Chapter 7
Martingale Approach to Pricing and Hedging

In this chapter we present the probabilistic martingale approach method to the pricing and hedging of options. In particular, this allows one to compute option prices as the expectations of the discounted option payoffs, and to determine the associated hedging portfolios.

7.1 Martingale Property of the Itô Integral

Recall (Definition 5.4) that an integrable process \( (X_t)_{t \in \mathbb{R}^+} \) is said to be a martingale with respect to the filtration \( (\mathcal{F}_t)_{t \in \mathbb{R}^+} \) if

\[
\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s, \quad 0 \leq s \leq t.
\]

Examples of martingales (i)

1. Given \( F \in L^2(\Omega) \) a square-integrable random variable and \( (\mathcal{F}_t)_{t \in \mathbb{R}^+} \) a filtration, the process \( (X_t)_{t \in \mathbb{R}^+} \) defined by

\[
X_t := \mathbb{E}[F \mid \mathcal{F}_t], \quad t \in \mathbb{R}^+,
\]

is an \( (\mathcal{F}_t)_{t \in \mathbb{R}^+} \)-martingale under \( \mathbb{P} \), as follows from the tower property:

\[
\mathbb{E}[X_t \mid \mathcal{F}_s] = \mathbb{E}[\mathbb{E}[F \mid \mathcal{F}_t] \mid \mathcal{F}_s] = \mathbb{E}[F \mid \mathcal{F}_s] = X_s, \quad 0 \leq s \leq t, \quad (7.1)
\]
by the tower property (22.38).

2. Any integrable stochastic process \((X_t)_{t \in \mathbb{R}_+}\) whose increments \((X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}})\) are independent and centered under \(P\) (i.e. \(\mathbb{E}[X_t] = 0, \ t \in \mathbb{R}_+\)) is a martingale with respect to its own filtration \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\):

\[
\mathbb{E}[X_t \mid \mathcal{F}_s] = \mathbb{E}[X_t - X_s + X_s \mid \mathcal{F}_s] \\
= \mathbb{E}[X_t - X_s \mid \mathcal{F}_s] + \mathbb{E}[X_s \mid \mathcal{F}_s] \\
= \mathbb{E}[X_t - X_s] + X_s \\
= X_s, \quad 0 \leq s \leq t. \tag{7.2}
\]

In particular, the standard Brownian motion \((B_t)_{t \in \mathbb{R}_+}\) is a martingale because it has centered and independent increments. This fact is also consequence of Proposition 7.1 below as \(B_t\) can be written as

\[
B_t = \int_0^t dB_s, \quad t \in \mathbb{R}_+.
\]

The following result shows that the indefinite Itô integral is a martingale with respect to the Brownian filtration \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\). It is the continuous-time analog of the discrete-time Proposition 2.10.

**Proposition 7.1.** The indefinite stochastic integral \(\left(\int_0^t u_s dB_s\right)_{t \in \mathbb{R}_+}\) of a square-integrable adapted process \(u \in L^2_{\text{ad}}(\Omega \times \mathbb{R}_+)\) is a martingale, i.e.:

\[
\mathbb{E} \left[ \int_0^t u_r dB_r \mid \mathcal{F}_s \right] = \int_0^s u_r dB_r, \quad 0 \leq s \leq t. \tag{7.3}
\]

In particular, \(\int_0^t u_s dB_s\) is \(\mathcal{F}_t\)-measurable, \(t \in \mathbb{R}_+\), and since \(\mathcal{F}_0 = \{\emptyset, \Omega\}\), Relation (7.3) applied with \(t = 0\) recovers the fact that the Itô integral is a centered random variable:

\[
\mathbb{E} \left[ \int_0^\infty u_s dB_s \right] = \mathbb{E} \left[ \int_0^\infty u_s dB_s \mid \mathcal{F}_0 \right] = \int_0^0 u_s dB_s = 0.
\]

**Proof.** The statement is first proved in case \(u\) is a simple predictable process, and then extended to the general case, cf. e.g. Proposition 2.5.7 in Privault (2009). For example, for \(u\) a predictable step process of the form

\[
u_s := F1_{[a,b]}(s) = \begin{cases} \ F \text{ if } s \in [a,b], \\ 0 \text{ if } s \not\in [a,b], \end{cases}
\]

with \(F\) an \(\mathcal{F}_a\)-measurable random variable and \(t \in [a,b]\), we have

\[
\mathbb{E} \left[ \int_0^\infty u_s dB_s \mid \mathcal{F}_t \right] = \mathbb{E} \left[ \int_0^\infty F1_{[a,b]}(s)dB_s \mid \mathcal{F}_t \right]
\]

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\begin{align*}
\mathbb{E}[F(B_b - B_a) | \mathcal{F}_t] &= F \mathbb{E}[B_b - B_a | \mathcal{F}_t] \\
&= F(B_t - B_a) \\
&= \int_a^t u_s dB_s \\
&= \int_0^t u_s dB_s, \quad a \leq t \leq b.
\end{align*}

On the other hand, when \( t \in [0, a] \) we have

\[ \int_0^t u_s dB_s = 0, \]

hence we check that

\begin{align*}
\mathbb{E} \left[ \int_0^\infty u_s dB_s \mid \mathcal{F}_t \right] &= \mathbb{E} \left[ \int_0^\infty F \mathbb{1}_{[a,b]}(s) dB_s \mid \mathcal{F}_t \right] \\
&= \mathbb{E}[F(B_b - B_a) | \mathcal{F}_t] \\
&= \mathbb{E}[\mathbb{E}[F(B_b - B_a) | \mathcal{F}_a] | \mathcal{F}_t] \\
&= \mathbb{E}[\mathbb{E}[B_b - B_a | \mathcal{F}_a] | \mathcal{F}_t] \\
&= 0, \quad 0 \leq t \leq a,
\end{align*}

where we used the tower property (22.38) of conditional expectations and the fact that \((B_t)_{t \in \mathbb{R}_+}\) is a martingale:

\[ \mathbb{E}[B_b - B_a | \mathcal{F}_a] = \mathbb{E}[B_b | \mathcal{F}_a] - B_a = B_a - B_a = 0. \]

The extension from simple processes to square-integrable processes in \(L^2_{ad}(\Omega \times \mathbb{R}_+)\) can be proved as in Proposition 4.16. Indeed, given \((u^{(n)})_{n \in \mathbb{N}}\) be a sequence of simple predictable processes converging to \(u\) in \(L^2(\Omega \times [0,T])\) cf. Lemma 1.1 of Ikeda and Watanabe (1989), pages 22 and 46, by Fatou’s Lemma 22.3 and Jensen’s inequality we have:

\begin{align*}
\mathbb{E} \left[ \left( \int_0^t u_s dB_s - \mathbb{E} \left[ \int_0^\infty u_s dB_s \mid \mathcal{F}_t \right] \right)^2 \right] \\
\leq \liminf_{n \to \infty} \mathbb{E} \left[ \left( \int_0^t u_{{s}^{(n)}} dB_s - \mathbb{E} \left[ \int_0^\infty u_s dB_s \mid \mathcal{F}_t \right] \right)^2 \right] \\
= \liminf_{n \to \infty} \mathbb{E} \left[ \left( \mathbb{E} \left[ \int_0^\infty u_{{s}^{(n)}} dB_s - \int_0^\infty u_s dB_s \mid \mathcal{F}_t \right] \right)^2 \right] \\
\leq \liminf_{n \to \infty} \mathbb{E} \left[ \mathbb{E} \left[ \left( \int_0^\infty u_{{s}^{(n)}} dB_s - \int_0^\infty u_s dB_s \right)^2 \mid \mathcal{F}_t \right] \right] \\
= \liminf_{n \to \infty} \mathbb{E} \left[ \left( \int_0^\infty (u_{{s}^{(n)}} - u_s) dB_s \right)^2 \right]
\end{align*}
\[ = \lim_{n \to \infty} \mathbb{E} \left[ \int_0^\infty |u_s^{(n)} - u_s|^2 \, ds \right] \]
\[ = \lim_{n \to \infty} \|u^{(n)} - u\|_{L^2(\Omega \times [0,T])}^2 \]
\[ = 0, \]
where we used the Itô isometry (4.15). We conclude that
\[ \mathbb{E} \left[ \int_0^\infty u_s \, dB_s \bigg| \mathcal{F}_t \right] = \int_0^t u_s \, dB_s, \quad t \in \mathbb{R}_+, \]
for \( u \in L^2_{\text{ad}}(\Omega \times \mathbb{R}_+) \) a square-integrable adapted process, which leads to (7.3) after applying this identity to the process \( (1_{[0,t]} u_s)_{s \in \mathbb{R}_+} \), i.e.,

\[ \mathbb{E} \left[ \int_0^t u_\tau \, dB_\tau \bigg| \mathcal{F}_s \right] = \mathbb{E} \left[ \int_0^\infty 1_{[0,t]}(\tau) u_\tau \, dB_\tau \bigg| \mathcal{F}_s \right] \]
\[ = \int_0^s 1_{[0,t]}(\tau) u_\tau \, dB_\tau \]
\[ = \int_0^s u_\tau \, dB_\tau, \quad 0 \leq s \leq t. \]

\[ \square \]

Examples of martingales (ii)

1. The driftless geometric Brownian motion

\[ X_t := X_0 e^{\sigma B_t - \sigma^2 t/2} \]

is a martingale. Indeed, we have

\[ \mathbb{E}[X_t \mid \mathcal{F}_s] = \mathbb{E}[X_0 e^{\sigma B_t - \sigma^2 t/2} \mid \mathcal{F}_s] \]
\[ = X_0 e^{-\sigma^2 t/2} \mathbb{E}[ e^{\sigma B_t} \mid \mathcal{F}_s] \]
\[ = X_0 e^{-\sigma^2 t/2} \mathbb{E}[ e^{(B_t - B_s)\sigma + \sigma B_s} \mid \mathcal{F}_s] \]
\[ = X_0 e^{-\sigma^2 t/2 + \sigma B_s} \mathbb{E}[ e^{(B_t - B_s)\sigma} \mid \mathcal{F}_s] \]
\[ = X_0 e^{-\sigma^2 t/2 + \sigma B_s} e^{\sigma^2 (t-s)/2} \]
\[ = X_0 e^{\sigma B_s - \sigma^2 s/2} \]
\[ = X_s, \quad 0 \leq s \leq t. \]

This fact can also be recovered from Proposition 7.1 since \((X_t)_{t \in \mathbb{R}_+}\) satisfies the equation

\[ dX_t = \sigma X_t dB_t, \]
which shows that \( X_t \) can be written using the Brownian stochastic integral
\[
X_t = X_0 + \sigma \int_0^t X_u dB_u, \quad t \in \mathbb{R}_+.
\]
The discounted asset price
\[
\tilde{S}_t = S_0 e^{(\mu - r)t + \sigma B_t - \frac{\sigma^2 t}{2}}
\]
is a martingale under \( \mathbb{P} \) when \( \mu = r \). The case \( \mu \neq r \) will be treated in Section 7.3 using risk-neutral probability measures and the Girsanov Theorem, cf. (7.12) below.

2. The discounted value
\[
\tilde{V}_t = e^{-rt} V_t
\]
of a self-financing portfolio is given by
\[
\tilde{V}_t = \tilde{V}_0 + \int_0^t \xi_u d\tilde{S}_u, \quad t \in \mathbb{R}_+,
\]
cf. Lemma 5.14 is a martingale when \( \mu = r \) by Proposition 7.1 because
\[
\tilde{V}_t = \tilde{V}_0 + \sigma \int_0^t \xi_u \tilde{S}_u dB_u, \quad t \in \mathbb{R}_+,
\]
since
\[
d\tilde{S}_t = \tilde{S}_t ((\mu - r) dt + \sigma dB_t).
\]
Since the Black-Scholes theory is in fact valid for any value of the parameter \( \mu \) we will look forward to including the case \( \mu \neq r \) in the sequel.

