Chapter 12
Lookback Options

In this chapter we price floating strike lookback options, whose payoff with exercise date $T$ is given by the functional

$$C = S_T - \min_{0 \leq t \leq T} S_t$$

of the underlying asset price $(S_t)_{t \in [0,T]}$ in the case of call options, and by

$$C = \left( \max_{0 \leq t \leq T} S_t \right) - S_T,$$

in the case of put options.

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12.1 The Lookback Put Option

The standard lookback put option gives its holder the right to sell the underlying asset at its historically highest price. In this case the strike price is $M_0^T$ and the payoff is given by the terminal value

$$C = M_0^T - S_T$$

of the drawdown process $(M_t^T - S_t)_{t \in [0,T]}$. The following pricing formula for lookback put options is a direct consequence of Proposition 10.6.

**Proposition 12.1.** The price at time $t \in [0,T]$ of the lookback put option with payoff $M_0^T - S_T$ is given by

\[ C = M_0^T - S_T \]
Figure 12.1 represents the lookback put option price as a function of $S_t$ and $M_t^0$, for different values of the time to maturity $T - t$.

From Figures 12.1 and 12.2 we note the following.

i) When the underlying asset price $S_t$ is close to $M_t^0$, an increase in the value $S_t$ results into a higher put option price, since in this case the variation of $S_t$ can increase the value of $M_t^0$.

ii) When the underlying asset price $S_t$ is far from $M_t^0$, an increase in $S_t$ is less likely to affect the value of $M_t^0$, and this results into a lower option price.

Figure 12.2 shows accordingly that, according to the Delta hedging strategy for lookback put options, see Proposition 12.2 below, one should short the underlying asset when $S_t$ is far from $M_t^0$, and long this asset when $S_t$ becomes closer to $M_t^0$.

* The animation works in Acrobat Reader on the entire pdf file.
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![Graph of the lookback put option price (2D) with $M_0^t = 60$.](image)

**Fig. 12.2:** Graph of the lookback put option price (2D) with $M_0^t = 60$.

**Proof of Proposition 12.1.** We have

$$
\mathbb{E}^* \left[ M_0^T - S_T \mid \mathcal{F}_t \right] = \mathbb{E}^* \left[ M_0^T \mid \mathcal{F}_t \right] - \mathbb{E}^* \left[ S_T \mid \mathcal{F}_t \right]
$$

$$
= \mathbb{E}^* \left[ M_0^T \mid \mathcal{F}_t \right] - e^{(T-t)T} S_t,
$$

hence Proposition 10.6 shows that

$$
e^{-\left(T-t\right)T} \mathbb{E}^* \left[ M_0^T - S_T \mid \mathcal{F}_t \right]
$$

$$
= e^{-\left(T-t\right)T} \mathbb{E}^* \left[ M_0^T \mid \mathcal{F}_t \right] - e^{-\left(T-t\right)T} \mathbb{E}^* \left[ S_T \mid \mathcal{F}_t \right]
$$

$$
= e^{-\left(T-t\right)T} \mathbb{E}^* \left[ M_0^T \mid M_0^t \right] - S_t
$$

$$
= M_0^t e^{-\left(T-t\right)T} \Phi \left( -\delta_{-t} T \left( \frac{S_t}{M_0^t} \right) \right) - S_t \Phi \left( -\delta_{+} T \left( \frac{S_t}{M_0^t} \right) \right)
$$

$$
+ S_t \sigma^2 \left( \frac{\delta_{-} T \left( \frac{S_t}{M_0^t} \right)}{2} \right) - S_t \sigma^2 \frac{\delta_{+} T \left( \frac{M_0^t}{S_t} \right)}{2} e^{-\left(T-t\right)T} \left( \frac{M_0^t}{S_t} \right)^{2r/\sigma^2} \Phi \left( -\delta_{-t} T \left( \frac{M_0^t}{S_t} \right) \right).
$$

$\square$

**12.2 PDE Method**

Since the couple $(S_t, M_0^t)$ is a Markov process, the price of the lookback put option at time $t \in [0, T]$ can be written as a function

$$
f(t, S_t, M_0^t) = e^{-\left(T-t\right)T} \mathbb{E}^* \left[ \phi(S_T, M_0^T) \mid \mathcal{F}_t \right]
$$

$$
= e^{-\left(T-t\right)T} \mathbb{E}^* \left[ \phi(S_T, M_0^T) \mid S_t, M_0^t \right]
$$

of $S_t$ and $M_0^t$, $0 \leq t \leq T$.  

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Black-Scholes PDE for lookback put option prices

In the next proposition we derive the partial differential equation (PDE) for the pricing function \( f(t,x,y) \) of a self-financing portfolio hedging a lookback put option.

**Proposition 12.2.** The function \( f(t,x,y) \) defined by
\[
f(t,x,y) = e^{-(T-t)r}E^*[S_T - M_0^T | S_t = x, M_0^T = y], \quad t \in [0,T], \ x,y > 0,
\]
is \( C^2((0,T) \times (0,\infty)^2) \) and satisfies the Black-Scholes PDE
\[
rf(t,x,y) = \frac{\partial f}{\partial t}(t,x,y) + rx\frac{\partial f}{\partial x}(t,x,y) + \frac{1}{2}x^2\sigma^2\frac{\partial^2 f}{\partial x^2}(t,x,y), \quad (12.2)
\]
0 \( \leq t \leq T, \ x,y > 0 \), under the boundary conditions
\[
\begin{align*}
\frac{\partial f}{\partial y}(t,x,y)_{y=x} &= 0, \quad 0 \leq t \leq T, \ y > 0, \quad (12.3b) \\
f(T,x,y) &= y - x, \quad 0 \leq x \leq y. \quad (12.3c)
\end{align*}
\]
The replicating portfolio of the lookback put option is given by
\[
\xi_t = \frac{\partial f}{\partial x}(t,S_t,M_0^t), \quad t \in [0,T]. \quad (12.4)
\]

