Chapter 7
Martingale Approach to Pricing and Hedging

In the *martingale approach* to the pricing and hedging of financial derivatives, option prices are expressed as the expected values of discounted option payoffs. This approach relies on the construction of risk-neutral probability measures by the Girsanov theorem, and the associated hedging portfolios are obtained via stochastic integral representations.

### 7.1 Martingale Property of the Itô Integral

Recall (Definition 5.5) that an integrable process $(X_t)_{t \in \mathbb{R}^+}$ is said to be a *martingale* with respect to the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$ if

$$\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s, \quad 0 \leq s \leq t.$$ 

**Examples of martingales (i)**

1. Given $F \in L^2(\Omega)$ a square-integrable random variable and $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$ a filtration, the process $(X_t)_{t \in \mathbb{R}^+}$ defined by

$$X_t := \mathbb{E}[F \mid \mathcal{F}_t], \quad t \in \mathbb{R}^+,$$

is an $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$-martingale under $\mathbb{P}$, as follows from the tower property (23.38) of conditional expectations:
\( \mathbb{E}[X_t \mid \mathcal{F}_s] = \mathbb{E}[\mathbb{E}[F \mid \mathcal{F}_t] \mid \mathcal{F}_s] = \mathbb{E}[F \mid \mathcal{F}_s] = X_s, \quad 0 \leq s \leq t. \) (7.1)

2. Any integrable stochastic process \((X_t)_{t \in \mathbb{R}^+}\) whose increments \((X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}})\) are independent and centered under \(\mathbb{P}\) (i.e. \(\mathbb{E}[X_t] = 0, \ t \in \mathbb{R}^+\)) is a martingale with respect to the filtration \((\mathcal{F}_t)_{t \in \mathbb{R}^+}\) generated by \((X_t)_{t \in \mathbb{R}^+}\), as we have

\[
\mathbb{E}[X_t \mid \mathcal{F}_s] = \mathbb{E}[X_t - X_s + X_s \mid \mathcal{F}_s] = \mathbb{E}[X_t - X_s] + \mathbb{E}[X_s \mid \mathcal{F}_s] = X_s, \quad 0 \leq s \leq t. \quad (7.2)
\]

In particular, the standard Brownian motion \((B_t)_{t \in \mathbb{R}^+}\) is a martingale because it has centered and independent increments. This fact is also consequence of Proposition 7.1 below as \(B_t\) can be written as

\[
B_t = \int_0^t dB_s, \quad t \in \mathbb{R}^+.
\]

The following result shows that the Itô integral yields a martingale with respect to the Brownian filtration \((\mathcal{F}_t)_{t \in \mathbb{R}^+}\). It is the continuous-time analog of the discrete-time Theorem 2.11.

**Proposition 7.1.** The stochastic integral process \(\left(\int_0^t u_s dB_s\right)_{t \in \mathbb{R}^+}\) of a square-integrable adapted process \(u \in L^2_{\text{ad}}(\Omega \times \mathbb{R}^+)\) is a martingale, i.e.:

\[
\mathbb{E} \left[ \int_0^t u_\tau dB_\tau \mid \mathcal{F}_s \right] = \int_0^s u_\tau dB_\tau, \quad 0 \leq s \leq t. \quad (7.3)
\]

In particular, \(\int_0^t u_s dB_s\) is \(\mathcal{F}_t\)-measurable, \(t \in \mathbb{R}^+\), and since \(\mathcal{F}_0 = \{\emptyset, \Omega\}\), Relation (7.3) applied with \(t = 0\) recovers the fact that the Itô integral is a centered random variable:

\[
\mathbb{E} \left[ \int_0^\infty u_s dB_s \right] = \mathbb{E} \left[ \int_0^\infty u_s dB_s \mid \mathcal{F}_0 \right] = \int_0^0 u_s dB_s = 0. \]

**Proof.** The statement is first proved in case \((u_t)_{t \in \mathbb{R}^+}\) is a simple predictable process, and then extended to the general case, cf. e.g. Proposition 2.5.7 in Privault (2009). For example, for \(u\) a predictable step process of the form

\[
u_s := F 1_{[a,b)}(s) = \begin{cases} 
F & \text{if } s \in [a,b], \\
0 & \text{if } s \notin [a,b],
\end{cases}
\]

with \(F\) an \(\mathcal{F}_a\)-measurable random variable and \(t \in [a,b]\), we have
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\[
\mathbb{E} \left[ \int_0^\infty u_s dB_s \mid \mathcal{F}_t \right] = \mathbb{E} \left[ \int_0^\infty F \mathbb{1}_{[a,b]}(s) dB_s \mid \mathcal{F}_t \right] = \mathbb{E} \left[ F(B_b - B_a) \mid \mathcal{F}_t \right] = F \mathbb{E} \left[ B_b - B_a \mid \mathcal{F}_t \right] = \int_a^t u_s dB_s = \int_0^t u_s dB_s, \quad a \leq t \leq b.
\]

On the other hand, when \( t \in [0, a] \) we have

\[
\int_0^t u_s dB_s = 0,
\]

and we check that

\[
\mathbb{E} \left[ \int_0^\infty u_s dB_s \mid \mathcal{F}_t \right] = \mathbb{E} \left[ \int_0^\infty F \mathbb{1}_{[a,b]}(s) dB_s \mid \mathcal{F}_t \right] = \mathbb{E} \left[ F(B_b - B_a) \mid \mathcal{F}_a \right] - B_a = 0.
\]

where we used the tower property (23.38) of conditional expectations and the fact that Brownian motion \((B_t)_{t \in \mathbb{R}^+}\) is a martingale:

\[
\mathbb{E}[B_b - B_a \mid \mathcal{F}_a] = \mathbb{E}[B_b \mid \mathcal{F}_a] - B_a = B_b - B_a = 0.
\]

The extension from simple processes to square-integrable processes in \(L^2_{\text{ad}}(\Omega \times \mathbb{R}^+)\) can be proved as in Proposition 4.20. Indeed, given \((u^{(n)})_{n \in \mathbb{N}}\) be a sequence of simple predictable processes converging to \(u\) in \(L^2(\Omega \times [0, T])\) cf. Lemma 1.1 of Ikeda and Watanabe (1989), pages 22 and 46, by Fatou’s Lemma 23.4 and Jensen’s inequality we have:

\[
\mathbb{E} \left[ \left( \int_0^t u_s dB_s - \mathbb{E} \left[ \int_0^\infty u_s dB_s \mid \mathcal{F}_t \right] \right)^2 \right] \leq \liminf_{n \to \infty} \mathbb{E} \left[ \left( \int_0^t u_s^{(n)} dB_s - \mathbb{E} \left[ \int_0^\infty u_s dB_s \mid \mathcal{F}_t \right] \right)^2 \right] = \liminf_{n \to \infty} \mathbb{E} \left[ \left( \mathbb{E} \left[ \int_0^\infty u_s^{(n)} dB_s - \int_0^\infty u_s dB_s \mid \mathcal{F}_t \right] \right)^2 \right] \leq \liminf_{n \to \infty} \mathbb{E} \left[ \left( \int_0^\infty u_s^{(n)} dB_s - \int_0^\infty u_s dB_s \right)^2 \mid \mathcal{F}_t \right].
\]
\[ \lim_{n \to \infty} \mathbb{E} \left[ \left( \int_0^\infty (u_s^{(n)} - u_s) dB_s \right)^2 \right] = \lim_{n \to \infty} \mathbb{E} \left[ \int_0^\infty |u_s^{(n)} - u_s|^2 ds \right] = \lim_{n \to \infty} \|u^{(n)} - u\|_{L^2(\Omega \times [0,T])}^2 = 0, \]

where we used the Itô isometry (4.16). We conclude that

\[ \mathbb{E} \left[ \int_0^\infty u_s dB_s \bigg| \mathcal{F}_t \right] = \int_0^t u_s dB_s, \quad t \in \mathbb{R}_+, \]

for \( u \in L^2_{ad}(\Omega \times \mathbb{R}_+) \) a square-integrable adapted process, which leads to (7.3) after applying this identity to the process \( \mathbbm{1}_{[0,t]} u_s \epsilon_{s} \), \( s \in \mathbb{R}_+ \), i.e.,

\[ \mathbb{E} \left[ \int_0^t u_\tau dB_\tau \bigg| \mathcal{F}_s \right] = \mathbb{E} \left[ \int_0^\infty \mathbbm{1}_{[0,t]}(\tau) u_\tau dB_\tau \bigg| \mathcal{F}_s \right] = \int_0^s \mathbbm{1}_{[0,t]}(\tau) u_\tau dB_\tau = \int_0^s u_\tau dB_\tau, \quad 0 \leq s \leq t. \]

□

Examples of martingales (ii)

1. The driftless geometric Brownian motion

\[ X_t := X_0 e^{\sigma B_t - \sigma^2 t/2} \]

is a martingale. Indeed, we have

\[ \mathbb{E}[X_t | \mathcal{F}_s] = \mathbb{E} \left[ X_0 e^{\sigma B_t - \sigma^2 t/2} | \mathcal{F}_s \right] = X_0 e^{-\sigma^2 t/2} \mathbb{E} \left[ e^{\sigma B_t} | \mathcal{F}_s \right] = X_0 e^{-\sigma^2 t/2} \mathbb{E} \left[ e^{(B_t-B_s)\sigma + \sigma B_s} | \mathcal{F}_s \right] = X_0 e^{-\sigma^2 t/2 + \sigma B_s} \mathbb{E} \left[ e^{(B_t-B_s)\sigma} | \mathcal{F}_s \right] = X_0 e^{-\sigma^2 t/2 + \sigma B_s} \mathbb{E} \left[ e^{(B_t-B_s)\sigma} \right] = X_0 e^{-\sigma^2 t/2 + \sigma B_s} e^{\sigma^2(t-s)/2} = X_0 e^{\sigma B_s - \sigma^2 s/2} = X_s, \quad 0 \leq s \leq t. \]
This fact can also be recovered from Proposition 7.1 since \((X_t)_{t \in \mathbb{R}_+}\) satisfies the equation
\[
dX_t = \sigma X_t dB_t,
\]
which shows that \(X_t\) can be written using the Brownian stochastic integral
\[
X_t = X_0 + \sigma \int_0^t X_u dB_u, \quad t \in \mathbb{R}_+.
\]

2. Consider an asset price process \((S_t)_{t \in \mathbb{R}_+}\) given by the stochastic differential equation
\[
dS_t = \mu S_t dt + \sigma S_t dB_t, \quad t \in \mathbb{R}_+.
\] (7.4)

By the Discounting Lemma 5.14, the discounted asset price process \(\tilde{S}_t := e^{-rt}S_t, t \in \mathbb{R}_+\), satisfies the stochastic differential equation
\[
d\tilde{S}_t = \tilde{S}_t((\mu - r)dt + \sigma dB_t).
\]

The discounted asset price
\[
\tilde{S}_t = e^{-rt}S_t = S_0 e^{(\mu - r)t + \sigma B_t - \sigma^2 t/2}
\]
is a martingale under \(\mathbb{P}\) when \(\mu = r\). The case \(\mu \neq r\) will be treated in Section 7.3 using risk-neutral probability measures and the Girsanov Theorem 7.3, see (7.14) below.

3. The discounted value
\[
\tilde{V}_t = e^{-rt}V_t
\]
of a self-financing portfolio is given by
\[
\tilde{V}_t = \tilde{V}_0 + \int_0^t \xi_u d\tilde{S}_u, \quad t \in \mathbb{R}_+,
\]
cf. Lemma 5.15 is a martingale when \(\mu = r\) by Proposition 7.1 because
\[
\tilde{V}_t = \tilde{V}_0 + \sigma \int_0^t \xi_u \tilde{S}_u dB_u, \quad t \in \mathbb{R}_+,
\]
since we have
\[
d\tilde{S}_t = \tilde{S}_t((\mu - r)dt + \sigma dB_t) = \sigma \tilde{S}_t dB_t
\]
by the Discounting Lemma 5.14. Since the Black-Scholes theory is in fact valid for any value of the parameter \(\mu\) we will look forward to including the case \(\mu \neq r\) in the sequel.
7.2 Risk-neutral Probability Measures

Recall that by definition, a risk-neutral measure is a probability measure $P^*$ under which the discounted asset price process

$$(\tilde{S}_t)_{t \in \mathbb{R}^+} := (e^{-rt}S_t)_{t \in \mathbb{R}^+}$$

is a martingale, cf. Proposition 5.6.

Consider an asset price process $(S_t)_{t \in \mathbb{R}^+}$ given by the stochastic differential equation (7.4). Note that when $\mu = r$, the discounted asset price process $(\tilde{S}_t)_{t \in \mathbb{R}^+} = (S_0 e^{\sigma B_t - \sigma^2 t/2})_{t \in \mathbb{R}^+}$ is a martingale under $P^* = P$, which is a risk-neutral probability measure.

In this section, we address the construction of a risk-neutral probability measure $P^*$ in the general case $\mu \neq r$ using the Girsanov Theorem 7.3 below. Note that the relation

$$d\tilde{S}_t = \tilde{S}_t((\mu - r)dt + \sigma dB_t)$$

where $\mu - r$ is the risk premium, can be rewritten as

$$d\tilde{S}_t = \sigma \tilde{S}_t d\tilde{B}_t,$$

where $(\tilde{B}_t)_{t \in \mathbb{R}^+}$ is a drifted Brownian motion given by

$$\tilde{B}_t := \frac{\mu - r}{\sigma} t + B_t, \quad t \in \mathbb{R}^+,$$

where the drift coefficient $\nu := (\mu - r)/\sigma$ is the “Market Price of Risk” (MPoR). It represents the difference between the return $\mu$ expected when investing in the risky asset $S_t$, and the risk-free interest rate $r$, measured in units of volatility $\sigma$.