### 7.2 Risk-neutral Probability Measures

Recall that by definition, a risk-neutral measure is a probability measure \( \mathbb{P}^* \) under which the discounted asset price process
\[
(\tilde{S}_t)_{t \in \mathbb{R}_+} := (e^{-rt} S_t)_{t \in \mathbb{R}_+}
\]
is a martingale, cf. Proposition 5.5. Note that when \( \mu = r \), the process \( (\tilde{S}_t)_{t \in \mathbb{R}_+} = (S_0 e^{\sigma B_t - \frac{\sigma^2 t}{2}})_{t \in \mathbb{R}_+} \) is a martingale under \( \mathbb{P}^* = \mathbb{P} \), which is a risk-neutral probability measure.

In this section we address the construction of a risk-neutral probability measure \( \mathbb{P}^* \) in the general case \( \mu \neq r \) using the Girsanov Theorem. Note that the relation
\[
d\tilde{S}_t = \tilde{S}_t ((\mu - r) dt + \sigma dB_t)
\]
where \( \mu - r \) is the risk premium, can be rewritten as
\[ d\tilde{S}_t = \sigma \tilde{S}_t d\tilde{B}_t, \]

where \((\tilde{B}_t)_{t \in \mathbb{R}_+}\) is a drifted Brownian motion given by
\[
\tilde{B}_t := \frac{\mu - r}{\sigma} t + B_t, \quad t \in \mathbb{R}_+,\]

where the drift coefficient \((\mu - r)/\sigma\) is the “Market Price of Risk” (MPoR). It represents the difference between the return \(\mu\) expected when investing in the risky asset \(S_t\), and the risk-free interest rate \(r\), measured in units of volatility \(\sigma\).

Therefore, the search for a risk-neutral probability measure can be replaced by the search for a probability measure \(P^*\) under which \((\tilde{B}_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion.

Let us come back to the informal interpretation of Brownian motion via its infinitesimal increments:
\[ \Delta B_t = \pm \sqrt{\Delta t}, \]

with
\[ \mathbb{P}(\Delta B_t = +\sqrt{\Delta t}) = \mathbb{P}(\Delta B_t = -\sqrt{\Delta t}) = \frac{1}{2}. \]

Clearly, given \(\nu \in \mathbb{R}\), the drifted process
\[ \tilde{B}_t := \nu t + B_t, \quad t \in \mathbb{R}_+,\]

is no longer a standard Brownian motion because it is not centered:
\[ \mathbb{E}[\nu t + B_t] = \nu t + \mathbb{E}[B_t] = \nu t \neq 0, \]

cf. Figure 7.1. This identity can be formulated in terms of infinitesimal increments as
\[ \mathbb{E}[\nu \Delta t + \Delta B_t] = \frac{1}{2} (\nu \Delta t + \sqrt{\Delta t}) + \frac{1}{2} (\nu \Delta t - \sqrt{\Delta t}) = \nu \Delta t \neq 0. \]
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Fig. 7.1: Drifted Brownian path \((\widehat{B}_t)_{t \in [0,T]}\).

In order to make \(\nu t + B_t\) a centered process \(i.e.\) a standard Brownian motion, since \(\nu t + B_t\) conserves all the other properties \((i)-(iii)\) in the definition of Brownian motion, one may change the probabilities of ups and downs, which have been fixed so far equal to 1/2.

That is, the problem is now to find two numbers \(p^*, q^* \in [0,1]\) such that

\[
\begin{align*}
  p^*(\nu \Delta t + \sqrt{\Delta t}) + q^*(\nu \Delta t - \sqrt{\Delta t}) &= 0 \\
  p^* + q^* &= 1.
\end{align*}
\]

The solution to this problem is given by

\[
p^* = \frac{1}{2}(1 - \nu \sqrt{\Delta t}) \quad \text{and} \quad q^* = \frac{1}{2}(1 + \nu \sqrt{\Delta t}).
\]

(7.4)

Coming back to Brownian motion considered as a discrete random walk with independent increments \(\pm \sqrt{\Delta t}\), we try to construct a new probability measure denoted \(\mathbb{P}^*\), under which the drifted process \(\widehat{B}_t := \nu t + B_t\) will be a standard Brownian motion. This probability measure will be defined through its density

\[
d\mathbb{P}^* = \frac{P^*(\Delta B_{t_1} = \epsilon_1 \sqrt{\Delta t}, \ldots, \Delta B_{t_N} = \epsilon_N \sqrt{\Delta t})}{P(\Delta B_{t_1} = \epsilon_1 \sqrt{\Delta t}, \ldots, \Delta B_{t_N} = \epsilon_N \sqrt{\Delta t})} = \frac{P^*(\Delta B_{t_1} = \epsilon_1 \sqrt{\Delta t}) \cdots P^*(\Delta B_{t_N} = \epsilon_N \sqrt{\Delta t})}{P(\Delta B_{t_1} = \epsilon_1 \sqrt{\Delta t}) \cdots P(\Delta B_{t_N} = \epsilon_N \sqrt{\Delta t})} = \frac{1}{(1/2)^N} P^*(\Delta B_{t_1} = \epsilon_1 \sqrt{\Delta t}) \cdots P^*(\Delta B_{t_N} = \epsilon_N \sqrt{\Delta t}),
\]

\(\epsilon_1, \epsilon_2, \ldots, \epsilon_N \in \{-1,1\}\), with respect to the historical probability measure \(\mathbb{P}\), obtained by taking the product of the above probabilities divided by the
reference probability $1/2^N$ corresponding to the symmetric random walk.

Interpreting $N = T/\Delta t$ as an (infinitely large) number of discrete time steps and under the identification $[0, T] \simeq \{0 = t_0, t_1, \ldots, t_N = T\}$, this density can be rewritten as

$$\frac{dP^*}{dP} \simeq \frac{1}{(1/2)^N} \prod_{0 < t < T} \left( \frac{1}{2} + \frac{1}{2} \nu \sqrt{\Delta t} \right) \quad (7.5)$$

where $2^N$ becomes a normalization factor. Using the expansion

$$\log (1 \pm \nu \sqrt{\Delta t}) = \pm \nu \sqrt{\Delta t} - \frac{1}{2} (\pm \nu \sqrt{\Delta t})^2 + o(\Delta t)$$

$$= \pm \nu \sqrt{\Delta t} - \frac{1}{2} (\nu \sqrt{\Delta t})^2 + o(\Delta t),$$

for small values of $\Delta t$, this density can be informally shown to converge as follows as $N$ tends to infinity, i.e. as the time step $\Delta t = T/N$ tends to zero:

$$2^N \prod_{0 < t < T} \left( \frac{1}{2} \pm \frac{1}{2} \nu \sqrt{\Delta t} \right) = \prod_{0 < t < T} \left( 1 \mp \nu \sqrt{\Delta t} \right)$$

$$= \exp \left( \log \prod_{0 < t < T} \left( 1 \mp \nu \sqrt{\Delta t} \right) \right)$$

$$= \exp \left( \sum_{0 < t < T} \log \left( 1 \mp \nu \sqrt{\Delta t} \right) \right)$$

$$\simeq \exp \left( \nu \sum_{0 < t < T} \mp \sqrt{\Delta t} - \frac{1}{2} \sum_{0 < t < T} (\mp \nu \sqrt{\Delta t})^2 \right)$$

$$= \exp \left( -\nu \sum_{0 < t < T} \pm \sqrt{\Delta t} - \frac{\nu^2}{2} \sum_{0 < t < T} \Delta t \right)$$

$$= \exp \left( -\nu \sum_{0 < t < T} \Delta B_t - \frac{\nu^2}{2} \sum_{0 < t < T} \Delta t \right)$$

$$= \exp \left( -\nu B_T - \frac{\nu^2}{2} T \right),$$

based on the identifications

$$B_T \simeq \sum_{0 < t < T} \pm \sqrt{\Delta t} \quad \text{and} \quad T \simeq \sum_{0 < t < T} \Delta t.$$

The following R code is rescaling probabilities as in (7.4) based on the value of the drift $\mu$.
The discretized illustration in Figure 7.2 displays the drifted Brownian motion \( \tilde{B}_t := \nu t + B_t \) under the shifted probability measure \( \mathbb{P}^* \) in (7.5) using the above R code with \( N = 100 \). The code makes big transitions less frequent than small transitions, resulting into a standard, centered Brownian motion under \( \mathbb{P}^* \).

![Fig. 7.2: Drifted Brownian motion paths under a shifted Girsanov measure.](image)

### 7.3 Change of Measure and the Girsanov Theorem

In this section we restate the Girsanov Theorem in a more rigorous way, using changes of probability measures.

**Definition 7.2.** We say that a probability measure \( \mathbb{Q} \) is absolutely continuous with respect to another probability measure \( \mathbb{P} \) if there exists a nonnegative random variable \( F : \Omega \rightarrow \mathbb{R} \) such that \( \mathbb{E}[F] = 1 \), and

\[
\frac{d\mathbb{Q}}{d\mathbb{P}} = F, \quad \text{i.e.} \quad d\mathbb{Q} = Fd\mathbb{P}. \tag{7.6}
\]

In this case, \( F \) is called the Radon-Nikodym density of \( \mathbb{Q} \) with respect to \( \mathbb{P} \).

Relation (7.6) is equivalent to the relation

\[
\mathbb{E}_{\mathbb{Q}}[G] = \int_{\Omega} G(\omega)d\mathbb{Q}(\omega) = \int_{\Omega} G(\omega)F(\omega)d\mathbb{P}(\omega) = \mathbb{E}[FG],
\]

for any integrable random variable \( G \).

The Girsanov Theorem can actually be extended to shifts by adapted processes \( (\psi_t)_{t \in [0,T]} \) as follows, cf. e.g. Theorem III-42, page 141 of Protter (2004). The extension of the Girsanov Theorem to jump processes will be
covered in Section 19.6. Recall also that here, \( \Omega = C_0([0,T]) \) is the Wiener space and \( \omega \in \Omega \) is a continuous function on \([0,T]\) starting at 0 in \( t = 0 \).

**Theorem 7.3.** Let \((\psi_t)_{t \in [0,T]}\) be an adapted process satisfying the Novikov integrability condition

\[
\mathbb{E}\left[ \exp \left( \frac{1}{2} \int_0^T |\psi_t|^2 dt \right) \right] < \infty,
\]

and let \(Q\) denote the probability measure defined by

\[
\frac{dQ}{dP} = \exp \left( - \int_0^T \psi_s dB_s - \frac{1}{2} \int_0^T |\psi_s|^2 ds \right).
\]

Then

\[
\hat{B}_t := B_t + \int_0^t \psi_s ds, \quad 0 \leq t \leq T,
\]

is a standard Brownian motion under \(Q\).

In the case of the simple shift

\[
\hat{B}_t := B_t + \nu t, \quad 0 \leq t \leq T,
\]

by a drift \(\nu t\) with constant \(\nu \in \mathbb{R}\), the process \((\hat{B}_t)_{t \in \mathbb{R}_+}\) is a standard (centered) Brownian motion under the probability measure \(Q\) defined by

\[
\frac{dQ}{dP}(\omega) = \exp \left( -\nu B_T - \frac{\nu^2}{2} T \right) dP(\omega).
\]

For example, the fact that \(\hat{B}_T\) has a standard (centered) Gaussian distribution under \(Q\) can be recovered as follows:

\[
\mathbb{E}_Q \left[ f(\hat{B}_T) \right] = \mathbb{E}_Q [f(\nu T + B_T)]
= \int_\Omega f(\nu T + B_T) dQ
= \int_\Omega f(\nu T + B_T) \exp \left( -\nu B_T - \frac{1}{2} \nu^2 T \right) dP
= \int_\mathbb{R} f(\nu T + y) \exp \left( -\nu y - \frac{1}{2} \nu^2 T \right) e^{-y^2/(2T)} \frac{dy}{\sqrt{2\pi T}}
= \int_\mathbb{R} f(\nu T + y) e^{-\frac{(y^2+\nu^2T)}{2T}} \frac{dy}{\sqrt{2\pi T}}
= \int_\mathbb{R} f(y) e^{-\frac{y^2}{2T}} \frac{dy}{\sqrt{2\pi T}}
= \mathbb{E}_P [f(B_T)],
\]

i.e.
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$$\mathbb{E}_Q[f(\nu T + B_T)] = \int_{\Omega} f(\nu T + B_T) dQ$$

$$= \int_{\Omega} f(B_T) d\mathbb{P}$$

$$= \mathbb{E}_\mathbb{P}[f(B_T)],$$

showing that, under $Q$, $\nu T + B_T$ has a centered Gaussian distribution with variance $T$.