**Proof.** The existence of \( f(t,x,y) \) follows from the Markov property, more precisely, the function \( f(t,x,y) \) satisfies
\[
\begin{align*}
f(t,x,y) &= e^{-(T-t)r} E^*\left[\phi(S_T,M_0^T) | S_t = x, M_0^T = y\right] \\
&= e^{-(T-t)r} E^*\left[\phi\left(\frac{S_T}{S_t}, \text{Max}(y,M_t^T)\right)\right] \\
&= e^{-(T-t)r} E^*\left[\phi\left(\frac{S_{T-t}}{S_0}, \text{Max}(y,M_0^{T-t})\right)\right], \quad t \in [0,T],
\end{align*}
\]
from the time homogeneity of the asset price process \( (S_t)_{t \in \mathbb{R}_+} \). Applying the change of variable formula to the discounted portfolio value
\[
\tilde{f}(t,x,y) := e^{-rt} f(t,x,y) = e^{-rT} E^*\left[\phi(S_T,M_0^T) | S_t = x, M_0^T = y\right]
\]
which is a martingale indexed by \( t \in [0,T] \), we have
\[
d\tilde{f}(t,S_t,M_0^t) = -r e^{-rt} f(t,S_t,M_0^t) dt + e^{-rt} df(t,S_t,M_0^t)
\]

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\[ = -re^{-rt}f(t, S_t, M^t_0)dt + e^{-rt}\frac{\partial f}{\partial t}(t, S_t, M^t_0)dt + r e^{-rt} S_t \frac{\partial f}{\partial x}(t, S_t, M^t_0)dt \]

\[ + \frac{1}{2} e^{-rt} \sigma^2 S_t^2 \frac{\partial^2 f}{\partial x^2}(t, S_t, M^t_0)dt + e^{-rt} \sigma S_t \frac{\partial f}{\partial x}(t, S_t, M^t_0)dB_t \]

\[ + e^{-rt} \frac{\partial f}{\partial y}(t, S_t, M^t_0) dM^t_0. \quad (12.5) \]

Since \( \tilde{f}(t, S_t, M^t_0)_{t \in [0, T]} = \left( e^{-rT} \mathbb{E}^* \left[ \phi(S_T, M^T_t) \mid \mathcal{F}_t \right] \right)_{t \in [0, T]} \) is a martingale under \( \mathbb{P} \) and \( \left( M^t_0 \right)_{t \in [0, T]} \) has finite variation (it is in fact a non-decreasing process), (12.5) yields:

\[ df(t, S_t, M^t_0) = \sigma S_t \frac{\partial \tilde{f}}{\partial x}(t, S_t, M^t_0) dB_t, \quad t \in [0, T], \quad (12.6) \]

and the function \( f(t, x, y) \) satisfies the equation

\[ \frac{\partial f}{\partial t}(t, S_t, M^t_0)dt + r S_t \frac{\partial f}{\partial x}(t, S_t, M^t_0)dt \]

\[ + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 f}{\partial x^2}(t, S_t, M^t_0)dt + \frac{\partial f}{\partial y}(t, S_t, M^t_0) dM^t_0 = r f(t, S_t, M^t_0)dt, \]

which implies

\[ \frac{\partial f}{\partial t}(t, S_t, M^t_0) + r S_t \frac{\partial f}{\partial x}(t, S_t, M^t_0) + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 f}{\partial x^2}(t, S_t, M^t_0) = r f(t, S_t, M^t_0), \]

which is (12.2), and

\[ \frac{\partial f}{\partial y}(t, S_t, M^t_0) dM^t_0 = 0, \]

because \( M^t_0 \) increases only on a set of zero measure (which has no isolated points), see the Lebesgue decomposition theorem and also the Cantor function. This implies

\[ \frac{\partial f}{\partial y}(t, S_t, M^t_0) = 0, \]

when \( dM^t_0 > 0 \), hence since

\[ \{ S_t = M^t_0 \} \iff dM^t_0 > 0 \]

and

\[ \{ S_t < M^t_0 \} \iff dM^t_0 = 0, \]

we have

\[ \frac{\partial f}{\partial y}(t, S_t, S_t) = \frac{\partial f}{\partial y}(t, x, y)_{x=S_t, y=S_t} = 0, \]
since \( M_t^k \) hits \( S_t \), i.e. \( M_t^k = S_t \), only when \( M_t^k \) increases at time \( t \), and this shows the boundary condition (12.3b).

On the other hand, (12.6) shows that

\[
\phi(S_T, M_0^T) = \mathbb{E}^*[\phi(S_T, M_0^T)] + \sigma \int_0^T S_t \left. \frac{\partial f}{\partial x}(t, x, M_0^t) \right|_{x=S_t} dB_t,
\]

\( 0 \leq t \leq T \), which implies (12.4) as in the proof of Proposition 6.1 or 11.3. □

In other words, the price of the lookback put option takes the form

\[
f(t, S_t, M_0^k) = e^{-(T-t)r} \mathbb{E}^* \left[ M_0^T - S_T \mid \mathcal{F}_t \right],
\]

where the function \( f(t, x, y) \) is given from Proposition 12.1 as

\[
f(t, x) = y e^{-(T-t)r} \Phi \left( -\delta_T^{-t}(x/y) \right) + x \left( 1 + \frac{\sigma^2}{2r} \right) \Phi \left( \delta_T^{-t}(x/y) \right) - x \left( \frac{\sigma^2}{2r} e^{-(T-t)r} \left( \frac{y}{x} \right)^{2r/\sigma^2} \Phi \left( -\delta_T^{-t}(y/x) \right) - x. \right.
\]

(12.7)

**Remark 12.3.** We have

\[
f(t, x, x) = xC(T-t),
\]

with

\[
C(\tau) = e^{-\tau r} \Phi (-\delta_T^{-\tau}(1)) + \left( 1 + \frac{\sigma^2}{2r} \right) \Phi (\delta_T^{-\tau}(1)) - \frac{\sigma^2}{2r} e^{-\tau r} \Phi (-\delta_T^{-\tau}(1)) - 1,
\]

\( \tau > 0 \), hence

\[
\left. \frac{\partial f}{\partial x} \right|_{t=x} = C(T-t), \quad t \in [0, T].
\]