Therefore, the search for a risk-neutral probability measure can be replaced by the search for a probability measure $P^*$ under which $(\tilde{B}_t)_{t \in \mathbb{R}^+}$ is a standard Brownian motion.

Let us come back to the informal interpretation of Brownian motion via its infinitesimal increments:

$$\Delta B_t = \pm \sqrt{\Delta t},$$

with

$$\mathbb{P}(\Delta B_t = +\sqrt{\Delta t}) = \mathbb{P}(\Delta B_t = -\sqrt{\Delta t}) = \frac{1}{2}.$$
Clearly, given \( \nu \in \mathbb{R} \), the drifted process

\[
\tilde{B}_t := \nu t + B_t, \quad t \in \mathbb{R}_+,
\]

is no longer a standard Brownian motion because it is not centered:

\[
\mathbb{E}[\tilde{B}_t] = \mathbb{E}[\nu t + B_t] = \nu t + \mathbb{E}[B_t] = \nu t \neq 0,
\]

cf. Figure 7.1. This identity can be formulated in terms of infinitesimal increments as

\[
\mathbb{E}[\nu \Delta t + \Delta B_t] = \frac{1}{2}(\nu \Delta t + \sqrt{\Delta t}) + \frac{1}{2}(\nu \Delta t - \sqrt{\Delta t}) = \nu \Delta t \neq 0.
\]

In order to make \( \nu t + B_t \) a centered process \( i.e. \) a standard Brownian motion, since \( \nu t + B_t \) conserves all the other properties \( (i)-(iii) \) in the definition of Brownian motion, one may change the probabilities of ups and downs, which have been fixed so far equal to \( 1/2 \).

That is, the problem is now to find two numbers \( p^*, q^* \in [0,1] \) such that

\[
\begin{cases}
    p^*(\nu \Delta t + \sqrt{\Delta t}) + q^*(\nu \Delta t - \sqrt{\Delta t}) = 0 \\
    p^* + q^* = 1.
\end{cases}
\]

The solution to this problem is given by

\[
p^* = \frac{1}{2}(1 - \nu \sqrt{\Delta t}) \quad \text{and} \quad q^* = \frac{1}{2}(1 + \nu \sqrt{\Delta t}).
\]

(7.5)

Coming back to Brownian motion considered as a discrete random walk with independent increments \( \pm \sqrt{\Delta t} \), we try to construct a new probability measure denoted \( \mathbb{P}^* \), under which the drifted process \( \tilde{B}_t := \nu t + B_t \) will be a standard Brownian motion. This probability measure will be defined through its Radon-Nikodym density.
\[ \frac{d\mathbb{P}^*}{d\mathbb{P}} := \frac{\mathbb{P}^*(\Delta B_{t_1} = \epsilon_1 \sqrt{\Delta t}, \ldots, \Delta B_{t_N} = \epsilon_N \sqrt{\Delta t})}{\mathbb{P}(\Delta B_{t_1} = \epsilon_1 \sqrt{\Delta t}, \ldots, \Delta B_{t_N} = \epsilon_N \sqrt{\Delta t})} \]

\[ = \frac{\mathbb{P}^*(\Delta B_{t_1} = \epsilon_1 \sqrt{\Delta t}) \cdots \mathbb{P}^*(\Delta B_{t_N} = \epsilon_N \sqrt{\Delta t})}{\mathbb{P}(\Delta B_{t_1} = \epsilon_1 \sqrt{\Delta t}) \cdots \mathbb{P}(\Delta B_{t_N} = \epsilon_N \sqrt{\Delta t})} \]

\[ = \frac{1}{(1/2)^N} \mathbb{P}^*(\Delta B_{t_1} = \epsilon_1 \sqrt{\Delta t}) \cdots \mathbb{P}^*(\Delta B_{t_N} = \epsilon_N \sqrt{\Delta t}), \quad (7.6) \]

\[ \epsilon_1, \epsilon_2, \ldots, \epsilon_N \in \{-1, 1\}, \text{ with respect to the historical probability measure } \mathbb{P}, \text{ obtained by taking the product of the above probabilities divided by the reference probability } 1/2^N \text{ corresponding to the symmetric random walk.} \]

Interpreting \( N = T/\Delta t \) as an (infinitely large) number of discrete time steps and under the identification \([0, T] \simeq \{0 = t_0, t_1, \ldots, t_N = T\}\), this Radon-Nikodym density \( (7.6) \) can be rewritten as

\[ \frac{d\mathbb{P}^*}{d\mathbb{P}} \simeq \frac{1}{(1/2)^N} \prod_{0 < t < T} \left( \frac{1}{2} \mp \frac{1}{2} \nu \sqrt{\Delta t} \right) \quad (7.7) \]

where \( 2^N \) becomes a normalization factor. Using the expansion

\[ \log \left( 1 \mp \nu \sqrt{\Delta t} \right) = \pm \nu \sqrt{\Delta t} - \frac{1}{2} (\pm \nu \sqrt{\Delta t})^2 + o(\Delta t) \]

\[ = \pm \nu \sqrt{\Delta t} - \frac{\nu^2}{2} \Delta t + o(\Delta t), \]

for small values of \( \Delta t \), this Radon-Nikodym density can be informally shown to converge as follows as \( N \) tends to infinity, \textit{i.e.} as the time step \( \Delta t = T/N \) tends to zero:

\[ 2^N \prod_{0 < t < T} \left( \frac{1}{2} \mp \frac{1}{2} \nu \sqrt{\Delta t} \right) = \prod_{0 < t < T} \left( 1 \mp \nu \sqrt{\Delta t} \right) \]

\[ = \exp \left( \log \prod_{0 < t < T} \left( 1 \mp \nu \sqrt{\Delta t} \right) \right) \]

\[ = \exp \left( \sum_{0 < t < T} \log \left( 1 \mp \nu \sqrt{\Delta t} \right) \right) \]

\[ \simeq \exp \left( \nu \sum_{0 < t < T} \mp \sqrt{\Delta t} - \frac{1}{2} \sum_{0 < t < T} (\mp \nu \sqrt{\Delta t})^2 \right) \]

\[ = \exp \left( -\nu \sum_{0 < t < T} \pm \sqrt{\Delta t} - \frac{\nu^2}{2} \sum_{0 < t < T} \Delta t \right) \]
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\[
\begin{align*}
&= \exp \left( -\nu \sum_{0 \leq t < T} \Delta B_t - \frac{\nu^2}{2} \sum_{0 \leq t < T} \Delta t \right) \\
&= \exp \left( -\nu B_T - \frac{\nu^2}{2} T \right),
\end{align*}
\]

based on the identifications

\[
B_T \simeq \sum_{0 < t < T} \pm \sqrt{\Delta t} \quad \text{and} \quad T \simeq \sum_{0 < t < T} \Delta t.
\]

Informally, the drifted process \((\hat{B}_t)_{t \in [0,T]} = (\nu t + B_t)_{t \in [0,T]}\) is a standard Brownian motion under the probability measure \(P^*\) defined by its Radon-Nikodym density

\[
\frac{dP^*}{dP} = \exp \left( -\nu B_T - \frac{\nu^2}{2} T \right).
\]

The following R code is rescaling probabilities as in (7.5) based on the value of the drift \(\mu\).

```r
N=1000; t <- 0:N; dt <- 1.0/N; nu=3; p=0.5*(1-nu*(dt)^0.5); nsim <- 10
X <- matrix((dt)^0.5*(rbinom( nsim * N, 1, p)-0.5)*2, nsim, N)
X <- cbind(rep(0, nsim), t(apply(X, 1, cumsum)))
for (i in 1:nsim){lines(t,t*nu*dt+X[i,],type="l",ylim=c(-2*N*dt,2*N*dt),col=i)}
```

The discretized illustration in Figure 7.2 displays the drifted Brownian motion \(\hat{B}_t := \nu t + B_t\) under the shifted probability measure \(P^*\) in (7.7) using the above R code with \(N = 100\). The code makes big transitions less frequent than small transitions, resulting into a standard, centered Brownian motion under \(P^*\).

![Fig. 7.2: Drifted Brownian motion paths under a shifted Girsanov measure.](https://personal.ntu.edu.sg/nprivault/indext.html)
7.3 Change of Measure and the Girsanov Theorem

In this section we restate the Girsanov Theorem in a more rigorous way, using changes of probability measures.

**Definition 7.2.** We say that a probability measure $Q$ is absolutely continuous with respect to another probability measure $P$ if there exists a nonnegative random variable $F : \Omega \rightarrow \mathbb{R}$ such that $\mathbb{E}[F] = 1$, and

$$\frac{dQ}{dP} = F, \quad \text{i.e.} \quad dQ = F dP. \quad (7.8)$$

In this case, $F$ is called the Radon-Nikodym density of $Q$ with respect to $P$.

Relation (7.8) is equivalent to the relation

$$\mathbb{E}_Q[G] = \int_{\Omega} G(\omega) dQ(\omega)$$

$$= \int_{\Omega} G(\omega) \frac{dQ(\omega)}{dP(\omega)} dP(\omega)$$

$$= \int_{\Omega} G(\omega) F(\omega) dP(\omega)$$

$$= \mathbb{E}[FG],$$

for any integrable random variable $G$.

The Girsanov Theorem can actually be extended to shifts by adapted processes $(\psi_t)_{t \in [0, T]}$ as follows, cf. e.g. Theorem III-42, page 141 of Protter (2004). An extension of the Girsanov Theorem to jump processes will be covered in Section 20.5. Recall also that here, $\Omega = C([0, T])$ is the Wiener space and $\omega \in \Omega$ is a continuous function on $[0, T]$ starting at 0 in $t = 0$.

The Girsanov Theorem 7.3 will be used in Section 7.4 for the construction of a unique risk-neutral probability measure $P^*$, showing absence of arbitrage and completeness in the Black-Scholes market, see Theorems 5.8 and 5.12.

**Theorem 7.3.** Let $(\psi_t)_{t \in [0, T]}$ be an adapted process satisfying the Novikov integrability condition

$$\mathbb{E} \left[ \exp \left( \frac{1}{2} \int_0^T |\psi_t|^2 dt \right) \right] < \infty, \quad (7.9)$$

and let $Q$ denote the probability measure defined by

$$\frac{dQ}{dP} = \exp \left( - \int_0^T \psi_s dB_s - \frac{1}{2} \int_0^T |\psi_s|^2 ds \right).$$

Then

$$\tilde{B}_t := B_t + \int_0^t \psi_s ds, \quad 0 \leq t \leq T,$$
is a standard Brownian motion under $Q$.

In the case of the simple shift

$$\hat{B}_t := B_t + \nu t, \quad 0 \leq t \leq T,$$

by a drift $\nu t$ with constant $\nu \in \mathbb{R}$, the process $(\hat{B}_t)_{t \in \mathbb{R}_+}$ is a standard (centered) Brownian motion under the probability measure $Q$ defined by

$$dQ(\omega) = \exp \left( -\nu B_T - \frac{\nu^2}{2} T \right) dP(\omega).$$

For example, the fact that $\hat{B}_T$ has a centered Gaussian distribution under $Q$ can be recovered as follows:

$$\mathbb{E}_Q [f(\hat{B}_T)] = \mathbb{E}_Q [f(\nu T + B_T)]$$

$$= \int_{\Omega} f(\nu T + B_T) dQ$$

$$= \int_{\Omega} f(\nu T + B_T) \exp \left( -\nu B_T - \frac{1}{2} \nu^2 T \right) dP$$

$$= \int_{-\infty}^{\infty} f(\nu T + y) \exp \left( -\nu y - \frac{1}{2} \nu^2 T \right) e^{-y^2/(2T)} \frac{dy}{\sqrt{2\pi T}}$$

$$= \int_{-\infty}^{\infty} f(y) e^{-y^2/(2T)} \frac{dy}{\sqrt{2\pi T}}$$

$$= \mathbb{E}_P [f(B_T)],$$

i.e.

$$\mathbb{E}_Q [f(\nu T + B_T)] = \int_{\Omega} f(\nu T + B_T) dQ$$

$$= \int_{\Omega} f(B_T) dP$$

$$= \mathbb{E}_P [f(B_T)],$$

showing that, under $Q$, $\nu T + B_T$ has the centered $\mathcal{N}(0, T)$ Gaussian distribution with variance $T$. For example, taking $f(x) = x$, Relation (7.10) recovers the fact that $\hat{B}_T$ is a centered random variable under $Q$, i.e.

$$\mathbb{E}_Q [\hat{B}_T] = \mathbb{E}_Q [\nu T + B_T] = \mathbb{E}_P [B_T] = 0.$$

The Girsanov Theorem 7.3 also allows us to extend (7.10) as

$$\mathbb{E}[F] = \mathbb{E} \left[ F \left( B_0 + \int_0^T \psi_s^2 ds \right) \exp \left( -\int_0^T \psi_s dB_s - \frac{1}{2} \int_0^T |\psi_s|^2 ds \right) \right],$$

This version: October 15, 2020
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for all random variables \( F \in L^1(\Omega) \), see also Exercise 7.21.

When applied to the (constant) market price of risk (or Sharpe ratio)

\[ \psi_t := \frac{\mu - r}{\sigma}, \]

the Girsanov Theorem 7.3 shows that

\[ \widehat{B}_t := \frac{\mu - r}{\sigma} t + B_t, \quad 0 \leq t \leq T, \quad (7.12) \]

is a standard Brownian motion under the probability measure \( \mathbb{P}^* \) defined by

\[ \frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp \left( -\frac{\mu - r}{\sigma} B_T - \frac{(\mu - r)^2}{2\sigma^2} T \right). \quad (7.13) \]

Hence by Proposition 7.1 the discounted price process \((\tilde{S}_t)_{t \in \mathbb{R}^+}\) solution of

\[ d\tilde{S}_t = (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t dB_t = \sigma \tilde{S}_t d\widehat{B}_t, \quad t \in \mathbb{R}^+, \quad (7.14) \]

is a martingale under \( \mathbb{P}^* \), therefore \( \mathbb{P}^* \) is a risk-neutral probability measure, and we obviously have \( \mathbb{P} = \mathbb{P}^* \) when \( \mu = r \).