The Girsanov Theorem allows us to extend (7.8) as

$$\mathbb{E}[F] = \mathbb{E}\left[F(B_t + \int_0^T \psi_s ds) \exp\left(-\int_0^T \psi_s dB_s - \frac{1}{2} \int_0^T |\psi_s|^2 ds\right)\right],$$

for all random variables $F \in L^1(\Omega)$, see also Exercise 7.19.

When applied to the (constant) market price of risk (or Sharpe ratio)

$$\psi_t := \frac{\mu - r}{\sigma},$$

the Girsanov Theorem shows that

$$\tilde{B}_t := \frac{\mu - r}{\sigma} t + B_t, \quad 0 \leq t \leq T,$$

is a standard Brownian motion under the probability measure $\mathbb{P}^*$ defined by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left(-\frac{\mu - r}{\sigma} B_T - \frac{(\mu - r)^2}{2\sigma^2} T\right).$$

Hence by Proposition 7.1 the discounted price process $(\tilde{S}_t)_{t \in \mathbb{R}_+}$ solution of

$$d\tilde{S}_t = (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t dB_t = \sigma \tilde{S}_t d\tilde{B}_t, \quad t \in \mathbb{R}_+,$$

is a martingale under $\mathbb{P}^*$, therefore $\mathbb{P}^*$ is a risk-neutral probability measure, and we obviously have $\mathbb{P} = \mathbb{P}^*$ when $\mu = r$.

In the sequel, we consider probability measures $Q$ that are equivalent to $\mathbb{P}$ in the sense that they have the same events of zero probability.

**Definition 7.4.** A probability measure $Q$ on $(\Omega, \mathcal{F})$ is said to be equivalent to another probability measure $\mathbb{P}$ when

$$Q(A) = 0 \quad \text{if and only if} \quad P(A) = 0, \quad \text{for all} \quad A \in \mathcal{F}.$$
7.4 Pricing by the Martingale Method

In this section we give the expression of the Black-Scholes price using expectations of discounted payoffs.

Recall that according to the first fundamental theorem of asset pricing Theorem 5.7, a continuous market is without arbitrage opportunities if and only if there exists (at least) an equivalent risk-neutral probability measure $P^*$ under which the discounted price process

$$\tilde{S}_t := e^{-rt} S_t, \quad t \in \mathbb{R}_+,$$

is a martingale under $P^*$. In addition, when the risk-neutral probability measure is unique, the market is said to be complete.

The equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \quad t \in \mathbb{R}_+,$$

satisfied by the price process $(S_t)_{t \in \mathbb{R}_+}$ can be rewritten using (7.10) as

$$\frac{dS_t}{S_t} = rdt + \sigma d\tilde{B}_t, \quad t \in \mathbb{R}_+, \quad (7.13)$$

with the solution

$$S_t = S_0 e^{\mu t + \sigma B_t - \sigma^2 t / 2} = S_0 e^{rt + \sigma \tilde{B}_t - \sigma^2 t / 2}, \quad t \in \mathbb{R}_+. $$

From by the discounting Lemma 5.13 we have

$$d\tilde{S}_t = (\mu - r) \tilde{S}_t dt + \sigma \tilde{S}_t dB_t$$

$$= \sigma \tilde{S}_t \left( \frac{\mu - r}{\sigma} dt + dB_t \right)$$

$$= \sigma \tilde{S}_t d\tilde{B}_t, \quad t \in \mathbb{R}_+, \quad (7.14)$$

hence the discounted price process

$$\tilde{S}_t := e^{-rt} S_t$$

$$= S_0 e^{(\mu - r)t + \sigma B_t - \sigma^2 t / 2}$$

$$= S_0 e^{\sigma \tilde{B}_t - \sigma^2 t / 2}, \quad t \in \mathbb{R}_+,$$

is a martingale under the probability measure $P^*$ defined by (7.11), i.e. $P^*$ is a risk-neutral probability measure, also called martingale measure.

Therefore, by Lemma 5.14 the discounted value $\tilde{V}_t$ of a self-financing portfolio can be written as

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\[ \tilde{V}_t = \tilde{V}_0 + \int_0^t \xi_u \tilde{S}_u \, \mathrm{d}u \]
\[ = \tilde{V}_0 + \sigma \int_0^t \xi_u \tilde{S}_u \, \mathrm{d}\tilde{B}_u, \quad t \in \mathbb{R}_+ \]

and by Proposition 7.1 it becomes a martingale under \( \mathbb{P}^* \).

As in Chapter 3, the value \( V_t \) at time \( t \) of a self-financing portfolio strategy \( (\xi_t)_{t \in [0,T]} \) hedging an attainable claim payoff \( C \) will be called an arbitrage price of the claim payoff \( C \) at time \( t \) and denoted by \( \pi_t(C) \), \( t \in [0,T] \).

Arbitrage prices can be used to ensure that financial derivatives are “marked” at their fair value (“mark to market”).

**Theorem 7.5.** Let \( (\xi_t, \eta_t)_{t \in [0,T]} \) be a portfolio strategy with price

\[ V_t = \eta_t A_t + \xi_t S_t, \quad t \in [0,T], \]

and let \( C \) be a contingent claim, such that

(i) \( (\xi_t, \eta_t)_{t \in [0,T]} \) is a self-financing portfolio, and

(ii) \( (\xi_t, \eta_t)_{t \in [0,T]} \) hedges the claim payoff \( C \), i.e. we have \( V_T = C \).

Then the arbitrage price of the claim payoff \( C \) is given by

\[ \pi_t(C) = V_t = e^{-(T-t)r} \mathbb{E}^*[C \mid \mathcal{F}_t], \quad 0 \leq t \leq T, \quad (7.15) \]

where \( \mathbb{E}^* \) denotes expectation under the risk-neutral probability measure \( \mathbb{P}^* \).

**Proof.** Since the portfolio strategy \( (\xi_t, \eta_t)_{t \in \mathbb{R}_+} \) is self-financing, by Lemma 5.14 and (7.14) the discounted portfolio value \( \tilde{V}_t = e^{-rt} V_t \) satisfies

\[ \tilde{V}_t = \tilde{V}_0 + \sigma \int_0^t \xi_u \tilde{S}_u \, \mathrm{d}\tilde{B}_u = V_0 + \int_0^t \xi_u \, \mathrm{d}\tilde{S}_u, \quad t \in \mathbb{R}_+, \]

which is a martingale under \( \mathbb{P}^* \) from Proposition 7.1, hence

\[ \tilde{V}_t = \mathbb{E}^*[\tilde{V}_T \mid \mathcal{F}_t] \]
\[ = e^{-rT} \mathbb{E}^*[V_T \mid \mathcal{F}_t] \]
\[ = e^{-rT} \mathbb{E}^*[C \mid \mathcal{F}_t], \]

which implies

\[ V_t = e^{rt} \tilde{V}_t = e^{-(T-t)r} \mathbb{E}^*[C \mid \mathcal{F}_t], \quad 0 \leq t \leq T. \]

\[ \square \]
Proposition 7.6. Assume that $\phi$ is a Lipschitz payoff function, and that 
$(S_t)_{t \in \mathbb{R}_+} = \left( S_0 e^{rB_t + (\sigma - \frac{\sigma^2}{2})t/2} \right)_{t \in \mathbb{R}_+}$ is a geometric Brownian motion. Then 
The function $C(t, x)$ defined in (7.16) is in $C^{1,2}([0, T] \times \mathbb{R}_+)$ and solves the Black-Scholes PDE 

$$
\begin{align*}
  rC(t, x) &= \frac{\partial C}{\partial t}(t, x) + r x \frac{\partial C}{\partial x}(t, x) + \frac{1}{2} x^2 \sigma^2 \frac{\partial^2 C}{\partial x^2}(t, x) \\
  C(T, x) &= \phi(x), \quad x > 0.
\end{align*}
$$

Proof. It can be checked by integrations by parts that the function $C(t, x)$ defined by 

$$
C(t, S_t) = e^{-(T-t)r} \mathbb{E}^*[\phi(S_T) \mid F_t] = e^{-(T-t)r} \mathbb{E}^*[\phi(xS_T / S_t)]_{x=S_t},
$$

$0 \leq t \leq T$, is in $C^{1,2}([0, T] \times \mathbb{R}_+)$ when $\phi$ is a Lipschitz function, from the properties of the lognormal distribution of $S_T$. We note that by (4.23), the application of Itô’s formula Theorem 4.19 to $V_t = C(t, S_t)$ and (7.13) leads to 

$$
d(e^{-rt}C(t, S_t)) = -r e^{-rt}C(t, S_t)dt e^{-rt}dC(t, S_t)$$

$$
= -r e^{-rt}C(t, S_t)dt + e^{-rt} \frac{\partial C}{\partial t}(t, S_t)dt$$

$$
+ e^{-rt} \frac{\partial C}{\partial x}(t, S_t) dS_t + \frac{1}{2} e^{-rt}(dS_t)^2 \frac{\partial^2 C}{\partial x^2}(t, S_t)$$

$$
= -r e^{-rt}C(t, S_t)dt + e^{-rt} \frac{\partial C}{\partial t}(t, S_t)dt$$

$$
+v_t e^{-rt} \frac{\partial C}{\partial x}(t, S_t) dt + u_t e^{-rt} \frac{\partial C}{\partial x}(t, S_t) d\tilde{B}_t + \frac{1}{2} e^{-rt}|u_t|^2 \frac{\partial^2 C}{\partial x^2}(t, S_t) dt$$

$$
= -r e^{-rt}C(t, S_t)dt + e^{-rt} \frac{\partial C}{\partial t}(t, S_t)dt$$

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\[ + r S_t e^{-rt} \frac{\partial C}{\partial x}(t, S_t) dt + \frac{1}{2} e^{-rt} \sigma^2 S_t^2 \frac{\partial^2 C}{\partial x^2}(t, S_t) dt + \sigma e^{-rt} S_t \frac{\partial C}{\partial x}(t, S_t) d\tilde{B}_t. \]

By Lemma 5.14 and Proposition 7.1, the discounted price \( \tilde{V}_t = e^{-rt} C(t, S_t) \) of a self-financing hedging portfolio is a martingale under the risk-neutral measure \( \mathbb{P}^* \), therefore from e.g. Corollary II-6-1, page 72 of Protter (2004), all terms in \( dt \) should vanish in the above expression of

\[ d(e^{-rt} g(t, S_t)) = -r e^{-rt} g(t, S_t) dt + e^{-rt} dg(t, S_t), \]

which shows that

\[ -r C(t, S_t) + \frac{\partial C}{\partial t}(t, S_t) + r S_t \frac{\partial C}{\partial x}(t, S_t) + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 C}{\partial x^2}(t, S_t) = 0, \]

and leads to the Black-Scholes PDE

\[ r C(t, x) = \frac{\partial C}{\partial t}(t, x) + r x \frac{\partial C}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 C}{\partial x^2}(t, x), \quad x > 0. \]

□

Forward contracts

The long forward contract with payoff \( C = S_T - K \) is priced as

\[ V_t = e^{-(T-t)r} \mathbb{E}^* [S_T - K \mid \mathcal{F}_t] \]

\[ = e^{-(T-t)r} \mathbb{E}^* [S_T \mid \mathcal{F}_t] - K e^{-(T-t)r} \]

\[ = S_t - K e^{-(T-t)r}, \quad 0 \leq t \leq T, \]

which recovers the Black-Scholes PDE solution (6.8), i.e.

\[ g(t, x) = x - K e^{-(T-t)r}, \quad x > 0, \quad t \in [0, T]. \]

European call options

In the case of European options with payoff function \( \phi(x) = (x - K)^+ \) we recover the Black-Scholes formula (6.9), cf. Proposition 6.8, by a probabilistic argument.

