We consider the pricing and hedging of options in a discrete-time financial model with $N + 1$ time instants $t = 0, 1, \ldots, N$. Vanilla options are treated using backward induction, and exotic options with arbitrary payoff functions are considered using the Clark-Ocone formula in discrete time.

3.1 Pricing Contingent Claims

Let us consider an attainable contingent claim with (random) claim payoff $C \geq 0$ and maturity $N$. Recall that by the Definition 2.14 of attainability there exists a (self-financing) portfolio strategy $(\xi_t)_{t=1,2,\ldots,N}$ that hedges the claim payoff $C$, in the sense that

$$\bar{\xi}_N \cdot \bar{S}_N = \sum_{k=0}^{d} \xi_N^{(k)} S_N^{(k)} = C \quad (3.1)$$

at time $N$. Clearly, if (3.1) holds, then investing the amount

$$V_0 = \bar{\xi}_1 \cdot \bar{S}_0 = \sum_{k=0}^{d} \xi_1^{(k)} S_0^{(k)} \quad (3.2)$$

at time $t = 0$, resp.
\[ V_t = \xi_t \cdot S_t = \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)} \]  

(3.3)

at times \( t = 1, 2, \ldots, N \) into a self-financing hedging portfolio will allow one to hedge the option and to obtain the perfect replication (3.1) at time \( N \).

**Definition 3.1.** The value (3.2)-(3.3) at time \( t \) of a self-financing portfolio strategy \((\xi_t)_{t=1,2,\ldots,N}\) hedging an attainable claim payoff \( C \) will be called an arbitrage price of the claim payoff \( C \) at time \( t \) and denoted by \( \pi_t(C) \), \( t = 0, 1, \ldots, N \).

Recall that arbitrage prices can be used to ensure that financial derivatives are “marked” at their fair value (mark to market).

Next we develop a second approach to the pricing of contingent claims, based on conditional expectations and martingale arguments. We will need the following lemma, in which \( \tilde{V}_t := V_t/(1+r)^t \) denotes the discounted portfolio value, \( t = 0, 1, \ldots, N \).

Relation (3.4) in the following lemma has a natural interpretation by saying that when a portfolio is self-financing the value \( \tilde{V}_t \) of the (discounted) portfolio at time \( t \) is given by summing up the (discounted) profits and losses registered over all time periods from time 0 to time \( t \). Note that in (3.4), the use of the vector of discounted asset prices allows us to add up the discounted profits and losses \( \xi_t \cdot (X_t - X_{t-1}) \) since they are expressed in units of currency “at time 0”. Indeed, in general, $1 at time \( t = 0 \) cannot be added to $1 at time \( t = 1 \) without proper discounting.

**Lemma 3.2.** The following statements are equivalent:

(i) The portfolio strategy \((\xi_t)_{t=1,2,\ldots,N}\) is self-financing.

(ii) \( \xi_t \cdot X_t = \xi_{t+1} \cdot X_t \) for all \( t = 1, 2, \ldots, N - 1 \).

(iii) the discounted portfolio value \( \tilde{V}_t \) can be written as the stochastic summation

\[ \tilde{V}_t = \tilde{V}_0 + \sum_{k=1}^{t} \xi_k \cdot (X_k - X_{k-1}), \quad t = 0, 1, \ldots, N, \]  

(3.4)

of discounted profits and losses.

**Proof.** First, the self-financing condition (i)

\[ \xi_t \cdot X_t = \xi_{t+1} \cdot X_t \]  

for all \( t = 1, 2, \ldots, N - 1 \) implies that the discounted portfolio value satisfies

\[ \tilde{V}_t = \tilde{V}_0 + \sum_{k=1}^{t} \xi_k \cdot (X_k - X_{k-1}), \quad t = 0, 1, \ldots, N, \]  

(3.4)

which is the statement (iii) of the lemma.

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\[ \xi_{t-1} \cdot \mathcal{S}_{t-1} = \xi_t \cdot \mathcal{S}_{t-1}, \quad t = 2, 3, \ldots, N, \]

is clearly equivalent to \((ii)\) by division of both sides by \((1 + r)^{t-1}\).

Next, assuming that \((ii)\) holds we have the telescoping identity

\[
\tilde{V}_t = \tilde{V}_0 + \sum_{k=1}^{t} (\tilde{V}_k - \tilde{V}_{k-1}) \\
= \tilde{V}_0 + \sum_{k=1}^{t} \xi_k \cdot \mathcal{X}_k - \xi_{k-1} \cdot \mathcal{X}_{k-1} \\
= \tilde{V}_0 + \sum_{k=1}^{t} \xi_k \cdot \mathcal{X}_k - \xi_k \cdot \mathcal{X}_{k-1} \\
= \tilde{V}_0 + \sum_{k=1}^{t} \xi_k \cdot (\mathcal{X}_k - \mathcal{X}_{k-1}), \quad t = 1, 2, \ldots, N.
\]

Finally, assuming that \((iii)\) holds we get

\[
\tilde{V}_t - \tilde{V}_{t-1} = \xi_t \cdot (\mathcal{X}_t - \mathcal{X}_{t-1}),
\]

which rewrites as

\[
\xi_t \cdot \mathcal{X}_t - \xi_{t-1} \cdot \mathcal{X}_{t-1} = \xi_t \cdot (\mathcal{X}_t - \mathcal{X}_{t-1}),
\]

or

\[
\xi_{t-1} \cdot \mathcal{X}_{t-1} = \xi_t \cdot \mathcal{X}_{t-1}, \quad t = 1, 2, \ldots, N,
\]

which implies \((ii)\).

In Relation (3.4), the term \(\xi_t \cdot (\mathcal{X}_t - \mathcal{X}_{t-1})\) represents the profit and loss

\[
\tilde{V}_t - \tilde{V}_{t-1} = \xi_t \cdot (\mathcal{X}_t - \mathcal{X}_{t-1}),
\]

of the self-financing portfolio strategy \((\xi_j)_{j=1,2,\ldots,N}\) over the time interval \((t-1, t]\), computed by multiplication of the portfolio allocation \(\xi_t\) with the change of price \(\mathcal{X}_t - \mathcal{X}_{t-1}, t = 1, 2, \ldots, N\).

The sum (3.4) is also referred to as a discrete-time stochastic integral of the portfolio strategy \((\xi_t)_{t=1,2,\ldots,N}\) with respect to the random process \((\mathcal{X}_t)_{t=0,1,\ldots,N}\).

**Remark 3.3.** As a consequence of the above Lemma 3.2, if a contingent claim payoff \(C\) with discounted payoff is attainable by a self-financing portfolio strategy \((\xi_t)_{t=1,2,\ldots,N}\) then the discounted claim payoff

\[
\circ
\]

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rewrites as the sum of discounted profits and losses
\[
\tilde{C} = \tilde{V}_N = \xi_N \cdot \bar{X}_N = \tilde{V}_0 + \sum_{t=1}^{N} \xi_t \cdot (\bar{X}_t - \bar{X}_{t-1}).
\]  

Remark 3.4. By Proposition 2.11, the process \((\bar{X}_t)_{t=0,1,\ldots,N}\) is a martingale under the risk-neutral probability measure \(P^*\), hence by Proposition 2.9 and Lemma 3.2, \((\tilde{V}_t)_{t=0,1,\ldots,N}\) in (3.4) is also martingale under \(P^*\), provided that \((\tilde{\xi}_t)_{t=1,2,\ldots,N}\) is a self-financing and predictable process.

The above remarks will be used in the proof of the next Theorem 3.5.

Theorem 3.5. The arbitrage price \(\pi_t(C)\) of any (integrable) attainable contingent claim payoff \(C\) is given by
\[
\pi_t(C) = \frac{1}{(1+r)^{N-t}} \mathbb{E}^*[C \mid \mathcal{F}_t], \quad t = 0, 1, \ldots, N,
\]  
where \(P^*\) denotes any risk-neutral probability measure.

Proof. a) Short proof. Since the claim payoff \(C\) is attainable, there exists a self-financing portfolio strategy \((\xi_t)_{t=1,2,\ldots,N}\) such that \(C = V_N\), i.e. \(\tilde{C} = \tilde{V}_N\).

In addition, by Lemma 3.2 and Proposition 2.9 the process \((\tilde{V}_t)_{t=0,1,\ldots,N}\) is a martingale under \(P^*\), hence we have
\[
\tilde{V}_t = \mathbb{E}^*[\tilde{V}_N \mid \mathcal{F}_t] = \mathbb{E}^*[\tilde{C} \mid \mathcal{F}_t], \quad t = 0, 1, \ldots, N,
\]  
which shows (3.8). To conclude, we note that by Definition 3.1 the arbitrage price \(\pi_t(C)\) of the claim at time \(t\) is equal to the value \(V_t\) of the self-financing hedging \(C\).

b) Long proof. For completeness, we include a self-contained, step by step derivation of (3.7) by following the argument of Proposition 2.9, as follows. By Remark 3.3 we have
\[
\mathbb{E}^*[\tilde{C} \mid \mathcal{F}_t] = \mathbb{E}^*[\tilde{V}_N \mid \mathcal{F}_t]
\]  
\[
= \mathbb{E}^*\left[\tilde{V}_0 + \sum_{k=1}^{N} \xi_k \cdot (\bar{X}_k - \bar{X}_{k-1}) \mid \mathcal{F}_t\right]
\]  
\[
= \mathbb{E}^*[\tilde{V}_0 \mid \mathcal{F}_t] + \sum_{k=1}^{N} \mathbb{E}^*[\xi_k \cdot (\bar{X}_k - \bar{X}_{k-1}) \mid \mathcal{F}_t]
\]  

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\[ \tilde{V}_0 + \sum_{k=1}^{t} \mathbb{E}^* [\tilde{\xi}_k \cdot (\bar{X}_k - \bar{X}_{k-1}) \mid \mathcal{F}_t] + \sum_{k=t+1}^{N} \mathbb{E}^* [\tilde{\xi}_k \cdot (\bar{X}_k - \bar{X}_{k-1}) \mid \mathcal{F}_t] \]

\[ = \tilde{V}_0 + \sum_{k=1}^{t} \tilde{\xi}_k \cdot (\bar{X}_k - \bar{X}_{k-1}) + \sum_{k=t+1}^{N} \mathbb{E}^* [\tilde{\xi}_k \cdot (\bar{X}_k - \bar{X}_{k-1}) \mid \mathcal{F}_t] \]

\[ = \tilde{V}_t + \sum_{k=t+1}^{N} \mathbb{E}^* [\tilde{\xi}_k \cdot (\bar{X}_k - \bar{X}_{k-1}) \mid \mathcal{F}_t], \]

where we used Relation (3.4) of Lemma 3.2. In order to obtain (3.8) we need to show that

\[ \sum_{k=t+1}^{N} \mathbb{E}^* [\tilde{\xi}_k \cdot (\bar{X}_k - \bar{X}_{k-1}) \mid \mathcal{F}_t] = 0, \]

or

\[ \mathbb{E}^* [\tilde{\xi}_j \cdot (\bar{X}_j - \bar{X}_{j-1}) \mid \mathcal{F}_t] = 0, \]

for all \( j = t + 1, \ldots, N \). Since \( 0 \leq t \leq j - 1 \) we have \( \mathcal{F}_t \subset \mathcal{F}_{j-1} \), hence by the “tower property” of conditional expectations we get

\[ \mathbb{E}^* [\tilde{\xi}_j \cdot (\bar{X}_j - \bar{X}_{j-1}) \mid \mathcal{F}_t] = \mathbb{E}^* [\mathbb{E}^* [\tilde{\xi}_j \cdot (\bar{X}_j - \bar{X}_{j-1}) \mid \mathcal{F}_{j-1}] \mid \mathcal{F}_t], \]

therefore it suffices to show that

\[ \mathbb{E}^* [\tilde{\xi}_j \cdot (\bar{X}_j - \bar{X}_{j-1}) \mid \mathcal{F}_{j-1}] = 0, \quad j = 1, 2, \ldots, N. \]

We note that the portfolio allocation \( \tilde{\xi}_j \) over the time period \( [j - 1, j] \) is predictable, i.e. it is decided at time \( j - 1 \), and it thus depends only on the information \( \mathcal{F}_{j-1} \) known up to time \( j - 1 \), hence

\[ \mathbb{E}^* [\tilde{\xi}_j \cdot (\bar{X}_j - \bar{X}_{j-1}) \mid \mathcal{F}_{j-1}] = \tilde{\xi}_j \cdot \mathbb{E}^* [\bar{X}_j - \bar{X}_{j-1} \mid \mathcal{F}_{j-1}]. \]

Finally we note that

\[ \mathbb{E}^* [\bar{X}_j - \bar{X}_{j-1} \mid \mathcal{F}_{j-1}] = \mathbb{E}^* [\bar{X}_j \mid \mathcal{F}_{j-1}] - \mathbb{E}^* [\bar{X}_{j-1} \mid \mathcal{F}_{j-1}] \]

\[ = \mathbb{E}^* [\bar{X}_j \mid \mathcal{F}_{j-1}] - \bar{X}_{j-1} \]

\[ = 0, \quad j = 1, 2, \ldots, N, \]

because \( (\bar{X}_t)_{t=0,1,\ldots,N} \) is a martingale under the risk-neutral probability measure \( \mathbb{P}^* \), and this concludes the proof of (3.7). Let

\[ \tilde{C} = \frac{C}{(1 + r)^N} \]

denote the discounted payoff of the claim payoff \( C \). We will show that under any risk-neutral probability measure \( \mathbb{P}^* \) the discounted value of any self-
financing portfolio hedging $C$ is given by
\[ \tilde{V}_t = \mathbb{E}^* [\tilde{C} | \mathcal{F}_t], \quad t = 0, 1, \ldots, N, \tag{3.8} \]
which shows that
\[ V_t = \frac{1}{(1+r)^{N-t}} \mathbb{E}^* [C | \mathcal{F}_t] \]
after multiplication of both sides by $(1+r)^t$. Next, we note that (3.8) follows from the martingale transform result of Proposition 2.9.

Note that (3.6) admits an interpretation in an insurance framework, in which $\pi_t(C)$ represents an insurance premium and $C$ represents the random value of an insurance claim made by a subscriber. In this context, the premium of the insurance contract reads as the average of the values (3.6) of the random claims after discounting for the time value of money.

**Remark 3.6.** The self-financing discounted portfolio price process
\[ (\tilde{V}_t)_{t=0,1,\ldots,N} = ((1+r)^{-t} \pi_t(C))_{t=0,1,\ldots,N} \]
is a martingale under $\mathbb{P}^*$. From Theorem 3.5, we can recover this fact as in Remark 3.3, since from the “tower property” (22.38) of conditional expectations we have
\[ \tilde{V}_t = \mathbb{E}^* [\tilde{C} | \mathcal{F}_t] \\
= \mathbb{E}^* [\mathbb{E}^* [\tilde{C} | \mathcal{F}_{t+1}] | \mathcal{F}_t] \\
= \mathbb{E}^* [\tilde{V}_{t+1} | \mathcal{F}_t], \quad t = 0, 1, \ldots, N - 1. \tag{3.9} \]
This also allows us to compute $V_t$ by backward induction on $t = 0, 1, \ldots, N - 1$, starting from $V_N = C$.