In the sequel, we consider probability measures \( Q \) that are equivalent to \( \mathbb{P} \) in the sense that they share the same events of zero probability.

**Definition 7.4.** A probability measure \( Q \) on \((\Omega, \mathcal{F})\) is said to be equivalent to another probability measure \( \mathbb{P} \) when

\[ Q(A) = 0 \quad \text{if and only if} \quad \mathbb{P}(A) = 0, \quad \text{for all} \quad A \in \mathcal{F}. \]

Note that when \( Q \) is defined by (7.8), it is equivalent to \( \mathbb{P} \) if and only if \( F > 0 \) with \( \mathbb{P} \)-probability one.

### 7.4 Pricing by the Martingale Method

In this section we give the expression of the Black-Scholes price using expectations of discounted payoffs.

Recall that according to the first fundamental theorem of asset pricing Theorem 5.8, a continuous market is without arbitrage opportunities if and only if there exists (at least) an equivalent risk-neutral probability measure \( \mathbb{P}^* \) under which the discounted price process

\[ \tilde{S}_t := e^{-rt} S_t, \quad t \in \mathbb{R}^+, \]

is a martingale under \( \mathbb{P}^* \).
Martingale Approach to Pricing and Hedging

is a martingale under $\mathbb{P}^*$. In addition, when the risk-neutral probability measure is unique, the market is said to be complete.

The equation
\[ \frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \quad t \in \mathbb{R}_+, \]
satisfied by the price process $(S_t)_{t \in \mathbb{R}_+}$ can be rewritten using (7.12) as
\[ \frac{dS_t}{S_t} = r dt + \sigma d\tilde{B}_t, \quad t \in \mathbb{R}_+, \tag{7.15} \]
with the solution
\[ S_t = S_0 e^{\mu t + \sigma B_t - \sigma^2 t/2} = S_0 e^{rt + \sigma \tilde{B}_t - \sigma^2 t/2}, \quad t \in \mathbb{R}_+. \]

By the discounting Lemma 5.14, we have
\[ d\tilde{S}_t = (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t dB_t \]
\[ = \sigma \tilde{S}_t \left( \frac{\mu - r}{\sigma} dt + dB_t \right) \]
\[ = \sigma \tilde{S}_t d\tilde{B}_t, \quad t \in \mathbb{R}_+, \tag{7.16} \]
hence the discounted price process
\[ \tilde{S}_t := e^{-rt} S_t \]
\[ = S_0 e^{(\mu - r)t + \sigma B_t - \sigma^2 t/2} \]
\[ = S_0 e^{\sigma \tilde{B}_t - \sigma^2 t/2}, \quad t \in \mathbb{R}_+, \]
is a martingale under the probability measure $\mathbb{P}^*$ defined by (7.13). We note that $\mathbb{P}^*$ is a risk-neutral probability measure equivalent to $\mathbb{P}$, also called martingale measure, whose existence and uniqueness ensure absence of arbitrage and completeness according to Theorems 5.8 and 5.12.

Therefore, by Lemma 5.15 the discounted value $\tilde{V}_t$ of a self-financing portfolio can be written as
\[ \tilde{V}_t = \tilde{V}_0 + \int_0^t \xi_u d\tilde{S}_u \]
\[ = \tilde{V}_0 + \sigma \int_0^t \xi_u \tilde{S}_u d\tilde{B}_u, \quad t \in \mathbb{R}_+, \]
and by Proposition 7.1 it becomes a martingale under $\mathbb{P}^*$.

As in Chapter 3, the value $V_t$ at time $t$ of a self-financing portfolio strategy $(\xi_t)_{t \in [0,T]}$ hedging an attainable claim payoff $C$ will be called an arbitrage
price of the claim payoff $C$ at time $t$ and denoted by $\pi_t(C)$, $t \in [0, T]$. Arbitrage prices can be used to ensure that financial derivatives are “marked” at their fair value (“mark to market”).

**Theorem 7.5.** Let $(\xi_t, \eta_t)_{t \in [0,T]}$ be a portfolio strategy with value

$$V_t = \eta_t A_t + \xi_t S_t, \quad t \in [0, T],$$

and let $C$ be a contingent claim payoff, such that

(i) $(\xi_t, \eta_t)_{t \in [0,T]}$ is a self-financing portfolio, and

(ii) $(\xi_t, \eta_t)_{t \in [0,T]}$ hedges the claim payoff $C$, i.e. we have $V_T = C$.

Then the arbitrage price of the claim payoff $C$ is given by the portfolio value

$$\pi_t(C) = V_t = e^{-(T-t)r} \mathbb{E}^*[C \mid \mathcal{F}_t], \quad 0 \leq t \leq T,$$

where $\mathbb{E}^*$ denotes expectation under the risk-neutral probability measure $\mathbb{P}^*$.

**Proof.** Since the portfolio strategy $(\xi_t, \eta_t)_{t \in \mathbb{R}_+}$ is self-financing, by Lemma 5.15 and (7.16) the discounted portfolio value $\tilde{V}_t = e^{-rt}V_t$ satisfies

$$\tilde{V}_t = V_0 + \int_0^t \xi_u d\tilde{S}_u = \tilde{V}_0 + \sigma \int_0^t \xi_u \tilde{S}_u d\tilde{B}_u, \quad t \in \mathbb{R}_+,$$

which is a martingale under $\mathbb{P}^*$ from Proposition 7.1, hence

$$\tilde{V}_t = \mathbb{E}^* [\tilde{V}_T \mid \mathcal{F}_t]$$

$$= e^{-rT} \mathbb{E}^*[V_T \mid \mathcal{F}_t]$$

$$= e^{-rT} \mathbb{E}^*[C \mid \mathcal{F}_t],$$

which implies

$$V_t = e^{rt}\tilde{V}_t = e^{-(T-t)r} \mathbb{E}^*[C \mid \mathcal{F}_t], \quad 0 \leq t \leq T.$$ 

\[ \square \]

**Black-Scholes PDE for vanilla options by the martingale method**

The martingale method can be used to recover the Black-Scholes PDE of Proposition 6.1. As the process $(S_t)_{t \in \mathbb{R}_+}$ has the Markov property, see Section 4.5, § V-6 of Protter (2004) and Definition 7.15 below, the value

$$V_t = e^{-(T-t)r} \mathbb{E}^*[\phi(S_T) \mid \mathcal{F}_t]$$

$$= e^{-(T-t)r} \mathbb{E}^*[\phi(S_T) \mid S_t], \quad 0 \leq t \leq T,$$

of the portfolio at time $t \in [0, T]$ can be written from (7.17) as a function

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of $t$ and $S_t$, $0 \leq t \leq T$.

**Proposition 7.6.** Assume that $\phi$ is a Lipschitz payoff function, and that
$(S_t)_{t \in \mathbb{R}^+} = \left(S_0 e^{\sigma B_t + (r-\sigma^2)t/2}\right)_{t \in \mathbb{R}^+}$ is a geometric Brownian motion. Then the function $C(t,x)$ defined in (7.18) is in $C^{1,2}([0,T] \times \mathbb{R}^+)$ and solves the Black-Scholes PDE

$$
\begin{align*}
&C(t,x) = \frac{\partial C}{\partial t}(t,x) + rx \frac{\partial C}{\partial x}(t,x) + \frac{1}{2} x^2 \sigma^2 \frac{\partial^2 C}{\partial x^2}(t,x) \\
&C(T,x) = \phi(x), \quad x > 0.
\end{align*}
$$

**Proof.** It can be checked by integrations by parts that the function $C(t,x)$ defined by

$$
C(t,S_t) = e^{-(T-t)r} \mathbb{E}^* [\phi(S_T) \mid S_t] = e^{-(T-t)r} \mathbb{E}^* [\phi(xS_T/S_t)]_{x=S_t},
$$

$0 \leq t \leq T$, is in $C^{1,2}([0,T] \times \mathbb{R}^+)$ when $\phi$ is a Lipschitz function, from the properties of the lognormal distribution of $S_T$. We note that by (4.24), the application of Itô’s formula Theorem 4.23 to $V_t = C(t,S_t)$ and (7.15) leads to

$$
d(e^{-rt}C(t,S_t)) = -r e^{-rt}C(t,S_t)dt e^{-rt}dC(t,S_t)
$$

$$
= -r e^{-rt}C(t,S_t)dt + e^{-rt} \frac{\partial C}{\partial t}(t,S_t)dt \\
+ e^{-rt} \frac{\partial C}{\partial x}(t,S_t) dS_t + \frac{1}{2} e^{-rt}(dS_t)^2 \frac{\partial^2 C}{\partial x^2}(t,S_t)
$$

$$
= -r e^{-rt}C(t,S_t)dt + e^{-rt} \frac{\partial C}{\partial t}(t,S_t)dt \\
+ v_t e^{-rt} \frac{\partial C}{\partial x}(t,S_t) dt + u_t e^{-rt} \frac{\partial C}{\partial x}(t,S_t) d\tilde{B}_t + \frac{1}{2} e^{-rt} u_t^2 \frac{\partial^2 C}{\partial x^2}(t,S_t)dt
$$

$$
= -r e^{-rt}C(t,S_t)dt + e^{-rt} \frac{\partial C}{\partial t}(t,S_t)dt \\
+ r S_t e^{-rt} \frac{\partial C}{\partial x}(t,S_t) dt + \frac{1}{2} e^{-rt} \sigma^2 S_t^2 \frac{\partial^2 C}{\partial x^2}(t,S_t)dt + \sigma e^{-rt} S_t \frac{\partial C}{\partial x}(t,S_t) d\tilde{B}_t.
$$

By Lemma 5.15 and Proposition 7.1, the discounted price $\tilde{V}_t = e^{-rt}C(t,S_t)$ of a self-financing hedging portfolio is a martingale under the risk-neutral probability measure $\mathbb{P}^*$, therefore from e.g. Corollary II-6-1, page 72 of Protter (2004), all terms in $dt$ should vanish in the above expression of

$$
d(e^{-rt}g(t,S_t)) = -r e^{-rt}g(t,S_t)dt + e^{-rt}dg(t,S_t),
$$

which shows that
and leads to the Black-Scholes PDE
\[ rC(t, x) = \frac{\partial C}{\partial t}(t, x) + r x \frac{\partial C}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 C}{\partial x^2}(t, x), \quad x > 0. \]

**Forward contracts**

The long forward contract with payoff \( C = S_T - K \) is priced as
\[ V_t = e^{-(T-t)r} \mathbb{E}^* [S_T - K | \mathcal{F}_t] \]
\[ = e^{-(T-t)r} \mathbb{E}^* [S_T | \mathcal{F}_t] - K e^{-(T-t)r} \]
\[ = S_t - K e^{-(T-t)r}, \quad 0 \leq t \leq T, \]

which recovers the Black-Scholes PDE solution (6.8), i.e.
\[ g(t, x) = x - K e^{-(T-t)r}, \quad x > 0, \quad t \in [0, T]. \]

**European call options**

In the case of European call options with payoff function \( \phi(x) = (x - K)^+ \) we recover the Black-Scholes formula (6.10), cf. Proposition 6.11, by a probabilistic argument.

**Proposition 7.7.** The price at time \( t \in [0, T] \) of the European call option with strike price \( K \) and maturity \( T \) is given by
\[ C(t, S_t) = e^{-(T-t)r} \mathbb{E}^* [(S_T - K)^+ | \mathcal{F}_t] \]
\[ = S_t \Phi(d_+(T-t)) - K e^{-(T-t)r} \Phi(d_-(T-t)), \quad 0 \leq t \leq T, \]

with
\[
\begin{align*}
  d_+(T-t) &:= \frac{\log(x/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}, \\
  d_-(T-t) &:= \frac{\log(x/K) + (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}, \quad 0 \leq t < T,
\end{align*}
\]
where “log” denotes the natural logarithm “ln” and $\Phi$ is the standard Gaussian Cumulative Distribution Function.

**Proof.** The proof of Proposition 7.7 is a consequence of (7.17) and Lemma 7.8 below. Using the relation

$$S_T = S_t e^{(T-t)r + (\hat{B}_T - \hat{B}_t)\sigma - (T-t)\sigma^2 / 2}, \quad 0 \leq t \leq T,$$

by Theorem 7.5 the value at time $t \in [0, T]$ of the portfolio hedging $C$ is given by

$$V_t = e^{- (T-t)r} \mathbb{E}^*[C \mid \mathcal{F}_t]$$

where

$$m(x) := (T-t)r - \frac{\sigma^2}{2} (T-t) + \log x$$

and

$$X := (\hat{B}_T - \hat{B}_t)\sigma \sim \mathcal{N}(0, (T-t)\sigma^2)$$

is a centered Gaussian random variable with variance

$$\text{Var}[X] = \text{Var}[(\hat{B}_T - \hat{B}_t)\sigma] = \sigma^2 \text{Var}[\hat{B}_T - \hat{B}_t] = (T-t)\sigma^2$$

under $\mathbb{P}^*$. Hence by Lemma 7.8 below we have

$$C(t, S_t) = V_t$$

$$= e^{- (T-t)r} \mathbb{E}^*[(e^{m(x)} + X - K)^+]_{x=S_t}$$

$$= e^{- (T-t)r} e^{m(S_t) + \sigma^2(T-t)/2} \Phi\left(\frac{m(S_t) - \log K}{\sigma}\right)$$

$$- K e^{- (T-t)r} \Phi\left(\frac{m(S_t) - \log K}{\sigma}\right)$$

$$= S_t \Phi\left(\frac{m(S_t) - \log K}{\sigma}\right) - K e^{- (T-t)r} \Phi\left(\frac{m(S_t) - \log K}{\sigma}\right)$$

$$= S_t \Phi(d_+(T-t)) - K e^{- (T-t)r} \Phi(d_-(T-t)),$$

$$0 \leq t \leq T.$$

Relation (7.19) can also be written as

$$\text{...}$$
Let $Call-put$ parity

Proof.