**Proposition 7.7.** The price at time \( t \) of a European call option with strike price \( K \) and maturity \( T \) is given by

\[ C(t, S_t) = S_t \Phi(d_+(T-t)) - K e^{-(T-t)r} \Phi(d_-(T-t)), \quad 0 \leq t \leq T, \quad (7.17) \]

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proof. The proof of Proposition 7.7 is a consequence of (7.15) and Lemma 7.8 below. Using the relation

\begin{align*}
S_T &= S_t e^{(T-t) r + (\hat{\beta}_T - \hat{\beta}_t) \sigma - (T-t) \sigma^2 / 2}, \quad 0 \leq t \leq T,
\end{align*}

by Theorem 7.5 the price of the portfolio hedging $C$ is given by

\begin{align*}
V_t &= e^{-(T-t) r} \mathbb{E}^* [C \mid \mathcal{F}_t] \\
&= e^{-(T-t) r} \mathbb{E}^* [(S_T - K)^+ \mid \mathcal{F}_t] \\
&= e^{-(T-t) r} \mathbb{E}^* [(S_t e^{(T-t) r + (\hat{\beta}_T - \hat{\beta}_t) \sigma - (T-t) \sigma^2 / 2} - K)^+ \mid \mathcal{F}_t] \\
&= e^{-(T-t) r} \mathbb{E}^* [(e^{m(x)} + X - K)^+]_{x=S_t}, \quad 0 \leq t \leq T,
\end{align*}

where

\[
m(x) := (T-t) r - \frac{\sigma^2}{2} (T-t) + \log x
\]

and

\[
X := (\hat{\beta}_T - \hat{\beta}_t) \sigma \sim \mathcal{N}(0, (T-t) \sigma^2)
\]
is a centered Gaussian random variable with variance

\[
\text{Var} [X] = \text{Var} [(\hat{\beta}_T - \hat{\beta}_t) \sigma] = \sigma^2 \text{Var} [\hat{\beta}_T - \hat{\beta}_t] = (T-t) \sigma^2
\]
under $\mathbb{P}^*$. Hence by Lemma 7.8 below we have

\[
C(t, S_t) = V_t \\
&= e^{-(T-t) r} \mathbb{E}^* [(e^{m(x)} + X - K)^+]_{x=S_t} \\
&= e^{-(T-t) r} e^{m(S_t) + \sigma^2(T-t)/2} \Phi \left( v + \frac{m(S_t) - \log K}{v} \right) \\
&\quad - K e^{-(T-t) r} \Phi \left( \frac{m(S_t) - \log K}{v} \right) \\
&= S_t \Phi \left( v + \frac{m(S_t) - \log K}{v} \right) - K e^{-(T-t) r} \Phi \left( \frac{m(S_t) - \log K}{v} \right) \\
&= S_t \Phi (d_+ (T-t)) - K e^{-(T-t) r} \Phi (d_- (T-t)),
\]
0 ≤ t ≤ T.

Relation (7.17) can also be written as
\[ e^{-(T-t)r} \mathbb{E}^* [(S_T - K)^+ | \mathcal{F}_t] = e^{-(T-t)r} \mathbb{E}^* [(S_T - K)^+ | S_t] \] (7.18)

\[ = S_t \Phi \left( \frac{\log(x/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right) - K e^{-(T-t)r} \Phi \left( \frac{\log(x/K) + (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right), \quad 0 ≤ t ≤ T. \]

**Lemma 7.8.** Let X be a centered Gaussian random variable with variance \( v^2 > 0 \). We have
\[ \mathbb{E} [(e^{m+X} - K)^+] = e^{m+v^2/2} \Phi (v + (m - \log K)/v) - K \Phi ((m - \log K)/v). \]

**Proof.** We have
\[
\mathbb{E} [(e^{m+X} - K)^+] = \frac{1}{\sqrt{2\pi v^2}} \int_{-\infty}^{\infty} (e^{m+x} - K)^+ e^{-x^2/(2v^2)} \, dx \\
= \frac{e^m}{\sqrt{2\pi v^2}} \int_{-\infty}^{\infty} e^{x^2/(2v^2)} \, dx \Phi (x + (m - \log K)/v) - K \Phi ((m - \log K)/v) \\
= \frac{e^{m+v^2/2}}{\sqrt{2\pi v^2}} \int_{-\infty}^{\infty} e^{-x^2/(2v^2)} \, dx \Phi (x + (m - \log K)/v) - K \Phi ((m - \log K)/v) \\
= e^{m+v^2/2} \Phi (v + (m - \log K)/v) - K \Phi ((m - \log K)/v). 
\]

\[ \square \]

**Call-put parity**

Let
\[ P(t, S_t) := e^{-(T-t)r} \mathbb{E}^* [(K - S_T)^+ | \mathcal{F}_t], \quad 0 ≤ t ≤ T, \]
denote the price of the put option with strike price \( K \) and maturity \( T \).

**Proposition 7.9.** Call-put parity. We have the relation
\[ C(t, S_t) - P(t, S_t) = S_t - e^{-(T-t)r} K \] (7.19)
between the Black-Scholes prices of call and put options, in terms of the forward contract price \( S_t - K e^{-(T-t)r} \).
Proof. From Theorem 7.5 we have

\[
C(t, S_t) - P(t, S_t) = e^{-(T-t)r} E^*[(S_T - K)^+ | \mathcal{F}_t] - e^{-(T-t)r} E^*[(K - S_T)^+ | \mathcal{F}_t]
\]

\[
= e^{-(T-t)r} E^*[S_T - K | \mathcal{F}_t] - e^{-(T-t)r} E^*[S_T | \mathcal{F}_t] - K e^{-(T-t)r}
\]

\[
= S_t - e^{-(T-t)r} K, \quad 0 \leq t \leq T,
\]

as we have \( E^*[S_T | \mathcal{F}_t] = e^{-(T-t)r} S_t, \ t \in [0, T], \) under the risk-neutral probability measure \( \mathbb{P}^*. \)

Using the call-put parity Relation (7.19) we can recover the European put option price (6.9) from the European call option price (6.9)-(7.17) as

\[
P(t, S_t) = C(t, S_t) - S_t + e^{-(T-t)r} K
\]

\[
= S_t \Phi(d_+(T-t)) + e^{-(T-t)r} K - S_t - e^{-(T-t)r} K \Phi(d_-(T-t))
\]

\[
= -S_t(1 - \Phi(d_+(T-t))) + e^{-(T-t)r} K(1 - \Phi(d_-(T-t)))
\]

\[
= -S_t \Phi(-d_+(T-t)) + e^{-(T-t)r} K \Phi(-d_-(T-t)).
\]

7.5 Hedging by the Martingale Method

Hedging exotic options

In the next Proposition 7.10 we compute a self-financing hedging strategy leading to an arbitrary square-integrable random claim payoff \( C \in L^2(\Omega) \) of an exotic option admitting a stochastic integral decomposition of the form

\[
C = E^*[C] + \int_0^T \zeta_t dB_t, \quad (7.20)
\]

where \( (\zeta_t)_{t \in [0,T]} \) is a square-integrable adapted process. Consequently, the mathematical problem of finding the stochastic integral decomposition (7.20) of a given random variable has important applications in finance. The process \( (\zeta_t)_{t \in [0,T]} \) can be computed using the Malliavin gradient on the Wiener space, see e.g. Nunno et al. (2009) or § 8.2 of Privault (2009).

Simple examples of stochastic integral decompositions include the relations

\[
B_T^2 = T + 2 \int_0^T B_t dB_t,
\]

cf. Exercise 7.1, and
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\[ B^3_T = 3 \int_0^T (T - t + B_t^2) dB_t, \]

see Exercise 4.5. In the sequel, recall that the risky asset follows the equation

\[
dS_t = \mu dt + \sigma dB_t, \quad t \in \mathbb{R}_+, \quad S_0 > 0,
\]

and as a consequence, the discounted asset price satisfies

\[
d\tilde{S}_t = \sigma \tilde{S}_t d\tilde{B}_t, \quad t \in \mathbb{R}_+, \quad \tilde{S}_0 = S_0 > 0, \tag{7.21}
\]

where \((\tilde{B}_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion under the risk-neutral probability measure \(\mathbb{P}^*\). The following proposition applies to arbitrary square-integrable payoff functions, in particular it covers exotic and path-dependent options.

**Proposition 7.10.** Consider a random claim payoff \(C \in L^2(\Omega)\) and the process \((\zeta_t)_{t \in [0, T]}\) given by (7.20), and let

\[
\xi_t = \frac{e^{-(T-t)r}}{\sigma S_t} \zeta_t, \tag{7.22}
\]

\[
\eta_t = \frac{e^{-(T-t)r}}{A_t} \mathbb{E}^*\left[ C \mid \mathcal{F}_t \right] - \xi_t S_t, \quad 0 \leq t \leq T. \tag{7.23}
\]

Then the portfolio allocation \((\xi_t, \eta_t)_{t \in [0, T]}\) is self-financing, and letting

\[
V_t = \eta_t A_t + \xi_t S_t, \quad 0 \leq t \leq T, \tag{7.24}
\]

we have

\[
V_t = e^{-(T-t)r} \mathbb{E}^*[C \mid \mathcal{F}_t], \quad 0 \leq t \leq T. \tag{7.25}
\]

In particular we have

\[
V_T = C, \tag{7.26}
\]

i.e. the portfolio allocation \((\xi_t, \eta_t)_{t \in [0, T]}\) yields a hedging strategy leading to \(C\), starting from the initial value

\[
V_0 = e^{-rT} \mathbb{E}^*[C].
\]

**Proof.** Relation (7.25) follows from (7.23) and (7.24), and it implies

\[
V_0 = e^{-rT} \mathbb{E}^*[C] = \eta_0 A_0 + \xi_0 S_0
\]

at \(t = 0\), and (7.26) at \(t = T\). It remains to show that the portfolio strategy \((\xi_t, \eta_t)_{t \in [0, T]}\) is self-financing. By (7.20) and Proposition 7.1 we have

\[
V_t = \eta_t A_t + \xi_t S_t = e^{-(T-t)r} \mathbb{E}^*[C \mid \mathcal{F}_t]
\]
\[ e^{-(T-t)r} E^* \left[ E^*[C] + \int_0^T \xi_u d\tilde{B}_u \mid \mathcal{F}_t \right] \]
\[ = e^{-(T-t)r} \left( E^*[C] + \int_0^t \xi_u d\tilde{B}_u \right) \]
\[ = e^{rt} V_0 + e^{-(T-t)r} \int_0^t \xi_u d\tilde{B}_u \]
\[ = e^{rt} V_0 + \sigma \int_0^t \xi_u s_u e^{(t-u)r} d\tilde{B}_u \]
\[ = e^{rt} V_0 + \sigma e^{rt} \int_0^t \xi_u \tilde{s}_u d\tilde{B}_u. \]

By (7.21) this shows that the portfolio strategy \((\xi_t, \eta_t)_{t \in [0,T]}\) given by (7.22)-(7.23) and its discounted portfolio value \(\tilde{V}_t = e^{-rt} V_t\) satisfy
\[ \tilde{V}_t = V_0 + \int_0^t \xi_u d\tilde{s}_u, \quad 0 \leq t \leq T, \]
which implies that \((\xi_t, \eta_t)_{t \in [0,T]}\) is self-financing by Lemma 5.14. \(\square\)

The above proposition shows that there always exists a hedging strategy starting from
\[ V_0 = E^*[C] e^{-rT}. \]
In addition, since there exists a hedging strategy leading to
\[ \tilde{V}_T = e^{-rT} C, \]
then \((\tilde{V}_t)_{t \in [0,T]}\) is necessarily a martingale, with
\[ \tilde{V}_t = E^*[\tilde{V}_T \mid \mathcal{F}_t] = e^{-rT} E^*[C \mid \mathcal{F}_t], \quad 0 \leq t \leq T, \]
and initial value
\[ \tilde{V}_0 = E^*[\tilde{V}_T] = e^{-rT} E^*[C]. \]

**Hedging vanilla options**

In practice, the hedging problem can now be reduced to the computation of the process \((\tilde{\xi}_t)_{t \in [0,T]}\) appearing in (7.20). This computation, called Delta hedging, can be performed by the application of the Itô formula and the Markov property, see e.g. Protter (2001). The next lemma allows us to compute the process \((\tilde{\xi}_t)_{t \in [0,T]}\) in case the payoff \(C\) is of the form \(C = \phi(S_T)\) for some function \(\phi\).