In particular, at $t = 0$ we obtain the price of the contingent claim payoff $C$ at time 0:
\[ \pi_0(C) = \mathbb{E}^* [\tilde{C} | \mathcal{F}_0] = \mathbb{E}^* [\tilde{C}] = \frac{1}{(1+r)^N} \mathbb{E}^*[C]. \]

### 3.2 Pricing Vanilla Options in the CRR Model

In this section we consider the pricing of contingent claims in the discrete-time Cox-Ross-Rubinstein model, with $d = 1$. More precisely we are concerned with vanilla options whose payoffs depend on the terminal value of the underlying asset, as opposed to exotic options whose payoffs may depend
on the whole path of the underlying asset price until expiration time.

Recall that the portfolio value process \((V_t)_{t=0,1,...,N}\) and the discounted portfolio value process respectively satisfy

\[
V_t = \xi_t \cdot S_t \quad \text{and} \quad \tilde{V}_t = \frac{1}{(1+r)^t} V_t = \frac{1}{(1+r)^t} \xi_t \cdot S_t = \xi_t \cdot X_t, \quad t = 0, 1, \ldots, N.
\]

Here we will be concerned with the pricing of vanilla options with payoffs of the form

\[
C = f(S_N^{(1)}),
\]
e.g. \(f(x) = (x - K)^+\) in the case of a European call. Equivalently, the discounted claim payoff

\[
\tilde{C} = \frac{C}{(1+r)^N}
\]
satisfies \(\tilde{C} = \tilde{f}(S_N^{(1)})\) with \(\tilde{f}(x) = f(x)/(1+r)^N\). For example in the case of a European call option with strike price \(K\) we have

\[
\tilde{f}(x) = \frac{1}{(1+r)^N} (x - K)^+.
\]

From Theorem 3.5, the discounted value of a portfolio hedging the attainable (discounted) claim payoff \(\tilde{C}\) is given by

\[
\tilde{V}_t = \mathbb{E}^* \left[ \tilde{f}(S_N^{(1)} \mid \mathcal{F}_t) \right], \quad t = 0, 1, \ldots, N,
\]

under the risk-neutral probability measure \(\mathbb{P}^*\). As a consequence, we have the following proposition.

**Proposition 3.7.** The arbitrage price \(\pi_t(C)\) at time \(t = 0, 1, \ldots, N\) of the contingent claim payoff \(C = f(S_N^{(1)})\) is given by

\[
\pi_t(C) = \frac{1}{(1+r)^N} \mathbb{E}^* \left[ f(S_N^{(1)} \mid \mathcal{F}_t) \right] = \frac{1}{(1+r)^N} \mathbb{E}^* \left[ f(S_N^{(1)} \mid S_t) \right], \quad t = 0, 1, \ldots, N.
\]

In the next proposition we implement the calculation of (3.10).

**Proposition 3.8.** The price \(\pi_t(C)\) of the contingent claim payoff \(C = f(S_N^{(1)})\) satisfies

\[
\pi_t(C) = v(t, S_t^{(1)}), \quad t = 0, 1, \ldots, N,
\]

* Download the corresponding (non-recursive) IPython notebook that can be run [here](http://www.ntu.edu.sg/home/nprivault/indext.html).
where the function \(v(t, x)\) is given by

\[
v(t, x) = \frac{1}{(1+r)^{N-t}} \mathbb{E}^* \left[ f \left( x \prod_{j=t+1}^{N} (1 + R_j) \right) \right] \tag{3.11}
\]

\[
= \frac{1}{(1+r)^{N-t}} \sum_{k=0}^{N-t} \binom{N-t}{k} (p^*)^k (1-p^*)^{N-t-k} f(x(1+b)^k(1+a)^{N-t-k}).
\]

Proof. From the relations

\[
S_N^{(1)} = S_t^{(1)} \prod_{j=t+1}^{N} (1 + R_j),
\]

and (3.10) we have, using Property (v) of the conditional expectation, see page 58, and the independence of the market returns \(\{R_1, \ldots, R_t\}\) and \(\{R_{t+1}, \ldots, R_N\}\),

\[
\begin{align*}
\pi_t(C) &= \frac{1}{(1+r)^{N-t}} \mathbb{E}^* \left[ f(S_N^{(1)}) \mid \mathcal{F}_t \right] \\
&= \frac{1}{(1+r)^{N-t}} \mathbb{E}^* \left[ f \left( S_t^{(1)} \prod_{j=t+1}^{N} (1 + R_j) \right) \mid S_t^{(1)} \right] \\
&= \frac{1}{(1+r)^{N-t}} \mathbb{E}^* \left[ f \left( x \prod_{j=t+1}^{N} (1 + R_j) \right) \right]_{x=S_t^{(1)}},
\end{align*}
\]

where we used Property (v) of the conditional expectation, see page 58, and the independence of the market returns. Next, we note that the number of times \(R_j\) is equal to \(b\) for \(j \in \{t+1, \ldots, N\}\), has a binomial distribution with parameter \((N-t, p^*)\), where

\[
p^* = \frac{r-a}{b-a} \quad \text{and} \quad 1-p^* = \frac{b-r}{b-a}, \tag{3.12}
\]

since the set of paths from time \(t+1\) to time \(N\) containing \(j\) times “\((1+b)\)” has cardinality \(\binom{N-t}{j}\) and each such path has probability

\[
(p^*)^j (1-p^*)^{N-t-j}, \quad j = 0, \ldots, N-t.
\]

Hence we have

\[
\pi_t(C) = \frac{1}{(1+r)^{N-t}} \mathbb{E}^* \left[ f(S_N^{(1)}) \mid \mathcal{F}_t \right]
\]
\[
\frac{1}{(1 + r)^{N-t}} \sum_{k=0}^{N-t} \binom{N-t}{k} (p^*)^k (1 - p^*)^{N-t-k} f(S_t^{(1)}(1 + b)^k (1 + a)^{N-t-k}).
\]

In the above proof we have also shown that \( \pi_t(C) \) is given by the conditional expected value

\[
\pi_t(C) = \frac{1}{(1 + r)^{N-t}} \mathbb{E}^*[f(S_N^{(1)}) \mid \mathcal{F}_t] = \frac{1}{(1 + r)^{N-t}} \mathbb{E}^*[f(S_t^{(1)}) \mid S_t^{(1)}]
\]
given the value of \( S_t^{(1)} \) at time \( t = 0, 1, \ldots, N \), due to the Markov property of \( (S_t^{(1)})_{t=0,1,\ldots,N} \). In particular, the price of the claim payoff \( C \) is written as the average (path integral) of the values of the contingent claim over all possible paths starting from \( S_t^{(1)} \).

**Market terms and data**

**Intrinsic value.** The *intrinsic value* at time \( t = 0, 1, \ldots, N \) of the option with payoff \( C = h(S_N^{(1)}) \) is given by the immediate exercise payoff \( h(S_t^{(1)}) \). The *extrinsic value* at time \( t = 0, 1, \ldots, N \) of the option is the remaining difference \( \pi_t(C) - h(S_t^{(1)}) \) between the option price \( \pi_t(C) \) and the immediate exercise payoff \( h(S_t^{(1)}) \). In general, the option price \( \pi_t(C) \) decomposes as

\[
\pi_t(C) = h(S_t^{(1)}) + \underbrace{\pi_t(C) - h(S_t^{(1)})}_{\text{extrinsic value}}, \quad t = 0, 1, \ldots, N.
\]

**Gearing.** The *gearing* at time \( t = 0, 1, \ldots, N \) of the option with payoff \( C = h(S_N^{(1)}) \) is defined as the ratio

\[
G_t := \frac{S_t^{(1)}}{\pi_t(C)} = \frac{S_t^{(1)}}{v(t, S_t^{(1)})}, \quad t = 0, 1, \ldots, N.
\]

**Break-even price.** The *break-even* price \( \text{BEP}_t \) of the underlying asset at time \( t = 0, 1, \ldots, N \) is the value of \( S \) for which the intrinsic option value \( h(S_t^{(1)}) \) equals the option price \( \pi_t(C) \). In other words, \( \text{BEP}_t \) represents the price of the underlying asset for which we would break even if the option was exercised immediately. For European call options it is given by

\[
\text{BEP}_t := K + \pi_t(C) = K + v(t, S_t^{(1)}), \quad t = 0, 1, \ldots, N.
\]

whereas for European put options it is given by
BEP_t := K - \pi_t(C) = K - v(t, S_t^{(1)}), \quad t = 0, 1, \ldots, N.

**Premium.** The option *premium* OP_t can be defined as the variation required from the underlying asset price in order to reach the break-even price for which the intrinsic option payoff equals the current option price, i.e. we have

\[
\text{OP}_t := \frac{\text{BEP}_t - S_t^{(1)}}{S_t^{(1)}} = \frac{K + v(t, S_t^{(1)}) - S_t^{(1)}}{S_t^{(1)}}, \quad t = 0, 1, \ldots, N,
\]

for European call options, and

\[
\text{OP}_t := \frac{S_t^{(1)} - \text{BEP}_t}{S_t^{(1)}} = \frac{S_t^{(1)} + v(t, S_t^{(1)}) - K}{S_t^{(1)}}, \quad t = 0, 1, \ldots, N,
\]

for European put options. The term “premium” is sometimes also used to denote the arbitrage price \(v(t, S_t^{(1)})\) of the option.

**Pricing by Backward Induction**

In the CRR model, the discounted portfolio price \(\tilde{V}_t\) can be computed by *backward induction* as in (3.9), using the martingale property of the discounted option price process \((\tilde{V}_t)_{t=0,1,\ldots,N}\) under the risk-neutral probability measure \(\mathbb{P}^\ast\). Namely, by the “tower property” of conditional expectations, letting

\[
\tilde{v}(t, S_t^{(1)}) := \frac{1}{(1+r)^t} v(t, S_t^{(1)}), \quad t = 0, 1, \ldots, N,
\]

we have

\[
\tilde{V}_t = \tilde{v}(t, S_t^{(1)})
\]

\[
= \mathbb{E}^\ast \left[ \tilde{f}(S_N^{(1)}) \mid \mathcal{F}_t \right]
\]

\[
= \mathbb{E}^\ast \left[ \mathbb{E}^\ast \left[ \tilde{f}(S_N^{(1)}) \mid \mathcal{F}_{t+1} \right] \mid \mathcal{F}_t \right]
\]

\[
= \mathbb{E}^\ast \left[ \tilde{V}_{t+1} \mid \mathcal{F}_t \right]
\]

\[
= \mathbb{E}^\ast \left[ \tilde{v}(t+1, S_{t+1}^{(1)}) \mid S_t \right]
\]

\[
= \tilde{v}(t + 1, (1 + a)S_{t+1}^{(1)}) \mathbb{P}^\ast (R_{t+1} = a) + \tilde{v}(t + 1, (1 + b)S_{t+1}^{(1)}) \mathbb{P}^\ast (R_{t+1} = b)
\]

\[
= (1 - p^\ast)\tilde{v}(t + 1, (1 + a)S_{t+1}^{(1)}) + p^\ast \tilde{v}(t + 1, (1 + b)S_{t+1}^{(1)}),
\]

which shows that \(\tilde{v}(t, x)\) satisfies the backward recursion*.

* Download the corresponding (recursive) **IPython notebook** that can be run [here](http://www.ntu.edu.sg/home/nprivault/index.html).
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$$\tilde{v}(t, x) = (1 - p^*)\tilde{v}(t + 1, x(1 + a)) + p^*\tilde{v}(t + 1, x(1 + b)), \quad (3.13)$$

while the terminal condition $\tilde{V}_N = \tilde{f}(S_N^{(1)})$ implies

$$\tilde{v}(N, x) = \tilde{f}(x), \quad x > 0.$$ 

For non-discounted option prices $v(t, S_t)$, the function $v(t, x)$ satisfies the relation

$$v(t, x) = \frac{1 - p^*}{1 + r}v(t + 1, x(1 + a)) + \frac{p^*}{1 + r}v(t + 1, x(1 + b)), \quad (3.14)$$

with the terminal condition

$$v(N, x) = f(x), \quad x > 0.$$ 

The next Figure 3.1 presents a tree-based implementation of the pricing recursion (3.14).

![Discrete-time call option pricing tree](image)

Fig. 3.1: Discrete-time call option pricing tree.

Note that the discrete-time recursion (3.14) can be connected to the continuous-time Black-Scholes PDE (6.2), cf. Exercises 6.14.

### 3.3 Hedging Contingent Claims

The basic idea of hedging is to allocate assets in a portfolio in order to protect oneself from a given risk. For example, a risk of increasing oil prices can be hedged by buying oil-related stocks, whose value should be positively correlated with the oil price. In this way, a loss connected to increasing oil prices could be compensated by an increase in the value of the corresponding portfolio.
In the setting of this chapter, hedging an attainable contingent claim payoff $C$ means computing a self-financing portfolio strategy $(\xi_t)_{t=1,2,\ldots,N}$ such that

$$\xi_N \cdot S_N = C, \quad \text{i.e.} \quad \xi_N \cdot X_N = \tilde{C}. \quad (3.15)$$

**Price, then hedge.**

The portfolio allocation $\xi_N$ can be computed by first solving (3.15) for $\xi_N$ from the payoff values $C$, based on the fact that the allocation $\xi_N$ depends only on information up to time $N - 1$, by the predictability of $(\xi_k)_{1 \leq k \leq N}$.

If the self-financing portfolio value $V_t$ is known, for example from (3.6), i.e.

$$V_t = \frac{1}{(1+r)^{N-t}} \mathbb{E}^*[C \mid \mathcal{F}_t], \quad t = 0, 1, \ldots, N, \quad (3.16)$$

we may similarly compute $\xi_t$ by solving $\xi_t \cdot S_t = V_t$ for all $t = 1, 2, \ldots, N - 1$.

**Hedge, then price.**

If $V_t = \pi_t(C)$ has not been computed, we can use backward induction to compute a self-financing portfolio strategy. Starting from the values of $\xi_N$ obtained by solving

$$\xi_N \cdot S_N = C,$$

we use the self-financing condition to solve for $\xi_{N-1}$, $\xi_{N-2}$, $\ldots$, $\xi_4$, down to $\xi_3$, $\xi_2$, and finally $\xi_1$.

In order to implement this algorithm we can use the $N - 1$ self-financing equations

$$\xi_t \cdot X_t = \xi_{t+1} \cdot X_t, \quad t = 1, 2, \ldots, N - 1, \quad (3.17)$$

allowing us in principle to compute the portfolio strategy $(\xi_t)_{t=1,2,\ldots,N}$.