Lemma 7.8. Let $X \sim N(0, v^2)$ be a centered Gaussian random variable with variance $v^2 > 0$. We have

$$\mathbb{E}[(e^{m+X} - K)^+] = e^{m+v^2/2} \Phi(v + (m - \log K)/v) - K \Phi((m - \log K)/v).$$

Proof. We have

$$\mathbb{E}[(e^{m+X} - K)^+] = \frac{1}{\sqrt{2\pi v^2}} \int_{-\infty}^{\infty} (e^{m+x} - K) e^{-x^2/(2v^2)} dx$$

$$= \frac{1}{\sqrt{2\pi v^2}} \int_{-m+\log K}^{\infty} (e^{m+x} - K) e^{-x^2/(2v^2)} dx$$

$$\frac{e^m}{\sqrt{2\pi v^2}} \int_{-m+\log K}^{\infty} e^{x-x^2/(2v^2)} dx - \frac{K}{\sqrt{2\pi v^2}} \int_{-m+\log K}^{\infty} e^{-x^2/(2v^2)} dx$$

$$= \frac{e^{m+v^2/2}}{\sqrt{2\pi v^2}} \int_{-m+\log K}^{\infty} e^{-(x^2-x)/2v^2} dx - \frac{K}{\sqrt{2\pi v^2}} \int_{-m+\log K}^{\infty} e^{-x^2/2v^2} dx$$

$$= \frac{e^{m+v^2/2}}{\sqrt{2\pi v^2}} \int_{-\infty}^{-m+\log K} e^{-y^2/(2v^2)} dy - K \Phi((m - \log K)/v)$$

$$= e^{m+v^2/2} \Phi(v + (m - \log K)/v) - K \Phi((m - \log K)/v).$$

Call-put parity

Let

$$P(t, S_t) := e^{-(T-t)r} \mathbb{E}^*[(K - S_T)^+ | \mathcal{F}_t], \quad 0 \leq t \leq T,$$

denote the price of the put option with strike price $K$ and maturity $T$.

Proposition 7.9. Call-put parity. We have the relation

$$C(t, S_t) - P(t, S_t) = S_t - e^{-(T-t)r} K$$

(7.21)

between the Black-Scholes prices of call and put options, in terms of the forward contract price $S_t - K e^{-(T-t)r}$.

Proof. From Theorem 7.5 we have

$$C(t, S_t) - P(t, S_t)$$

$$= e^{-(T-t)r} \mathbb{E}^*[(S_T - K)^+ | \mathcal{F}_t] - e^{-(T-t)r} \mathbb{E}^*[(K - S_T)^+ | \mathcal{F}_t]$$

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Using the call-put parity Relation (7.21) we can recover the European put option price (6.10) from the European call option price (6.10)-(7.19).

**Proposition 7.10.** The price at time $t \in [0, T]$ of the European put option with strike price $K$ and maturity $T$ is given by

$$P(t, S_t) = e^{-(T-t)r} \mathbb{E}^*[K - S_T]^+ | \mathcal{F}_t]$$

with

$$d_+(T-t) := \frac{\log(x/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}},$$

$$d_-(T-t) := \frac{\log(x/K) + (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}, \quad 0 \leq t < T,$$

where “log” denotes the natural logarithm “ln” and $\Phi$ is the standard Gaussian Cumulative Distribution Function.

**Proof.** By the call-put parity (7.21), we have

$$P(t, S_t) = C(t, S_t) - S_t + e^{-(T-t)r}K$$

$$= S_t \Phi(d_+(T-t)) + e^{-(T-t)r}K - S_t - e^{-(T-t)r}K \Phi(d_+(T-t))$$

$$= -S_t (1 - \Phi(d_+(T-t))) + e^{-(T-t)r}K (1 - \Phi(d_+(T-t)))$$

$$= -S_t \Phi(-d_+(T-t)) + e^{-(T-t)r}K \Phi(-d_+(T-t)).$$

□
7.5 Hedging by the Martingale Method

Hedging exotic options

In the next Proposition 7.11 we compute a self-financing hedging strategy leading to an arbitrary square-integrable random claim payoff $C \in L^2(\Omega)$ of an exotic option admitting a stochastic integral decomposition of the form

$$ C = \mathbb{E}^*[C] + \int_0^T \zeta_t \, d\tilde{B}_t, \quad (7.22) $$

where $(\zeta_t)_{t \in [0,T]}$ is a square-integrable adapted process. Consequently, the mathematical problem of finding the stochastic integral decomposition (7.22) of a given random variable has important applications in finance. The process $(\zeta_t)_{t \in [0,T]}$ can be computed using the Malliavin gradient on the Wiener space, see e.g. Nunno et al. (2009) or § 8.2 of Privault (2009).

Simple examples of stochastic integral decompositions include the relations

$$ (B_T)^2 = T + 2 \int_0^T B_t \, dB_t, $$

cf. Exercise 7.1, and

$$ (B_T)^3 = 3 \int_0^T (T - t + B_t^2) \, dB_t, $$

see Exercise 4.11. In the sequel, recall that the risky asset follows the equation

$$ \frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \quad t \in \mathbb{R}_+, \quad S_0 > 0, $$

and by (7.14), the discounted asset price $\tilde{S}_t := e^{-rt}S_t$

$$ d\tilde{S}_t = \sigma \tilde{S}_t d\tilde{B}_t, \quad t \in \mathbb{R}_+, \quad \tilde{S}_0 = S_0 > 0, \quad (7.23) $$

where $(\tilde{B}_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$. The following proposition applies to arbitrary square-integrable payoff functions, in particular it covers exotic and path-dependent options.

**Proposition 7.11.** Consider a random claim payoff $C \in L^2(\Omega)$ and the process $(\zeta_t)_{t \in [0,T]}$ given by (7.22), and let

$$ \xi_t = \frac{e^{-(T-t)r}}{\sigma S_t} \zeta_t, \quad (7.24) $$

$$ \eta_t = \frac{e^{-(T-t)r}}{A_t} \mathbb{E}^*[C \mid \mathcal{F}_t] - \xi_t S_t, \quad 0 \leq t \leq T. \quad (7.25) $$

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Then the portfolio allocation \((\xi_t, \eta_t)_{t \in [0,T]}\) is self-financing, and letting

\[
V_t = \eta_tA_t + \xi_tS_t, \quad 0 \leq t \leq T, \quad (7.26)
\]

we have

\[
V_t = e^{-(T-t)r} \mathbb{E}^*[C | \mathcal{F}_t], \quad 0 \leq t \leq T. \quad (7.27)
\]

In particular we have

\[
V_T = C, \quad (7.28)
\]

i.e. the portfolio allocation \((\xi_t, \eta_t)_{t \in [0,T]}\) yields a hedging strategy leading to the claim payoff \(C\) at maturity, after starting from the initial value

\[
V_0 = e^{-rT} \mathbb{E}^*[C].
\]

**Proof.** Relation (7.27) follows from (7.25) and (7.26), and it implies

\[
V_0 = e^{-rT} \mathbb{E}^*[C] = \eta_0A_0 + \xi_0S_0
\]

at \(t = 0\), and (7.28) at \(t = T\). It remains to show that the portfolio strategy \((\xi_t, \eta_t)_{t \in [0,T]}\) is self-financing. By (7.22) and Proposition 7.1 we have

\[
V_t = \eta_tA_t + \xi_tS_t
\]

\[
= e^{-(T-t)r} \mathbb{E}^*[C | \mathcal{F}_t]
\]

\[
= e^{-(T-t)r} \mathbb{E}^* \left[ \mathbb{E}^*[C] + \int_0^T \zeta_udB_u | \mathcal{F}_t \right]
\]

\[
= e^{-(T-t)r} \left( \mathbb{E}^*[C] + \int_0^t \zeta_udB_u \right)
\]

\[
= e^{rt}V_0 + e^{-(T-t)r} \int_0^t \zeta_udB_u
\]

\[
= e^{rt}V_0 + \sigma \int_0^t \xi_uS_u e^{(t-u)r}d\tilde{B}_u
\]

\[
= e^{rt}V_0 + \sigma e^{rt} \int_0^t \xi_u\tilde{S}_u d\tilde{B}_u.
\]

By (7.23) this shows that the portfolio strategy \((\xi_t, \eta_t)_{t \in [0,T]}\) given by (7.24)-(7.25) and its discounted portfolio value \(\tilde{V}_t := e^{-rt}V_t\) satisfy

\[
\tilde{V}_t = V_0 + \int_0^t \xi_u\tilde{S}_u, \quad 0 \leq t \leq T,
\]

which implies that \((\xi_t, \eta_t)_{t \in [0,T]}\) is self-financing by Lemma 5.15. □

The above proposition shows that there always exists a hedging strategy starting from

\[
V_0 = \mathbb{E}^*[C] e^{-rT}.
\]

In addition, since there exists a hedging strategy leading to
$V_T = e^{-rT} C,$

then $(\tilde{V}_t)_{t \in [0,T]}$ is necessarily a martingale, with

$$\tilde{V}_t = \mathbb{E}^*[\tilde{V}_T | \mathcal{F}_t] = e^{-rT} \mathbb{E}^*[C | \mathcal{F}_t], \quad 0 \leq t \leq T,$$

and initial value

$$\tilde{V}_0 = \mathbb{E}^*[\tilde{V}_T] = e^{-rT} \mathbb{E}^*[C].$$

### Hedging vanilla options

In practice, the hedging problem can now be reduced to the computation of the process $(\zeta_t)_{t \in [0,T]}$ appearing in (7.22). This computation, called Delta hedging, can be performed by the application of the Itô formula and the Markov property, see e.g. Protter (2001). The next lemma allows us to compute the process $(\zeta_t)_{t \in [0,T]}$ in case the payoff $C$ is of the form $C = \phi(S_T)$ for some function $\phi$.

**Lemma 7.12.** Assume that $\phi$ is a Lipschitz payoff function. Then the function $C(t, x)$ defined by

$$C(t, S_t) = \mathbb{E}^*[\phi(S_T) | S_t]$$

is in $C^{1,2}([0, T] \times \mathbb{R})$, and the stochastic integral decomposition

$$\phi(S_T) = \mathbb{E}^*[\phi(S_T)] + \int_0^T \zeta_t d\tilde{B}_t \quad (7.29)$$

is given by

$$\zeta_t = \sigma S_t \frac{\partial C}{\partial x}(t, S_t), \quad 0 \leq t \leq T. \quad (7.30)$$

In addition, the hedging strategy $(\xi_t)_{t \in [0,T]}$ satisfies

$$\xi_t = e^{-(T-t)r} \frac{\partial C}{\partial x}(t, S_t), \quad 0 \leq t \leq T. \quad (7.31)$$

**Proof.** It can be checked as in the proof of Proposition 7.6 the function $C(t, x)$ is in $C^{1,2}([0, T] \times \mathbb{R})$. Therefore, we can apply the Itô formula to the process

$$t \mapsto C(t, S_t) = \mathbb{E}^*[\phi(S_T) | \mathcal{F}_t],$$

which is a martingale from the tower property (23.38) of conditional expectations as in (7.38). From the fact that the finite variation term in the Itô formula vanishes when $(C(t, S_t))_{t \in [0,T]}$ is a martingale, (see e.g. Corollary II-6-1 page 72 of Protter (2004)), we obtain:

$$C(t, S_t) = C(0, S_0) + \sigma \int_0^t S_u \frac{\partial C}{\partial x}(u, S_u) d\tilde{B}_u, \quad 0 \leq t \leq T. \quad (7.32)$$
with \(C(0, S_0) = \mathbb{E}^* \left[ \phi(S_T) \right] \). Letting \(t = T\), we obtain (7.30) by uniqueness of the stochastic integral decomposition (7.29) of \(C = \phi(S_T)\). Finally, (7.31) follows from (7.24) and (7.30).

By (7.39) we also have

\[
\zeta_t = \sigma S_t \frac{\partial}{\partial x} \mathbb{E}^* [\phi(S_T) | S_t = x]_{x=S_t}
\]

\[= \sigma S_t \frac{\partial}{\partial x} \mathbb{E}^* \left[ \phi \left( \frac{S_T}{S_t} x \right) \right]_{x=S_t}, \quad 0 \leq t \leq T,
\]

hence

\[
\xi_t = \frac{1}{\sigma S_t} e^{-(T-t)r} \zeta_t
\]

\[= e^{-(T-t)r} \frac{\partial}{\partial x} \mathbb{E}^* \left[ \phi \left( \frac{S_T}{S_t} x \right) \right]_{x=S_t}, \quad 0 \leq t \leq T,
\]

which recovers the formula (6.3) for the Delta of a vanilla option. As a consequence we have \(\xi_t \geq 0\) and there is no short selling when the payoff function \(\phi\) is non-decreasing.

In the case of European options, the process \(\zeta\) can be computed via the next proposition which follows from Lemma 7.12 and the relation

\[
C(t, x) = \mathbb{E}^* \left[ f(x, S_t, T) \right], \quad 0 \leq t \leq T, \ x > 0.
\]

**Corollary 7.13.** Assume that \(C = (S_T - K)^+\). Then, for \(0 \leq t \leq T\) we have

\[
\zeta_t = \sigma S_t \mathbb{E}^* \left[ \frac{S_T}{S_t} \mathbb{1}_{[K, \infty)} \left( \frac{S_T}{S_t} x \right) \right]_{x=S_t}, \quad 0 \leq t \leq T,
\]

and

\[
\xi_t = e^{-(T-t)r} \mathbb{E}^* \left[ \frac{S_T}{S_t} \mathbb{1}_{[K, \infty)} \left( \frac{S_T}{S_t} x \right) \right]_{x=S_t}, \quad 0 \leq t \leq T.
\]

By evaluating the expectation (7.34) in Corollary 7.13 we can recover the formula (6.15) in Proposition 6.4 for the Delta of the European call option in the Black-Scholes model. In that sense, the next proposition provides another proof of the result of Proposition 6.4.