**Scaling property of lookback put option prices**

From (12.7) and the following argument we note the scaling property

\[
f(t, x, y) = e^{-(T-t)r} \mathbb{E}^* \left[ M_0^T - S_T \mid S_t = x, \ M_0^t = y \right]
\]

\[
= e^{-(T-t)r} \mathbb{E}^* \left[ \text{Max} \left( M_0^t, M_T^t \right) - S_T \mid S_t = x, \ M_0^t = y \right]
\]

\[
= e^{-(T-t)r} \mathbb{E}^* \left[ \text{Max} \left( \frac{M_0^t}{S_t}, \frac{M_T^t}{S_t} \right) - S_T \mid S_t = x, \ M_0^t = y \right]
\]

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\[ = e^{-(T-t) r x} \mathbb{E}^* \left[ \max \left( \frac{y}{x}, \frac{M_t^T}{x} \right) - \frac{S_T}{x} \mid S_t = x, \ M_0^t = y \right] \]

\[ = e^{-(T-t) r x} \mathbb{E}^* \left[ \max (M_0^t, M_t^T) - S_T \mid S_t = 1, \ M_0^t = \frac{y}{x} \right] \]

\[ = e^{-(T-t) r x} \mathbb{E}^* \left[ M_0^T - S_T \mid S_t = 1, \ M_0^t = \frac{y}{x} \right] \]

\[ = x f(t, 1, y/x) \]

\[ = x g(T - t, x/y), \]

where we let

\[ g(\tau, z) := \]

\[ \frac{1}{z} e^{-\tau r} \Phi (-\delta^- (z)) + \left( 1 + \frac{\sigma^2}{2r} \right) \Phi (\delta^- (z)) - \frac{\sigma^2}{2r} e^{-\tau r} \left( \frac{1}{z} \right)^{2r/\sigma^2} \Phi \left( -\delta^- \left( \frac{1}{z} \right) \right) - 1, \]

with the boundary condition

\[ \begin{cases} 
\frac{\partial g}{\partial z} (\tau, 1) = 0, & \tau > 0, \\
g(0, z) = \frac{1}{z} - 1, & z \in (0, 1].
\end{cases} \]

The next Figure 12.3 shows a graph of the function \( g(\tau, z) \).

![Graph of the normalized lookback put option price.](https://www.ntu.edu.sg/home/nprivault/indext.html)

Fig. 12.3: Graph of the normalized lookback put option price.

**Black-Scholes approximation of lookback put option prices**

Letting

\[ B_{lp}(x, K, r, \sigma, \tau) := K e^{-\tau r} \Phi \left( -\delta^- \left( \frac{x}{K} \right) \right) - x \Phi \left( -\delta^+ \left( \frac{x}{K} \right) \right) \]
denote the standard Black-Scholes formula for the price of the European put option, we observe that when \( S_t < M_0 \), the lookback put option price satisfies

\[
e^{-r(T-t)} E^* \left[ M_0^T - S_T \mid F_t \right] = Bl_p(S_t, M_0^t, r, \sigma, T-t) + S_t \frac{\sigma^2}{2r} \left( \Phi \left( \delta_{+}^T \left( \frac{S_t}{M_0^t} \right) \right) - e^{-r(T-t)} \left( M_0^t \right) \Phi \left( -\delta_{-}^T \left( \frac{M_0^t}{S_t} \right) \right) \right),
\]

i.e.

\[
e^{-r(T-t)} E^* \left[ M_0^T - S_T \mid F_t \right] = Bl_p(S_t, M_0^t, r, \sigma, T-t) + S_t h_p \left( T-t, \frac{S_t}{M_0^t} \right)
\]

where the function

\[
h_p(\tau, z) = \frac{\sigma^2}{2r} \left( \Phi \left( \delta_{+}^\tau \left( z \right) \right) - e^{-r\tau}z^{-2r/\sigma^2} \Phi \left( -\delta_{-}^\tau \left( \frac{1}{z} \right) \right) \right),
\]

depends only on time \( \tau \) and \( z = S_t/M_0^t \). In other words, due to the relation

\[
Bl_p(x, y, r, \sigma, \tau) = ye^{-r\tau} \Phi \left( -\delta_{-}^\tau \left( \frac{x}{y} \right) \right) - x \Phi \left( \delta_{+}^\tau \left( \frac{x}{y} \right) \right)
\]

for the standard Black-Scholes put option price formula, we observe that \( f(t, x, y) \) satisfies

\[
f(t, x, y) = xBl_p(1, y/x, r, \sigma, T-t) + xh(T-t, x/y),
\]

i.e.

\[
f(t, x, y) = xg(T-t, x/y),
\]

with

\[
g(\tau, z) = Bl_p \left( 1, \frac{1}{z}, r, \sigma, \tau \right) + h_p(\tau, z),
\]

where the function \( h_p(\tau, z) \) is a correction term given by \( (12.9) \) which is small when \( z = x/y \) or \( \tau \) become small.

Note that \( (x, y) \mapsto xh_p(T-t, x/y) \) also satisfies the Black-Scholes PDE \( (12.2) \), in particular \( (\tau, z) \mapsto Bl_p(1, 1/z, r, \sigma, \tau) \) and \( h_p(\tau, z) \) both satisfy the PDE

\[
\frac{\partial h_p(\tau, z)}{\partial \tau} = z \left( r + \frac{\sigma^2}{2} \right) \frac{\partial h_p(\tau, z)}{\partial z} + \frac{1}{2} \sigma^2 z^2 \frac{\partial^2 h_p(\tau, z)}{\partial z^2},
\]

\( \tau \in [0, T], z \in [0, 1] \), under the boundary condition

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\[ h_p(0, z) = 0, \quad 0 \leq z \leq 1. \]

The next Figures 12.4 and 12.5 illustrate the decomposition (12.10) of the normalized lookback put option price \( g(\tau, z) \) in Figure 12.3 into the Black-Scholes put price function \( B_p(1, 1/z, r, \sigma, \tau) \) and \( h_p(\tau, z) \).

Fig. 12.4: Black-Scholes put price in the decomposition (12.10).

Fig. 12.5: Correction term \( h_p(\tau, z) \) in the decomposition (12.10).

Note that in Figures 12.4-12.5 the condition \( h_p(0, z) = 0 \) is not fully respected as \( z \to 1 \) due to numerical error in the approximation of the function \( \Phi \).