Based on the values of $\xi_N$ we can solve

$$\xi_{N-1} \cdot S_{N-1} = \xi_N \cdot S_{N-1}$$

for $\xi_{N-1}$, then

$$\xi_{N-2} \cdot S_{N-2} = \xi_{N-1} \cdot S_{N-2}$$

for $\xi_{N-2}$, and successively $\xi_2$ down to $\xi_1$. In Section 3.4 the backward induction (3.17) will be implemented in the CRR model, see the proof of Proposition 3.9, and Exercises 3.13 and 3.4 for an application in a two-step model.
The discounted value $\tilde{V}_t$ at time $t$ of the portfolio claim can then be obtained from

$$\tilde{V}_0 = \xi_1 \cdot X_0 \quad \text{and} \quad \tilde{V}_t = \xi_t \cdot X_t, \quad t = 1, 2, \ldots, N. \quad (3.18)$$

In addition we have shown in the proof of Theorem 3.5 that the price $\pi_t(C)$ of the claim payoff $C$ at time $t$ coincides with the value $V_t$ of any self-financing portfolio hedging the claim payoff $C$, i.e.

$$\pi_t(C) = V_t, \quad t = 0, 1, \ldots, N,$$

as given by (3.18). Hence the price of the claim can be computed either algebraically by solving (3.15) and (3.17) using backward induction and then using (3.18), or by a probabilistic method by a direct evaluation of the discounted expected value (3.16).

The development of hedging algorithms has increased credit exposure and counterparty risk when one party is unable to deliver the option payoff stated in the contract.

### 3.4 Hedging Vanilla Options in the CRR model

In this section we implement the backward induction (3.17) of Section 3.3 for the hedging of contingent claims in the discrete-time Cox-Ross-Rubinstein model. Our aim is to compute a self-financing portfolio strategy hedging a vanilla option with payoff of the form

$$C = h(S_N^{(1)}).$$

Since the discounted price $\tilde{S}_t^{(0)}$ of the riskless asset satisfies

$$\tilde{S}_t^{(0)} = (1 + r)^{-t} S_t^{(0)} = S_0^{(0)},$$

we may sometimes write $S_0^{(0)}$ in place of $\tilde{S}_t^{(0)}$. In Propositions 3.9* and 3.11 we present two different approaches to hedging and to the computation of the predictable process $(\xi_t^{(1)})_{t=1,2,\ldots,N}$, which is also called the Delta.

**Proposition 3.9.** Price, then hedge. The self-financing replicating portfolio strategy $(\xi_t^{(0)}, \xi_t^{(1)})_{t=1,2,\ldots,N} = (\xi_t^{(0)}(S_t^{(1)}), \xi_t^{(1)}(S_{t-1}^{(1)}))_{t=1,2,\ldots,N}$ hedging the contingent claim payoff $C = h(S_N^{(1)})$ is given by

* Download the corresponding pricing and hedging IPython notebook that can be run [here](http://www.ntu.edu.sg/home/nprivault/indext.html).

This version: June 10, 2019

http://www.ntu.edu.sg/home/nprivault/indext.html
\[ \xi^{(1)}_t(S^{(1)}_{t-1}) = \frac{v(t, (1 + b)S^{(1)}_{t-1}) - v(t, (1 + a)S^{(1)}_{t-1})}{(b - a)S^{(1)}_{t-1}} \]
\[ = \frac{\tilde{v}(t, (1 + b)S^{(1)}_{t-1}) - \tilde{v}(t, (1 + a)S^{(1)}_{t-1})}{(b - a)\tilde{S}^{(1)}_{t-1}/(1 + r)}, \quad (3.19) \]

where the function \( v(t, x) \) is given by (3.11), and

\[ \xi^{(0)}_t(S^{(1)}_{t-1}) = \frac{(1 + b)v(t, (1 + a)S^{(1)}_{t-1}) - (1 + a)v(t, (1 + b)S^{(1)}_{t-1})}{(b - a)S^{(0)}_t} \]
\[ = \frac{(1 + b)\tilde{v}(t, (1 + a)S^{(1)}_{t-1}) - (1 + a)\tilde{v}(t, (1 + b)S^{(1)}_{t-1})}{(b - a)S^{(0)}_0}, \quad (3.20) \]

\( t = 1, 2, \ldots, N \), where the function \( \tilde{v}(t, x) = (1 + r)^{-t}v(t, x) \) is given by (3.11).

**Proof.** We first compute the self-financing hedging strategy \( (\xi_t)_{t=1,2,\ldots,N} \) by solving

\[ \xi_t \cdot \bar{X}_t = \tilde{V}_t, \quad t = 1, 2, \ldots, N, \]

from which we deduce the two equations

\[ \begin{cases} 
\xi^{(0)}_t(S^{(1)}_{t-1})S^{(0)}_0 + \xi^{(1)}_t(S^{(1)}_{t-1})\frac{1 + a}{1 + r}S^{(1)}_{t-1} = \tilde{v}(t, (1 + a)S^{(1)}_{t-1}) \\
\xi^{(0)}_t(S^{(1)}_{t-1})S^{(0)}_0 + \xi^{(1)}_t(S^{(1)}_{t-1})\frac{1 + b}{1 + r}S^{(1)}_{t-1} = \tilde{v}(t, (1 + b)S^{(1)}_{t-1}),
\end{cases} \]

which can be solved as

\[ \begin{cases} 
\xi^{(0)}_t(S^{(1)}_{t-1}) = \frac{(1 + b)\tilde{v}(t, (1 + a)S^{(1)}_{t-1}) - (1 + a)\tilde{v}(t, (1 + b)S^{(1)}_{t-1})}{(b - a)S^{(0)}_0} \\
\xi^{(1)}_t(S^{(1)}_{t-1}) = \frac{\tilde{v}(t, (1 + b)S^{(1)}_{t-1}) - \tilde{v}(t, (1 + a)S^{(1)}_{t-1})}{(b - a)\tilde{S}^{(1)}_{t-1}/(1 + r)},
\end{cases} \]

\( t = 1, 2, \ldots, N \), which only depends on \( S^{(1)}_{t-1} \), as expected. This is consistent with the fact that \( \xi^{(1)}_t \) represents the (possibly fractional) quantity of the risky asset to be present in the portfolio over the time period \([t - 1, t]\) in order to hedge the claim payoff \( C \) at time \( N \), and is decided at time \( t - 1 \).

By applying (3.19) to the function \( v(t, x) \) in (3.11) we find
Pricing and Hedging in Discrete Time

\[
\xi_{t_1}^{(1)}(S_{t_1}^{(1)}) = \frac{1}{(1 + r)^{N-t}} \sum_{k=0}^{N-t} \binom{N-t}{k} (p^*)^k (1 - p^*)^{N-t-k} \\
\times \frac{f(S_{t_1}^{(1)}(1+b)^{k+1}(1+a)^{N-t-k}) - f(S_{t_1}^{(1)}(1+b)^{k}(1+a)^{N-t-k+1})}{(b-a)S_{t_1}^{(1)}}
\]

\[t = 0, 1, \ldots, N.\]

The next Figure 3.2 presents a tree-based implementation of the risky hedging component (3.19).

**Fig. 3.2:** Discrete-time call option hedging strategy (risky component).

The next Figure 3.3 presents a tree-based implementation of the riskless hedging component (3.20).

**Fig. 3.3:** Discrete-time call option hedging strategy (riskless component).
Market terms and data

Effective gearing. The effective gearing at time \( t = 1, 2, \ldots, N \) of the option with payoff \( C = h(S_{N}^{(1)}) \) is defined as the ratio

\[
G_{t}^{e} := G_{t}^{\xi^{(1)}} = \frac{S_{t}^{(1)}}{\pi_{t}(C) S_{t}^{(1)}}
\]

\[
= \frac{S_{t}^{(1)}(v(t, (1 + b)S_{t-1}^{(1)}) - v(t, (1 + a)S_{t-1}^{(1)}))}{S_{t-1}^{(1)}(b - a)/S_{t}^{(1)}}
\]

\[
= \frac{(v(t, (1 + b)S_{t-1}^{(1)}) - v(t, (1 + a)S_{t-1}^{(1)}))}{v(t, S_{t}^{(1)})}, \quad t = 1, 2, \ldots, N.
\]

The effective gearing \( G_{t}^{e} = \xi_{t}S_{t}^{(1)}/\pi_{t}(C) \) can be interpreted as the hedge ratio, i.e. the percentage of the portfolio which is invested on the risky asset. It also represents the ratio between the percentage change \( (v(t, (1 + b)S_{t-1}^{(1)}) - v(t, (1 + a)S_{t-1}^{(1)}))/v(t, S_{t}^{(1)}) \) in the option price and the potential percentage change \( S_{t-1}^{(1)}(b - a)/S_{t}^{(1)} \) in the underlying asset price when the market return switches from \( a \) to \( b \).

**Remark 3.10.**

i) If the function \( x \mapsto h(x) \) is non-decreasing, e.g. in the case of European call options, then the function \( x \mapsto \bar{v}(t, x) \) is also non-decreasing for all fixed \( t = 0, 1, \ldots, N \), hence the portfolio strategy \( (\xi_{t}^{(0)}, \xi_{t}^{(1)})_{t=1,2,\ldots,N} \) defined by (3.11) or (3.19) satisfies \( \xi_{t}^{(1)} \geq 0, \quad t = 1, 2, \ldots, N \) and there is not short selling.

ii) Similarly, we can show that when \( x \mapsto h(x) \) is a non-increasing function, e.g in the case of European put options, the portfolio allocation \( \xi_{t}^{(1)} \leq 0 \) is negative, \( t = 1, 2, \ldots, N \), i.e. short selling always occurs.

iii) We can check that the portfolio strategy

\[
(\bar{\xi}_{t})_{t=1,2,\ldots,N} = (\xi_{t}^{(0)}, \xi_{t}^{(1)})_{t=1,2,\ldots,N} = (\xi_{t}^{(0)}(S_{t-1}^{(1)}), \xi_{t}^{(1)}(S_{t-1}^{(1)}))_{t=1,2,\ldots,N}
\]

is self-financing, as we have

\[
\bar{\xi}_{t+1} \cdot \bar{x}_{t} = \xi_{t+1}^{(0)}(S_{t}^{(1)})S_{0}^{(0)} + \xi_{t+1}^{(1)}(S_{t}^{(1)})\tilde{S}_{t}^{(1)}
\]

\[
= S_{0}^{(0)} \frac{(1 + b)\bar{v}(t + 1, (1 + a)S_{t}^{(1)}) - (1 + a)\bar{v}(t + 1, (1 + b)S_{t}^{(1)})}{(b - a)S_{0}^{(0)}}
\]
we also obtain

\[ + \tilde{S}_t^{(1)} \frac{\tilde{v}(t+1, (1+b)S_t^{(1)}) - \tilde{v}(t+1, (1+a)S_t^{(1)})}{(b-a)\tilde{S}_t^{(1)}/(1+r)} \]

\[ = \frac{(1+b)\tilde{v}(t+1, (1+a)S_t^{(1)}) - (1+a)\tilde{v}(t+1, (1+b)S_t^{(1)})}{(b-a)} \]

\[ + \frac{\tilde{v}(t+1, (1+b)S_t^{(1)}) - \tilde{v}(t+1, (1+a)S_t^{(1)})}{(b-a)/(1+r)} \]

\[ = \frac{r-a}{b-a} \tilde{v}(t+1, (1+b)S_t^{(1)}) + \frac{b-r}{b-a} \tilde{v}(t+1, (1+a)S_t^{(1)}) \]

\[ = p^*\tilde{v}(t+1, (1+b)S_t^{(1)}) + q^*\tilde{v}(t+1, (1+a)S_t^{(1)}) \]

\[ = \tilde{v}(t, S_t^{(1)}) \]

\[ = \xi_t^{(0)}(S_t^{(1)}) S_0^{(0)} + \xi_t^{(1)}(S_t^{(1)}) \tilde{S}_t^{(1)} \]

\[ = \xi_t \cdot \tilde{X}_t, \quad t = 0, 1, \ldots, N - 1, \]

where we used (3.13) or the martingale property of the discounted price process \((\tilde{v}(t, S_t^{(1)}))_{t=0,1,\ldots,N}\), cf. Lemma 3.2.

As a consequence of (3.20), the discounted amounts \(\xi_t^{(0)} S_0^{(0)}\) and \(\xi_t^{(1)} \tilde{S}_t^{(1)}\) respectively invested on the riskless and risky assets are given by

\[ S_0^{(0)} \xi_t^{(0)}(S_{t-1}^{(1)}) = \frac{(1+b)\tilde{v}(t, (1+a)S_{t-1}^{(1)}) - (1+a)\tilde{v}(t, (1+b)S_{t-1}^{(1)})}{b-a} \]

and

\[ \tilde{S}_t^{(1)} \xi_t^{(1)}(S_{t-1}^{(1)}) = (1+R_t) \frac{\tilde{v}(t, (1+b)S_{t-1}^{(1)}) - \tilde{v}(t, (1+a)S_{t-1}^{(1)})}{b-a}, \]

\( t = 1, 2, \ldots, N. \)

Regarding the quantity \(\xi_t^{(0)}\) of the riskless asset in the portfolio at time \(t\), from the relation

\[ \tilde{V}_t = \xi_t \cdot \tilde{X}_t = \xi_t^{(0)} \tilde{S}_t^{(0)} + \xi_t^{(1)} \tilde{S}_t^{(1)}, \quad t = 1, 2, \ldots, N, \]

we also obtain

\[ \xi_t^{(0)} = \frac{\tilde{V}_t - \xi_t^{(1)} \tilde{S}_t^{(1)}}{\tilde{S}_t^{(0)}} \]

\[ = \frac{\tilde{V}_t - \xi_t^{(1)} \tilde{S}_t^{(1)}}{S_t^{(0)}} \]
and strategy

\[ \xi_N^{(1)}(S_{N-1}) = \frac{h((1 + b)S_{N-1}^{(1)}) - h((1 + a)S_{N-1}^{(1)})}{(b - a)S_{N-1}^{(1)}}, \]

t = 1, 2, \ldots, N. In the next proposition we compute the hedging strategy by backward induction, starting from the relation

\[ \xi_N^{(1)}(S_{N-1}) = \frac{h((1 + b)S_{N-1}^{(1)}) - h((1 + a)S_{N-1}^{(1)})}{(b - a)S_{N-1}^{(1)}}, \]

and

\[ \xi_N^{(0)}(S_{N-1}) = \frac{(1 + b)h((1 + a)S_{N-1}^{(1)}) - (1 + a)h((1 + b)S_{N-1}^{(1)})}{(b - a)S_{N-1}^{(1)}}, \]

that follow from (3.19) and (3.20) applied to the payoff function \( h(\cdot) \).