**Proposition 7.14.** The Delta of the European call option with payoff function \(f(x) = (x - K)^+\) is given by

\[
\xi_t = \Phi(d_+(T-t)) = \Phi \left( \frac{\log(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right), \quad 0 \leq t \leq T.
\]

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Proof. By Proposition 7.11 and Corollary 7.13, we have

\[ \xi_t = \frac{1}{\sigma S_t} e^{-(T-t)r} \zeta_t \]

\[ = e^{-(T-t)r} \mathbb{E}^* \left[ \frac{S_T}{S_t} 1_{[K,\infty)} \left( x S_T \right) \right]_{x=S_t} \]

\[ = e^{-(T-t)r} \times \mathbb{E}^* \left[ e^{(\hat{B}_T-\hat{B}_t)\sigma-(T-t)\sigma^2/2+(T-t)r} 1_{[K,\infty)} (x e^{(\hat{B}_T-\hat{B}_t)\sigma-(T-t)\sigma^2/2+(T-t)r}) \right]_{x=S_t} \]

\[ = \frac{1}{\sqrt{2(T-t)\pi}} \int_{(T-t)/\sigma+(T-t)r/\sigma+\sigma^{-1} \log(K/S_t)}^{\infty} e^{\sigma y-(T-t)\sigma^2/2-y^2/(2(T-t))} dy \]

\[ = \frac{1}{\sqrt{2(T-t)\pi}} \int_{d_-(T-t)/\sqrt{T-t}}^{\infty} e^{-(y-(T-t)\sigma)^2/(2(T-t))} dy \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{d_-(T-t)}^{\infty} e^{-(y-(T-t)\sigma)^2/2} dy \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{d_+(T-t)}^{-\infty} dy \]

\[ = \Phi(d_+(T-t)). \]

\[ \square \]

Proposition 7.14, combined with Proposition 7.7, shows that the Black-Scholes self-financing hedging strategy is to hold a (possibly fractional) quantity

\[ \xi_t = \Phi(d_+(T-t)) = \Phi \left( \frac{\log(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right) \geq 0 \quad (7.36) \]

of the risky asset, and to borrow a quantity

\[ -\eta_t = K e^{-rT} \Phi \left( \frac{\log(S_t/K) + (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right) \geq 0 \quad (7.37) \]

of the riskless (savings) account, see also Corollary 16.17 in Chapter 16.

As noted above, the result of Proposition 7.14 recovers (6.16) which is obtained by a direct differentiation of the Black-Scholes function as in (6.3) or (7.33).
Markovian semi-group

**Definition 7.15.** The Markov semi-group \((P_t)_{0 \leq t \leq T}\) associated to \((S_t)_{t \in [0,T]}\) is the mapping \(P_t\) defined on functions \(f \in C^2_b(\mathbb{R})\) as

\[ P_t f(x) := \mathbb{E}^*[f(S_t) \mid S_0 = x], \quad t \in \mathbb{R}_+. \]

By the Markov property and time homogeneity of \((S_t)_{t \in [0,T]}\) we also have

\[ P_t f(S_u) := \mathbb{E}^*[f(S_{t+u}) \mid \mathcal{F}_u] = \mathbb{E}^*[f(S_{t+u}) \mid S_u], \quad t, u \in \mathbb{R}_+, \]

and the semi-group \((P_t)_{0 \leq t \leq T}\) satisfies the composition property

\[ P_s P_t = P_t P_s = P_{s+t} = P_{t+s}, \quad s, t \in \mathbb{R}_+, \]

as we have, using the Markov property and the tower property (23.38) of conditional expectations as in (7.38),

\[ P_s P_t f(x) = \mathbb{E}^*[P_t f(S_s) \mid S_0 = x] \]
\[ = \mathbb{E}^*[\mathbb{E}^*[f(S_t) \mid S_0 = y]_{y=S_s} \mid S_0 = x] \]
\[ = \mathbb{E}^*[\mathbb{E}^*[f(S_{t+s}) \mid S_0 = x]_{y=S_s} \mid S_0 = x] \]
\[ = \mathbb{E}^*[\mathbb{E}^*[f(S_{t+s}) \mid \mathcal{F}_s] \mid S_0 = x] \]
\[ = \mathbb{E}^*[f(S_{t+s}) \mid \mathcal{F}_s \mid S_0 = x] \]
\[ = P_{t+s} f(x), \quad s, t \geq 0. \]

Similarly we can show that the process \((P_{T-t} f(S_t))_{t \in [0,T]}\) is an \(\mathcal{F}_t\)-martingale as in Example (7.1), i.e.:

\[ \mathbb{E}^*[P_{T-t} f(S_t) \mid \mathcal{F}_u] = \mathbb{E}^*[\mathbb{E}^*[f(S_T) \mid \mathcal{F}_t] \mid \mathcal{F}_u] \]
\[ = \mathbb{E}^*[f(S_T) \mid \mathcal{F}_u] \]
\[ = P_{T-u} f(S_u), \quad 0 \leq u \leq t \leq T, \quad (7.38) \]

and we have

\[ P_{t-u} f(x) = \mathbb{E}^*[f(S_t) \mid S_u = x] = \mathbb{E}^*[f \left(x \frac{S_t}{S_u}\right)], \quad 0 \leq u \leq t. \quad (7.39) \]

**Exercises**

**Exercise 7.1** (Exercise 6.1 continued). Consider a market made of a riskless asset priced \(A_t = A_0\) with zero interest rate, \(t \in \mathbb{R}_+\), and a risky asset whose price modeled by a standard Brownian motion as \(S_t = B_t, \ t \in \mathbb{R}_+\).
the vanilla option with payoff $C = (B_T)^2$, and deduce the solution of the Black-Scholes PDE of Exercise 6.1.

Exercise 7.2 Given the price process $(S_t)_{t \in \mathbb{R}_+}$ defined as

$$S_t := S_0 e^{\sigma B_t + (r - \sigma^2/2) t}, \quad t \in \mathbb{R}_+,$$

price the option with payoff function $\phi(S_T)$ by writing $e^{-rT} \mathbb{E}^* [\phi(S_T)]$ as an integral.

Exercise 7.3 Consider an asset price $(S_t)_{t \in \mathbb{R}_+}$ which is a martingale under the risk-neutral probability measure $\mathbb{P}^*$ in a market with interest rate $r = 0$, and let $\phi(x) = (x - K)^+$ be the (convex) European call payoff function.

Show that, for any two maturities $T_1 < T_2$ and $p, q \in [0, 1]$ such that $p + q = 1$, the price of the option on average with payoff $\phi(pS_{T_1} + qS_{T_2})$ is upper bounded by the price of the European call option with maturity $T_2$, i.e. show that

$$\mathbb{E}^* [\phi(pS_{T_1} + qS_{T_2})] \leq \mathbb{E}^* [\phi(S_{T_2})].$$

Hints:

i) For $\phi$ a convex function we have $\phi(px + qy) \leq p\phi(x) + q\phi(y)$ for any $x, y \in \mathbb{R}$ and $p, q \in [0, 1]$ such that $p + q = 1$.

ii) Any convex function $\phi(S_t)$ of a martingale $S_t$ is a submartingale.

Exercise 7.4 Consider a price process $(S_t)_{t \in \mathbb{R}_+}$ and a risk-neutral measure $\mathbb{P}^*$.

a) Does the European call option price $C(K) := e^{-rT} \mathbb{E}^*[ (S_T - K)^+]$ increase or decrease with the strike price $K$? Justify your answer.

b) Does the European put option price $C(K) := e^{-rT} \mathbb{E}^*[ (K - S_T)^+]$ increase or decrease with the strike price $K$? Justify your answer.

Exercise 7.5 Consider an underlying asset price process $(S_t)_{t \in \mathbb{R}_+}$.

a) Show that the price at time $t$ of the European call option with strike price $K$ and maturity $T$ is lower bounded by the positive part $(S_t - Ke^{-(T-t)r})^+$ of the corresponding forward contract price, i.e.

$$e^{-(T-t)r} \mathbb{E}^* [(S_T - K)^+ | \mathcal{F}_t] \geq (S_t - Ke^{-(T-t)r})^+, \quad 0 \leq t \leq T.$$

b) Show that the price at time $t$ of the European put option with strike price $K$ and maturity $T$ is lower bounded by $Ke^{-(T-t)r} - S_t$, i.e.

$$e^{-(T-t)r} \mathbb{E}^* [(K - S_T)^+ | \mathcal{F}_t] \geq (Ke^{-(T-t)r} - S_t)^+, \quad 0 \leq t \leq T.$$

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Exercise 7.6  The following two graphs describe the payoff functions $\phi$ of *bull spread* and *bear spread* options with payoff $\phi(S_N)$ on an underlying asset priced $S_N$ at maturity time $N$.

(i) Bull spread payoff.  
(ii) Bear spread payoff.  

Fig. 7.3: Payoff functions of bull spread and bear spread options.

(a) Show that in each case (i) and (ii) the corresponding option can be realized by purchasing and/or short selling standard European call and put options with strike prices to be specified.

(b) Price the bull spread option in cases (i) and (ii).

*Hint:* An option with payoff $\phi(S_N)$ is priced $(1 + r)^{-N} \mathbb{E}^* [\phi(S_N)]$ at time 0. The payoff of the European call (resp. put) option with strike price $K$ is $(S_N - K)^+$, resp. $(K - S_N)^+$.

Exercise 7.7 Butterfly options. A butterfly option is designed to deliver a limited payoff when the future volatility of the underlying asset is expected to be low. The payoff function of a butterfly option is plotted in Figure 7.4, with $K_1 := 50$ and $K_2 := 150$.

Fig. 7.4: Butterfly payoff function.

(a) Show that the butterfly option can be realized by purchasing and/or issuing standard European call or put options with strike prices to be specified.

(b) Does the hedging strategy of the butterfly option involve holding or shorting the underlying stock?
Hints: Recall that an option with payoff $\phi(S_N)$ is priced in discrete time as $(1+r)^{-N} \mathbb{E}^* [\phi(S_N)]$ at time 0. The payoff of the European call (resp. put) option with strike price $K$ is $(S_N - K)^+$, resp. $(K - S_N)^+$. 

Exercise 7.8 Forward contracts revisited. Consider a risky asset whose price $S_t$ is given by $S_t = S_0 e^{\sigma B_t + rt - \sigma^2 t/2}$, $t \in \mathbb{R}_+$, where $(B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion. Consider a forward contract with maturity $T$ and payoff $S_T - \kappa$.

a) Compute the price $C_t$ of this claim at any time $t \in [0, T]$.

b) Compute a hedging strategy for the option with payoff $S_T - \kappa$.

Exercise 7.9 Option pricing with dividends (Exercise 6.3 continued). Consider an underlying asset price process $(S_t)_{t \in \mathbb{R}_+}$ paying dividends at the continuous-time rate $\delta > 0$, and modeled as

$$dS_t = (\mu - \delta)S_t dt + \sigma S_t dB_t,$$

where $(B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion.

a) Show that as in Lemma 5.15, if $(\eta_t, \xi_t)_{t \in \mathbb{R}_+}$ is a portfolio strategy with value

$$V_t = \eta_t A_t + \xi_t S_t, \quad t \in \mathbb{R}_+,$$

where the dividend yield $\delta S_t$ per share is continuously reinvested in the portfolio, then the discounted portfolio value $\tilde{V}_t$ can be written as the stochastic integral

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \xi_u dS_u, \quad t \in \mathbb{R}_+,$$

b) Show that, as in Theorem 7.5, if $(\xi_t, \eta_t)_{t \in [0,T]}$ hedges the claim payoff $C$, i.e. if $V_T = C$, then the arbitrage price of the claim payoff $C$ is given by

$$\pi_t(C) = V_t = e^{-(T-t)r} \tilde{\mathbb{E}}[C \mid \mathcal{F}_t], \quad 0 \leq t \leq T,$$

where $\tilde{\mathbb{E}}$ denotes expectation under a suitably chosen risk-neutral probability measure $\tilde{P}$.

c) Compute the price at time $t \in [0, T]$ of a European call option in a market with dividend rate $\delta$ by the martingale method.

Exercise 7.10 Forward start options (Rubinstein (1991)). A forward start European call option is an option whose holder receives at time $T_1$ (e.g. your birthday) the value of a standard European call option at the money and with maturity $T_2 > T_1$. Price this birthday present at any time $t \in [0, T_1]$, i.e. compute the price.
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\[ e^{-(T_1-t)r} \mathbb{E}^* [ e^{-(T_2-T_1)r} \mathbb{E}^* \left[ (S_{T_2} - S_{T_1})^+ \mid \mathcal{F}_{T_1} \right] \mid \mathcal{F}_t] \]

at time \( t \in [0, T_1] \), of the forward start European call option using the Black-Scholes formula

\[
\text{Bl}(K, x, \sigma, r, T-t) = x \Phi \left( \frac{\log(x/K) + (r + \sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}} \right) - Ke^{-(T-t)r} \Phi \left( \frac{\log(x/K) + (r - \sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}} \right),
\]

for \( 0 \leq t < T \).

Exercise 7.11 Log-contracts. (Exercise 6.9 continued), see also Exercise 8.4. Consider the price process \((S_t)_{t \in [0, T]}\) given by

\[
\frac{dS_t}{S_t} = rdt + \sigma dB_t
\]

and a riskless asset valued \( A_t = A_0 e^{rt}, t \in [0, T] \), with \( r > 0 \). Compute the arbitrage price

\[
C(t, S_t) = e^{-(T-t)r} \mathbb{E}^* \left[ \log S_T \mid \mathcal{F}_t \right],
\]

at time \( t \in [0, T] \), of the log-contract with payoff \( \log S_T \).