12.3 The Lookback Call Option

The following result gives the value of the average minimum \( \mathbb{E}^* \left[ m^T_0 \mid \mathcal{F}_t \right] \) of \( (S_t)_{t \in [0,T]} \) over the interval \([0,T] \).

**Proposition 12.4.** The average minimum value of \((S_t)_{t \in [0,T]} \) over \([0,T] \) is given by

\[ \text{Proposition 12.4.} \text{ The average minimum value of } (S_t)_{t \in [0,T]} \text{ over } [0,T] \text{ is given by} \]
We note a certain symmetry between the expressions of (10.15) and (12.12).

When \( t = 0 \) we have \( S_0 = m_0^0 \), and given (10.16) the formula (12.12) simplifies to

\[
\mathbb{E}^* \left[ m_T^0 \mid F_t \right] = \mathbb{E}^* \left[ \min \left( m_t^0, m_T^0 \right) \mid F_t \right] + \mathbb{E}^* \left[ m_T^0 \mathbb{1}_{\{m_0^0 < m_T^0\}} \mid F_t \right] \\
= m_t^0 \mathbb{P}(m_t^0 < m_T^0 \mid F_t) + \mathbb{E}^* \left[ m_T^0 \mathbb{1}_{\{m_0^0 > m_T^0\}} \mid F_t \right].
\]

(12.13)

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Next, by integration with respect to the probability density function (10.12) of the minimum $m_0^{T-t}$ of $(S_t)_{t \in \mathbb{R}_+}$ over the time interval $[0, T-t]$, hence

$$
P(m_0^t < m_0^{T-t} | \mathcal{F}_t) = \mathbb{P} \left( \frac{m_0^t}{S_t} < \frac{m_0^{T-t}}{S_t} \right) = \mathbb{P} \left( x < \frac{m_0^{T-t}}{S_t} \right)_{x = m_0^t/S_t}
$$

$$
= \mathbb{P} \left( \frac{m_0^{T-t}}{S_0} > x \right)_{x = m_0^t/S_t}
$$

$$
= \Phi \left( \delta_{-t} \left( \frac{S_t}{m_0^t} \right) \right) - \left( \frac{m_0^t}{S_t} \right)^{-1+2r/\sigma^2} \Theta \left( \delta_{-t} \left( \frac{m_0^t}{S_t} \right) \right).
$$

Next, by integration with respect to the probability density function (10.12) as in (10.17) in the proof of Proposition 10.6, we find

$$
\mathbb{E}^* \left[ m_0^{T} \mathbbm{1}_{\{m_0^t > m_0^T \}} | \mathcal{F}_t \right] = S_t \mathbb{E}^* \left[ \min_{u \in [t,T]} \frac{S_u}{S_t} \mathbbm{1}_{\{m_0^t > m_0^T \}} \right]_{x = m_0^t, y = S_t}
$$

$$
= S_t \mathbb{E}^* \left[ \min_{u \in [t,T]} \frac{S_u}{S_t} \mathbbm{1}_{\{\min_{u \in [t,T]} S_u / S_t < x \}} \right]_{x = m_0^t / S_t}
$$

$$
= 2S_t e^{(T-t)r} \Phi \left( -\delta_{+} \left( \frac{S_t}{m_0^t} \right) \right) - S_t \frac{\mu \sigma}{r} e^{(T-t)r} \Phi \left( -\delta_{+} \left( \frac{S_t}{m_0^t} \right) \right)
$$

$$
+ S_t \frac{\mu \sigma}{r} \left( \frac{m_0^t}{S_t} \right)^{2r/\sigma^2} \Phi \left( \delta_{-t} \left( \frac{m_0^t}{S_t} \right) \right).
$$

Given the relation $\mu \sigma / r = 1 - \sigma^2 / (2r)$, this yields

$$
\mathbb{E}^* \left[ m_0^{T} | \mathcal{F}_t \right] = m_0^t \mathbb{E} \left( \frac{m_0^{T-t}}{S_0} > x \right)_{x = m_0^t / S_t}
$$

$$
+ S_t \mathbb{E}^* \left[ \min_{u \in [0,T-t]} \frac{S_u}{S_0} \mathbbm{1}_{\{\min_{u \in [0,T-t]} S_u / S_0 < x \}} \right]_{x = m_0^t / S_t}
$$

$$
= m_0^t \Phi \left( \delta_{-t} \left( \frac{S_t}{m_0^t} \right) \right) - m_0^t \left( \frac{m_0^t}{S_t} \right)^{-1+2r/\sigma^2} \Phi \left( \delta_{-t} \left( \frac{m_0^t}{S_t} \right) \right).
$$
The standard Lookback call option gives the right to buy the underlying asset at its historically lowest price. In this case the strike price is \( m_0^T \) and the payoff is

\[
C = S_T - m_0^T.
\]

The following result gives the price of the lookback call option, cf. e.g. Proposition 9.5.1, page 270 of Dana and Jeanblanc (2007).

**Proposition 12.5.** The price at time \( t \in [0, T] \) of the lookback call option with payoff \( S_T - m_0^T \) is given by

\[
e^{-(T-t)r} \mathbb{E}^* \left[ S_T - m_0^T \mid \mathcal{F}_t \right] = S_t \Phi \left( \delta_{+}^{T-t} \left( \frac{S_t}{m_0^T} \right) \right) - m_0^t e^{-(T-t)r} \Phi \left( \delta_{-}^{T-t} \left( \frac{S_t}{m_0^T} \right) \right) + e^{-(T-t)r} S_t \frac{\sigma^2}{2r} \left( \frac{m_0^t}{S_t} \right)^{2r/\sigma^2} \Phi \left( \delta_{-}^{T-t} \left( \frac{m_0^t}{S_t} \right) \right) - S_t \frac{\sigma^2}{2r} \Phi \left( \delta_{+}^{T-t} \left( \frac{m_0^t}{S_t} \right) \right).
\]

**Proof.** We have

\[
e^{-(T-t)r} \mathbb{E}^* \left[ S_T - m_0^T \mid \mathcal{F}_t \right] = S_t - e^{-(T-t)r} \mathbb{E}^* \left[ m_0^T \mid \mathcal{F}_t \right] = S_t \Phi \left( \delta_{+}^{T-t} \left( \frac{S_t}{m_0^T} \right) \right) - e^{-(T-t)r} m_0^t \Phi \left( \delta_{-}^{T-t} \left( \frac{S_t}{m_0^T} \right) \right) + e^{-(T-t)r} S_t \frac{\sigma^2}{2r} \left( \frac{m_0^t}{S_t} \right)^{2r/\sigma^2} \Phi \left( \delta_{-}^{T-t} \left( \frac{m_0^t}{S_t} \right) \right) - e^{(T-t)r} \Phi \left( -\delta_{+}^{T-t} \left( \frac{S_t}{m_0^T} \right) \right).
\]

Figure 12.6 represents the price of the lookback call option as a function of \( m_0^t \) and \( S_t \) for different values of the time to maturity \( T - t \).