**Proposition 3.11.** Hedge, then price. The self-financing replicating portfolio strategy \((\xi_t^{(0)}, \xi_t^{(1)})_{t=1,2,\ldots,N} = (\xi_t^{(0)}(S_t^{(1)}), \xi_t^{(1)}(S_t^{(1)}))_{t=1,2,\ldots,N}\) hedging the contingent claim payoff \( C = h(S_N^{(1)}) \) is given from (3.19) at time \( t = N \) by

\[ \xi_N^{(1)}(S_{N-1}) = \frac{h((1 + b)S_{N-1}^{(1)}) - h((1 + a)S_{N-1}^{(1)})}{(b - a)S_{N-1}^{(1)}}, \]  

(3.22)

where the function \( v(t, x) \) is given by (3.11), and

\[ \xi_N^{(0)}(S_{N-1}) = \frac{(1 + b)h((1 + a)S_{N-1}^{(1)}) - (1 + a)h((1 + b)S_{N-1}^{(1)})}{(b - a)S_N^{(0)}}, \]  

(3.23)

and then inductively by

\[ \xi_t^{(1)}(S_{t-1}) = \frac{(1 + b)\xi_{t+1}^{(1)}((1 + b)S_{t-1}^{(1)}) - (1 + a)\xi_{t+1}^{(1)}((1 + a)S_{t-1}^{(1)})}{b - a} \]

\[ + S_0^{(0)} \xi_{t+1}^{(0)}((1 + b)S_{t-1}^{(1)}) - \xi_{t+1}^{(0)}((1 + a)S_{t-1}^{(1)}) \]

\[ (b - a)\tilde{S}_{t-1}^{(1)}/(1 + r), \]

and

\[ \xi_t^{(0)}(S_{t-1}) = \frac{(1 + a)(1 + b)\tilde{S}_{t-1}^{(1)}(\xi_{t+1}^{(1)}((1 + a)S_{t-1}^{(1)}) - \xi_{t+1}^{(1)}((1 + b)S_{t-1}^{(1)}))}{(b - a)(1 + r)S_0^{(0)}} \]

\[ + \frac{(1 + b)\xi_{t+1}^{(0)}((1 + a)S_{t-1}^{(1)}) - (1 + a)\xi_{t+1}^{(0)}((1 + b)S_{t-1}^{(1)})}{b - a}, \]

(3.24)
Proof. The pricing function \( \tilde{v}(t, x) = (1 + r)^{-t}v(t, x) \) is then given by

\[
\tilde{v}(t, S^{(1)}_t) = S^{(0)}_0 \xi^{(0)}_t (S^{(1)}_{t-1}) + \tilde{S}^{(1)}_t \xi_t (S^{(1)}_{t-1}), \quad t = 1, 2, \ldots, N.
\]

Relations (3.22)-(3.23) follow from (3.19)-(3.20) at time \( t = N \). Next, by the self-financing condition (3.17) we have

\[
\tilde{\xi}_t \cdot X_t = \tilde{\xi}_{t+1} \cdot X_t
\]

i.e.

\[
\begin{align*}
S^{(0)}_0 \xi^{(0)}_t (S^{(1)}_{t-1}) + \tilde{S}^{(1)}_t \xi_t (S^{(1)}_{t-1}) & = \xi^{(0)}_{t+1} ((1 + b)S^{(1)}_{t-1}) S^{(0)}_0 + \xi^{(1)}_{t+1} ((1 + b)S^{(1)}_{t-1}) \tilde{S}^{(1)}_{t-1} \frac{1 + b}{1 + r}, \\
S^{(0)}_0 \xi^{(1)}_t (S^{(1)}_{t-1}) + \tilde{S}^{(1)}_t \xi_t (S^{(1)}_{t-1}) & = \xi^{(0)}_{t+1} ((1 + a)S^{(1)}_{t-1}) S^{(0)}_0 + \xi^{(1)}_{t+1} ((1 + a)S^{(1)}_{t-1}) \tilde{S}^{(1)}_{t-1} \frac{1 + a}{1 + r},
\end{align*}
\]

which can be solved as

\[
\xi^{(1)}_t (S^{(1)}_{t-1}) = \frac{(1 + b) \xi^{(1)}_{t+1} ((1 + b)S^{(1)}_{t-1}) - (1 + a) \xi^{(1)}_{t+1} ((1 + a)S^{(1)}_{t-1})}{b - a} + (1 + r) S^{(0)}_0 \frac{\xi^{(0)}_{t+1} ((1 + b)S^{(1)}_{t-1}) - \xi^{(0)}_{t+1} ((1 + a)S^{(1)}_{t-1})}{(b - a) \tilde{S}^{(1)}_{t-1}},
\]

and

\[
\xi^{(0)}_t (S^{(1)}_{t-1}) = \frac{(1 + a)(1 + b) \tilde{S}^{(1)}_{t-1} (\xi^{(1)}_{t+1} ((1 + a)S^{(1)}_{t-1}) - \xi^{(1)}_{t+1} ((1 + b)S^{(1)}_{t-1}))}{(b - a)(1 + r) S^{(0)}_0} + \frac{(1 + b) \xi^{(0)}_{t+1} ((1 + a)S^{(1)}_{t-1}) - (1 + a) \xi^{(0)}_{t+1} ((1 + b)S^{(1)}_{t-1})}{b - a},
\]

\( t = 1, 2, \ldots, N - 1 \). □

Remark 3.12. We can check that the corresponding discounted price process

\[
(\tilde{V}_t)_{t=1,2,\ldots,N} = (\tilde{\xi}_t \cdot X_t)_{t=1,2,\ldots,N}
\]

is a martingale under \( \mathbb{P}^* \):

\[
\tilde{V}_t = \tilde{\xi}_t \cdot X_t
\]
\[
\begin{align*}
&= S_0^{(0)} \xi_t^{(0)} (S_t^{(1)} - S_{t-1}^{(1)}) + \tilde{S}_t^{(1)} \xi_{t+1}^{(1)} (S_{t-1}^{(1)}) \\
&= \frac{(1 + a)(1 + b)S_t^{(1)}(\xi_{t+1}^{(1)}((1 + a)S_t^{(1)}) - \xi_{t+1}^{(1)}((1 + b)S_{t-1}^{(1)}))}{(b - a)(1 + r)} \\
&\quad + S_0^{(0)} (1 + b)\xi_t^{(0)}((1 + a)S_t^{(1)}) - (1 + a)\xi_{t+1}^{(1)}((1 + b)S_{t-1}^{(1)}) \\
&\quad + \tilde{S}_t^{(1)} (1 + b)\xi_{t+1}^{(1)}((1 + b)S_t^{(1)}) - (1 + a)\xi_{t+1}^{(1)}((1 + a)S_{t-1}^{(1)}) \\
&\quad + (1 + r)\tilde{S}_t^{(1)} S_0^{(0)} (\xi_{t+1}^{(1)}((1 + b)S_{t-1}^{(1)}) - \xi_{t+1}^{(1)}((1 + a)S_{t-1}^{(1)})) \\
&= \frac{r - a}{b - a} S_0^{(0)} \xi_t^{(0)} (S_t^{(1)}) + \frac{b - r}{b - a} S_0^{(0)} \xi_{t+1}^{(1)} (S_t^{(1)}) \\
&\quad + \frac{(r - a)(1 + b)}{(b - a)(1 + r)} \tilde{S}_t^{(1)} \xi_{t+1}^{(1)} (S_t^{(1)}) + \frac{(b - r)(1 + a)}{(b - a)(1 + r)} \tilde{S}_t^{(1)} \xi_{t+1}^{(1)} (S_t^{(1)}) \\
&= p^* S_0^{(0)} \xi_{t+1}^{(1)} (S_t^{(1)}) + q^* S_0^{(0)} \xi_{t+1}^{(1)} (S_t^{(1)}) \\
&\quad + p \frac{1 + b}{1 + r} \tilde{S}_t^{(1)} \xi_{t+1}^{(1)} (S_t^{(1)}) + q \frac{1 + a}{1 + r} \tilde{S}_t^{(1)} \xi_{t+1}^{(1)} (S_t^{(1)}) \\
&= \mathbb{E}^*[S_0^{(0)} \xi_{t+1}^{(1)} (S_t^{(1)}) + \tilde{S}_t^{(1)} \xi_{t+1}^{(1)} (S_t^{(1)}) \mid \mathcal{F}_t] \\
&= \mathbb{E}^* [\tilde{V}_{t+1} \mid \mathcal{F}_t],
\end{align*}
\]

\(t = 1, 2, \ldots, N - 1.\)

The next Figure 3.4 presents a tree-based implementation of the riskless hedging component (3.20).*

\[\text{Fig. 3.4: Tree of asset prices in the CRR model.}\]

* Download the corresponding pricing and hedging IPython notebook that can be run here.
The next Figure 3.5 presents a tree-based implementation of option prices in the CRR model.

The next Figure 3.6 presents a tree-based implementation of risky hedging portfolio allocation in the CRR model.

3.5 Hedging Exotic Options in the CRR Model

In this section we take $p = p^*$ given by (3.12) and we consider the hedging of path-dependent options. Here we choose to use the finite difference gradient and the discrete Clark-Ocone formula of stochastic analysis, see also Föllmer and Schied (2004), Lamberton and Lapeyre (1996), Privault (2008), Chapter 1 of Privault (2009), Ruiz de Chávez (2001), or § 15-1 of Williams (1991). See Nunno et al. (2009) and Section 8.2 of Privault (2009) for a similar approach in continuous time. Given
\[ \omega = (\omega_1, \omega_2, \ldots, \omega_N) \in \Omega = \{-1, 1\}^N, \]
and \( r = 1, 2, \ldots, N, \) let
\[ \omega^+_t := (\omega_1, \omega_2, \ldots, \omega_{t-1}, +1, \omega_{t+1}, \ldots, \omega_N) \]
and
\[ \omega^-_t := (\omega_1, \omega_2, \ldots, \omega_{t-1}, -1, \omega_{t+1}, \ldots, \omega_N). \]
We also assume that the return \( R_t(\omega) \) is constructed as
\[
R_t(\omega^{\pm}_t) = \begin{cases} 
  b & \omega = +1, \\
  a & \omega = -1,
\end{cases} \quad t = 1, 2, \ldots, N, \quad \omega \in \Omega.
\]

**Definition 3.13.** The operator \( D_t \) is defined on any random variable \( F \) by
\[
D_t F(\omega) = F(\omega^{t}_+) - F(\omega^{t}_-), \quad t = 1, 2, \ldots, N. \tag{3.25}
\]
We define the centered and normalized return \( Y_t \) by
\[
Y_t := \frac{R_t - r}{b - a} = \begin{cases} 
  \frac{b - r}{b - a} = q, & \omega_t = +1, \\
  \frac{a - r}{b - a} = -p, & \omega_t = -1,
\end{cases} \quad t = 1, 2, \ldots, N.
\]
Note that under the risk-neutral probability measure \( \mathbb{P}^* \) we have
\[
\mathbb{E}^*[Y_t] = \mathbb{E}^* \left[ \frac{R_t - r}{b - a} \right] \\
= \frac{a - r}{b - a} \mathbb{P}^*(R_t = a) + \frac{b - r}{b - a} \mathbb{P}^*(R_t = b) \\
= \frac{a - r}{b - a} \times \frac{b - r}{b - a} \times \frac{r - a}{b - a} \\
= 0,
\]
and
\[
\text{Var}[Y_t] = pq^2 + qp^2 = pq, \quad t = 1, 2, \ldots, N.
\]
In addition, the discounted asset price increment reads
\[
\tilde{S}^{(1)}_t - \tilde{S}^{(1)}_{t-1} = \tilde{S}^{(1)}_{t-1} \frac{1 + R_t}{1 + r} - \tilde{S}^{(1)}_{t-1} \\
= \frac{R_t - r}{1 + r} \tilde{S}^{(1)}_{t-1} \\
= \frac{b - a}{1 + r} Y_t \tilde{S}^{(1)}_{t-1}, \quad t = 1, 2, \ldots, N.
\]
We also have
\[ D_t Y_t = \frac{b - r}{b - a} + \frac{r - a}{b - a} = 1, \quad t = 1, 2, \ldots, N, \]

and

\[ D_t S_N^{(1)} = S_0^{(1)} (1 + b) \prod_{k=1 \atop k \neq t}^N (1 + R_k) - S_0^{(1)} (1 + a) \prod_{k=1 \atop k \neq t}^N (1 + R_k) \]

\[ = (b - a) S_0^{(1)} \prod_{k=1 \atop k \neq t}^N (1 + R_k) \]

\[ = S_0^{(1)} \frac{b - a}{1 + R_t} \prod_{k=1}^N (1 + R_k) \]

\[ = \frac{b - a}{1 + R_t} S_N^{(1)}, \quad t = 1, 2, \ldots, N. \]

The following stochastic integral decomposition formula for the functionals of the binomial process is known as the Clark-Ocone formula in discrete time, cf. e.g. Privault (2009), Proposition 1.7.1.

**Proposition 3.14.** For any square-integrable random variables \( F \) on \( \Omega \) we have

\[ F = \mathbb{E}^*[F] + \sum_{k=1}^{\infty} Y_k \mathbb{E}^*[D_k F \mid \mathcal{F}_{k-1}]. \]  

(3.26)

The Clark-Ocone formula has the following consequence.

**Corollary 3.15.** Assume that \((M_k)_{k \in \mathbb{N}}\) is a square-integrable \((\mathcal{F}_k)_{k \in \mathbb{N}}\)-martingale. Then we have

\[ M_N = \mathbb{E}^*[M_N] + \sum_{k=1}^N Y_k D_k M_k, \quad N \geq 0. \]

**Proof.** We have

\[ M_N = \mathbb{E}^*[M_N] + \sum_{k=1}^{\infty} Y_k \mathbb{E}^*[D_k M_N \mid \mathcal{F}_{k-1}] \]

\[ = \mathbb{E}^*[M_N] + \sum_{k=1}^{\infty} Y_k D_k \mathbb{E}^*[M_N \mid \mathcal{F}_k] \]

\[ = \mathbb{E}^*[M_N] + \sum_{k=1}^{\infty} Y_k D_k M_k \]

\[ = \mathbb{E}^*[M_N] + \sum_{k=1}^N Y_k D_k M_k. \]
In addition to the Clark-Ocone formula we also state a discrete-time analog of Itô's change of variable formula, which can be useful for option hedging. The next result extends Proposition 1.13.1 of Privault (2009) by removing the unnecessary martingale requirement on $(M_t)_{n \in \mathbb{N}}$.