Exercise 7.12 Power option. (Exercise 6.5 continued). Consider the price process \((S_t)_{t \in [0, T]}\) given by

\[
\frac{dS_t}{S_t} = rdt + \sigma dB_t
\]

and a riskless asset valued \( A_t = A_0 e^{rt}, t \in [0, T] \), with \( r > 0 \). In this problem, \((\eta_t, \xi_t)_{t \in [0, T]}\) denotes a portfolio strategy with value

\[
V_t = \eta_t A_t + \xi_t S_t, \quad 0 \leq t \leq T.
\]

a) Compute the arbitrage price

\[
C(t, S_t) = e^{-(T-t)r} \mathbb{E}^* \left[ |S_T|^2 \mid \mathcal{F}_t \right],
\]

at time \( t \in [0, T] \), of the power option with payoff \( |S_T|^2 \).

b) Compute a self-financing hedging strategy \((\eta_t, \xi_t)_{t \in [0, T]}\) hedging the claim payoff \( |S_T|^2 \).

Exercise 7.13 Bachelier (1900) model (Exercise 6.11 continued).

a) Consider the solution \((S_t)_{t \in \mathbb{R}^+}\) of the stochastic differential equation

\[
\frac{dS_t}{S_t} = \sigma dB_t
\]

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\[ dS_t = \alpha S_t dt + \sigma dB_t. \]

For which value \( \alpha_M \) of \( \alpha \) is the discounted price process \( \tilde{S}_t = e^{-rt}S_t \), \( 0 \leq t \leq T \), a martingale under \( \mathbb{P} \)?

b) For each value of \( \alpha \), build a probability measure \( \mathbb{P}_\alpha \) under which the discounted price process \( \tilde{S}_t = e^{-rt}S_t \), \( 0 \leq t \leq T \), is a martingale.

c) Compute the arbitrage price

\[ C(t, S_t) = e^{-(T-t)r} \mathbb{E}_\alpha [e^{S_T} \mid \mathcal{F}_t] \]

at time \( t \in [0, T] \) of the contingent claim with payoff \( \exp(S_T) \), and recover the result of Exercise 6.11.

d) Explicitly compute the portfolio strategy \((\eta_t, \xi_t)_{t \in [0,T]}\) that hedges the contingent claim with payoff \( \exp(S_T) \).

e) Check that this strategy is self-financing.

Exercise 7.14 Compute the arbitrage price

\[ C(t, S_t) = e^{-(T-t)r} \mathbb{E}_\alpha [(S_T)^2 \mid \mathcal{F}_t] \]

at time \( t \in [0, T] \) of the power option with payoff \((S_T)^2\) in the framework of the Bachelier (1900) model of Exercise 7.13.

Exercise 7.15 (Exercise 6.2 continued). Price the option with vanilla payoff \( C = \phi(S_T) \) using the noncentral Chi square probability density function (17.5) of the Cox et al. (1985) (CIR) model.

Exercise 7.16 Let \((B_t)_{t \in \mathbb{R}_+}\) be a standard Brownian motion generating a filtration \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\). Recall that for \( f \in C^2(\mathbb{R}_+ \times \mathbb{R}) \), Itô’s formula for \((B_t)_{t \in \mathbb{R}_+}\) reads

\[
 f(t, B_t) = f(0, B_0) + \int_0^t \frac{\partial f}{\partial s}(s, B_s) ds \\
 + \int_0^t \frac{\partial f}{\partial x}(s, B_s) dB_s + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s) ds.
\]

a) Let \( r \in \mathbb{R}, \sigma > 0 \), \( f(x, t) = e^{rt+\sigma x-\sigma^2 t/2} \), and \( S_t = f(t, B_t) \). Compute \( df(t, B_t) \) by Itô’s formula, and show that \( S_t \) solves the stochastic differential equation

\[ dS_t = rS_t dt + \sigma S_t dB_t, \]

where \( r > 0 \) and \( \sigma > 0 \).

b) Show that

\[
 \mathbb{E}[e^{\sigma B_T} \mid \mathcal{F}_t] = e^{\sigma B_t + (T-t)\sigma^2/2}, \quad 0 \leq t \leq T.
\]
Hint: Use the independence of increments of \((B_t)_{t \in [0,T]}\) in the time splitting decomposition

\[ B_T = (B_t - B_0) + (B_T - B_t), \]

and the Gaussian moment generating function \(\mathbb{E} \left[ e^{\alpha X} \right] = e^{\alpha^2 \eta^2 / 2}\) when \(X \sim \mathcal{N}(0, \eta^2)\).

c) Show that the process \((S_t)_{t \in \mathbb{R}^+}\) satisfies

\[ \mathbb{E} \left[ S_T \mid \mathcal{F}_t \right] = e^{(T-t)r} S_t, \quad 0 \leq t \leq T. \]

d) Let \(C = S_T - K\) denote the payoff of a forward contract with exercise price \(K\) and maturity \(T\). Compute the discounted expected payoff

\[ V_t := e^{-(T-t)r} \mathbb{E} \left[ C \mid \mathcal{F}_t \right]. \]

e) Find a self-financing portfolio strategy \((\xi_t, \eta_t)_{t \in \mathbb{R}^+}\) such that

\[ V_t = \xi_t S_t + \eta_t A_t, \quad 0 \leq t \leq T, \]

where \(A_t = A_0 e^{rt}\) is the price of a riskless asset with fixed interest rate \(r > 0\). Show that it recovers the result of Exercise 6.7-(c).

f) Show that the portfolio allocation \((\xi_t, \eta_t)_{t \in [0,T]}\) found in Question (e) hedges the payoff \(C = S_T - K\) at time \(T\), i.e. show that \(V_T = C\).

Exercise 7.17 Binary options. Consider a price process \((S_t)_{t \in \mathbb{R}^+}\) given by

\[ \frac{dS_t}{S_t} = rdt + \sigma dB_t, \quad S_0 = 1, \]

under the risk-neutral probability measure \(\mathbb{P}^*\). A binary (or digital) call, resp. put, option is a contract with maturity \(T\), strike price \(K\), and payoff

\[ C_d := \begin{cases} \begin{array}{ll} \$1 & \text{if } S_T \geq K, \quad & \text{resp. } P_d := \begin{cases} \begin{array}{ll} \$1 & \text{if } S_T \leq K, \quad & 0 & \text{if } S_T < K, \quad & 0 & \text{if } S_T > K. \end{array} \end{cases} \end{array} \end{cases} \]

Recall that the prices \(\pi_t(C_d)\) and \(\pi_t(P_d)\) at time \(t\) of the binary call and put options are given by the discounted expected payoffs

\[ \pi_t(C_d) = e^{-(T-t)r} \mathbb{E} \left[ C_d \mid \mathcal{F}_t \right] \quad \text{and} \quad \pi_t(P_d) = e^{-(T-t)r} \mathbb{E} \left[ P_d \mid \mathcal{F}_t \right]. \]

(7.40)

a) Show that the payoffs \(C_d\) and \(P_d\) can be rewritten as

\[ C_d = 1_{[K,\infty)}(S_T) \quad \text{and} \quad P_d = 1_{[0,K]}(S_T). \]
b) Using Relation (7.40), Question (a), and the relation
\[ \mathbb{E} [1_{[K,\infty)}(S_T) \mid S_t = x] = P^*(S_T \geq K \mid S_t = x), \]
show that the price \( \pi_t(C_d) \) is given by
\[ \pi_t(C_d) = C_d(t, S_t), \]
where \( C_d(t, x) \) is the function defined by
\[ C_d(t, x) := e^{-(T-t)r}P^*(S_T \geq K \mid S_t = x). \]

c) Using the results of Exercise 5.9-(d) and of Question (b), show that the price \( \pi_t(C_d) \) of the binary call option is given by
\[ C_d(t, x) = e^{-(T-t)r} \Phi \left( \frac{(r - \sigma^2/2)(T-t) + \log(x/K)}{\sigma \sqrt{T-t}} \right) = e^{-(T-t)r} \Phi(d_-(T-t)), \]
where
\[ d_-(T-t) = \frac{(r - \sigma^2/2)(T-t) + \log(S_t/K)}{\sigma \sqrt{T-t}}. \]

d) Assume that the binary option holder is entitled to receive a “return amount” \( \alpha \in [0, 1] \) in case the underlying asset price ends out of the money at maturity. Compute the price at time \( t \in [0, T] \) of this modified contract.

e) Using Relation (7.40) and Question (a), prove the call-put parity relation
\[ \pi_t(C_d) + \pi_t(P_d) = e^{-(T-t)r}, \quad 0 \leq t \leq T. \] (7.41)
If needed, you may use the fact that \( P^*(S_T = K) = 0. \)
f) Using the results of Questions (e) and (c), show that the price \( \pi_t(P_d) \) of the binary put option is given by
\[ \pi_t(P_d) = e^{-(T-t)r} \Phi(-d_-(T-t)). \]
g) Using the result of Question (c), compute the Delta
\[ \xi_t := \frac{\partial C_d}{\partial x}(t, S_t) \]
of the binary call option. Does the Black-Scholes hedging strategy of such a call option involve short selling? Why?
h) Using the result of Question (f), compute the Delta
\[ \xi_t := \frac{\partial P_d}{\partial x}(t, S_t) \]
of the binary put option. Does the Black-Scholes hedging strategy of such a put option involve short selling? Why?

Exercise 7.18 Computation of Greeks. Consider an underlying asset whose price \((S_t)_{t \in \mathbb{R}^+}\) is given by a stochastic differential equation of the form
\[
dS_t = rS_t dt + \sigma(S_t) dW_t,
\]
where \(\sigma(x)\) is a Lipschitz coefficient, and an option with payoff function \(\phi\) and price
\[
C(x, T) = e^{-rT} \mathbb{E} \left[ \phi(S_T) \mid S_0 = x \right],
\]
where \(\phi(x)\) is a twice continuously differentiable \((C^2)\) function, with \(S_0 = x\). Using the Itô formula, show that the sensitivity
\[
\text{Theta}_T = \frac{\partial}{\partial T} \left( e^{-rT} \mathbb{E} \left[ \phi(S_T) \mid S_0 = x \right] \right)
\]
of the option price with respect to maturity \(T\) can be expressed as
\[
\text{Theta}_T = -r e^{-rT} \mathbb{E} \left[ \phi(S_T) \mid S_0 = x \right] + r e^{-rT} \mathbb{E} \left[ S_t \phi'(S_T) \mid S_0 = x \right] + \frac{1}{2} e^{-rT} \mathbb{E} \left[ \phi''(S_T) \sigma^2(S_T) \mid S_0 = x \right].
\]

Problem 7.19 Chooser options. In this problem we denote by \(C(t, S_t, K, T)\), resp. \(P(t, S_t, K, T)\), the price at time \(t\) of the European call, resp. put, option with strike price \(K\) and maturity \(T\), on an underlying asset priced \(S_t = S_0 e^{\sigma B_t + rt - \sigma^2 t/2}, t \in \mathbb{R}^+\), where \((B_t)_{t \in \mathbb{R}^+}\) is a standard Brownian motion under the risk-neutral probability measure.

a) Prove the call-put parity formula
\[
C(t, S_t, K, T) - P(t, S_t, K, T) = S_t - K e^{-(T-t)r}, \quad 0 \leq t \leq T. \quad (7.42)
\]

b) Consider an option contract with maturity \(T\), which entitles its holder to receive at time \(T\) the value of the European put option with strike price \(K\) and maturity \(U > T\).
Write down the price this contract at time \(t \in [0, T]\) using a conditional expectation under the risk-neutral probability measure \(\mathbb{P}^*\).

c) Consider now an option contract with maturity \(T\), which entitles its holder to receive at time \(T\) either the value of a European call option or a European put option, whichever is higher. The European call and put options have same strike price \(K\) and same maturity \(U > T\).

Show that at maturity \(T\), the payoff of this contract can be written as
\[
P(T, S_T, K, U) + \text{Max}\left(0, S_T - K e^{-\left(U - T\right)t}\right).
\]

**Hint:** Use the call-put parity formula (7.42).

d) Price the contract of Question (c) at any time \( t \in [0, T] \) using the call and put option pricing functions \( C(t, x, K, T) \) and \( P(t, x, K, U) \).

e) Using the Black-Scholes formula, compute the self-financing hedging strategy \( (\xi_t, \eta_t)_{t \in [0, T]} \) with portfolio value

\[
V_t = \xi_t S_t + \eta_t e^{rt}, \quad 0 \leq t \leq T,
\]

for the option contract of Question (c).

f) Consider now an option contract with maturity \( T \), which entitles its holder to receive at time \( T \) the value of either a European call or a European put option, whichever is lower. The two options have same strike price \( K \) and same maturity \( U > T \).

Show that the payoff of this contract at maturity \( T \) can be written as

\[
C(T, S_T, K, U) - \text{Max}\left(0, S_T - K e^{-\left(U - T\right)t}\right).
\]

g) Price the contract of Question (f) at any time \( t \in [0, T] \).

h) Using the Black-Scholes formula, compute the self-financing hedging strategy \( (\xi_t, \eta_t)_{t \in [0, T]} \) with portfolio value

\[
V_t = \xi_t S_t + \eta_t e^{rt}, \quad 0 \leq t \leq T,
\]

for the option contract of Question (f).

i) Give the price and hedging strategy of the contract that yields the sum of the payoffs of Questions (c) and (f).

j) What happens when \( U = T \)? Give the payoffs of the contracts of Questions (c), (f) and (i).