**Lemma 7.11.** Assume that \(\phi\) is a Lipschitz payoff function. Then the function \(C(t, x)\) defined by
\[ C(t, S_t) = E^*[\phi(S_T) \mid S_t] \]
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is in $C^{1,2}([0, T] \times \mathbb{R})$, and the stochastic integral decomposition

$$
\phi(S_T) = \mathbb{E}^*[\phi(S_T)] + \int_0^T \zeta_t dB_t
$$

is given by

$$
\zeta_t = \sigma S_t \frac{\partial C}{\partial x}(t, S_t), \quad 0 \leq t \leq T.
$$

Proof. It can be checked as in the proof of Proposition 7.6 the function $C(t, x)$ is in $C^{1,2}([0, T] \times \mathbb{R})$. Therefore, we can apply the Itô formula to the process

$$
t \mapsto C(t, S_t) = \mathbb{E}^*[\phi(S_T) \mid \mathcal{F}_t],
$$

which is a martingale from the tower property (22.38) of conditional expectations as in (7.34). From the fact that the finite variation term in the Itô formula vanishes when $(C(t, S_t))_{t \in [0, T]}$ is a martingale, (see e.g. Corollary II-6-1 page 72 of Protter (2004)), we obtain:

$$
C(t, S_t) = C(0, S_0) + \sigma \int_0^t S_u \frac{\partial C}{\partial x}(u, S_u) d\tilde{B}_u, \quad 0 \leq t \leq T, \quad (7.29)
$$

with $C(0, S_0) = \mathbb{E}^*[\phi(S_T)]$. Letting $t = T$, we obtain (7.28) by uniqueness of the stochastic integral decomposition (7.27) of $C = \phi(S_T)$.

By (7.35) we also have

$$
\zeta_t = \sigma S_t \frac{\partial}{\partial x} \mathbb{E}^*[\phi(S_T) \mid S_t = x]_{x=S_t}
$$

$$
= \sigma S_t \frac{\partial}{\partial x} \mathbb{E}^* \left[ \phi \left( \frac{S_T}{S_t} \right) \right]_{x=S_t}, \quad 0 \leq t \leq T,
$$

hence

$$
\xi_t = \frac{1}{\sigma S_t} e^{-(T-t)r} \zeta_t
$$

$$
= e^{-(T-t)r} \frac{\partial}{\partial x} \mathbb{E}^* \left[ \phi \left( \frac{S_T}{S_t} \right) \right]_{x=S_t}, \quad 0 \leq t \leq T,
$$

which recovers the formula (6.3) for the Delta of a vanilla option. As a consequence we have $\xi_t \geq 0$ and there is no short selling when the payoff function $\phi$ is non-decreasing.

In the case of European options, the process $\zeta$ can be computed via the next proposition which follows from Lemma 7.11 and the relation

$$
C(t, x) = \mathbb{E}^*[f(S^x_{t,T})], \quad 0 \leq t \leq T, \quad x > 0.
$$

**Corollary 7.12.** Assume that $C = (S_T - K)^+$. Then for $0 \leq t \leq T$ we have

$$
\xi_t
$$
\[ \zeta_t = \sigma S_t \mathbb{E}^*[\frac{S_T}{S_t} \mathbb{1}_{[K,\infty)}(S_T / S_t)] \quad \text{for} \quad x = S_t. \quad (7.31) \]

By evaluating the expectation (7.31) in Corollary 7.12 we can recover the formula (6.12) in Proposition 6.3 for the Delta of a European call option in the Black-Scholes model. In that sense, the next proposition provides another proof of the result of Proposition 6.3.

**Proposition 7.13.** The Delta of a European call option with payoff function \( f(x) = (x - K)^+ \) is given by

\[
\xi_t = \Phi(d_+(T-t)) = \Phi \left( \frac{\log(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right), \quad 0 \leq t \leq T.
\]

**Proof.** By Proposition 7.10 and Corollary 7.12, we have

\[
\xi_t = \frac{1}{\sigma S_t} e^{-(T-t)r} \zeta_t
\]

\[
= e^{-(T-t)r} \mathbb{E}^* \left[ \frac{S_T}{S_t} \mathbb{1}_{[K,\infty)} \left( x \frac{S_T}{S_t} \right) \right]_{x=S_t}
\]

\[
= e^{-(T-t)r} \mathbb{E}^* \left[ e^{(\hat{B}_T - \hat{B}_t) \sigma - (T-t) \sigma^2/2 + (T-t)r} \mathbb{1}_{[K,\infty)} \left( x e^{(\hat{B}_T - \hat{B}_t) \sigma - (T-t) \sigma^2/2 + (T-t)r} \right) \right]_{x=S_t}
\]

\[
= \frac{1}{\sqrt{2(T-t)\pi}} \int_{-\infty}^{\infty} e^{\sigma y - (T-t)\sigma^2/2 - y^2/(2(T-t))} dy
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-d_-(T-t)}^{d_-(T-t)} e^{-y^2/2} dy
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-d_+(T-t)}^{d_+(T-t)} e^{-y^2/2} dy
\]

\[
= \Phi(d_+(T-t)).
\]

\[ \square \]

Proposition 7.13, combined with Proposition 7.7, shows that the Black-Scholes self-financing hedging strategy is to hold a (possibly fractional) quantity

\[
\xi_t = \Phi(d_+(T-t)) = \Phi \left( \frac{\log(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right) \geq 0 \quad (7.32)
\]

of the risky asset, and to borrow a quantity
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\[- \eta_t = K e^{-rT} \Phi \left( \frac{\log(S_t/K) + (r - 0.5\sigma^2)(T-t)}{\sigma \sqrt{T-t}} \right) \geq 0 \quad (7.33)\]

of the riskless (savings) account, cf. also Corollary 15.15 in Chapter 15.

As noted above, the result of Proposition 7.13 recovers (6.13) which is obtained by a direct differentiation of the Black-Scholes function as in (6.3) or (7.30).

**Markovian semi-group**

**Definition 7.14.** The Markov semi-group \((P_t)_{0 \leq t \leq T}\) associated to \((S_t)_{t \in [0,T]}\) is the mapping \(P_t\) defined on functions \(f \in C_b^2(\mathbb{R})\) as

\[
P_t f(x) := \mathbb{E}^*[f(S_t) \mid S_0 = x], \quad t \in \mathbb{R}_+.
\]

By the Markov property and time homogeneity of \((S_t)_{t \in [0,T]}\) we also have

\[
P_t f(S_u) := \mathbb{E}^*[f(S_{t+u}) \mid \mathcal{F}_u] = \mathbb{E}^*[f(S_{t+u}) \mid S_u], \quad t, u \in \mathbb{R}_+,
\]

and the semi-group \((P_t)_{0 \leq t \leq T}\) satisfies the composition property

\[
P_s P_t = P_{t+s} = P_{t+s}, \quad s, t \in \mathbb{R}_+,
\]

as we have, using the Markov property and the tower property \((22.38)\) of conditional expectations as in \((7.34)\),

\[
P_s P_t f(x) = \mathbb{E}^* [P_t f(S_s) \mid S_0 = x]
  = \mathbb{E}^* \left[ \mathbb{E}^* \left[ f(S_t) \mid S_0 = y \right]_{y=S_s} \mid S_0 = x \right]
  = \mathbb{E}^* \left[ \mathbb{E}^* \left[ f(S_{t+s}) \mid S = y \right]_{y=S_s} \mid S_0 = x \right]
  = \mathbb{E}^* \left[ \mathbb{E}^* \left[ f(S_{t+s}) \mid \mathcal{F}_s \right] \mid S_0 = x \right]
  = \mathbb{E}^* \left[ f(S_{t+s}) \mid \mathcal{F}_s \mid S_0 = x \right]
  = P_{t+s} f(x), \quad s, t \geq 0.
\]

Similarly we can show that the process \((P_{T-t} f(S_t))_{t \in [0,T]}\) is an \(\mathcal{F}_t\)-martingale as in Example (7.1), *i.e.*:

\[
\mathbb{E}^* \left[ P_{T-t} f(S_t) \mid \mathcal{F}_u \right] = \mathbb{E}^* \left[ \mathbb{E}^* \left[ f(S_T) \mid \mathcal{F}_t \right] \mid \mathcal{F}_u \right]
  = \mathbb{E}^* \left[ f(S_T) \mid \mathcal{F}_u \right]
  = P_{T-u} f(S_u), \quad 0 \leq u \leq t \leq T, \quad (7.34)
\]

and we have
\[ P_{t-u} f(x) = \mathbb{E}^*[f(S_t) \mid S_u = x] = \mathbb{E}^*[f \left( \frac{S_t}{S_u} \right)], \quad 0 \leq u \leq t. \quad (7.35) \]

Exercises

Exercise 7.1  (Exercise 6.1 continued). Consider a market made of a riskless asset priced \( A_t = A_0 \) with zero interest rate, \( t \in \mathbb{R}_+ \), and a risky asset whose price modeled by a standard Brownian motion as \( S_t = B_t, \ t \in \mathbb{R}_+ \). Price the vanilla option with payoff \( C = B_t^2 \), and deduce the solution of the Black-Scholes PDE of Exercise 6.1.

Exercise 7.2  Given the price process \((S_t)_{t \in \mathbb{R}_+}\) defined as
\[ S_t := S_0 e^{\sigma B_t + (r - \sigma^2/2)t}, \quad t \in \mathbb{R}_+, \]
price the option with payoff function \( \phi(S_T) \) by writing \( e^{-rT} \mathbb{E}^*[\phi(S_T)] \) as an integral.

Exercise 7.3  Consider an asset price \((S_t)_{t \in \mathbb{R}_+}\) which is a martingale under the risk-neutral probability measure \( \mathbb{P}^* \) in a market with interest rate \( r = 0 \), and let \( \phi(x) = (x - K)^+ \) be the (convex) European call payoff function.

Show that, for any two maturities \( T_1 < T_2 \) and \( p, q \in [0, 1] \) such that \( p + q = 1 \), the price of the average option with payoff \( \phi(pS_{T_1} + qS_{T_2}) \) is upper bounded by the price of the European call option with maturity \( T_2 \), i.e. show that
\[ \mathbb{E}^* [\phi(pS_{T_1} + qS_{T_2})] \leq \mathbb{E}^* [\phi(S_{T_2})]. \]

Hint 1: For \( \phi \) a convex function we have \( \phi(px + qy) \leq p\phi(x) + q\phi(y) \) for any \( x, y \in \mathbb{R} \) and \( p, q \in [0, 1] \) such that \( p + q = 1 \).

Hint 2: Any convex function \( \phi(S_t) \) of a martingale \( S_t \) is a submartingale.

Exercise 7.4  Consider an underlying asset price process \((S_t)_{t \in \mathbb{R}_+}\).

a) Show that the price at time \( t \) of a European call option with strike price \( K \) and maturity \( T \) is lower bounded by the positive part \( (S_t - K e^{-(T-t)r})^+ \) of the corresponding forward contract price, i.e.
\[ e^{-(T-t)r} \mathbb{E}^* [(S_T - K)^+ \mid \mathcal{F}_t] \geq (S_t - K e^{-(T-t)r})^+, \quad 0 \leq t \leq T. \]

b) Show that the price at time \( t \) of a European put option with strike price \( K \) and maturity \( T \) is lower bounded by \( K e^{-(T-t)r} - S_t \), i.e.
\[ e^{-(T-t)r} \mathbb{E}^* [(K - S_T)^+ \mid \mathcal{F}_t] \geq (K e^{-(T-t)r} - S_t)^+, \quad 0 \leq t \leq T. \]
Exercise 7.5  The following two graphs describe the payoff functions $\phi$ of 
bull spread and bear spread options with payoff $\phi(S_N)$ on an underlying asset 
priced $S_N$ at maturity time $N$.

![Fig. 7.3: Payoff functions of bull spread and bear spread options.]

(a) Show that in each case (i) and (ii) the corresponding option can be real-
ized by purchasing and/or short selling standard European call and put 
options with strike prices to be specified.

(b) Price the bull spread option in cases (i) and (ii).

*Hint:* An option with payoff $\phi(S_N)$ is priced $(1 + r)^{-N} \mathbb{E}^*[\phi(S_N)]$ at time 0.
The payoff of the European call (resp. put) option with strike price $K$ is 
$(S_N - K)^+$, resp. $(K - S_N)^+$.

Exercise 7.6 Forward contracts revisited. Consider a risky asset whose price 
$S_t$ is given by $S_t = S_0 e^{\sigma B_t + r t - \sigma^2 t / 2}$, $t \in \mathbb{R}_+$, where $(B_t)_{t \in \mathbb{R}_+}$ is a standard 
Brownian motion. Consider a forward contract with maturity $T$ and payoff 
$S_T - \kappa$.