**Proposition 3.16.** Let $(Z_n)_{n \in \mathbb{N}}$ be an $(\mathcal{F}_n)_{n \in \mathbb{N}}$-adapted process and let $f : \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{R}$ be a given function. We have

$$f(Z_t, t) = f(Z_0, 0) + \sum_{k=1}^{t} D_k f(Z_k, k) Y_k + \sum_{k=1}^{t} \left( \mathbb{E}^*[f(Z_k, k) \mid \mathcal{F}_{k-1}] - f(Z_{k-1}, k - 1) \right).$$ (3.27)

**Proof.** First, we note that the process

$$t \mapsto f(Z_t, t) - \sum_{k=1}^{t} \left( \mathbb{E}^*[f(Z_k, k) \mid \mathcal{F}_{k-1}] - f(Z_{k-1}, k - 1) \right)$$

is a martingale under $\mathbb{P}^*$. Indeed, we have

$$\mathbb{E}^* \left[ f(Z_t, t) - \sum_{k=1}^{t} \left( \mathbb{E}^*[f(Z_k, k) \mid \mathcal{F}_{k-1}] - f(Z_{k-1}, k - 1) \right) \bigg| \mathcal{F}_{t-1} \right]$$

$$= \mathbb{E}^*[f(Z_t, t) \mid \mathcal{F}_{t-1}] - \sum_{k=1}^{t} \left( \mathbb{E}^*[\mathbb{E}^*[f(Z_k, k) \mid \mathcal{F}_{k-1}] \mid \mathcal{F}_{t-1}] - \mathbb{E}^*[\mathbb{E}^*[f(Z_{k-1}, k - 1) \mid \mathcal{F}_{k-1}] \mid \mathcal{F}_{t-1}] \right)$$

$$= \mathbb{E}^*[f(Z_t, t) \mid \mathcal{F}_{t-1}] - \sum_{k=1}^{t} \left( \mathbb{E}^*[f(Z_k, k) \mid \mathcal{F}_{k-1}] - f(Z_{k-1}, k - 1) \right)$$

$$= f(Z_{t-1}, t - 1) - \sum_{k=1}^{t-1} \left( \mathbb{E}^*[f(Z_k, k) \mid \mathcal{F}_{k-1}] - f(Z_{k-1}, k - 1) \right), \quad t \geq 1.$$ 

Note that if $(Z_t)_{t \in \mathbb{N}}$ is a discrete-time $(\mathcal{F}_t)_{t \in \mathbb{N}}$-martingale in $L^2(\Omega)$ written as

$$Z_t = Z_0 + \sum_{k=1}^{t} u_k Y_k, \quad t \in \mathbb{N},$$

where $(u_t)_{t \in \mathbb{N}}$ is an $(\mathcal{F}_t)_{t \in \mathbb{N}}$-predictable process locally in $L^2(\Omega \times \mathbb{N})$, (i.e. $u(\cdot) \mathbb{1}_{[0,N]}(\cdot) \in L^2(\Omega \times \mathbb{N})$ for all $N > 0$), then we have
\[ D_t f(Z_t, t) = f(Z_{t-1} + qu_t, t) - f(Z_{t-1} - pu_t, t), \quad (3.28) \]

\( t = 1, 2, \ldots, N \). On the other hand, the term

\[ \mathbb{E}[ f(Z_t, t) - f(Z_{t-1}, t - 1) \mid \mathcal{F}_{t-1}] \]

is analog to the finite variation part in the continuous-time Itô formula, and can be written as

\[ pf(Z_{t-1} + qu_t, t) + qf(Z_{t-1} - pu_t, t) - f(Z_{t-1}, t - 1). \]

Naturally, if \((f(Z_t, t))_{t \in \mathbb{N}}\) is a martingale we recover the decomposition

\[ f(Z_t, t) = f(Z_0, 0) \]
\[ + \sum_{k=1}^{t} \left( f(Z_{k-1} + qu_k, k) - f(Z_{k-1} - pu_k, k) \right) Y_k \]
\[ = f(Z_0, 0) + \sum_{k=1}^{t} Y_k D_k f(Z_k, k). \quad (3.29) \]

This identity follows from Corollary 3.15 as well as from Proposition 3.14. In this case the Clark-Ocone formula (3.26) and the change of variable formula (3.29) both coincide and we have in particular

\[ D_k f(Z_k, k) = \mathbb{E}[D_k f(Z_N, N) \mid \mathcal{F}_{k-1}], \]

\( k = 1, 2, \ldots, N \). For example this recovers the martingale representation

\[ \widetilde{S}^{(1)}_t = S^{(1)}_0 + \sum_{k=1}^{t} Y_k D_k \widetilde{S}^{(1)}_k \]
\[ = S^{(1)}_0 + \frac{b - a}{1 + r} \sum_{k=1}^{t} \widetilde{S}^{(1)}_{k-1} Y_k \]
\[ = S^{(1)}_0 + \sum_{k=1}^{t} \widetilde{S}^{(1)}_{k-1} \frac{R_k - r}{1 + r} \]
\[ = S^{(1)}_0 + \sum_{k=1}^{t} \left( \widetilde{S}^{(1)}_k - \widetilde{S}^{(1)}_{k-1} \right), \]

of the discounted asset price.

Our goal is to hedge an arbitrary claim payoff \( C \) on \( \Omega \), i.e. given an \( \mathcal{F}_N \)-measurable random variable \( C \) we search for a portfolio strategy \( (\xi^{(0)}_t, \xi^{(1)}_t)_{t=1,2,\ldots,N} \) such that the equality
\[ C = V_N = \xi_N^{(0)} S_N^{(0)} + \xi_N^{(1)} S_N^{(1)} \] (3.30)

holds, where \( S_N^{(0)} = S_0^{(0)}(1 + r)^N \) denotes the value of the riskless asset at time \( N \in \mathbb{N} \).

The next proposition is the main result of this section, and provides a solution to the hedging problem under the constraint (3.30).

**Proposition 3.17.** Given a contingent claim payoff \( C \), let

\[
\xi_t^{(1)} = \frac{(1 + r)^{-(N-t)}}{(b - a)S_{t-1}^{(1)}} \mathbb{E}^*[D_tC \mid \mathcal{F}_{t-1}], \quad t = 1, 2, \ldots, N, \tag{3.31}
\]

and

\[
\xi_t^{(0)} = \frac{1}{S_t^{(0)}} \left( (1 + r)^{-(N-t)} \mathbb{E}^*[C \mid \mathcal{F}_t] - \xi_t^{(1)} S_t^{(1)} \right), \tag{3.32}
\]

\( t = 1, 2, \ldots, N \). Then the portfolio strategy \( (\xi_t^{(0)}, \xi_t^{(1)})_{t=1,2,\ldots,N} \) is self-financing and satisfies

\[ V_t = \xi_t^{(0)} S_t^{(0)} + \xi_t^{(1)} S_t^{(1)} = (1 + r)^{-(N-t)} \mathbb{E}^*[C \mid \mathcal{F}_t], \quad t = 1, 2, \ldots, N, \]

in particular we have \( V_N = C \), hence \( (\xi_t^{(0)}, \xi_t^{(1)})_{t=1,2,\ldots,N} \) is a hedging strategy leading to \( C \).

**Proof.** Let \( (\xi_t^{(1)})_{t=1,2,\ldots,N} \) be defined by (3.31), and consider the process \( (\xi_t^{(0)})_{t=0,1,\ldots,N} \) defined by

\[
\xi_0^{(0)} = (1 + r)^{-N} \frac{\mathbb{E}^*[C]}{S_0^{(1)}} \quad \text{and} \quad \xi_{t+1}^{(0)} = \xi_t^{(0)} - \frac{(\xi_{t+1}^{(1)} - \xi_t^{(1)}) S_t^{(1)}}{S_t^{(0)}}, \quad t = 0, 1, \ldots, N - 1.
\]

Then \( (\xi_t^{(0)}, \xi_t^{(1)})_{t=1,2,\ldots,N} \) satisfies the self-financing condition

\[ S_t^{(0)}(\xi_{t+1}^{(0)} - \xi_t^{(0)}) + S_t^{(1)}(\xi_{t+1}^{(1)} - \xi_t^{(1)}) = 0, \quad t = 1, 2, \ldots, N - 1. \]

Let now

\[ V_0 := \frac{1}{(1 + r)^N} \mathbb{E}^*[C], \quad V_t := \xi_t^{(0)} S_t^{(0)} + \xi_t^{(1)} S_t^{(1)}, \quad t = 1, 2, \ldots, N, \]

and

\[ \tilde{V}_t = \frac{V_t}{(1 + r)^t} \quad t = 0, 1, \ldots, N. \]
Since \((\xi_t^{(0)}, \xi_t^{(1)})_{t=1,2,...,N}\) is self-financing, by Lemma 3.2 we have

\[
\tilde{V}_t = \tilde{V}_0 + (b - a) \sum_{k=1}^{t} \frac{1}{(1 + r)^k} Y_k \xi_k^{(1)} S_{k-1}^{(1)},
\]

\[(3.33)\]

\(t = 1, 2, \ldots, N\). On the other hand, from the Clark-Ocone formula (3.26) and the definition of \((\xi_t^{(1)})_{t=1,2,...,N}\) we have

\[
\frac{1}{(1 + r)^N} \mathbb{E}^*[C \mid \mathcal{F}_t] = \frac{1}{(1 + r)^N} \mathbb{E}^* \left[ \mathbb{E}^*[C] + \sum_{k=0}^{N} Y_k \mathbb{E}^*[D_k C \mid \mathcal{F}_{k-1}] \mid \mathcal{F}_t \right]
\]

\[
= \frac{1}{(1 + r)^N} \mathbb{E}^*[C] + \frac{1}{(1 + r)^N} \sum_{k=0}^{t} \mathbb{E}^*[D_k C \mid \mathcal{F}_{k-1}] Y_k
\]

\[
= \frac{1}{(1 + r)^N} \mathbb{E}^*[C] + (b - a) \sum_{k=0}^{t} \frac{1}{(1 + r)^k} \xi_k^{(1)} S_{k-1}^{(1)} Y_k
\]

from (3.33). Hence

\[
\tilde{V}_t = \frac{1}{(1 + r)^N} \mathbb{E}^*[C \mid \mathcal{F}_t], \quad t = 0, 1, \ldots, N,
\]

and

\[
V_t = (1 + r)^{-(N-t)} \mathbb{E}^*[C \mid \mathcal{F}_t], \quad t = 0, 1, \ldots, N. \quad (3.34)
\]

In particular, (3.34) shows that we have \(V_N = C\). To conclude the proof we note that from the relation \(V_t = \xi_t^{(0)} S_t^{(0)} + \xi_t^{(1)} S_t^{(1)}\), \(t = 1, 2, \ldots, N\), the process \((\xi_t^{(0)})_{t=1,2,...,N}\) coincides with \((\xi_t^{(0)})_{t=1,2,...,N}\) defined by (3.32). □

From Proposition 3.8, the price \(\pi_t(C)\) of the contingent claim payoff \(C = f(S_N^{(1)})\) is given by

\[
\pi_t(C) = v(t, S_t^{(1)}),
\]

where the function \(v(t, x)\) is given by

\[
v(t, S_t^{(1)}) = \frac{1}{(1 + r)^{N-t}} \mathbb{E}^*[C \mid \mathcal{F}_t]
\]

\[
= \frac{1}{(1 + r)^{N-t}} \mathbb{E}^* \left[ f \left( x \prod_{j=t+1}^{N} (1 + R_j) \right) \right]_{x=S_t^{(1)}}.
\]
Note that in this case we have \( C = v(N, S_N^{(1)}) \), \( \mathbb{E}[C] = v(0, M_0) \), and the discounted claim payoff \( \tilde{C} = C/(1+r)^N = \tilde{v}(N, S_N^{(1)}) \) satisfies

\[
\tilde{C} = \mathbb{E}[\tilde{C}] + \sum_{t=1}^{N} Y_t \mathbb{E}[D_t \tilde{v}(N, S_N^{(1)}) | \mathcal{F}_{t-1}]
\]

\[
= \mathbb{E}[\tilde{C}] + \sum_{t=1}^{N} Y_t D_t \tilde{v}(t, S_t^{(1)})
\]

\[
= \mathbb{E}[\tilde{C}] + \sum_{t=1}^{N} \frac{1}{(1+r)t} Y_t D_t \tilde{v}(t, S_t^{(1)})
\]

\[
= \mathbb{E}[\tilde{C}] + \sum_{t=1}^{N} Y_t D_t \mathbb{E}[\tilde{v}(N, S_N^{(1)}) | \mathcal{F}_t]
\]

\[
= \mathbb{E}[\tilde{C}] + \frac{1}{(1+r)^N} \sum_{t=1}^{N} Y_t D_t \mathbb{E}[C | \mathcal{F}_t],
\]

hence we have

\[
\mathbb{E}[D_t v(N, S_N^{(1)}) | \mathcal{F}_{t-1}] = (1+r)^{N-t} D_t v(t, S_t^{(1)}), \quad t = 1, 2, \ldots, N,
\]

and by Proposition 3.17 the hedging strategy for \( C = f(S_N^{(1)}) \) is given by

\[
\xi_t^{(1)} = \frac{(1+r)^{-(N-t)}}{(b-a)S_t^{(1)}} \mathbb{E}[D_t v(N, S_N^{(1)}) | \mathcal{F}_{t-1}]
\]

\[
= \frac{1}{(b-a)S_t^{(1)}} D_t v(t, S_t^{(1)})
\]

\[
= \frac{1}{(b-a)S_t^{(1)}} \left( v(t, S_{t-1}^{(1)}(1+b)) - v(t, S_{t-1}^{(1)}(1+a)) \right)
\]

\[
= \frac{1}{(b-a)\tilde{S}_{t-1}^{(1)}(1+r)} \left( \tilde{v}(t, S_{t-1}^{(1)}(1+b)) - \tilde{v}(t, S_{t-1}^{(1)}(1+a)) \right),
\]

\( t = 1, 2, \ldots, N \), which recovers Proposition 3.9 as a particular case. Note that \( \xi_t^{(1)} \) is nonnegative (i.e. there is no short selling) when \( f \) is a non-decreasing function, because \( a < b \). This is in particular true in the case of a European call option for which we have \( f(x) = (x - K)^+ \).
3.6 Convergence of the CRR Model

As the pricing formulas (3.11) in the CRR model can be difficult to implement for large values on \( N \), in this section we consider the convergence of the discrete-time model to the continuous-time Black Scholes model.

**Continuous compounding - riskless asset**

Consider the discretization

\[
\left[ 0, \frac{T}{N}, \frac{2T}{N}, \ldots, \frac{(N-1)T}{N}, T \right]
\]

of the time interval \([0, T]\) into \( N \) time steps.