**Problem 7.20** (Peng (2010)) Consider a risky asset priced

\[
S_t = S_0 e^{\sigma B_t + (\mu - \sigma^2/2) t}, \quad \text{i.e.} \quad dS_t = \mu S_t dt + \sigma S_t dB_t, \quad t \in \mathbb{R}_+,
\]
a riskless asset valued \( A_t = A_0 e^{rt} \), and a self-financing portfolio allocation \( (\eta_t, \xi_t)_{t \in \mathbb{R}_+} \) with value

\[
V_t := \eta_t A_t + \xi_t S_t, \quad t \in \mathbb{R}_+.
\]

a) Using the portfolio self-financing condition \( dV_t = \eta_t dA_t + \xi_t dS_t \), show that we have

\[
V_T = V_t + \int_t^T \left(rV_s + (\mu - r)\xi_s S_s\right) ds + \sigma \int_t^T \xi_s S_s dB_s.
\]
b) Show that under the risk-neutral probability measure $\mathbb{P}^*$ the portfolio value $V_t$ satisfies the Backward Stochastic Differential Equation (BSDE)

$$V_t = V_T - \int_t^T rV_s ds - \int_t^T \pi_s d\hat{B}_s,$$  \hspace{1cm} (7.43)

where $\pi_t := \sigma \xi_t S_t$ is the risky amount invested on the asset $S_t$, multiplied by $\sigma$, and $(\hat{B}_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$.

*Hint:* the Girsanov Theorem 7.3 states that

$$\hat{B}_t := B_t + \frac{(\mu - r)t}{\sigma}, \quad t \in \mathbb{R}_+,$$

is a standard Brownian motion under $\mathbb{P}^*$.

c) Show that under the risk-neutral probability measure $\mathbb{P}^*$, the discounted portfolio value $\tilde{V}_t := e^{-rt}V_t$ can be rewritten as

$$\tilde{V}_T = \tilde{V}_0 + \int_0^T e^{-rs} \pi_s d\hat{B}_s.$$  \hspace{1cm} (7.44)

d) Express $dv(t, S_t)$ by the Itô formula, where $v(t, x)$ is a $C^2$ function of $t$ and $x$.

e) Consider now a more general BSDE of the form

$$V_t = V_T - \int_t^T f(s, S_s, V_s, \pi_s) ds - \int_t^T \pi_s dB_s,$$  \hspace{1cm} (7.45)

with terminal condition $V_T = g(S_T)$. By matching (7.45) to the Itô formula of Question (d), find the PDE satisfied by the function $v(t, x)$ defined as $V_t = v(t, S_t)$.

f) Show that when

$$f(t, x, v, z) = rv + \frac{\mu - r}{\sigma} z,$$

the PDE of Question (e) recovers the standard Black-Scholes PDE.

g) Assuming again $f(t, x, v, z) = rv + \frac{\mu - r}{\sigma} z$ and taking the terminal condition

$$V_T = (S_0 e^{\sigma B_T + (\mu - \sigma^2/2)T} - K)^+,$$

give the process $(\pi_t)_{t \in [0, T]}$ appearing in the stochastic integral representation (7.44) of the discounted claim payoff $e^{-rT}(S_0 e^{\sigma B_T + (\mu - \sigma^2/2)T} - K)^+$.

h) From now on we assume that short selling is penalized\(^\dagger\) at a rate $\gamma > 0$, i.e. $\gamma S_t |\xi_t| dt$ is subtracted from the portfolio value change $dV_t$ whenever

---

\(^\dagger\) General Black-Scholes knowledge can be used for this question.

\(^\dagger\) SGX started to penalize naked short sales with an interim measure in September 2008.
\( \xi_t < 0 \) over the time interval \( [t, t+dt] \). Rewrite the self-financing condition using \( (\xi_t)^- := \min(\xi_t, 0) \).

i) Find the BSDE of the form (7.45) satisfied by \((V_t)_{t \in \mathbb{R}_+}\), and the corresponding function \( f(t, x, v, z) \).

j) Under the above penalty on short selling, find the PDE satisfied by the function \( u(t, x) \) when the portfolio value \( V_t \) is given as \( V_t = u(t, S_t) \).

k) Differential interest rate. Assume that one can borrow only at a rate \( R \) which is higher\(^*\) than the risk-free interest rate \( r > 0 \), i.e. we have

\[
dV_t = R \eta_t A_t dt + \xi_t dS_t
\]

when \( \eta_t < 0 \), and

\[
dV_t = r \eta_t A_t dt + \xi_t dS_t
\]

when \( \eta_t > 0 \). Find the PDE satisfied by the function \( u(t, x) \) when the portfolio value \( V_t \) is given as \( V_t = u(t, S_t) \).

l) Assume that the portfolio differential reads

\[
dV_t = \eta_t dA_t + \xi_t dS_t - dU_t,
\]

where \((U_t)_{t \in \mathbb{R}_+}\) is a non-decreasing process. Show that the corresponding portfolio strategy \((\xi_t)_{t \in \mathbb{R}_+}\) is superhedging the claim payoff \( V_T = C \).

Exercise 7.21 Girsanov Theorem. Assume that the Novikov integrability condition (7.9) is not satisfied. How does this modify the statement (7.11) of the Girsanov Theorem 7.3?

Problem 7.22 The Capital Asset Pricing Model (CAPM) of W.F. Sharpe (1990 Nobel Prize in Economics) is based on a linear decomposition

\[
\frac{dS_t}{S_t} = (r + \alpha) dt + \beta \times \left( \frac{dM_t}{M_t} - r dt \right)
\]

of stock returns \( dS_t / S_t \) into:

- a risk-free interest rate\(^†\) \( r \),
- an excess return \( \alpha \),
- a risk premium given by the difference between a benchmark market index return \( dM_t / M_t \) and the risk free rate \( r \).

The coefficient \( \beta \) measures the sensitivity of the stock return \( dS_t / S_t \) with respect to the market index returns \( dM_t / M_t \). In other words, \( \beta \) is the relative volatility of \( dS_t / S_t \) with respect to \( dM_t / M_t \), and it measures the risk of

---

\(^*\) Regular savings account usually pays \( r=0.05\% \) per year. Effective Interest Rates (EIR) for borrowing could be as high as \( R=20.61\% \) per year.

\(^†\) The risk-free interest rate \( r \) is typically the yield of the 10-year Treasury bond.
$(S_t)_{t \in \mathbb{R}_+}$ in comparison to the market index $(M_t)_{t \in \mathbb{R}_+}$.

If $\beta > 1$, resp. $\beta < 1$, then the stock price $S_t$ is more volatile (i.e. more risky), resp. less volatile (i.e. less risky), than the benchmark market index $M_t$. For example, if $\beta = 2$ then $S_t$ goes up (or down) twice as much as the index $M_t$. Inverse Exchange-Traded Funds (IETFs) have a negative value of $\beta$. On the other hand, a fund which has a $\beta = 1$ can track the index $M_t$.

Vanguard 500 Index Fund (VFINX) has a $\beta = 1$ and can be considered as replicating the variations of the S&P 500 index $M_t$, while Invesco S&P 500 (SPHB) has a $\beta = 1.42$, and Xtrackers Low Beta High Yield Bond ETF (HYDW) has a $\beta$ close to 0.36 and $\alpha = 6.36$.

In the sequel we assume that the benchmark market is represented by an index fund $(M_t)_{t \in \mathbb{R}_+}$ whose value is modeled according to

$$
\frac{dM_t}{M_t} = \mu dt + \sigma_M dB_t, \quad (7.46)
$$

where $(B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion. The asset price $(S_t)_{t \in \mathbb{R}_+}$ is modeled in a stochastic version of the CAPM as

$$
\frac{dS_t}{S_t} = r dt + \alpha dt + \beta \left( \frac{dM_t}{M_t} - r dt \right) + \sigma_S dW_t, \quad (7.47)
$$

with an additional stock volatility term $\sigma_S dW_t$, where $(W_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion independent of $(B_t)_{t \in \mathbb{R}_+}$, with

$$
\text{Cov}(B_t, W_t) = 0 \quad \text{and} \quad dB_t \cdot dW_t = 0, \quad t \in \mathbb{R}_+.
$$

The following 10 questions are interdependent and should be treated in sequence.

a) Show that $\beta$ coincides with the regression coefficient

$$
\beta = \frac{\text{Cov}(dS_t/S_t, dM_t/M_t)}{\text{Var}[dM_t/M_t]}.
$$

Hint: We have

$$
\text{Cov}(dW_t, dW_t) = dt, \quad \text{Cov}(dB_t, dB_t) = dt, \quad \text{and} \quad \text{Cov}(dW_t, dB_t) = 0.
$$

b) Show that the evolution of $(S_t)_{t \in \mathbb{R}_+}$ can be written as

$$
\frac{dS_t}{S_t} = (r + \beta(\mu - r))dt + S_t \sqrt{\beta^2 \sigma_M^2 + \sigma_S^2} dZ_t
$$

$\Box$

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https://personal.ntu.edu.sg/nprivault/index.html
where \((Z_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion.

**Hint:** The standard Brownian motion \((Z_t)_{t \in \mathbb{R}_+}\) can be characterized as the only continuous (local) martingale such that \((dZ_t)^2 = dt\), see e.g. Theorem 7.36 page 203 of Klebaner (2005).

From now on, we assume that \(\beta\) is allowed to depend locally on the state of the benchmark market index \(M_t\), as \(\beta(M_t), t \in \mathbb{R}_+\).

**c)** Rewrite the equations (7.46)-(7.47) into the system

\[
\begin{align*}
\frac{dM_t}{M_t} &= r dt + \sigma_M dB_t^*, \\
\frac{dS_t}{S_t} &= r dt + \sigma_M \beta(M_t) dB_t^* + \sigma_S dW_t^*,
\end{align*}
\]

where \((B_t^*)_{t \in \mathbb{R}_+}\) and \((W_t^*)_{t \in \mathbb{R}_+}\) have to be determined explicitly.

**d)** Using the Girsanov Theorem 7.3, construct a probability measure \(P^*\) under which \((B_t^*)_{t \in \mathbb{R}_+}\) and \((W_t^*)_{t \in \mathbb{R}_+}\) are independent standard Brownian motions.

**Hint:** Only the expression of the Radon-Nikodym density \(dP^*/dP\) is needed here.

**e)** Show that the market based on the assets \(S_t\) and \(M_t\) is without arbitrage opportunities.

**f)** Consider a portfolio strategy \((\xi_t, \zeta_t, \eta_t)_{t \in [0,T]}\) based on the three assets \((S_t, M_t, A_t)_{t \in [0,T]}\), with value

\[V_t = \xi_t S_t + \zeta_t M_t + \eta_t A_t, \quad t \in [0,T],\]

where \((A_t)_{t \in \mathbb{R}_+}\) is a riskless asset given by \(A_t = A_0 e^{rt}\). Write down the self-financing condition for the portfolio strategy \((\xi_t, \zeta_t, \eta_t)_{t \in [0,T]}\).

**g)** Consider an option with payoff \(C = h(S_T, M_T)\), priced as

\[f(t, S_t, M_t) = e^{-(T-t)r} \mathbb{E}^*[h(S_T, M_T) | \mathcal{F}_t], \quad 0 \leq t \leq T.\]

Assuming that the portfolio \((V_t)_{t \in [0,T]}\) replicates the option price process \((f(t, S_t, M_t))_{t \in [0,T]}\), derive the pricing PDE satisfied by the function \(f(t, x, y)\) and its terminal condition.

**Hint:** The following version of the Itô formula with two variables can be used for the function \(f(t, x, y)\), see (4.26):
\[
df(t, S_t, M_t) = \frac{\partial f}{\partial t}(t, S_t, M_t)dt + \frac{\partial f}{\partial S}(t, S_t, M_t)dS_t + \frac{1}{2}(dS_t)^2 \frac{\partial^2 f}{\partial x^2}(t, S_t, M_t) \\
+ \frac{\partial f}{\partial y}(t, S_t, M_t)dM_t + \frac{1}{2}(dM_t)^2 \frac{\partial^2 f}{\partial y^2}(t, S_t, M_t) + dS_t \cdot dM_t \frac{\partial^2 f}{\partial x \partial y}(t, S_t, M_t).
\]

h) Find the self-financing hedging portfolio strategy \((\xi_t, \zeta_t, \eta_t)_{t \in [0, T]}\) replicating the vanilla payoff \(h(S_T, M_T)\).
i) Solve the PDE of Question (g) and compute the replicating portfolio of Question (h) when \(\beta(M_t) = \beta\) is a constant and \(C\) is the European call option payoff on \(S_T\) with strike price \(K\).
j) Solve the PDE of Question (g) and compute the replicating portfolio of Question (h) when \(\beta(M_t) = \beta\) is a constant and \(C\) is the European put option payoff on \(S_T\) with strike price \(K\).

Problem 7.23 Quantile hedging (Föllmer and Leukert (1999), §6.2 of Mel’nikov et al. (2002)). Recall that given two probability measures \(\mathbb{P}\) and \(\mathbb{Q}\), the Radon-Nikodym density \(\frac{d\mathbb{Q}}{d\mathbb{P}}\) links the expectations of random variables \(F\) under \(\mathbb{P}\) and under \(\mathbb{Q}\) via the relation
\[
\mathbb{E}_\mathbb{Q}[F] = \int \Omega F(\omega)d\mathbb{Q}(\omega) = \int \Omega F(\omega) \frac{d\mathbb{Q}}{d\mathbb{P}}(\omega)d\mathbb{P}(\omega) = \mathbb{E}_\mathbb{P} \left[ F \frac{d\mathbb{Q}}{d\mathbb{P}} \right].
\]
a) Neyman-Pearson Lemma. Given \(\mathbb{P}\) and \(\mathbb{Q}\) two probability measures, consider the event
\[
A_\alpha := \left\{ \frac{d\mathbb{P}}{d\mathbb{Q}} > \alpha \right\}, \quad \alpha \geq 0.
\]
Show that for \(A\) any event, \(\mathbb{Q}(A) \leq \mathbb{Q}(A_\alpha)\) implies \(\mathbb{P}(A) \leq \mathbb{P}(A_\alpha)\).