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https://www.ntu.edu.sg/home/nprivault/index.html
From Figures 12.6 and 12.7 we note the following.

i) When the underlying asset price \( S_t \) is far from \( m_{t_0} \), an increase in the value \( S_t \) clearly results into a higher call option price.

ii) When the underlying asset price \( S_t \) is close to \( m_{t_0} \), a decrease in \( S_t \) could lead to a decrease in the value of \( m_{t_0} \), however on average this is not sufficient to increase the option payoff.

Figure 12.7 shows accordingly that, according to the Delta hedging strategy for lookback call options, see Propositions 12.6 and 12.7, one should long the underlying asset in order to hedge a lookback call option.

**Black-Scholes PDE for lookback call option prices**

Since the couple \( (S_t, m_{t_0}) \) is also a Markov process, the price of the lookback call option at time \( t \in [0, T] \) can be written as a function
of \( S_t \) and \( m^t_0 \), \( 0 \leq t \leq T \). By the same argument as in the proof of Proposition 12.2 we obtain the following result.

**Proposition 12.6.** The function \( f(t, x, y) \) defined by

\[
f(t, x, y) = e^{-(T-t)r} \mathbb{E}^* \left[ S_T - m^T_0 \mid S_t = x, \ m^T_0 = y \right], \quad t \in [0, T], \ x, y > 0,
\]

is \( C^2((0, T) \times (0, \infty)^2) \) and satisfies the Black-Scholes PDE

\[
rf(t, x, y) = \frac{\partial f}{\partial t}(t, x, y) + rx \frac{\partial f}{\partial x}(t, x, y) + \frac{1}{2} x^2 \sigma^2 \frac{\partial^2 f}{\partial x^2}(t, x, y),
\]

\( 0 \leq t \leq T, \ x > 0 \), under the boundary conditions

\[
\begin{align*}
\lim_{y \searrow 0} f(t, x, y) &= x, \quad 0 \leq t \leq T, \ x > 0, \quad (12.13a) \\
\frac{\partial f}{\partial y}(t, x, y)_{y=x} &= 0, \quad 0 \leq t \leq T, \ y > 0, \quad (12.13b) \\
f(T, x, y) &= x - y, \quad 0 < y \leq x, \quad (12.13c)
\end{align*}
\]

and the corresponding self-financing hedging strategy is given by

\[
\xi_t = \frac{\partial f}{\partial x}(t, S_t, m^t_0), \quad t \in [0, T],
\]

which represents the quantity of the risky asset \( S_t \) to be held at time \( t \) in the hedging portfolio.

In other words, the price of the lookback call option takes the form

\[
f(t, S_t, m^t_t) = e^{-(T-t)r} \mathbb{E}^* \left[ S_T - m^T_0 \mid \mathcal{F}_t \right],
\]

where the function \( f(t, x, y) \) is given by
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\[
f(t, x, y) = x \Phi \left( \delta_{+}^{T-t} \left( \frac{x}{y} \right) \right) - e^{-(T-t)r} y \Phi \left( \delta_{-}^{T-t} \left( \frac{x}{y} \right) \right) \\
+ e^{-(T-t)r} x \frac{\sigma^2}{2r} \left( \frac{y}{x} \right)^{2r/\sigma^2} \Phi \left( \delta_{-}^{T-t} \left( \frac{y}{x} \right) \right) - e^{(T-t)r} \Phi \left( -\delta_{+}^{T-t} \left( \frac{x}{y} \right) \right)
\]

\[
= x - ye^{-(T-t)r} \Phi \left( \delta_{-}^{T-t} \left( \frac{x}{y} \right) \right) - x \left( 1 + \frac{\sigma^2}{2r} \right) \Phi \left( -\delta_{+}^{T-t} \left( \frac{x}{y} \right) \right)
\]

\[
+ xe^{-(T-t)r} \frac{\sigma^2}{2r} \left( \frac{y}{x} \right)^{2r/\sigma^2} \Phi \left( \delta_{-}^{T-t} \left( \frac{y}{x} \right) \right).
\]

Black-Scholes approximation of lookback call option prices

Letting

\[
Bl_c(S, K, r, \sigma, \tau) = S \Phi \left( \delta_{+}^{T} \left( \frac{S}{K} \right) \right) - K e^{-r\tau} \Phi \left( \delta_{-}^{T} \left( \frac{S}{K} \right) \right)
\]

denote the standard Black-Scholes formula for the price of the European call option, we observe that the lookback call option price satisfies

\[
e^{-(T-t)r} E^* \left[ S_T - m_T^{t} \mid \mathcal{F}_t \right] = Bl_c(S_t, m_0^{t}, r, \sigma, T-t)
\]

\[
- S_t \frac{\sigma^2}{2r} \left( \Phi \left( -\delta_{+}^{T-t} \left( \frac{S_t}{m_0^{t}} \right) \right) - e^{-(T-t)r} \left( \frac{m_0^{t}}{S_t} \right)^{2r/\sigma^2} \Phi \left( \delta_{-}^{T-t} \left( \frac{m_0^{t}}{S_t} \right) \right) \right),
\]

i.e.

