rS_t dt + \eta S_t - (dN_t - \alpha dt),
\]

where \( \eta > 0 \).

a) Find the value of \( \alpha \in \mathbb{R} \) such that the discounted process \( (e^{-rt}S_t)_{t \in \mathbb{R}_+} \) is a martingale under \( P \).

b) Compute the price at time \( t \in [0, T] \) of a power option with payoff \( |S_T|^2 \) at maturity \( T \).

Exercise 20.2 Consider a long forward contract with payoff \( S_T - K \) on a jump diffusion risky asset \( (S_t)_{t \in \mathbb{R}_+} \) given by

\[
dS_t = \mu S_t dt + \sigma S_t dB_t + S_t dY_t.
\]

a) Show that the forward claim admits a unique arbitrage price to be computed in a market with risk-free rate \( r > 0 \).

b) Show that the forward claim admits an exact replicating portfolio strategy based on the two assets \( S_t \) and \( e^{rt} \).
c) Recover portfolio strategy of Question (b) using the optimal portfolio strategy formula (20.16).

Exercise 20.3 Consider $(B_t)_{t \in \mathbb{R}_+}$ a standard Brownian motion and $(N_t)_{t \in \mathbb{R}_+}$ a standard Poisson process with intensity $\lambda > 0$, independent of $(B_t)_{t \in \mathbb{R}_+}$, under a probability measure $\mathbb{P}^*$. Let $(S_t)_{t \in \mathbb{R}_+}$ be defined by the stochastic differential equation

$$dS_t = \mu S_t dt + \eta S_t - dN_t + \sigma S_t dB_t.$$  

(20.17)

a) Solve the equation (20.17).

b) We assume that $\mu$, $\eta$ and the risk-free rate $r > 0$ are chosen such that the discounted process $(e^{-rt} S_t)_{t \in \mathbb{R}_+}$ is a martingale under $\mathbb{P}^*$. What relation does this impose on $\mu$, $\eta$, $\lambda$ and $r$?

c) Under the relation of Question (b), compute the price at time $t \in [0, T]$ of a European call option on $S_T$ with strike price $\kappa$ and maturity $T$, using a series expansion of Black-Scholes functions.

Exercise 20.4 Consider $(N_t)_{t \in \mathbb{R}_+}$ a standard Poisson process with intensity $\lambda > 0$ under a probability measure $\mathbb{P}$. Let $(S_t)_{t \in \mathbb{R}_+}$ be defined by the stochastic differential equation

$$dS_t = r S_t dt + Y N_t S_t - (dN_t - \alpha dt),$$

where $(Y_k)_{k \geq 1}$ is an i.i.d. sequence of uniformly distributed random variables on $[-1, 1]$.

a) Show that the discounted process $(e^{-rt} S_t)_{t \in \mathbb{R}_+}$ is a martingale under $\mathbb{P}$.

b) Compute the price at time $0$ of a European call option on $S_T$ with strike price $\kappa$ and maturity $T$, using a series of multiple integrals.

Exercise 20.5 Consider a standard Poisson process $(N_t)_{t \in \mathbb{R}_+}$ with intensity $\lambda > 0$ under a probability measure $\mathbb{P}$. Let $(S_t)_{t \in \mathbb{R}_+}$ be defined by the stochastic differential equation

$$dS_t = r S_t dt + Y N_t S_t - (dN_t - \alpha dt),$$

where $(Y_k)_{k \geq 1}$ is an i.i.d. sequence of uniformly distributed random variables on $[0, 1]$.

a) Find the value of $\alpha \in \mathbb{R}$ such that the discounted process $(e^{-rt} S_t)_{t \in \mathbb{R}_+}$ is a martingale under $\mathbb{P}$.

b) Compute the price at time $t \in [0, T]$ of the long forward contract with maturity $T$ and payoff $S_T - \kappa$. 

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https://www.ntu.edu.sg/home/nprivault/index.html
Exercise 20.6 Consider \((N_t)_{t \in \mathbb{R}_+}\), a standard Poisson process with intensity \(\lambda > 0\) under a risk-neutral probability measure \(\mathbb{P}^*\). Let \((S_t)_{t \in \mathbb{R}_+}\) be defined by the stochastic differential equation

\[
dS_t = rS_t \, dt + \alpha S_t - (dN_t - \lambda \, dt),
\]

(20.18)

where \(\alpha > 0\). Consider a portfolio with value

\[
V_t = \eta_t e^{rt} + \xi_t S_t
\]
at time \(t \in [0, T]\), and satisfying the self-financing condition

\[
dV_t = r\eta_t e^{rt} \, dt + \xi_t dS_t.
\]

We assume that the portfolio hedges a claim payoff \(C = \phi(S_T)\), and that the portfolio value can be written as a function \(V_t = f(t, S_t)\) of \(t\) and \(S_t\) for all times \(t \in [0, T]\).

a) Solve the stochastic differential equation (20.18). (10 marks)

b) Price the claim \(C = \phi(S_T)\) at time \(t \in [0, T]\) using a series expansion. (10 marks)

c) Show that under self-financing, the variation \(dV_t\) of the portfolio value \(V_t\) satisfies

\[
dV_t = rf(t, S_t) \, dt + \alpha \xi_t S_t - (dN_t - \lambda \, dt).
\]

(20.19)
d) Show that the claim payoff \(C = \phi(S_T)\) can be exactly replicated by the hedging strategy

\[
\xi_t = \frac{1}{\alpha S_t} \left( f(t, S_t - (1 + \alpha)) - f(t, S_t^-) \right).
\]

Exercise 20.7 Pricing by the Esscher transform (Gerber and Shiu (1994)). Consider a compound Poisson process \((Y_t)_{t \in [0, T]}\) with \(\mathbb{E} \left[ e^{(Y_t - Y_s)\theta} \right] = e^{(t-s)m(\theta)}\), \(\theta \in \mathbb{R}, 0 \leq s \leq t\), and the asset price process \(S_t := e^{rt + Y_t}\), \(t \in [0, T]\). Given \(\theta \in \mathbb{R}\), let

\[
N_t := \frac{e^{\theta Y_t}}{\mathbb{E} [e^{\theta Y_t}]} = e^{\theta Y_t - tm(\theta)} = S_t^\theta e^{-r\theta t - tm(\theta)},
\]

and consider the probability measure \(\mathbb{P}^\theta\) defined as

\[
\frac{d\mathbb{P}^\theta}{d\mathbb{P}} |_{\mathcal{F}_t} := \frac{N_T}{N_t} = e^{(Y_T - Y_t)\theta - (T-t)m(\theta)}, \quad 0 \leq t \leq T.
\]

a) Check that \((N_t)_{t \in \mathbb{R}_+}\) is a martingale under \(\mathbb{P}\).
b) Find a condition on $\theta$ such that the discounted price process $(e^{-rt}S_t)_{t \in [0,T]} = (e^{Y_t})_{t \in [0,T]}$ is a martingale under $\mathbb{P}^\theta$.

c) Price the European call option with payoff $(S_T - K)^+$ by taking $\mathbb{P}^\theta$ as risk-neutral measure.