\textbf{Hint:} Start by proving that we always have
\[
\left( \frac{d\mathbb{P}}{d\mathbb{Q}} - \alpha \right) (2\mathbb{1}_{A_\alpha} - 1) \geq \left( \frac{d\mathbb{P}}{d\mathbb{Q}} - \alpha \right) (2\mathbb{1}_A - 1). \quad (7.48)
\]
b) Let \(C \geq 0\) denote a nonnegative claim payoff on a financial market with risk-neutral measure \(\mathbb{P}^*\). Show that the Radon-Nikodym density
\[
\frac{d\mathbb{Q}^*}{d\mathbb{P}^*} := \frac{C}{\mathbb{E}_{\mathbb{P}^*}[C]} \quad (7.49)
\]
defines a probability measure \(\mathbb{Q}^*\).
Hint: Check first that \( dQ^*/dP^* \geq 0 \), and then that \( Q^*(\Omega) = 1 \). In the following questions we consider a nonnegative contingent claim with payoff \( C \geq 0 \) and maturity \( T > 0 \), priced \( e^{-rT} E_{P^*}[C] \) at time 0 under the risk-neutral measure \( P^* \).

Budget constraint. In the sequel we will assume that no more than a certain fraction \( \beta \in (0,1] \) of the claim price \( e^{-rT} E_{P^*}[C] \) is available to construct the initial hedging portfolio \( V_0 \) at time 0.

Since a self-financing portfolio process \((V_t)_{t \in \mathbb{R}_+}\) started at \( V_0 := \beta e^{-rT} E_{P^*}[C] \) may fall short of hedging the claim \( C \) when \( \beta < 1 \), we will attempt to maximize the probability \( P(V_T \geq C) \) of successful hedging, or, equivalently, to minimize the shortfall probability \( P(V_T < C) \).

For this, given \( A \) an event we consider the self-financing portfolio process \((V_t^A)_{t \in [0,T]}\) hedging the claim \( C1_A \), priced \( V_0^A = e^{-rT} E_{P^*}[C1_A] \) at time 0, and such that \( V_T^A = C1_A \) at maturity \( T \).

c) Show that if \( \alpha \) satisfies \( Q^*(A_\alpha) = \beta \), the event

\[
A_\alpha = \left\{ \frac{dP}{dQ^*} > \alpha \right\} = \left\{ \frac{dP}{dP^*} > \alpha \frac{dQ^*}{dP^*} \right\} = \left\{ \frac{dP}{dP^*} > \frac{\alpha C}{E_{P^*}[C]} \right\}
\]

maximizes \( P(A) \) over all possible events \( A \), under the condition

\[
e^{-rT} E_{P^*}[V_T^A] = e^{-rT} E_{P^*}[C1_A] \leq \beta e^{-rT} E_{P^*}[C]. \tag{7.50}
\]

Hint: Rewrite Condition (7.50) using the probability measure \( Q^* \), and apply the Neyman-Pearson Lemma of Question (a) to \( P \) and \( Q^* \).

d) Show that \( P(A_\alpha) \) coincides with the successful hedging probability \( P(V_T^A \geq C) = P(C1_{A_\alpha} \geq C) \), i.e. show that

\[
P(A_\alpha) = P(V_T^A \geq C) = P(C1_{A_\alpha} \geq C).
\]

Hint: To prove an equality \( x = y \) we can show first that \( x \leq y \), and then that \( x \geq y \). One inequality is obvious, and the other one follows from Question (c).

e) Check that the self-financing portfolio process \((V_t^{A_\alpha})_{t \in [0,T]}\) hedging the claim with payoff \( C1_{A_\alpha} \) uses only the initial budget \( \beta e^{-rT} E_{P^*}[C] \), and that \( P(V_T^{A_\alpha} \geq C) \) maximizes the successful hedging probability.

In the next Questions (f)-(j) we assume that \( C = (S_T - K)^+ \) is the payoff of a European option in the Black-Scholes model

\[
dS_t = rS_t dt + \sigma S_t dB_t, \tag{7.51}
\]

with \( P = P^* \), \( dP/dP^* = 1 \), and
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\[ S_0 := 1 \quad \text{and} \quad r = \frac{\sigma^2}{2} := \frac{1}{2}. \]  

(7.52)

f) Solve the stochastic differential equation (7.51) with the parameters
(7.52).

g) Compute the successful hedging probability

\[ \mathbb{P}(V_T^{A} \geq C) = \mathbb{P}(C_{1, A} \geq C) = \mathbb{P}(A_{\alpha}) \]

for the claim \( C =: (S_T - K)^+ \) in terms of \( K, T, \mathbb{E}_{\mathbb{P}^*}[C] \) and the parameter \( \alpha > 0 \).

h) From the result of Question (g), express the parameter \( \alpha \) using \( K, T, \mathbb{E}_{\mathbb{P}^*}[C] \), and the successful hedging probability \( \mathbb{P}(V_T^{A} \geq C) \) for the claim \( C =: (S_T - K)^+ \).

i) Compute the minimal initial budget \( e^{-rt} \mathbb{E}_{\mathbb{P}^*}[C_{1, A}] \) required to hedge the claim \( C = (S_T - K)^+ \) in terms of \( \alpha > 0, K, T \) and \( \mathbb{E}_{\mathbb{P}^*}[C] \).

j) Taking \( K := 1, T := 1 \) and assuming a successful hedging probability of 90%, compute numerically:

i) The European call price \( e^{-rt} \mathbb{E}_{\mathbb{P}^*}[(S_T - K)^+] \) from the Black-Scholes formula.

ii) The value of \( \alpha > 0 \) obtained from Question (h).

iii) The minimal initial budget needed to successfully hedge the European claim \( C = (S_T - K)^+ \) with probability 90% from Question (i).

iv) The value of \( \beta \), i.e. the budget reduction ratio which suffices to successfully hedge the claim \( C =: (S_T - K)^+ \) with 90% probability.

Problem 7.24 Log options.

a) Consider a market model made of a risky asset with price \( (S_t)_{t \in \mathbb{R}_+} \) as in Exercise 4.22-(d) and a riskless asset with price \( A_t = \$1 \times e^{rt} \) and risk-free interest rate \( r = \sigma^2/2 \). From the answer to Exercise 4.22-(b), show that the arbitrage price

\[ V_t = e^{-(T-t)r} \mathbb{E} [ (\log S_T)^+ | \mathcal{F}_t ] \]

at time \( t \in [0, T] \) of a log call option with payoff \( (\log S_T)^+ \) is equal to

\[ V_t = \sigma e^{-(T-t)r} \sqrt{\frac{T-t}{2\pi}} e^{-B_t^2/(2(T-t))} + \sigma e^{-(T-t)r} B_t \Phi \left( \frac{B_t}{\sqrt{T-t}} \right). \]

b) Show that \( V_t \) can be written as

\[ V_t = g(T-t, S_t), \]

where \( g(\tau, x) = e^{-r\tau} f(\tau, \log x) \), and

\( \Diamond \)
\[ f(\tau, y) = \sigma \sqrt{\frac{\tau}{2\pi}} e^{-y^2/(2\sigma^2\tau)} + y \Phi \left( \frac{y}{\sigma\sqrt{\tau}} \right). \]

(c) Figure 7.5 represents the graph of \((\tau, x) \mapsto g(\tau, x)\), with \(r = 0.05 = 5\% \text{ per year}\) and \(\sigma = 0.1\). Assume that the current underlying asset price is $1 and there remains 700 days to maturity. What is the price of the option?

![Figure 7.5: Option price as a function of underlying asset price and time to maturity.](image)

(d) Show* that the (possibly fractional) quantity \(\xi_t = \frac{\partial g}{\partial x}(T - t, S_t)\) of \(S_t\) at time \(t\) in a portfolio hedging the payoff \((\log S_T)^+\) is equal to

\[ \xi_t = e^{-(T-t)r} \frac{1}{S_t} \Phi \left( \frac{\log S_t}{\sigma\sqrt{T-t}} \right), \quad 0 \leq t \leq T. \]

(e) Figure 7.6 represents the graph of \((\tau, x) \mapsto \frac{\partial g}{\partial x}(\tau, x)\). Assuming that the current underlying asset price is $1 and that there remains 700 days to maturity, how much of the risky asset should you hold in your portfolio in order to hedge one log option?

* Recall the chain rule of derivation \(\frac{\partial}{\partial x} f(\tau, \log x) = \frac{1}{x} \frac{\partial f}{\partial y}(\tau, y)|_{y=\log x}\).
Fig. 7.6: Delta as a function of underlying asset price and time to maturity.

f) Based on the framework and answers of Questions (c) and (e), should you borrow or lend the riskless asset \( A_t = 1 \times e^{rt} \), and for what amount?

g) Show that the Gamma of the portfolio, defined as \( \Gamma_t = \frac{\partial^2 g}{\partial x^2} (T-t,S_t) \), equals

\[
\Gamma_t = e^{-(T-t)r} \frac{1}{S_t^2} \left( \frac{1}{\sigma \sqrt{2(T-t)}} \right) e^{-(\log S_t)^2/(2(T-t)\sigma^2)} - \Phi \left( \frac{\log S_t}{\sigma \sqrt{T-t}} \right),
\]

\( 0 \leq t < T \).

h) Figure 7.7 represents the graph of Gamma. Assume that there remains 60 days to maturity and that \( S_t \), currently at $1, is expected to increase. Should you buy or (short) sell the underlying asset in order to hedge the option?

Fig. 7.7: Gamma as a function of underlying asset price and time to maturity.

i) Let now \( \sigma = 1 \). Show that the function \( f(\tau,y) \) of Question (b) solves the heat equation.
Problem 7.25 Log put options with a given strike price.

a) Consider a market model made of a risky asset with price $(S_t)_{t \in \mathbb{R}^+}$ as in Exercise 5.9, a riskless asset valued $A_t = \$1 \times e^{rt}$, risk-free interest rate $r = \sigma^2/2$ and $S_0 = 1$. From the answer to Exercise A.4-(c), show that the arbitrage price

$$V_t = e^{-(T-t)r} \mathbb{E}^* \left[ (K - \log S_T)^+ | \mathcal{F}_t \right]$$

at time $t \in [0, T]$ of a log call option with strike price $K$ and payoff $(K - \log S_T)^+$ is equal to

$$V_t = \sigma e^{-(T-t)r} \sqrt{\frac{T-t}{2\pi}} e^{-(B_t-K/\sigma)^2/(2(T-t))} + e^{-(T-t)r} (K - \sigma B_t) \Phi \left( \frac{K - \sigma B_t}{\sqrt{T-t}} \right).$$

b) Show that $V_t$ can be written as

$$V_t = g(T-t, S_t),$$

where $g(\tau, x) = e^{-r\tau} f(\tau, \log x)$, and

$$f(\tau, y) = \sigma \sqrt{\frac{\tau}{2\pi}} e^{-(K-y)^2/(2\sigma^2\tau)} + (K - y) \Phi \left( \frac{K - y}{\sigma \sqrt{\tau}} \right).$$

c) Figure 7.8 represents the graph of $(\tau, x) \mapsto g(\tau, x)$, with $r = 0.125 \text{ per year}$ and $\sigma = 0.5$. Assume that the current underlying asset price is $\$3$, that $K = 1$, and that there remains 700 days to maturity. What is the price of the option?

Fig. 7.8: Option price as a function of underlying asset price and time to maturity.
d) Show* that the quantity \( \xi_t = \frac{\partial g}{\partial x} (T - t, S_t) \) of \( S_t \) at time \( t \) in a portfolio hedging the payoff \( (K - \log S_T)^+ \) is equal to
\[
\xi_t = -e^{-(T-t)r} \frac{1}{S_t} \Phi \left( \frac{K - \log S_t}{\sigma \sqrt{T-t}} \right), \quad 0 \leq t \leq T.
\]

e) Figure 7.9 represents the graph of \( (\tau, x) \mapsto \frac{\partial g}{\partial x} (\tau, x) \). Assuming that the current underlying asset price is $3 and that there remains 700 days to maturity, how much of the risky asset should you hold in your portfolio in order to hedge one log option?

\[
\text{Fig. 7.9: Delta as a function of underlying asset price and time to maturity.}
\]

f) Based on the framework and answers of Questions (c) and (e), should you borrow or lend the riskless asset \( A_t = \$1 \times e^{rt} \), and for what amount?

\[
\text{g) Show that the Gamma of the portfolio, defined as } \Gamma_t = \frac{\partial^2 g}{\partial x^2} (T - t, S_t), \text{ equals}
\]
\[
\Gamma_t = e^{-(T-t)r} \frac{1}{S_t^2} \left( \frac{1}{\sigma \sqrt{2(T-t)\pi}} e^{-(K-\log S_t)^2/(2(T-t)\sigma^2)} + \Phi \left( \frac{K - \log S_t}{\sigma \sqrt{T-t}} \right) \right),
\]
\[
0 \leq t \leq T.
\]

h) Figure 7.10 represents the graph of Gamma. Assume that there remains 10 days to maturity and that \( S_t \), currently at $3, is expected to increase. Should you buy or (short) sell the underlying asset in order to hedge the option?

* Recall the chain rule of derivation \( \frac{\partial}{\partial x} f(\tau, \log x) = \frac{1}{x} \frac{\partial f}{\partial y}(\tau, y)|_{y=\log x} \).
Fig. 7.10: Gamma as a function of underlying asset price and time to maturity.

i) Show that the function $f(\tau, y)$ of Question (b) solves the heat equation

$$
\begin{align*}
\frac{\partial f}{\partial \tau}(\tau, y) &= \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial y^2}(\tau, y) \\
\left. f(\tau, y) \right|_{\tau=0} &= (K - y)^+. 
\end{align*}
$$