\[
e^{-(T-t)r} E^* \left[ S_T - m_0^{t} \mid \mathcal{F}_t \right] := Bl_c(S_t, m_0^{t}, r, \sigma, T-t) + S_t h_c \left( T-t, \frac{S_t}{m_0^{t}} \right)
\]

where the correction term

\[
h_c(\tau, z) = - \frac{\sigma^2}{2r} \left( \Phi \left( \delta_{-}^{T} \left( \frac{1}{z} \right) \right) - e^{-r\tau} z^{-2r/\sigma^2} \Phi \left( \delta_{+}^{T} \left( \frac{1}{z} \right) \right) \right), \quad (12.16)
\]

is small when \( z = S_t/m_0^{t} \) becomes large or \( \tau \) becomes small. In addition, \( h_p(\tau, z) \) is linked to \( h_c(\tau, z) \) by the relation

\[
h_c(\tau, z) = h_p(\tau, z) - \frac{\sigma^2}{2r} \left( 1 - e^{-r\tau} z^{-2r/\sigma^2} \right), \quad \tau \in \mathbb{R}_+, \quad z \in \mathbb{R}_+
\]

where \( (z, \tau) \mapsto e^{-r\tau} z^{-2r/\sigma^2} \) also solves the PDE (12.11).
Scaling property of lookback call option prices

We note the scaling property

\[
f(t, x, y) = e^{-(T-t)r} E^* \left[ S_T - m_0^T \mid S_t = x, m_0^t = y \right]
\]

\[
= e^{-(T-t)r} E^* \left[ S_T - \min \left( m_0^t, m_t^T \right) \mid S_t = x, m_0^t = y \right]
\]

\[
= e^{-(T-t)r} x E^* \left[ \frac{S_T}{S_t} - \min \left( \frac{m_0^t}{S_t}, \frac{m_t^T}{S_t} \right) \mid S_t = x, m_0^t = y \right]
\]

\[
= e^{-(T-t)r} x E^* \left[ \frac{S_T}{x} - \min \left( \frac{y}{x}, \frac{m_t^T}{x} \right) \mid S_t = x, m_0^t = y \right]
\]

\[
= e^{-(T-t)r} x E^* \left[ S_T - \min \left( m_0^t, m_t^T \right) \mid S_t = 1, m_0^t = \frac{y}{x} \right]
\]

\[
= e^{-(T-t)r} x E^* \left[ S_T - m_0^T \mid S_t = 1, m_0^t = \frac{y}{x} \right]
\]

\[
= xf(t, 1, y/x)
\]

\[
= xg \left( T - t, \frac{1}{z} \right),
\]

where

\[
g(\tau, z) := 1 - \frac{1}{z} e^{-\tau} \Phi \left( \delta^- \left( \frac{z}{x} \right) \right) - \left( 1 + \frac{\sigma^2}{2r} \right) \Phi \left( -\delta^+ \left( \frac{z}{x} \right) \right) + \frac{\sigma^2}{2r} e^{-\tau} \frac{z^{-2r/\sigma^2}}{z} \Phi \left( \delta^- \left( \frac{1}{z} \right) \right),
\]

with \( g(\tau, 1) = C(T - t) \), and

\[
f(t, x, y) = xg \left( T - t, \frac{x}{y} \right)
\]

and the boundary condition

\[
\begin{cases}
\frac{\partial g}{\partial z}(\tau, 1) = 0, & \tau > 0, \\
g(0, z) = 1 - \frac{1}{z}, & z \geq 1.
\end{cases} \tag{12.17a}
\]

\[
\begin{cases}
g(0, z) = 1 - \frac{1}{z}, & z \geq 1.
\end{cases} \tag{12.17b}
\]

The next Figure 12.8 shows a graph of the function \( g(\tau, z) \).
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Fig. 12.8: Normalized lookback call option price.

The next Figure 12.9 represents the path of the underlying asset price used in Figure 12.8.

Fig. 12.9: Graph of underlying asset prices.

The next Figure 12.10 represents the corresponding underlying asset price and its running minimum.

Fig. 12.10: Running minimum of the underlying asset price.
Next, we represent the option price as a function of time, together with the process $(S_t - m_0^t)_{t \in \mathbb{R}^+}$.

**Fig. 12.11:** Graph of the lookback call option price.

### Black-Scholes approximation

Due to the relation

$$\text{Bl}_c(x, y, r, \sigma, \tau) = x \Phi\left(\frac{\delta_+}{y}(\frac{x}{y})\right) - y e^{-r\tau} \Phi\left(\frac{\delta_-}{y}(\frac{x}{y})\right)$$

$$= x \text{Bl}_c\left(1, \frac{y}{x}, r, \sigma, \tau\right)$$

for the standard Black-Scholes call price formula, recall that $f(t, x, y)$ can be decomposed as

$$f(t, x, y) = x \text{Bl}_c\left(1, \frac{y}{x}, r, \sigma, T - t\right) + xh_c\left(T - t, \frac{x}{y}\right),$$

where $h_c(\tau, z)$ is the function given by (12.16), i.e.

$$f(t, x, y) = xg\left(T - t, \frac{x}{y}\right),$$

with

$$g(\tau, z) = \text{Bl}_c\left(1, \frac{1}{z}, r, \sigma, \tau\right) + h_c(\tau, z), \quad (12.18)$$

where $(x, y) \mapsto xh_c(T - t, x/y)$ also satisfies the Black-Scholes PDE (12.2), i.e. $(\tau, z) \mapsto \text{Bl}_c(1, 1/z, r, \sigma, \tau)$ and $h_c(\tau, z)$ both satisfy the PDE (12.11) under the boundary condition

$$h_c(0, z) = 0, \quad z \geq 1.$$
The next Figures 12.12 and 12.13 show the decomposition of $g(t,z)$ in (12.18) and Figures 12.8-12.9 into the sum of the Black-Scholes call price function $B_{1c}(1,1/z,r,\sigma,\tau)$ and $h(t,z)$.