Note that

\[
\lim_{N \to \infty} (1 + r)^N = \infty,
\]

when \( r > 0 \), thus we need to renormalize \( r \) so that the interest rate on each time interval becomes \( r_N \), with \( \lim_{N \to \infty} r_N = 0 \). It turns out that the correct renormalization is

\[
r_N := \frac{r T}{N},
\]

so that for \( T \geq 0 \),

\[
\lim_{N \to \infty} (1 + r_N)^N = \lim_{N \to \infty} \left(1 + r \frac{T}{N}\right)^N
\]

\[
= \lim_{N \to \infty} \exp \left( N \log \left(1 + r \frac{T}{N}\right) \right)
\]

\[
= e^{rT}.
\]

Hence the price \( S_t^{(0)} \) of the riskless asset is given by

\[
S_t^{(0)} = S_0^{(0)} e^{rt}, \quad t \in \mathbb{R}_+,
\]

which solves the differential equation

\[
\frac{dS_t^{(0)}}{dt} = rS_t^{(0)}, \quad S_0^{(0)} = 1, \quad t \in \mathbb{R}_+.
\]
\[ ds^{(0)}_t = rS^{(0)}_t \, dt, \quad \text{or} \quad \frac{dS^{(0)}_t}{S^{(0)}_t} = r \, dt, \quad (3.39) \]

and using \( dS^{(0)}_t \simeq S^{(0)}_{t+dt} - S^{(0)}_t \) we can discretize this equation by saying that the return \( (S^{(0)}_{t+dt} - S^{(0)}_t) / S^{(0)}_t \) of the riskless asset equals \( r \, dt \) on the small time interval \([t, t + dt]\), i.e.

\[ \frac{S^{(0)}_{t+dt} - S^{(0)}_t}{S^{(0)}_t} = r \, dt. \]

In this sense, the rate \( r \) is the instantaneous interest rate per unit of time.

The same equation rewrites in integral form as

\[ S^{(0)}_T - S^{(0)}_0 = \int_0^T dS^{(0)}_t = r \int_0^T S^{(0)}_t \, dt. \]

**Continuous compounding - risky asset**

The Galton board simulation of Figure 3.7 shows the convergence of the binomial random walk to a Gaussian distribution in large time.

---

* Figure 3.7: Galton board simulation.

Figure 3.8 pictures a real-life Galton board.

* The animation works in Acrobat Reader on the entire pdf file.
In the CRR model we need to replace the standard Galton board by its multiplicative version, which shows that as $N$ tends to infinity the distribution of $S_N^{(1)}$ converges to the lognormal distribution with probability density function of the form

$$
x \mapsto f(x) = \frac{1}{x\sigma\sqrt{2\pi T}} \exp \left( -\frac{\left( -\left( r - \frac{\sigma^2}{2} \right) T + \log(x/S_0^{(1)}) \right)^2}{2\sigma^2 T} \right),
$$

$x > 0$, with location parameter $(r - \sigma^2/2)T + \log S_0^{(1)}$ and scale parameter $\sigma\sqrt{T}$, or log-variance $\sigma^2 T$, as illustrated in the modified Galton board of Figure 3.9.
In addition to the renormalization (3.35) for the interest rate \( r_N := rT/N \), we need to apply a similar renormalization to the coefficients \( a \) and \( b \) of the CRR model. Let \( \sigma > 0 \) denote a positive parameter called the volatility and let \( a_N, b_N \) be defined from

\[
\frac{1 + a_N}{1 + r_N} = 1 - \sigma \sqrt{\frac{T}{N}} + o\left(\sqrt{\frac{T}{N}}\right) \quad \text{and} \quad \frac{1 + b_N}{1 + r_N} = 1 + \sigma \sqrt{\frac{T}{N}} + o\left(\sqrt{\frac{T}{N}}\right),
\]

i.e.

\[
a_N \simeq (1 + r_N)(1 - \sigma\sqrt{T/N}) - 1 \quad \text{and} \quad b_N \simeq (1 + r_N)(1 + \sigma\sqrt{T/N}) - 1,
\]

where \( \sigma > 0 \) quantifies the range of random fluctuations. Consider the random return \( R_k^{(N)} \in \{a_N, b_N\} \) and the price process defined as

\[
S_{t, N}^{(1)} = S_{0, N}^{(1)} \prod_{k=1}^{t} (1 + R_k^{(N)}), \quad t = 1, 2, \ldots, N.
\]

Note that the risk-neutral probabilities are given by

\[
\mathbb{P}^*(R_t = a_N) = \frac{b_N - r_N}{b_N - a_N} = \frac{1}{2}, \quad t = 1, 2, \ldots, N,
\]

and

\[
\mathbb{P}^*(R_t = b_N) = \frac{r_N - a_N}{b_N - a_N} = \frac{1}{2}, \quad t = 1, 2, \ldots, N,
\]

which both converge to \( 1/2 \) as \( N \) goes to infinity.

* The animation works in Acrobat Reader on the entire pdf file.
Continuous-time limit in distribution

We have the following convergence result.

**Proposition 3.18.** Let $f$ be a continuous and bounded function on $\mathbb{R}$. The price at time $t = 0$ of a contingent claim with payoff $C = f(S_0^{(1), N})$ converges as follows:

$$
\lim_{N \to \infty} \frac{1}{(1 + r T / N)^N} \mathbb{E}^* \left[ f\left(S_0^{(1), N}\right) \right] = e^{-r T} \mathbb{E} \left[ f\left(S_0^{(1)} e^{\sigma X + r T - \sigma^2 T/2}\right) \right]
$$

(3.40)

where $X \approx \mathcal{N}(0, T)$ is a centered Gaussian random variable with variance $T > 0$.

**Proof.** This result is consequence of the weak convergence in distribution of the sequence $(S_0^{(1), N})_{N \geq 1}$ to a lognormal distribution, see e.g. Theorem 5.53 page 261 of Föllmer and Schied (2004). Informally, we note that

$$
\log S_N^{(1)} = \log S_0^{(1)} + \sum_{k=1}^{N} \log(1 + R_k)
$$

$$
= \log S_0^{(1)} + \sum_{k=1}^{N} \log(1 + r_N) + \sum_{k=1}^{N} \log \left(1 + \frac{R_k}{1 + r_N}\right)
$$

$$
= \log S_0^{(1)} + \sum_{k=1}^{N} \log \left(1 + \frac{r_T}{N} + o\left(\frac{T}{N}\right)\right) + \sum_{k=1}^{N} \log \left(1 \pm \sigma \sqrt{\frac{T}{N}} + o\left(\sqrt{\frac{T}{N}}\right)\right)
$$

$$
= \log S_0^{(1)} + r T - \frac{\sigma^2 T}{2} - \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \pm \sqrt{\sigma^2 T} + o(T),
$$

and we note that by the Central Limit Theorem (CLT), the normalized sum

$$
\frac{1}{\sqrt{N}} \sum_{k=1}^{N} \pm \sqrt{\sigma^2 T}
$$

of Bernoulli random variables converges in distribution to a centered $\mathcal{N}(0, \sigma^2 T)$ normal random variable with variance $\sigma^2 T$. The convergence of the discount factor $(1 + r T / N)^N$ to $e^{-r T}$ follows from (3.36). □

Note that the expectation (3.40) can be written as the Gaussian integral

$$
e^{-r T} \mathbb{E} \left[ f\left(S_0^{(1)} e^{\sigma X + r T - \sigma^2 T/2}\right) \right] = e^{-r T} \int_{-\infty}^{\infty} f\left(S_0^{(1)} e^{\sigma \sqrt{T} x + r T - \sigma^2 T/2}\right) \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx,
$$

see also Lemma 7.8 in Chapter 7, hence we have
\[
\lim_{N \to \infty} \frac{1}{(1 + rT/N)^N} \mathbb{E}^* \left[ f(S_{N,N}^{(1)}) \right] = e^{-rT} \int_{-\infty}^{\infty} f(S_0^{(1)} e^{\sigma \sqrt{T} x + rT - \sigma^2 T/2}) \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx.
\]

It is a remarkable fact that in case \( f(x) = (x - K)^+ \), i.e. when

\[
C = (S_T^{(1)} - K)^+
\]
is the payoff of a European call option with strike price \( K \), the above integral can be computed according to the Black-Scholes formula, as

\[
e^{-rT} \mathbb{E} \left[ (S_0^{(1)} e^{\sigma X + rT - \sigma^2 T/2} - K)^+ \right] = S_0^{(1)} \Phi(d_+) - K e^{-rT} \Phi(d_-),
\]

where

\[
d_- = \frac{(r - \sigma^2/2) T + \log \left( S_0^{(1)} / K \right)}{\sigma \sqrt{T}}, \quad d_+ = d_- + \sigma \sqrt{T},
\]

and

\[
\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy, \quad x \in \mathbb{R},
\]
is the Gaussian cumulative distribution function.

The Black-Scholes formula will be derived explicitly in the subsequent chapters using both PDE and probabilistic methods, cf. Propositions 6.8 and 7.7. It can be considered as a building block for the pricing of financial derivatives, and its importance is not restricted to the pricing of options on stocks. Indeed, the complexity of the interest rate models makes it in general difficult to obtain closed-form expressions, and in many situations one has to rely on the Black-Scholes framework in order to find pricing formulas, for example in the case of interest rate derivatives as in the Black caplet formula of the BGM model, cf. Proposition 18.4 in Section 18.3.

Our aim later on will be to price and hedge options directly in continuous-time using stochastic calculus, instead of applying the limit procedure described in the previous section. In addition to the construction of the riskless asset price \((A_t)_{t \in \mathbb{R}^+}\) via (3.37) and (3.38) we now need to construct a mathematical model for the price of the risky asset in continuous time.

The return of the risky asset \(S_t^{(1)}\) over the time period \([t, d + dt]\) will be defined as

\[
\frac{dS_t^{(1)}}{S_t^{(1)}} = \mu dt + \sigma dB_t,
\]

where in comparison with (3.39), we add a “small” Gaussian random perturbation \(\sigma dB_t\) which accounts for market volatility. Here, the Brownian
increment $dB_t$ is multiplied by the volatility parameter $\sigma > 0$. In the next Chapter 4 we will turn to the formal definition of the stochastic process $(B_t)_{t \in \mathbb{R}_+}$ which will be used for the modeling of risky assets in continuous time.

**Exercises**

Exercise 3.1  
(Exercise 2.3 continued) Consider a two-step trinomial market model $(S_t^{(1)})_{t=0,1,2}$ with $r = 0$ and three return rates $R_t = -1, 0, 1$. Taking $S_0^{(1)} = 1$, price the European put option with strike price $K = 1$ and maturity $N = 2$ at times $t = 0$ and $t = 1$.

Exercise 3.2  
Consider a two-step binomial market model $(S_t)_{t=0,1,2}$ with two return rates $a = 0$, $b = 1$ and $S_0 = 1$, and with the riskless account $A_t = (1 + r)^t$ where $r = 0.5$. Price and hedge the vanilla option whose payoff $C$ at time 2 is given by

$$C = \begin{cases} 3 & \text{if } S_2 = 4, \\ 1 & \text{if } S_2 = 2, \\ 3 & \text{if } S_2 = 1. \end{cases}$$

Exercise 3.3  
In a two-step trinomial market model $(S_t)_{t=0,1,2}$ with interest rate $r = 0$ and three return rates $R_t = -0.5, 0, 1$, we consider a down-an-out barrier call option with exercise date $N = 2$, strike price $K$ and barrier level $B$, whose payoff $C$ is given by

$$C = (S_N - K)^+ \mathbb{1}_{\{\min_{t=1,2,\ldots,N} S_t > B\}} = \begin{cases} (S_N - K)^+ & \text{if } \min_{t=1,2,\ldots,N} S_t > B, \\ 0 & \text{if } \min_{t=1,2,\ldots,N} S_t \leq B. \end{cases}$$

a) Show that $P^*$ given by $r^* = P^*(R_t = -0.5) := 1/2$, $q^* = P^*(R_t = 0) := 1/4$, $p^* = P^*(R_t = 1) := 1/4$ is risk-neutral.

b) Taking $S_0 = 1$, compute the possible values of the down-an-out barrier call option payoff $C$ with strike price $K = 1.5$ and barrier level $B = 1$, at maturity $N = 2$.

c) Price the down-an-out barrier call option with exercise date $N = 2$, strike price $K = 1.5$ and barrier level $B = 1$, at time $t = 0$ and $t = 1$.

*Hint:* Use the formula
\[ \pi_t(C) = \frac{1}{(1+r)^{N-t}} \mathbb{E}^*[C \mid S_t], \quad t = 0, 1, \ldots, N, \]

where \( N \) denotes maturity time and \( C \) is the option payoff.

d) Is this market complete? Is every contingent claim attainable?

Exercise 3.4 Consider a two-step binomial random asset model \((S_k)_{k=0,1,2}\) with possible returns \( a = 0 \) and \( b = 200\% \), and a riskless asset \( A_k = A_0(1+r)^k \), \( k = 0, 1, 2 \) with interest rate \( r = 100\% \), and \( S_0 = A_0 = 1 \), under the risk-neutral probability measure \( p^* = (r-a)/(b-a) = 1/2 \).

a) Draw a binomial tree for the possible values of \((S_k)_{k=0,1,2}\) and compute the values \( V_k \) of the hedging portfolio at times \( k = 0, 1, 2 \) of a European call option on \( S_T \) with strike price \( K = 8 \) and maturity \( T = 2 \).

\textit{Hint:} Consider three cases when \( k = 2 \), and two cases when \( k = 1 \).

b) Compute the self-financing hedging portfolio strategy \((\xi_k, \eta_k)_{k=1,2}\) with price
\[ V_k = \xi_kS_k + \eta_kA_k = \xi_{k+1}S_k + \eta_{k+1}A_k, \]
at \( k = 1 \), hedging the European call option with strike price \( K = 8 \) and maturity \( T = 2 \).

\textit{Hint:} Consider two separate cases for \( k = 2 \) and one case for \( k = 1 \).

Exercise 3.5 Let \( \mathbb{P}^* \) be a risk-neutral probability measure for a discrete-time asset price process \((S_n)_{n \in \mathbb{N}}\). Compute the arbitrage price \( V_k \) at time \( k = 0, 1, \ldots, N \) of the claim \( C \) with maturity time \( N \) and affine payoff function
\[ C = h(S_N) = \alpha + \beta S_N \]
where \( \alpha, \beta \in \mathbb{R} \) are constants, in a discrete-time market with risk free rate \( r \).

Exercise 3.6 Consider a two-step binomial random asset model \((S_k)_{k=0,1,2}\) with possible returns \( a = -50\% \) and \( b = 150\% \), and a riskless asset \( A_k = A_0(1+r)^k \), \( k = 0, 1, 2 \) with interest rate \( r = 100\% \), and \( S_0 = A_0 = 1 \), under the risk-neutral probability measure \( p^* = (r-a)/(b-a) = 3/4 \).

a) Draw a binomial tree for the values of \((S_k)_{k=0,1,2}\).

b) Compute the values \( V_k \) at times \( k = 0, 1, 2 \) of the hedging portfolio of a European put option with strike price \( K = 5/4 \) and maturity \( T = 2 \) on \( S_T \).
c) Compute the self-financing hedging portfolio strategy \((\xi_k, \eta_k)_{k=1,2}\) with price

\[ V_k = \xi_k S_k + \eta_k A_k = \xi_{k+1} S_k + \eta_{k+1} A_k, \]

at \(k = 1\), hedging the European put option with strike price \(K = 5/4\) and maturity \(T = 2\).

Exercise 3.7 Consider a two-step binomial model for a stock paying a dividend at the rate \(\alpha \in (0,1)\) at times \(k = 1\) and \(k = 2\), and the following recombining tree represents the ex-dividend* prices \(S_k\) at times \(k = 1,2\), starting from \(S_0 = $1\).