(a) Compute the price $C_t$ of this claim at any time $t \in [0, T]$.

(b) Compute a hedging strategy for the option with payoff $S_T - \kappa$.

Exercise 7.7 Option pricing with dividends (Exercise 6.3 continued). Con-
sider an underlying asset price process $(S_t)_{t \in \mathbb{R}_+}$ paying dividends at the 
continuous-time rate $\delta > 0$, and modeled as

$$dS_t = (\mu - \delta)S_t dt + \sigma S_t dB_t,$$

where $(B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion.

(a) Show that as in Lemma 5.14, if $(\eta_t, \xi_t)_{t \in \mathbb{R}_+}$ is a portfolio strategy with 
value
where the dividend yield $\delta S_t$ per share is continuously reinvested in the portfolio, then the discounted portfolio value $\tilde{V}_t$ can be written as the stochastic integral

$$
\tilde{V}_t = \tilde{V}_0 + \int_0^t \xi_u d\tilde{S}_u, \quad t \in \mathbb{R}_+,
$$

b) Show that as in Theorem 7.5, if $(\xi_t, \eta_t)_{t \in [0, T]}$ hedges the claim payoff $C$, i.e. if $V_T = C$, then the arbitrage price of the claim payoff $C$ is given by

$$
\pi_t(C) = V_t = e^{-(T-t)r} \mathbb{E}^*[C | \mathcal{F}_t], \quad 0 \leq t \leq T,
$$

where $\mathbb{E}$ denotes expectation under a suitably chosen risk-neutral probability measure $\mathbb{P}$.

c) Compute the price at time $t \in [0, T]$ of the European call option in a market with dividend rate $\delta$ by the martingale method.

Exercise 7.8 Forward start options (Rubinstein (1991)). A forward start European call option is an option whose holder receives at time $T_1$ (e.g. your birthday) the value of a standard European call option at the money and with maturity $T_2 > T_1$. Price this birthday present at any time $t \in [0, T_1]$, i.e. compute the price

$$
e^{-(T_1-t)r} \mathbb{E}^*[e^{-(T_2-T_1)r} \mathbb{E}^*[(S_{T_2} - S_{T_1})^+ | \mathcal{F}_{T_1}] | \mathcal{F}_t]
$$
ad time $t \in [0, T_1]$, of the forward start European call option using the Black-Scholes formula.

Exercise 7.9 Log-contracts. (Exercise 6.9 continued), see also Exercise 8.3. Consider the price process $(S_t)_{t \in [0, T]}$ given by

$$
\frac{dS_t}{S_t} = rdt + \sigma dB_t
$$
and a riskless asset valued $A_t = A_0 e^{rt}, t \in [0, T]$, with $r > 0$. Compute the arbitrage price

$$
C(t, S_t) = e^{-(T-t)r} \mathbb{E}^*[\log S_T | \mathcal{F}_t],
$$
at time $t \in [0, T]$, of the log-contract with payoff $\log S_T$.

Exercise 7.10 Power option. (Exercise 6.5 continued). Consider the price process $(S_t)_{t \in [0, T]}$ given by
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\[ \frac{dS_t}{S_t} = rdt + \sigma dB_t \]

and a riskless asset valued \( A_t = A_0 e^{rt}, t \in [0, T], \) with \( r > 0. \) In this problem, \((\eta_t, \xi_t)_{t \in [0, T]}\) denotes a portfolio strategy with value

\[ V_t = \eta_t A_t + \xi_t S_t, \quad 0 \leq t \leq T. \]

a) Compute the arbitrage price

\[ C(t, S_t) = e^{-(T-t)r} E^* [ |S_T|^2 | \mathcal{F}_t ], \]

at time \( t \in [0, T], \) of the power option with payoff \( |S_T|^2. \)
b) Compute a self-financing hedging strategy \((\eta_t, \xi_t)_{t \in [0, T]}\) hedging the claim payoff \( |S_T|^2. \)

Exercise 7.11 Bachelier model (Exercise 6.11 continued).

a) Consider the solution \((S_t)_{t \in \mathbb{R}_+}\) of the stochastic differential equation

\[ dS_t = \alpha S_t dt + \sigma dB_t. \]

For which value \( \alpha_M \) of \( \alpha \) is the discounted price process \( \tilde{S}_t = e^{-rt}S_t, \)
\( 0 \leq t \leq T, \) a martingale under \( \mathbb{P}? \)
b) For each value of \( \alpha, \) build a probability measure \( \mathbb{P}_\alpha \) under which the discounted price process \( \tilde{S}_t = e^{-rt}S_t, \) \( 0 \leq t \leq T, \) is a martingale.
c) Compute the arbitrage price

\[ C(t, S_t) = e^{-(T-t)r} E_\alpha [ e^{S_T} | \mathcal{F}_t ] \]

at time \( t \in [0, T], \) of the contingent claim with payoff \( \exp(S_T), \) and recover the result of Exercise 6.11.
d) Explicitly compute the portfolio strategy \((\eta_t, \xi_t)_{t \in [0, T]}\) that hedges the contingent claim payoff \( \exp(S_T). \)
e) Check that this strategy is self-financing.

Exercise 7.12 Compute the arbitrage price

\[ C(t, S_t) = e^{-(T-t)r} E_\alpha [ S_T^2 | \mathcal{F}_t ] \]

at time \( t \in [0, T], \) of the power option with payoff \( S_T^2, \) in the framework of the Bachelier model of Exercise 7.11.

Exercise 7.13 (Exercise 6.2 continued). Price the option with vanilla payoff \( C = \phi(S_T) \) using the noncentral Chi square probability density function.
(16.5) of the Cox et al. (1985) (CIR) model.

Exercise 7.14 Let \((B_t)_{t \in \mathbb{R}_+}\) be a standard Brownian motion generating a filtration \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\). Recall that for \(f \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{R})\), Itô’s formula for \((B_t)_{t \in \mathbb{R}_+}\) reads

\[
f(t, B_t) = f(0, B_0) + \int_0^t \frac{\partial f}{\partial s}(s, B_s)ds + \int_0^t \frac{\partial f}{\partial x}(s, B_s)dB_s + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s)ds.
\]

a) Let \(r \in \mathbb{R}, \sigma > 0, f(x, t) = e^{rt + \sigma x - \sigma^2 t/2}, \) and \(S_t = f(t, B_t)\). Compute \(df(t, B_t)\) by Itô’s formula, and show that \(S_t\) solves the stochastic differential equation

\[
dS_t = rS_t dt + \sigma S_t dB_t,
\]

where \(r > 0\) and \(\sigma > 0\).

b) Show that

\[
\mathbb{E} \left[ e^{\sigma B_T} \mid \mathcal{F}_t \right] = e^{\sigma B_t + (T-t)\sigma^2/2}, \quad 0 \leq t \leq T.
\]

*Hint:* Use the independence of increments of \((B_t)_{t \in [0, T]}\) in the time splitting decomposition

\[
B_T = (B_t - B_0) + (B_T - B_t),
\]

and the Gaussian moment generating function \(\mathbb{E} \left[ e^{\alpha X} \right] = e^{\alpha^2 \eta^2/2}\) when \(X \simeq \mathcal{N}(0, \eta^2)\).

c) Show that the process \((S_t)_{t \in \mathbb{R}_+}\) satisfies

\[
\mathbb{E} \left[ S_T \mid \mathcal{F}_t \right] = e^{(T-t)r} S_t, \quad 0 \leq t \leq T.
\]

d) Let \(C = S_T - K\) denote the payoff of a forward contract with exercise price \(K\) and maturity \(T\). Compute the discounted expected payoff

\[
V_t := e^{-(T-t)r} \mathbb{E} \left[ C \mid \mathcal{F}_t \right].
\]

e) Find a self-financing portfolio strategy \((\xi_t, \eta_t)_{t \in \mathbb{R}_+}\) such that

\[
V_t = \xi_t S_t + \eta_t A_t, \quad 0 \leq t \leq T,
\]

where \(A_t = A_0 e^{rt}\) is the price of a riskless asset with fixed interest rate \(r > 0\). Show that it recovers the result of Exercise 6.7-(c).

f) Show that the portfolio allocation \((\xi_t, \eta_t)_{t \in [0, T]}\) found in Question (e) hedges the payoff \(C = S_T - K\) at time \(T\), i.e. show that \(V_T = C\).
Exercise 7.15 Binary options. Consider a price process \((S_t)_{t \in \mathbb{R}_+}\) given by
\[
\frac{dS_t}{S_t} = r dt + \sigma dB_t, \quad S_0 = 1,
\]
under the risk-neutral probability measure \(P^*\). A binary (or digital) call, resp. put, option is a contract with maturity \(T\), strike price \(K\), and payoff
\[
C_d := \begin{cases} 
1 & \text{if } S_T \geq K, \\
0 & \text{if } S_T < K,
\end{cases} \quad \text{resp.} \quad P_d := \begin{cases} 
1 & \text{if } S_T \leq K, \\
0 & \text{if } S_T > K.
\end{cases}
\]
Recall that the prices \(\pi_t(C_d)\) and \(\pi_t(P_d)\) at time \(t\) of the binary call and put options are given by the discounted expected payoffs
\[
\pi_t(C_d) = e^{-(T-t)r} \mathbb{E} \left[ C_d | \mathcal{F}_t \right] \quad \text{and} \quad \pi_t(P_d) = e^{-(T-t)r} \mathbb{E} \left[ P_d | \mathcal{F}_t \right].
\]

(a) Show that the payoffs \(C_d\) and \(P_d\) can be rewritten as
\[
C_d = \mathbb{1}_{[K,\infty)}(S_T) \quad \text{and} \quad P_d = \mathbb{1}_{[0,K]}(S_T).
\]
(b) Using Relation (7.36), Question (a), and the relation
\[
\mathbb{E} \left[ \mathbb{1}_{[K,\infty)}(S_T) \right] | S_t = x = \mathbb{P}^*(S_T \geq K | S_t = x),
\]
show that the price \(\pi_t(C_d)\) is given by
\[
\pi_t(C_d) = C_d(t, S_t),
\]
where \(C_d(t, x)\) is the function defined by
\[
C_d(t, x) := e^{-(T-t)r} \mathbb{P}^*(S_T \geq K | S_t = x).
\]
(c) Using the results of Exercise 5.8-(d) and of Question (b), show that the price \(\pi_t(C_d)\) of the binary call option is given by
\[
C_d(t, x) = e^{-(T-t)r} \Phi \left( \frac{(r - \sigma^2/2)(T-t) + \log(x/K)}{\sigma \sqrt{T-t}} \right)
= e^{-(T-t)r} \Phi \left( d_-(T-t) \right),
\]
where
\[
d_-(T-t) = \frac{(r - \sigma^2/2)(T-t) + \log(S_t/K)}{\sigma \sqrt{T-t}}.
\]
(d) Assume that the binary option holder is entitled to receive a “return amount” \(\alpha \in [0,1]\) in case the underlying asset price ends out of the
money at maturity. Compute price at time $t \in [0, T]$ of this modified contract.

e) Using Relation (7.36) and Question (a), prove the call-put parity relation

$$\pi_t(C_d) + \pi_t(P_d) = e^{-(T-t)r}, \quad 0 \leq t \leq T.$$  \hfill (7.37)

If needed, you may use the fact that $\mathcal{P}^*(S_T = K) = 0$.

f) Using the results of Questions (e) and (c), show that the price $\pi_t(P_d)$ of the binary put option is given by

$$\pi_t(P_d) = e^{-(T-t)r} \Phi(-d_-(T-t)).$$

g) Using the result of Question (c), compute the Delta

$$\xi_t := \frac{\partial C_d}{\partial x}(t, S_t)$$

of the binary call option. Does the Black-Scholes hedging strategy of such a call option involve short selling? Why?

h) Using the result of Question (f), compute the Delta

$$\xi_t := \frac{\partial P_d}{\partial x}(t, S_t)$$

of the binary put option. Does the Black-Scholes hedging strategy of such a put option involve short selling? Why?