![Fig. 12.12: Black-Scholes call price in the decomposition (12.18) of the normalized lookback call option price $g(\tau,z)$.](image1)

![Fig. 12.13: Function $h_{c}(\tau,z)$ in the decomposition (12.18) of the normalized lookback call option price $g(\tau,z)$.](image2)

We also note that

$$\begin{align*}
\mathbb{E}^* \left[ M_0^T - m_0^T \mid S_0 = x \right] &= x - xe^{-(T-t)r}\Phi(\delta_-^{t-T}(1)) \\
&- x \left(1 + \frac{\sigma^2}{2r}\right)\Phi(-\delta_+^{t-T}(1)) + xe^{-(T-t)r}\frac{\sigma^2}{2r}\Phi(\delta_-^{t-T}(1)) \\
&+ xe^{-(T-t)r}\Phi(-\delta_-^{t-T}(1)) + x \left(1 + \frac{\sigma^2}{2r}\right)\Phi(\delta_+^{t-T}(1)) \\
&- \frac{\sigma^2}{2r}e^{-(T-t)r}\Phi(-\delta_-^{t-T}(1)) - x \\
&= x \left(1 + \frac{\sigma^2}{2r}\right) \left(\Phi(\delta_+^{t-T}(1)) - \Phi(-\delta_+^{t-T}(1))\right) \\
\end{align*}$$
\[ +x e^{-(T-t)r} \left( \frac{\sigma^2}{2r} - 1 \right) \left( \Phi(\delta_{-}^{T-t}(1)) - \Phi(-\delta_{-}^{T-t}(1)) \right). \]

12.4 Hedging Lookback Call Options


**Proposition 12.7.** The hedging strategy of the lookback call option is given by

\[ \xi_t = \Phi \left( \delta_{+}^{T-t} \left( \frac{S_t}{m_0} \right) \right) - \frac{\sigma^2}{2r} \Phi \left( -\delta_{+}^{T-t} \left( \frac{S_t}{m_0} \right) \right) \]

\[ + e^{-(T-t)r} \left( \frac{m_0}{S_t} \right)^{2r/\sigma^2} \left( \frac{\sigma^2}{2r} - 1 \right) \Phi \left( \delta_{-}^{T-t} \left( \frac{m_0}{S_t} \right) \right), \quad 0 \leq t \leq T. \] (12.19)

**Proof.** By (12.14), we need to differentiate

\[ f(t, x, y) = Bl_c(x, y, r, \sigma, T - t) + xh_c \left( T - t, \frac{x}{y} \right) \]

with respect to the variable \( x \), where

\[ h_c(\tau, z) = -\frac{\sigma^2}{2r} \left( \Phi(-\delta_{+}^{\tau}(z)) - e^{-r\tau}z^{-2r/\sigma^2} \Phi(\delta_{-}^{\tau} \left( \frac{1}{z} \right)) \right) \]

is given by (12.16) First, we note that the relation

\[ \frac{\partial}{\partial x} Bl_c(x, y, r, \sigma, \tau) = \Phi \left( \delta_{+}^{\tau} \left( \frac{x}{y} \right) \right) \]

is known, cf. Propositions 6.3 and 7.13. Next, we have

\[ \frac{\partial}{\partial x} \left( xh_c \left( \tau, \frac{x}{y} \right) \right) = h_c \left( \tau, \frac{x}{y} \right) + \frac{x}{y} \frac{\partial h_c}{\partial z} \left( \tau, \frac{x}{y} \right), \]

and

\[ \frac{\partial h_c}{\partial z} (\tau, z) = -\frac{\sigma^2}{2r} \left( \frac{\partial}{\partial x} \left( \Phi(-\delta_{+}^{\tau}(z)) \right) - e^{-r\tau}z^{-2r/\sigma^2} \frac{\partial}{\partial z} \left( \Phi \left( \delta_{-}^{\tau} \left( \frac{1}{z} \right) \right) \right) \right) \]

\[ - \frac{\sigma^2}{2r} \left( \frac{2r}{\sigma^2} e^{-r\tau}z^{-1-2r/\sigma^2} \Phi \left( \delta_{-}^{\tau} \left( \frac{1}{z} \right) \right) \right). \]
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\[ \frac{1}{2} \left( \frac{\sigma^2}{r} \right) \sqrt{2\pi r} \exp \left( \frac{-1}{2} \left( \frac{1}{z} \right)^2 \right) \]

\[ -e^{-r\tau} z^{-2r/\sigma^2} \frac{\sigma}{2r \sqrt{2\pi r}} \exp \left( \frac{-1}{2} \left( \frac{1}{z} \right)^2 \right) + \frac{2r}{\sigma^2} e^{-r\tau} z^{-1-2r/\sigma^2} \Phi \left( \frac{1}{z} \right) \right) . \]

Next, we note that

\[ e^{-\left( \frac{1}{2} \left( \frac{1}{z} \right)^2 \right) / 2} = \exp \left( \frac{-1}{2} \left( \frac{1}{z} \right)^2 \right) - \frac{1}{2} \left( \frac{4r^2}{\sigma^2} - \frac{4r}{\sigma} \frac{1}{z} \right) \right) \]

\[ = \exp \left( \frac{-1}{2} \left( \frac{4r^2}{\sigma^2} - \frac{4r}{\sigma} \right) \left( \log z + (r + \frac{1}{2}\sigma^2) \right) \right) \]

\[ = \exp \left( \frac{-2r^2}{\sigma^2} - \frac{2r}{\sigma^2} \log z + \frac{2r^2}{\sigma^2} + r\tau \right) \]

\[ = e^{r\tau} z^{2r/\sigma^2} e^{-\left( \frac{1}{2} \left( \frac{1}{z} \right)^2 \right) / 2} \quad (12.20) \]

as in the proof of Proposition 6.3, hence

\[ \frac{\partial h_c}{\partial z} \left( \tau, x, y \right) = -e^{-r\tau} z^{-1-2r/\sigma^2} \Phi \left( \frac{1}{z} \right) \right) , \]

and

\[ \frac{\partial}{\partial x} \left( x h_c \left( \tau, x, y \right) \right) = h_c \left( \tau, x \right) - e^{-r\tau} \left( \frac{y}{x} \right)^{2r/\sigma^2} \Phi \left( \frac{1}{x} \right) , \]

which concludes the proof. \( \square \)

We note that at maturity \( t = T \) the hedging strategy satisfies

\[ \xi_T = \begin{cases} 
1 & \text{if } M_0^T < S_T, \\
\frac{1}{2} - \frac{\sigma^2}{4r} + \frac{1}{2} \left( \frac{\sigma^2}{2r} - 1 \right) & = 0 \text{ if } M_0^T = S_T .
\end{cases} \]

In Figure 12.14 we represent the Delta of the lookback call option, as given by (12.19).
The above scaling procedure can be applied to the Delta as well, by noting that \( \xi_t \) can be written as

\[
\xi_t = \zeta \left( t, \frac{S_t}{m_0^t} \right),
\]

where the function \( \zeta(t, z) \) is given by

\[
\zeta(t, z) = \Phi \left( \delta_+^{T-t}(z) \right) - \frac{\sigma^2}{2r} \Phi \left( -\delta_+^{T-t}(z) \right) \\
+ e^{-(T-t)r} z^{-2r/\sigma^2} \left( \frac{\sigma^2}{2r} - 1 \right) \Phi \left( \delta_-^{T-t} \left( \frac{1}{z} \right) \right),
\]

\( t \in [0, T], \ z \in [0, 1] \). The graph of the function \( \zeta(t, x) \) is given in Figure 12.15.