![Recombining Tree Diagram]

\[\begin{align*}
S_0 &= 1 \\
S_1 &= 1 \\
S_1 &= 3 \\
S_2 &= 3 \\
S_2 &= 9 \\
\end{align*}\]

* “Ex-dividend” means after dividend payment.
The difference between the closing price on Dec 17 ($3.06) and the opening price on Dec 18 ($2.99) is $3.06 − $2.99 = $0.07. The adjusted price on Dec 17 ($2.992) is the closing price ($3.06) minus the dividend ($0.068).

<table>
<thead>
<tr>
<th>Year-Month-Day</th>
<th>Open</th>
<th>High</th>
<th>Low</th>
<th>Close</th>
<th>Volume</th>
<th>Adjusted (ex-dividend)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2018-12-17</td>
<td>3.05</td>
<td>3.08</td>
<td>3.05</td>
<td><strong>3.06</strong></td>
<td>17441000</td>
<td>2.992</td>
</tr>
<tr>
<td>2018-12-18</td>
<td><strong>2.99</strong></td>
<td>2.99</td>
<td>2.96</td>
<td>2.96</td>
<td>28456400</td>
<td>2.960</td>
</tr>
</tbody>
</table>

The dividend rate $\alpha$ is given by $\alpha = 0.068 / 3.06 = 2.22\%$.

We consider a riskless asset $A_k = A_0 (1 + r)^k$, $k = 0, 1, 2$ with interest rate $r = 100\%$ and $A_0 = 1$, and two portfolio allocations $(\xi_1, \eta_1)$ at time $k = 0$ and $(\xi_2, \eta_2)$ at time $k = 1$, with prices

$$V_1 = \xi_2 S_1 + \eta_2 A_1$$

(3.41)

and

$$V_0 = \xi_1 S_0 + \eta_1 A_0.$$  

(3.42)

We make the following three assumptions:

[A] All dividends are reinvested.

[B] The portfolio strategies are self-financing.

[C] The portfolio value $V_2$ at time $k = 2$ hedges a European call option with payoff $C = (S_T - K)^+$, strike price $K = 8$, and maturity $T = 2$.

a) Using (3.41) and [A], express $V_2$ in terms of $\xi_2, \eta_2, S_2, A_2$ and $\alpha$.
b) Using (3.42) and [A]-[B], express $V_1$ in terms of $\xi_1, \eta_1, S_1, A_1$ and $\alpha$.
c) Using [C] and the result of Question (a), compute the portfolio allocation $(\xi_2, \eta_2)$ in cases $S_1 = 1$ and $S_1 = 3$.
d) Using (3.41) and the portfolio allocation $(\xi_2, \eta_2)$ obtained in Question (c), compute the portfolio price $V_1$ in cases $S_1 = 1$ and $S_1 = 3$.
e) From the results of Questions (b) and (d), compute the initial portfolio allocation $(\xi_1, \eta_1)$. 

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f) Compute the initial portfolio price $V_0$ from the result of Question (e).

g) Knowing that the dividend rate is $\alpha = 25\%$, draw the tree of asset prices $(\tilde{S}_k)_{k=1,2}$ before the dividend payments.

\[ \tilde{S}_0 = S_0 = 1 \]

\[ \tilde{S}_1 = ? \]

\[ \tilde{S}_2 = ? \]

\[ p^* \]

\[ q^* \]

\[ p^* \]

\[ q^* \]

\[ p^* \]

\[ q^* \]

\[ \tilde{S}_1 = ? \]

\[ \tilde{S}_2 = ? \]

\[ \tilde{S}_1 = ? \]

\[ \tilde{S}_2 = ? \]

h) Compute the risk-neutral probabilities $p^*$ and $q^*$ under which the conditional expected return of $(\tilde{S}_k)_{k=0,1,2}$ is the risk-free interest rate $r = 100\%$.

i) Check that the portfolio price $V_1$ found in Question (d) satisfies

\[ V_1 = \frac{1}{1+r} \mathbb{E}^* [(S_2 - K)^+ \mid S_1]. \]

j) Check that the portfolio price $V_0$ found in Question (f) satisfies

\[ V_0 = \frac{1}{(1+r)^2} \mathbb{E}^* [(S_2 - K)^+] \quad \text{and} \quad V_0 = \frac{1}{1+r} \mathbb{E}^* [V_1]. \]

Exercise 3.8 Analysis of a binary option trading website.

a) In a one-step model with risky asset prices $S_0, S_1$ at times $t = 0$ and $t = 1$, compute the price at time $t = 0$ of the binary call option with payoff

\[ C = \mathbb{1}_{[K,\infty)}(S_1) = \begin{cases} $1 & \text{if } S_1 \geq K, \\ 0 & \text{if } S_1 < K, \end{cases} \]

in terms of the probability $p^* = \mathbb{P}^*(S_1 \geq K)$ and of the risk-free interest rate $r$.

b) Compute the two potential net returns obtained by purchasing one binary call option.

c) Compute the corresponding expected return.

d) A website proposes to pay a return of $86\%$ in case the binary call option matures “in the money”, i.e. when $S_1 \geq K$. Compute the corresponding expected return. What do you conclude?
Exercise 3.9 A put spread collar option requires its holder to sell an asset at the price \( f(x) \) when its market price is \( x \), where \( f(x) \) is the function plotted in Figure 3.11, with \( K_1 := 80 \), \( K_2 := 90 \), and \( K_3 := 110 \).

![Put spread collar price graph](image)

**Fig. 3.11:** Put spread collar price graph.

a) Draw the payoff function of the put spread collar as a function of the underlying asset price at maturity. See e.g. [http://optioncreator.com/](http://optioncreator.com/).

b) Show that this put spread collar option can be realized by purchasing and/or issuing standard European call and put options with strike prices to be specified.

**Hints:** Recall that an option with payoff \( \phi(S_N) \) is priced \((1+r)^{-N} \mathbb{E}^* [\phi(S_N)]\) at time 0. The payoff of the European call (resp. put) option with strike price \( K \) is \((S_N - K)^+\), resp. \((K - S_N)^+\).

Exercise 3.10 Consider an asset price \( (S_n)_{n=0,1,...,N} \) which is a martingale under the risk-neutral probability measure \( \mathbb{P}^* \), with respect to the filtration \( (\mathcal{F}_n)_{n=0,1,...,N} \). Given the (convex) function \( \phi(x) := (x - K)^+ \), show that the price of an Asian option with payoff

\[
\phi \left( \frac{S_1 + \cdots + S_N}{N} \right)
\]

and maturity \( N \geq 1 \) is always lower than the price of the corresponding European call option, i.e. show that

\[
\mathbb{E}^* \left[ \phi \left( \frac{S_1 + S_2 + \cdots + S_N}{N} \right) \right] \leq \mathbb{E}^*[\phi(S_N)].
\]

**Hint:** Use in the following order:

(i) the convexity inequality \( \phi(x_1/N + \cdots + x_N/N) \leq \phi(x_1)/N + \cdots + \phi(x_N)/N \),

(ii) the martingale property \( S_k = \mathbb{E}^*[S_N \mid \mathcal{F}_k], k = 1, 2, \ldots, N \).
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(iii) the Jensen inequality

\[ \phi(\mathbb{E}^*[S_N | \mathcal{F}_k]) \leq \mathbb{E}^*[\phi(S_N) | \mathcal{F}_k], \quad k = 1, 2, \ldots, N, \]

(iv) the tower property \( \mathbb{E}^*[\mathbb{E}^*[\phi(S_N) | \mathcal{F}_k]] = \mathbb{E}^*[\phi(S_N)] \) of conditional expectations, \( k = 1, 2, \ldots, N \).

Exercise 3.11 (Exercise 2.5 continued)

a) We consider a forward contract on \( S_N \) with strike price \( K \) and payoff

\[ C := S_N - K. \]

Find a portfolio allocation \((\eta_N, \xi_N)\) with price

\[ V_N = \eta_N \pi_N + \xi_N S_N \]

at time \( N \), such that

\[ V_N = C, \]  \hspace{1cm} (3.43)

by writing Condition (3.43) as a \( 2 \times 2 \) system of equations.

b) Find a portfolio allocation \((\eta_{N-1}, \xi_{N-1})\) with price

\[ V_{N-1} = \eta_{N-1} \pi_{N-1} + \xi_{N-1} S_{N-1} \]

at time \( N - 1 \), and verifying the self-financing condition

\[ V_{N-1} = \eta_N \pi_{N-1} + \xi_N S_{N-1}. \]

Next, at all times \( t = 1, 2, \ldots, N - 1 \), find a portfolio allocation \((\eta_t, \xi_t)\) with price \( V_t = \eta_t \pi_t + \xi_t S_t \) verifying (3.43) and the self-financing condition

\[ V_t = \eta_{t+1} \pi_t + \xi_{t+1} S_t, \]

where \( \eta_t \), resp. \( \xi_t \), represents the quantity of the riskless, resp. risky, asset in the portfolio over the time period \([t-1, t], t = 1, 2, \ldots, N\).

c) Compute the arbitrage price \( \pi_t(C) = V_t \) of the forward contract \( C \), at time \( t = 0, 1, \ldots, N \).

d) Check that the arbitrage price \( \pi_t(C) \) satisfies the relation

\[ \pi_t(C) = \frac{1}{(1 + r)^{N-t}} \mathbb{E}^*[C | \mathcal{F}_t], \quad t = 0, 1, \ldots, N. \]

Exercise 3.12 Power option. Let \((S_n)_{n \in \mathbb{N}}\) denote a binomial price process with returns \(-50\%\) and \(+50\%\), and let the riskless asset be valued \( A_k = \$1, \)
We consider a power option with payoff $C := (S_N)^2$, and a predictable self-financing portfolio strategy $(\xi_k, \eta_k)_{k=1,2,\ldots,N}$ with price

$$V_k = \xi_k S_k + \eta_k A_0, \quad k = 1, 2, \ldots, N.$$ 

a) Find the portfolio allocation $(\xi_N, \eta_N)$ that matches the payoff $C = (S_N)^2$ at time $N$, i.e. that satisfies

$$V_N = (S_N)^2.$$

*Hint:* We have $\eta_N = -3(S_{N-1})^2/4$.

b) Compute the portfolio price under the risk-neutral probability measure $p^* = 1/2$

$$V_{N-1} = \mathbb{E}^*[C \mid \mathcal{F}_{N-1}].$$

c) Find the portfolio allocation $(\eta_{N-1}, \xi_{N-1})$ at time $N - 1$ from the relation

$$V_{N-1} = \xi_{N-1} S_{N-1} + \eta_{N-1} A_0.$$

*Hint:* We have $\eta_{N-1} = -15(S_{N-2})^2/16$.

d) Check that the portfolio satisfies the self-financing condition

$$V_{N-1} = \xi_{N-1} S_{N-1} + \eta_{N-1} A_0 = \xi_N S_{N-1} + \eta_N A_0.$$

Exercise 3.13 Consider the discrete-time Cox-Ross-Rubinstein model with $N + 1$ time instants $t = 0, 1, \ldots, N$. The price $S_t^0$ of the riskless asset evolves as $S_t^0 = \pi^0 (1 + r)^t, t = 0, 1, \ldots, N$. The return of the risky asset, defined as

$$R_t := \frac{S_t - S_{t-1}}{S_{t-1}}, \quad t = 1, 2, \ldots, N,$$

is random and allowed to take only two values $a$ and $b$, with $-1 < a < r < b$.

The discounted asset price is given by $\tilde{S}_t := S_t / (1 + r)^t, t = 0, 1, \ldots, N$.

a) Show that this model admits a unique risk-neutral probability measure $\mathbb{P}^*$ and explicitly compute $\mathbb{P}^*(R_t = a)$ and $\mathbb{P}(R_t = b)$ for all $t = 1, 2, \ldots, N$, with $a = 2\%$, $b = 7\%$, $r = 5\%$.

b) Does there exist arbitrage opportunities in this model? Explain why.

c) Is this market model complete? Explain why.

d) Consider a contingent claim with payoff $C = (S_N)^2$.

* This is the payoff of a power call option with strike price $K = 0$. 

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Compute the discounted arbitrage price \( \tilde{V}_t \), \( t = 0, 1, \ldots, N \), of a self-financing portfolio hedging the claim payoff \( C \), \textit{i.e.} such that

\[
\tilde{V}_N = \tilde{C} = \frac{(S_N)^2}{(1 + r)^N}.
\]

e) Compute the portfolio strategy

\[
(\xi_t)_{t=1,2,\ldots,N} = (\xi_0, \xi^1_t)_{t=1,2,\ldots,N}
\]

associated to \( \tilde{V}_t \), \textit{i.e.} such that

\[
\tilde{V}_t = \xi_t \cdot X_t = \xi_0 X^0_t + \xi^1_t X^1_t, \quad t = 1, 2, \ldots, N.
\]

f) Check that the above portfolio strategy is self-financing, \textit{i.e.}

\[
\xi_t \cdot S_t = \xi_{t+1} \cdot S_t, \quad t = 1, 2, \ldots, N - 1.
\]

Exercise 3.14 We consider the discrete-time Cox-Ross-Rubinstein model with \( N + 1 \) time instants \( t = 0, 1, \ldots, N \).

The price \( \pi_t \) of the riskless asset evolves as \( \pi_t = \pi_0 (1 + r)^t \), \( t = 0, 1, \ldots, N \). The evolution of \( S_{t-1} \) to \( S_t \) is given by

\[
S_t = \begin{cases} 
(1 + b)S_{t-1} & \text{if } R_t = b, \\
(1 + a)S_{t-1} & \text{if } R_t = a,
\end{cases}
\]

with \(-1 < a < r < b\). The return of the risky asset is defined as

\[
R_t := \frac{S_t - S_{t-1}}{S_{t-1}}, \quad t = 1, 2, \ldots, N.
\]

Let \( \xi_t \), resp. \( \eta_t \), denote the (possibly fractional) quantities of the risky, resp. riskless, asset held over the time period \([t - 1, t]\) in the portfolio with value

\[
V_t = \xi_t S_t + \eta_t \pi_t, \quad t = 0, 1, \ldots, N. \tag{3.44}
\]

a) Show that

\[
V_t = (1 + R_t)\xi_t S_{t-1} + (1 + r)\eta_t \pi_{t-1}, \quad t = 1, 2, \ldots, N. \tag{3.45}
\]

b) Show that under the probability \( \mathbb{P}^* \) defined by

\[
\mathbb{P}^*(R_t = a \mid \mathcal{F}_{t-1}) = \frac{b - r}{b - a}, \quad \mathbb{P}^*(R_t = b \mid \mathcal{F}_{t-1}) = \frac{r - a}{b - a},
\]

\[\Diamond\]
where $\mathcal{F}_{t-1}$ represents the information generated by \( \{R_1, R_2, \ldots, R_{t-1}\} \), we have
\[
\mathbb{E}^*[R_t \mid \mathcal{F}_{t-1}] = r.
\]

c) Under the self-financing condition
\[
V_{t-1} = \xi_t S_{t-1} + \eta_t \pi_{t-1}, \quad t = 1, 2, \ldots, N,
\]
recover the martingale property
\[
V_{t-1} = \frac{1}{1+r} \mathbb{E}^*[V_t \mid \mathcal{F}_{t-1}],
\]
using the result of Question (a).

d) Let \( a = 5\% \), \( b = 25\% \) and \( r = 15\% \). Assume that the price \( V_t \) at time \( t \)
of the portfolio is $3 if \( R_t = a \) and $8 if \( R_t = b \), and compute the price \( V_{t-1} \) of the portfolio at time \( t-1 \).