Exercise 7.16 Computation of Greeks. Consider an underlying asset whose price $(S_t)_{t \in \mathbb{R}^+}$ is given by a stochastic differential equation of the form

$$dS_t = rS_t dt + \sigma(S_t) dW_t,$$

where $\sigma(x)$ is a Lipschitz coefficient, and an option with payoff function $\phi$ and price

$$C(x, T) = e^{-rT} \mathbb{E} \left[ \phi(S_T) \mid S_0 = x \right],$$

where $\phi(x)$ is a twice continuously differentiable ($C^2$) function, with $S_0 = x$. Using the Itô formula, show that the sensitivity

$$\text{Theta}_T = \frac{\partial}{\partial T} \left( e^{-rT} \mathbb{E} \left[ \phi(S_T) \mid S_0 = x \right] \right)$$

of the option price with respect to maturity $T$ can be expressed as

$$\text{Theta}_T = -r e^{-rT} \mathbb{E} \left[ \phi(S_T) \mid S_0 = x \right] + e^{-rT} \mathbb{E} \left[ S_t \phi'(S_T) \mid S_0 = x \right] + \frac{1}{2} e^{-rT} \mathbb{E} \left[ \phi''(S_T) \sigma^2(S_T) \mid S_0 = x \right].$$
Problem 7.17  Chooser options. In this problem we denote by $C(t, S_t, K, T)$, resp. $P(t, S_t, K, T)$, the price at time $t$ of a European call, resp. put, option with strike price $K$ and maturity $T$, on an underlying asset priced $S_t = S_0 e^{\sigma B_t + rt - \sigma^2 t/2}$, $t \in \mathbb{R}_+$, where $(B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure.

a) Prove the call-put parity formula

$$C(t, S_t, K, T) - P(t, S_t, K, T) = S_t - K e^{-(T-t)r}, \quad 0 \leq t \leq T. \quad (7.38)$$

b) Consider an option contract with maturity $T$, which entitles its holder to receive at time $T$ the value of a European put option with strike price $K$ and maturity $U > T$.
Write down the price this contract at time $t \in [0, T]$ using a conditional expectation under the risk-neutral probability measure $P^*$.

c) Consider now an option contract with maturity $T$, which entitles its holder to receive at time $T$ either the value of a European call option or a European put option whichever is higher. The European call and put options have same strike price $K$ and same maturity $U > T$.

Show that at maturity $T$, the payoff of this contract can be written as

$$P(T, S_T, K, U) + \max\left(0, S_T - K e^{-(U-T)r}\right).$$

Hint: Use the call-put parity formula (7.38).

d) Price the contract of Question (c) at any time $t \in [0, T]$ using the call and put option pricing functions $C(t, x, K, T)$ and $P(t, x, K, U)$.

e) Using the Black-Scholes formula, compute the self-financing hedging strategy $(\xi_t, \eta_t)_{t \in [0, T]}$ with portfolio value

$$V_t = \xi_t S_t + \eta_t e^{rt}, \quad 0 \leq t \leq T,$$

for the option contract of Question (c).

f) Consider now an option contract with maturity $T$, which entitles its holder to receive at time $T$ the value of either a European call or a European put option, whichever is lower. The two options have same strike price $K$ and same maturity $U > T$.

Show that the payoff of this contract at maturity $T$ can be written as

$$C(T, S_T, K, U) - \max\left(0, S_T - K e^{-(U-T)r}\right).$$

g) Price the contract of Question (f) at any time $t \in [0, T]$.

h) Using the Black-Scholes formula, compute the self-financing hedging strategy $(\xi_t, \eta_t)_{t \in [0, T]}$ with portfolio price
\[ V_t = \xi_t S_t + \eta_t e^{rt}, \quad 0 \leq t \leq T, \]

for the option contract of Question (f).
i) Give the price and hedging strategy of the contract that yields the sum of the payoffs of Questions (c) and (f).
j) What happens when \( U = T \)? Give the payoffs of the contracts of Questions (c), (f) and (i).

Problem 7.18 (Peng (2010)) Consider a risky asset priced

\[ S_t = S_0 e^{\sigma B_t + \mu t - \sigma^2 t/2}, \]

i.e. \( dS_t = \mu S_t dt + \sigma S_t dB_t \), \( t \in \mathbb{R}_+ \),
a riskless asset valued \( A_t = A_0 e^{rt} \), and a self-financing portfolio allocation \((\eta_t, \xi_t)_{t \in \mathbb{R}_+}\) with value

\[ V_t := \eta_t A_t + \xi_t S_t, \quad t \in \mathbb{R}_+. \]

a) Using the portfolio self-financing condition \( dV_t = \eta_t dA_t + \xi_t dS_t \), show that we have

\[ V_T = V_t + \int_t^T (rV_s + (\mu - r)\xi_s S_s) ds + \sigma \int_t^T \xi_s S_s dB_s. \]

b) Show that under the risk-neutral probability measure \( \mathbb{P}^* \) the portfolio value \( V_t \) satisfies the Backward Stochastic Differential Equation (BSDE)

\[ V_t = V_T - \int_t^T rV_s ds - \int_t^T \pi_s d\widehat{B}_s, \quad (7.39) \]

where \( \pi_t := \sigma \xi_t S_t \) is the risky amount invested on the asset \( S_t \), multiplied by \( \sigma \), and \((\widehat{B}_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion under \( \mathbb{P}^* \).

Hint: the Girsanov Theorem 7.3 states that

\[ \widehat{B}_t := B_t + \left( \frac{\mu - r}{} \right) t, \quad t \in \mathbb{R}_+, \]

is a standard Brownian motion under \( \mathbb{P}^* \).

c) Show that under the risk-neutral probability measure \( \mathbb{P}^* \), the discounted portfolio value \( \widetilde{V}_t := e^{-rt}V_t \) can be rewritten as

\[ \widetilde{V}_T = \widetilde{V}_0 + \int_0^T e^{-rs} \pi_s d\widehat{B}_s. \quad (7.40) \]

d) Express \( dv(t, S_t) \) by the Itô formula, where \( v(t, x) \) is a \( C^2 \) function of \( t \) and \( x \).
e) Consider now a more general BSDE of the form
Martingale Approach to Pricing and Hedging

\[ V_t = V_T - \int_t^T f(s, S_s, V_s, \pi_s) ds - \int_t^T \pi_s dB_s, \quad (7.41) \]

with terminal condition \( V_T = g(S_T) \). By matching (7.41) to the Itô formula of Question (d), find the PDE satisfied by the function \( v(t, x) \) defined as \( V_t = v(t, S_t) \).

f) Show that when \( f(t, x, v, z) = rv + \frac{\mu - r}{\sigma} z \), the PDE of Question (e) recovers the standard Black-Scholes PDE.

g) Assuming again \( f(t, x, v, z) = rv + \frac{\mu - r}{\sigma} z \) and taking the terminal condition \( V_T = (S_0 e^{\sigma B_T + (\mu - \sigma^2/2)T} - K)^+ \), give the process \( (\pi_t)_{t \in [0, T]} \) appearing in the stochastic integral representation (7.40) of the discounted claim payoff \( e^{-rT}(S_0 e^{\sigma B_T + (\mu - \sigma^2/2)T} - K) \).

h) From now on we assume that short selling is penalized\(^\dagger\) at a rate \( \gamma > 0 \), i.e. \( \gamma S_t |\xi_t|dt \) is subtracted from the portfolio value change \( dV_t \) whenever \( \xi_t < 0 \) over the time period \([t, t + dt]\). Rewrite the self-financing condition using \((\xi_t)^- := -\min(\xi_t, 0)\).

i) Find the BSDE of the form (7.41) satisfied by \( (V_t)_{t \in \mathbb{R}_+} \), and the corresponding function \( f(t, x, v, z) \).

j) Under the above penalty on short selling, find the PDE satisfied by the function \( u(t, x) \) when the portfolio value \( V_t \) is given as \( V_t = u(t, S_t) \).

k) **Differential interest rate.** Assume that one can borrow only at a rate \( R \) which is higher\(^\ddagger\) than the risk-free interest rate \( r > 0 \), i.e. we have

\[ dV_t = R \eta_t A_t dt + \xi_t dS_t \]

when \( \eta_t < 0 \), and

\[ dV_t = r \eta_t A_t dt + \xi_t dS_t \]

when \( \eta_t > 0 \). Find the PDE satisfied by the function \( u(t, x) \) when the portfolio value \( V_t \) is given as \( V_t = u(t, S_t) \).

l) Assume that the portfolio differential reads

\[ dV_t = \eta_t dA_t + \xi_t dS_t - dU_t, \]

where \( (U_t)_{t \in \mathbb{R}_+} \) is a non-decreasing process. Show that the corresponding portfolio strategy \( (\xi_t)_{t \in \mathbb{R}_+} \) is superhedging the claim payoff \( V_T = C \).

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\(^*\) General Black-Scholes knowledge can be used for this question.

\(^\dagger\) SGX started to penalize naked short sales with an interim measure in September 2008.

\(^\ddagger\) Regular savings account usually pays \( r=0.05\% \) per year. Effective Interest Rates (EIR) for borrowing could be as high as \( R=20.61\% \) per year.
Exercise 7.19 Girsanov Theorem. Assume that the Novikov integrability condition (7.7) is not satisfied. How does this modify the statement (7.9) of the Girsanov Theorem 7.3?

Problem 7.20 The Capital Asset Pricing Model (CAPM) of W.F. Sharpe (1990 Nobel Prize in Economics) is based on a linear decomposition

$$\frac{dS_t}{S_t} = (r + \alpha)dt + \beta \left( \frac{dM_t}{M_t} - rdt \right)$$

of stock returns $dS_t/S_t$ into:

- a risk-free interest rate* $r$,
- an excess return $\alpha$,
- a risk premium given by the difference between a benchmark market index return $dM_t/M_t$ and the risk free rate $r$.

The coefficient $\beta$ measures the sensitivity of the stock return $dS_t/S_t$ with respect to the market index returns $dM_t/M_t$. In other words, $\beta$ is the relative volatility of $dS_t/S_t$ with respect to $dM_t/M_t$, and it measures the risk of $(S_t)_{t \in \mathbb{R}^+}$ in comparison to the market index $(M_t)_{t \in \mathbb{R}^+}$.

If $\beta > 1$, resp. $\beta < 1$, then the stock price $S_t$ is more volatile (i.e. more risky), resp. less volatile (i.e. less risky), than the benchmark market index $M_t$. For example, if $\beta = 2$ then $S_t$ goes up (or down) twice as much as the index $M_t$. Inverse Exchange-Traded Funds (IETFs) have a negative value of $\beta$. On the other hand, a fund which has a $\beta = 1$ can track the index $M_t$.

Vanguard 500 Index Fund (VFINX) has a $\beta = 1$ and can be considered as replicating the variations of the S&P 500 index $M_t$, while Invesco S&P 500 (SPHB) has a $\beta = 1.42$, and Xtrackers Low Beta High Yield Bond ETF (HYDW) has a $\beta$ close to 0.36 and $\alpha = 6.36$.

In the sequel we assume that the benchmark market is represented by an index fund $(M_t)_{t \in \mathbb{R}^+}$ whose value is modeled according to

$$\frac{dM_t}{M_t} = \mu dt + \sigma dB_t, \quad (7.42)$$

where $(B_t)_{t \in \mathbb{R}^+}$ is a standard Brownian motion. The asset price $(S_t)_{t \in \mathbb{R}^+}$ is modeled in a stochastic version of the CAPM as

$$\frac{dS_t}{S_t} = r dt + \alpha dt + \beta \left( \frac{dM_t}{M_t} - rdt \right) + \sigma dW_t, \quad (7.43)$$

* The risk-free interest rate $r$ is typically the yield of the 10-year Treasury bond.
with an additional stock volatility term \( \sigma_S dW_t \), where \((W_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion independent of \((B_t)_{t \in \mathbb{R}_+}\), with
\[
\text{Cov}(B_t, W_t) = 0 \quad \text{and} \quad dB_t \cdot dW_t = 0, \quad t \in \mathbb{R}_+.
\]
The following 10 questions are interdependent and should be treated in sequence.

a) Show that \( \beta \) coincides with the regression coefficient
\[
\beta = \frac{\text{Cov}(dS_t/S_t, dM_t/M_t)}{\text{Var}[dM_t/M_t]}.
\]

*Hint:* We have
\[
\text{Cov}(dW_t, dW_t) = dt, \quad \text{Cov}(dB_t, dB_t) = dt, \quad \text{and} \quad \text{Cov}(dW_t, dB_t) = 0.
\]

b) Show that the evolution of \((S_t)_{t \in \mathbb{R}_+}\) can be written as
\[
dS_t = (r + \alpha + \beta(\mu - r))S_t dt + S_t \sqrt{\beta^2 \sigma_M^2 + \sigma_S^2} dZ_t
\]
where \((Z_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion.