Similar calculations using (12.4) can be carried out for other types of lookback options, such as options on extrema and partial lookback options, cf. El Khatib (2003). As a consequence of (12.21) we have

* The animation works in Acrobat Reader on the entire pdf file.
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\[ e^{-(T-t)r} \mathbb{E}^* \left[ S_T - m_T^0 \mid \mathcal{F}_t \right] \]

\[ = S_t \Phi \left( \frac{\delta^+ - t}{m^0_t} \right) - m^0_t e^{-(T-t)r} \Phi \left( \frac{\delta^+ - t}{m^0_t} \right) \]

\[ + e^{-(T-t)r} S_t \frac{\sigma^2}{2r} \Phi \left( \frac{\delta^- - t}{m^0_t} \right) - S_t \frac{\sigma^2}{2r} \Phi \left( -\frac{\delta^- - t}{m^0_t} \right) \]

\[ = \xi_t S_t - m^0_t e^{-(T-t)r} \left( \Phi \left( \frac{\delta^- - t}{m^0_t} \right) + \frac{S_t}{m^0_t} \right)^{1-2r/\sigma^2} \Phi \left( \frac{\delta^- - t}{m^0_t} \right) , \]

and the quantity of the riskless asset \( e^{rt} \) in the portfolio is given by

\[ \eta_t = -m^0_t e^{-rT} \left( \Phi \left( \frac{\delta^- - t}{m^0_t} \right) + \frac{S_t}{m^0_t} \right)^{1-2r/\sigma^2} \Phi \left( \frac{\delta^- - t}{m^0_t} \right) \leq 0, \]

so that the portfolio value \( V_t \) at time \( t \) satisfies

\[ V_t = \xi_t S_t + \eta_t e^{rt}, \quad t \in \mathbb{R}_+, \]

and one has to constantly borrow from the risk-free account in order to hedge the lookback option.

Exercises

Exercise 12.1

a) Give the probability density function of the maximum \( \max_{t \in [0,1]} (B_t + \sigma t/2) \) of drifted Brownian motion.

b) Taking \( S_t := e^{\sigma B_t - \sigma^2 t/2} \), compute the expected value

\[ \mathbb{E} \left[ \min_{t \in [0,1]} S_t \right] = \mathbb{E} \left[ \min_{t \in [0,1]} e^{\sigma B_t - \sigma^2 t/2} \right] \]

\[ = \mathbb{E} \left[ e^{-\sigma \max_{t \in [0,1]} (B_t + \sigma t/2)} \right] . \]

c) Compute the “optimal exercise” price \( E \left[ (K - S_0 \min_{t \in [0,1]} e^{\sigma B_t - \sigma^2 t/2})^+ \right] \)

of a finite expiration American put option with \( S_0 < K \).

Exercise 12.2 Let \( (B_t)_{t \in \mathbb{R}_+} \) denote a standard Brownian motion.

a) Compute the expected value
b) Compute the “optimal exercise” price

$$E \left[ \max_{t \in [0,1]} S_t \right] = E \left[ e^{\sigma \max_{t \in [0,1]} (B_t - \sigma t/2)} \right].$$

of a finite expiration American call option with $S_0 > K$.

Exercise 12.3 Consider a risky asset whose price $S_t$ is given by

$$dS_t = \sigma S_t dB_t + \sigma^2 S_t dt/2,$$

where $(B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion.

a) Give the probability distribution (distribution function and probability density function) of the minimum $\min_{t \in [0,T]} B_t$ over the interval $[0,T]$?

b) Compute the price value

$$e^{-\sigma^2 T/2} \mathbb{E}^* \left[ S_T - \min_{t \in [0,T]} S_t \right]$$

of a lookback call option on $S_T$ with maturity $T$.

Exercise 12.4 Compute the hedging strategy of the lookback put option priced in Proposition 10.6.

Exercise 12.5 Dassios and Lim (2019) The digital drawdown call option with qualifying period pays a unit amount when the drawdown period reaches one unit of time, if this happens before fixed maturity $T$, but only if the size of drawdown at this stopping time is larger than a prespecified $K$. This provides an insurance against a prolonged drawdown, if the drawdown amount is large. Specifically, the digital drawdown call option is priced as

$$\mathbb{E}^* \left[ e^{-r \tau} \mathbb{1}_{\{\tau \leq T\}} \mathbb{1}_{\{M^T_{\tau} - S_T \geq K\}} \right],$$

where $M^T_{\tau} := \max_{u \in [0,\tau]} S_u$, $U_t := t - \sup\{0 \leq u \leq t : M^t_{u} = S_u\}$, and $\tau := \inf\{t \in \mathbb{R}_+ : U_t = 1\}$. Write the price of the drawdown option as a triple integral using the joint probability density function $f_{(\tau,S_{\tau},M_{\tau})}(t,x,y)$ of $(\tau,S_{\tau},M_{\tau})$ under the risk-neutral probability measure $\mathbb{P}^*$.

Exercise 12.6

a) Check explicitly that the boundary conditions (12.3a)-(12.3b)-(12.3c) are satisfied.

b) Check explicitly that the boundary conditions (12.13a)-(12.13b) are satisfied.
Exercise 12.7  Compute the Delta hedging portfolio strategy for lookback put options according to (12.4) in Proposition 12.2.