Problem 3.15 We consider a riskless asset priced as
\[
S_k^{(0)} = S_0^{(0)} (1 + r)^k, \quad k = 0, 1, \ldots, N,
\]
with \( r > -1 \), and a risky asset \( S^{(1)} \) whose return is given by
\[
R_k := \frac{S_k^{(1)} - S_{k-1}^{(1)}}{S_{k-1}^{(1)}}, \quad k = 1, 2, \ldots, N,
\]
with \( N + 1 \) time instants \( k = 0, 1, \ldots, N \) and \( d = 1 \). In the CRR model the return \( R_k \) is random and allowed to take two values \( a \) and \( b \) at each time step, i.e.
\[
R_k \in \{a, b\}, \quad k = 1, 2, \ldots, N,
\]
with \(-1 < a < 0 < b\), and the random evolution of \( S_{k-1}^{(1)} \) to \( S_k^{(1)} \) is given by
\[
S_k^{(1)} = \begin{cases} 
(1 + b) S_{k-1}^{(1)} & \text{if } R_k = b \\
(1 + a) S_{k-1}^{(1)} & \text{if } R_k = a
\end{cases} = (1 + R_k) S_{k-1}^{(1)}, \quad k = 1, 2, \ldots, N,
\]
according to the tree
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\[
\begin{align*}
(1 + b)S_{k-1}^{(1)} & \\
S_{k-1}^{(1)} & \\
& \\
(1 + a)S_{k-1}^{(1)}
\end{align*}
\]

and we have

\[
S_k^{(1)} = S_0^{(1)} \prod_{i=1}^{k} (1 + R_i), \quad k = 0, 1, \ldots, N.
\]

The information \( \mathcal{F}_k \) known to the market up to time \( k \) is given by the knowledge of \( S_1^{(1)}, S_2^{(1)}, \ldots, S_k^{(1)} \), i.e. we write

\[
\mathcal{F}_k = \sigma(S_1^{(1)}, S_2^{(1)}, \ldots, S_k^{(1)}) = \sigma(R_1, R_2, \ldots, R_k),
\]

\( k = 0, 1, \ldots, N \), where \( S_0^{(1)} \) is a constant and \( \mathcal{F}_0 = \{\emptyset, \Omega\} \) contains no information.

Under the risk-neutral probability measure \( \mathbb{P}^* \) defined by

\[
p^* := \mathbb{P}^*(R_k = b) = \frac{r - a}{b - a} > 0, \quad q^* := \mathbb{P}^*(R_k = a) = \frac{b - r}{b - a} > 0,
\]

\( k = 1, 2, \ldots, N \), the market returns \( (R_k)_{k=1,2,\ldots,N} \) form a sequence of independent identically distributed random variables.

In the sequel we assume that the stock \( S_k \) pays a dividend rate \( \alpha > 0 \) at times \( k = 1, 2, \ldots, N \). At the beginning of every time step \( k = 1, 2, \ldots, N \), the price \( S_k \) is immediately adjusted to its ex-dividend level by losing \( \alpha\% \) of its value.

The following 10 questions are interdependent and should be treated in sequence.

a) Rewrite the evolution (3.47) of \( S_{k-1}^{(1)} \) to \( S_k^{(1)} \) in the presence of a daily dividend rate \( \alpha > 0 \).

b) Express the dividend amount as a percentage of the ex-dividend price \( S_k^{(1)} \), and show that under the risk-neutral probability measure the return of the risky asset satisfies
c) We consider a predictable portfolio strategy \((\xi_k, \eta_k)_{k=1,2,...,N}\) with price process
\[
V_k = \xi_{k+1}S_k^{(1)} + \eta_{k+1}S_k^{(0)}
\]
at time \(k = 0,1,...,N-1\). Write down the self-financing condition for the portfolio price process \((V_k)_{k=0,1,...,N}\) by taking into account the reinvested dividends, and give the expression of \(V_N\).

d) Show that under the self-financing condition, the discounted portfolio price process \(\tilde{V}_k := \frac{V_k}{S_k^{(0)}}, \quad k = 0,1,...,N\)
is a martingale under the risk-neutral probability measure \(\mathbb{P}^*\).

e) Show that the price at time \(k = 0,1,...,N\) of a claim with random payoff \(C\) can be written as
\[
V_k = \frac{1}{(1+r)^{N-k}} \mathbb{E}^*[C | \mathcal{F}_k], \quad k = 0,1,...,N,
\]
assuming that the claim \(C\) is attained at time \(N\) by the portfolio strategy \((\xi_k, \eta_k)_{k=1,2,...,N}\).

f) Compute the price at time \(t = 0,1,...,N\) of a vanilla option with payoff \(h(S_N^{(1)})\) using the pricing function
\[
C_0(k, x, N, a, b, r) := \frac{1}{(1+r)^{N-k}} \sum_{l=0}^{N-k} \binom{N-k}{l} (p^*)^l (q^*)^{N-k-l} h \left( x(1+b)^l (1+a)^{N-k-l} \right)
\]
of a vanilla claim with payoff \(h(S_N^{(1)})\) and zero dividends.

g) Show that the price at time \(t = 0,1,...,N\) of a vanilla option with payoff function \(h(S_N^{(1)})\) can be rewritten as
\[
V_k = C_{\alpha}(k, S_k^{(1)}, N, a_\alpha, b_\alpha, r_\alpha) := (1-\alpha)^{N-k} C_0(k, S_k^{(1)}, N, a_\alpha, b_\alpha, r_\alpha),
\]
\(k = 0,1,...,N\), where the coefficients \(a_\alpha, b_\alpha, r_\alpha\) will be determined explicitly.

h) Find a recurrence relation between the functions \(C_{\alpha}(k, x, N, a_\alpha, b_\alpha, r_\alpha)\) and \(C_{\alpha}(k+1, x, N, a_\alpha, b_\alpha, r_\alpha)\) using the martingale property of the discounted portfolio price process \((\tilde{V}_k)_{k=0,1,...,N}\) under the risk-neutral probability measure \(\mathbb{P}^*\).
i) Using the function \( C_0(k, x, N, a_\alpha, b_\alpha, r_\alpha) \), compute the quantity \( \xi_k \) of risky asset \( S_k^{(1)} \) allocated on the time interval \([k - 1, k)\) in a self-financing portfolio hedging the claim \( C = h(S_N^{(1)}) \).

j) How are the dividends reinvested in the self-financing hedging portfolio?

Problem 3.16 We consider a ternary tree (or trinomial) model with \( N + 1 \) time instants \( k = 0, 1, \ldots, N \) and \( d = 1 \) risky asset. The price \( S_k^{(0)} \) of the riskless asset evolves as

\[
S_k^{(0)} = S_0^{(0)} (1 + r)^k, \quad k = 0, 1, \ldots, N,
\]

with \( r > -1 \). Let the return of the risky asset \( S^{(1)} \) be defined as

\[
R_k := \frac{S_k^{(1)} - S_{k-1}^{(1)}}{S_{k-1}^{(1)}}, \quad k = 1, 2, \ldots, N.
\]

In this ternary tree model, the return \( R_k \) is random and allowed to take only three values \( a, 0 \) and \( b \) at each time step, i.e.

\[
R_k \in \{a, 0, b\}, \quad k = 1, 2, \ldots, N,
\]

with \(-1 < a < 0 < b\). That means, the evolution of \( S_{k-1}^{(1)} \) to \( S_k^{(1)} \) is random and given by

\[
S_k^{(1)} = \begin{cases} 
(1 + b)S_{k-1}^{(1)} & \text{if } R_k = b \\
S_{k-1}^{(1)} & \text{if } R_k = 0 \\
(1 + a)S_{k-1}^{(1)} & \text{if } R_k = a
\end{cases} = (1 + R_k)S_{k-1}^{(1)}, \quad k = 1, 2, \ldots, N,
\]

and

\[
S_k^{(1)} = S_0^{(1)} \prod_{i=1}^{k} (1 + R_i), \quad k = 0, 1, \ldots, N.
\]

The price process \((S_k^{(1)})_{k=0,1,\ldots,N}\) evolves on a ternary tree of the following form:
The information $F_k$ known to the market up to time $k$ is given by the knowledge of $S_1^{(1)}, S_2^{(1)}, \ldots, S_k^{(1)}$, i.e., we write

$$F_k = \sigma(S_1^{(1)}, S_2^{(1)}, \ldots, S_k^{(1)}) = \sigma(R_1, R_2, \ldots, R_k),$$

$k = 0, 1, \ldots, N$, where, as a convention, $S_0^{(1)}$ is a constant and $F_0 = \{\emptyset, \Omega\}$ contains no information. In the sequel we will consider that $(R_k)_{k=1,2,\ldots,N}$ is a sequence of independent identically distributed random variables under any risk-neutral probability measure $P^*$, and we denote

$$\begin{align*}
p^* &:= P^*(R_k = b) > 0, \\
\theta^* &:= P^*(R_k = 0) > 0, \\
q^* &:= P^*(R_k = a) > 0, \quad k = 1, 2, \ldots, N.
\end{align*}$$

a) Determine all possible risk-neutral probability measures $P^*$ equivalent to $P$ in terms of the parameter $\theta^* \in (0, 1)$.

b) Give a necessary and sufficient condition for absence of arbitrage in this ternary tree model.

Hint: Use your intuition of the market to find what the condition should be, and then prove that it is necessary and sufficient. Note that we have $a < 0$ and $b > 0$, and the condition should only depend on the model parameters $a$, $b$ and $r$.

c) When the model parameters allow for arbitrage opportunities, explain how you would exploit them if you joined the market with zero money to invest.

d) Is this ternary tree market model complete?

e) In this question we assume that the conditional variance

$$\text{Var}^* \left[ \frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}} \left| F_k \right. \right] = \sigma^2 > 0$$

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http://www.ntu.edu.sg/home/nprivault/index.html
of the asset return \( \frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}} \) given \( \mathcal{F}_k \) is constant and equal to \( \sigma^2 \), \( k = 0, 1, \ldots, N - 1 \). Show that this condition determines a unique value of \( \theta^* \) and a unique risk-neutral probability measure \( \mathbb{P}^*_\sigma \) to be written explicitly, under a certain condition on \( a, b, r \) and \( \sigma \).

f) In this question and in the following we impose the condition \((1 + a)(1 + b) = 1\), \(i.e.\) we let \( a := -b/(b + 1) \). What does this imply on this ternary tree model and on the risk-neutral probability measure \( \mathbb{P}^* \)?

g) We consider a vanilla financial claim with payoff \( C = h(S_N) \) and maturity \( N \), priced as time \( k \) as

\[
\begin{align*}
    f(k, S_k^{(1)}) &= \frac{1}{(1 + r)^{N-k}} \mathbb{E}_{\theta}^{*}[h(S_N) \mid \mathcal{F}_k] \\
    &= \frac{1}{(1 + r)^{N-k}} \mathbb{E}_{\theta}^{*}[h(S_N) \mid S_k^{(1)}],
\end{align*}
\]

\( k = 0, 1, \ldots, N, \) under the risk-neutral probability measure \( \mathbb{P}^*_\theta \). Find a recurrence equation between the functions \( f(k, \cdot) \) and \( f(k+1, \cdot) \), \( k = 0, \ldots, N - 1 \).

Hint: Use the “tower property” of conditional expectations.

h) Assuming that \( C \) is the payoff of a European put option with strike price \( K \), give the expression of \( f(N, x) \).

i) Modify the attached binomial Python code in order to make it deal with the trinomial model (attach a printout of your modified code).

j) Taking \( S_0^{(1)} = 1, r = 0.1, b = 1, (1 + a)(1 + b) = 1 \), compute the price at time \( k = 0 \) of the European put option with strike price \( K = 1 \) and maturity \( N = 2 \) using the code of Question (i) with \( \theta = 0.5 \).

Download* and install the Anaconda distribution from https://www.anaconda.com/distribution/ or try it online at https://jupyter.org/try.

* Download the corresponding IPython notebook that can be run here.
import networkx as nx
import numpy as np
import matplotlib
import matplotlib.pyplot as plt

N=2;S0=1
r = 0.1; a=-0.5; b=1; # change
# add definition of theta
p = (r-a)/(b-a) # change
q = (b-r)/(b-a) # change
def plot_tree(g):
    pos={}
    lab={}
    for n in g.nodes():
        pos[n]=(n[0],n[1])
        lab[n]=float("{0:.2f}".format(g.node[n]['value']))
    elarge=g.edges(data=True)
    nx.draw_networkx_edges(g,pos,edgelist=elarge)
    nx.draw_networkx_labels(g,pos,lab,font_size=15,font_family=’sans-serif’)
    plt.autoscale(enable=True)
    plt.show()
def graph_stock():
    S=nx.Graph()
    for k in range(0,N):
        for l in range(-k,k+2,2): # change range and step size
            S.add_edge((k,l),(k+1,l+1)) # add edge
            S.add_edge((k,l),(k+1,l-1))
    for n in S.nodes():
        k=n[0]
        l=n[1]
        S.node[n]['value']=S0*((1.0+b)**((k+l)/2))*((1.0+a)**((k-l)/2))
    return S
plot_tree(graph_stock())

def European_call_price(K):
    price = nx.Graph()
    hedge = nx.Graph()
    S = graph_stock()
    for k in range(0,N):
        for l in range(-k,k+2,2): # change range and step size
            price.add_edge((k,l),(k+1,l+1)) # add edge
            price.add_edge((k,l),(k+1,l-1))
    for l in range(-N,N+2,2): # change range and step size
        price.node[(N,l)]['value'] = np.maximum(S.node[(N,l)]['value']-K,0)
    for k in reversed(range(0,N)):
        for l in range(-k,k+2,2): # change range and step size
            price.node[(k,l)]['value'] = (price.node[(k+1,l+1)]['value']*p+price.node[(k+1,l-1)]['value']*q)/(1+r) # add theta
    return price

K = input("Strike K=")
call_price = European_call_price(float(K))
print('Underlying asset prices:')
plot_tree(graph_stock())
print('European call option prices:')
plot_tree(call_price)
print('Price at time 0 of the European call option:',float("{0:.4f}".format(call_price.node[(0,0)]['value'])))