*Hint:* The standard Brownian motion \((Z_t)_{t \in \mathbb{R}_+}\) can be characterized as the only continuous (local) martingale such that \((dZ_t)^2 = dt\), see e.g. Theorem 7.36 page 203 of Klebaner (2005).

From now on, we assume that \( \beta \) is allowed to depend locally on the state of the benchmark market index \(M_t\), as \( \beta(M_t), t \in \mathbb{R}_+ \).

c) Rewrite the equations (7.42)-(7.43) into the system
\[
\begin{cases}
\frac{dM_t}{M_t} = rdt + \sigma_M dB_t^*, \\
\frac{dS_t}{S_t} = rdt + \sigma_M \beta(M_t) dB_t^* + \sigma_S dW_t^*,
\end{cases}
\]
where \((B_t^*)_{t \in \mathbb{R}_+}\) and \((W_t^*)_{t \in \mathbb{R}_+}\) have to be determined explicitly.

d) Using the Girsanov theorem, construct a probability measure \( \mathbb{P}^* \) under which \((B_t^*)_{t \in \mathbb{R}_+}\) and \((W_t^*)_{t \in \mathbb{R}_+}\) are independent standard Brownian motions.

*Hint:* Only the expression of the density \( d\mathbb{P}^*/d\mathbb{P} \) is needed here.

e) Show that the market based on the assets \( S_t \) and \( M_t \) is without arbitrage opportunities.

f) Consider a portfolio strategy \((\xi_t, \zeta_t, \eta_t)_{t \in [0,T]}\) based on the three assets \((S_t, M_t, A_t)_{t \in [0,T]}\), with price

\[\Diamond\]
\[ V_t = \xi_t S_t + \zeta_t M_t + \eta_t A_t, \quad t \in [0, T], \]

where \((A_t)_{t \in \mathbb{R}^+}\) is a riskless asset given by \(A_t = A_0 e^{rt}\). Write down the self-financing condition for the portfolio strategy \((\xi_t, \zeta_t, \eta_t)_{t \in [0, T]}\).

g) Consider an option with payoff \(C = h(S_T, M_T)\), priced as

\[
 f(t, S_t, M_t) = e^{-(T-t)r} \mathbb{E}^*[h(S_T, M_T) \mid \mathcal{F}_t], \quad 0 \leq t \leq T.
\]

Assuming that the portfolio \((V_t)_{t \in [0,T]}\) replicates the option price process \((f(t, S_t, M_t))_{t \in [0,T]}\), derive the pricing PDE satisfied by the function \(f(t, x, y)\) and its terminal condition.

**Hint:** The following version of the Itô formula with two variables can be used for the function \(f(t, x, y)\):

\[
 df(t, S_t, M_t) = \frac{\partial f}{\partial t}(t, S_t, M_t)dt + \frac{\partial f}{\partial x}(t, S_t, M_t)dS_t + \frac{1}{2}(dS_t)^2 \frac{\partial^2 f}{\partial x^2}(t, S_t, M_t) \\
+ \frac{\partial f}{\partial y}(t, S_t, M_t)dM_t + \frac{1}{2}(dM_t)^2 \frac{\partial^2 f}{\partial y^2}(t, S_t, M_t) + dS_t \cdot dM_t \frac{\partial^2 f}{\partial x \partial y}(t, S_t, M_t).
\]

h) Find the self-financing hedging portfolio strategy \((\xi_t, \zeta_t, \eta_t)_{t \in [0,T]}\) replicating the vanilla payoff \(h(S_T, M_T)\).

i) Solve the PDE of Question (g) and compute the replicating portfolio of Question (h) when \(\beta(M_t) = \beta\) is a constant and \(C\) is a European call option payoff on \(S_T\) with strike price \(K\).

j) Solve the PDE of Question (g) and compute the replicating portfolio of Question (h) when \(\beta(M_t) = \beta\) is a constant and \(C\) is a European put option payoff on \(S_T\) with strike price \(K\).

**Problem 7.21 Log options.**

a) Consider a market model made of a risky asset with price \((S_t)_{t \in \mathbb{R}^+}\) as in Exercise 4.17-(d) and a riskless asset with price \(A_t = 1 \times e^{rt}\) and risk-free interest rate \(r = \sigma^2/2\). From the answer to Exercise 4.17-(b), show that the arbitrage price

\[
 V_t = e^{-(T-t)r} \mathbb{E}^*[(\log S_T)^+ \mid \mathcal{F}_t]
\]

at time \(t \in [0, T]\) of a log call option with payoff \((\log S_T)^+\) is equal to

\[
 V_t = \sigma e^{-(T-t)r} \sqrt{\frac{T-t}{2\pi}} e^{-B_t^2/(2(T-t))}
\]

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\[ + \sigma e^{-(T-t)r} B_t \Phi \left( \frac{B_t}{\sqrt{T-t}} \right). \]

b) Show that \( V_t \) can be written as

\[ V_t = g(T-t, S_t), \]

where \( g(\tau, x) = e^{-r\tau} f(\tau, \log x) \), and

\[ f(\tau, y) = \sigma \sqrt{\frac{\tau}{2\pi}} \exp\left(-\frac{y^2}{2\sigma^2\tau}\right) + y \Phi\left(\frac{y}{\sigma\sqrt{\tau}}\right). \]

c) Figure 7.4 represents the graph of \((\tau, x) \mapsto g(\tau, x)\), with \( r = 0.05 = 5\% \) \textit{per year} and \( \sigma = 0.1 \). Assume that the current underlying asset price is $1 and there remains 700 days to maturity. What is the price of the option?

![Option price graph](https://example.com/option_price_graph.png)

Fig. 7.4: Option price as a function of underlying asset price and time to maturity.

d) Show\(^*\) that the (possibly fractional) quantity \( \xi_t = \frac{\partial g}{\partial x}(T-t, S_t) \) of \( S_t \) at time \( t \) in a portfolio hedging the payoff \((\log S_T)^+\) is equal to

\[ \xi_t = e^{-(T-t)r} \frac{1}{S_t} \Phi \left( \frac{\log S_t}{\sigma\sqrt{T-t}} \right), \quad 0 \leq t \leq T. \]

e) Figure 7.5 represents the graph of \((\tau, x) \mapsto \frac{\partial g}{\partial x}(\tau, x)\). Assuming that the current underlying asset price is $1 and that there remains 700 days to maturity, how much of the risky asset should you hold in your portfolio in order to hedge one log option?

\(^*\) Recall the chain rule of derivation \( \frac{\partial}{\partial x} f(\tau, \log x) = \frac{1}{x} \frac{\partial f}{\partial y}(\tau, y)|_{y=\log x}. \)
Based on the framework and answers of Questions (c) and (e), should you borrow or lend the riskless asset $A_t = \$1 \times e^{rt}$, and for what amount?

Show that the Gamma of the portfolio, defined as $\Gamma_t = \frac{\partial^2 g}{\partial x^2}(T-t, S_t)$, equals

$$\Gamma_t = e^{-(T-t)r} \frac{1}{S_t^2} \left( \frac{1}{\sigma \sqrt{2(T-t)\pi}} e^{-(\log S_t)^2/(2(T-t)\sigma^2)} - \Phi \left( \frac{\log S_t}{\sigma \sqrt{T-t}} \right) \right),$$

$0 \leq t < T$.

Figure 7.6 represents the graph of Gamma. Assume that there remains 60 days to maturity and that $S_t$, currently at $\$1$, is expected to increase. Should you buy or (short) sell the underlying asset in order to hedge the option?

Let now $\sigma = 1$. Show that the function $f(\tau, y)$ of Question (b) solves the heat equation.
Problem 7.22 Log put options with a given strike price.

a) Consider a market model made of a risky asset with price $(S_t)_{t \in \mathbb{R}^+}$ as in Exercise 5.8, a riskless asset valued $A_t = 1 \times e^{rt}$, risk-free interest rate $r = \sigma^2/2$ and $S_0 = 1$. From the answer to Exercise A.4-(b), show that the arbitrage price

$$V_t = e^{-(T-t)r} \mathbb{E}^* [(K - \log S_T)^+ | \mathcal{F}_t]$$

at time $t \in [0,T]$ of a log call option with strike price $K$ and payoff $(K - \log S_T)^+$ is equal to

$$V_t = \sigma e^{-(T-t)r} \sqrt{\frac{T-t}{2\pi}} \left[ e^{-\left(\frac{(B_t-K/\sigma)^2}{2(T-t)}\right)} + e^{-(T-t)r} (K - \sigma B_t) \Phi \left( \frac{K/\sigma - B_t}{\sqrt{T-t}} \right) \right].$$

b) Show that $V_t$ can be written as

$$V_t = g(T-t, S_t),$$

where $g(\tau, x) = e^{-\tau} f(\tau, \log x)$, and

$$f(\tau, y) = \sigma \sqrt{\frac{\tau}{2\pi}} e^{-\left(\frac{(K-y)^2}{2\sigma^2}\right)} + (K - y) \Phi \left( \frac{K - y}{\sigma \sqrt{\tau}} \right).$$

c) Figure 7.7 represents the graph of $(\tau, x) \mapsto g(\tau, x)$, with $r = 0.125 \text{ per year}$ and $\sigma = 0.5$. Assume that the current underlying asset price is $3$, that $K = 1$, and that there remains 700 days to maturity. What is the price of the option?

![Fig. 7.7: Option price as a function of underlying asset price and time to maturity.](https://www.ntu.edu.sg/home/nprivault/indext.html)
d) Show* that the quantity \( \xi_t = \frac{\partial g}{\partial x}(T-t, S_t) \) of \( S_t \) at time \( t \) in a portfolio hedging the payoff \( (K - \log S_T)^+ \) is equal to

\[
\xi_t = -e^{-(T-t)r} \frac{1}{S_t} \Phi \left( \frac{K - \log S_t}{\sigma \sqrt{T-t}} \right), \quad 0 \leq t \leq T.
\]

e) Figure 7.8 represents the graph of \( (\tau, x) \mapsto \frac{\partial g}{\partial x}(\tau, x) \). Assuming that the current underlying asset price is \( \$3 \) and that there remains 700 days to maturity, how much of the risky asset should you hold in your portfolio in order to hedge one log option?

\[\text{Fig. 7.8: Delta as a function of underlying asset price and time to maturity.}\]

f) Based on the framework and answers of Questions (e) and (e), should you borrow or lend the riskless asset \( A_t = \$1 \times e^{rt} \), and for what amount?

g) Show that the Gamma of the portfolio, defined as \( \Gamma_t = \frac{\partial^2 g}{\partial x^2}(T-t, S_t) \), equals

\[
\Gamma_t = e^{-(T-t)r} \frac{1}{S_t^2} \left( \frac{1}{\sigma \sqrt{2(T-t)\pi}} e^{-(K-\log S_t)^2/(2(T-t)\sigma^2)} + \Phi \left( \frac{K - \log S_t}{\sigma \sqrt{T-t}} \right) \right),
\]

\( 0 \leq t \leq T \).

h) Figure 7.9 represents the graph of Gamma. Assume that there remains 10 days to maturity and that \( S_t \), currently at \( \$3 \), is expected to increase. Should you buy or (short) sell the underlying asset in order to hedge the option?

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* Recall the chain rule of derivation \( \frac{\partial}{\partial x} f(\tau, \log x) = \frac{1}{x} \frac{\partial f(\tau, y)}{\partial y} |_{y=\log x} \).
Fig. 7.9: Gamma as a function of underlying asset price and time to maturity.

i) Show that the function \( f(\tau, y) \) of Question (b) solves the heat equation

\[
\begin{align*}
\frac{\partial f}{\partial \tau}(\tau, y) &= \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial y^2}(\tau, y) \\
 f(0, y) &= (K - y)^+.
\end{align*}
\]