Chapter 17
Forward Rate Modeling

This deals with the modeling of forward rates and swap rates in the HJM and BGM models. We also consider the Nelson-Siegel and Svensson yield curve parametrizations, as well as two-factor models.

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17.1 Forward Rates

A forward interest rate contract (or Forward Rate Agreement, FRA) gives to its holder the possibility to lock an interest rate denoted by \( f(t,T,S) \) at present time \( t \) for a loan to be delivered over a future period of time \([T, S]\), with \( t \leq T \leq S \).

\[
\begin{array}{c}
0 & t & T & S \\
\end{array}
\]

The rate \( f(t,T,S) \) is called a forward interest rate. When \( T = t \), the \textit{spot} forward rate \( f(t,t,T) \) coincides with the \textit{yield}, see Relation (17.3) below.

Figure 17.1 presents a typical yield curve on the LIBOR (London Interbank Offered Rate) market with \( t = 07 \) May 2003.
Maturity transformation, i.e., the ability to transform short-term borrowing (debt with short maturities, such as deposits) into long term lending (credits with very long maturities, such as loans), is among the roles of banks. Profitability is then dependent on the difference between long rates and short rates.

Another example of market data is given in the next Figure 17.2, in which the red and blue curves refer respectively to July 21 and 22 of year 2011.

Long maturities usually correspond to higher rates as they carry an increased risk. The dip observed with short maturities can correspond to a lower motivation to lend/invest in the short-term.
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Forward rates from bond prices

Let us determine the arbitrage or “fair” value of the forward interest rate \( f(t, T, S) \) by implementing the Forward Rate Agreement using the instruments available in the market, which are bonds priced at \( P(t, T) \) for various maturity dates \( T > t \).

The loan can be realized using the available instruments (here, bonds) on the market, by proceeding in two steps:

1) At time \( t \), borrow the amount \( P(t, S) \) by issuing (or short selling) one bond with maturity \( S \), which means refunding $1 at time \( S \).

2) Since the money is only needed at time \( T \), the rational investor will invest the amount \( P(t, S) \) over the period \( [t, T] \) by buying a (possibly fractional) quantity \( P(t, S)/P(t, T) \) of a bond with maturity \( T \) priced \( P(t, T) \) at time \( t \). This will yield the amount

\[
$1 \times \frac{P(t, S)}{P(t, T)}
\]

at time \( T > 0 \).

As a consequence, the investor will actually receive \( P(t, S)/P(t, T) \) at time \( T \), to refund $1 at time \( S \).

The corresponding forward rate \( f(t, T, S) \) is then given by the relation

\[
\frac{P(t, S)}{P(t, T)} \exp \left( (S - T) f(t, T, S) \right) = 1, \quad 0 \leq t \leq T \leq S, \quad (17.1)
\]

where we used exponential compounding, which leads to the following definition (17.2).

**Definition 17.1.** The forward rate \( f(t, T, S) \) at time \( t \) for a loan on \([T, S]\) is given by

\[
f(t, T, S) = \frac{\log P(t, T) - \log P(t, S)}{S - T}. \quad (17.2)
\]

The spot forward rate \( f(t, t, T) \) coincides with the yield \( y(t, T) \), with
\[ f(t, t, T) = y(t, T) = -\frac{\log P(t, T)}{T - t}, \quad \text{or} \quad P(t, T) = e^{-(T-t)f(t,t,T)}, \]
\[ 0 \leq t \leq T. \] (17.3)

### Instantaneous Forward Rate

**Proposition 17.2.** The instantaneous forward rate \( f(t, T) = f(t, T, T) \) is defined by taking the limit of \( f(t, T, S) \) as \( S \downarrow T \), and satisfies

\[ f(t, T) := \lim_{S \searrow T} f(t, T, S) = -\frac{1}{P(t, T)} \frac{\partial P(t, T)}{\partial T}. \] (17.4)

**Proof.** We have

\[
\begin{align*}
    f(t, T) &:= \lim_{S \searrow T} f(t, T, S) \\
    &= -\lim_{S \searrow T} \frac{\log P(t, S) - \log P(t, T)}{S - T} \\
    &= -\lim_{\varepsilon \downarrow 0} \frac{\log P(t, T + \varepsilon) - \log P(t, T)}{\varepsilon} \\
    &= -\frac{\partial \log P(t, T)}{\partial T} \\
    &= -\frac{1}{P(t, T)} \frac{\partial P(t, T)}{\partial T}.
\end{align*}
\]

\[ \square \]

The above equation (17.4) can be viewed as a differential equation to be solved for \( \log P(t, T) \) under the initial condition \( P(T, T) = 1 \), which yields the following proposition.

**Proposition 17.3.** The bond price \( P(t, T) \) can be recovered from the instantaneous forward rate \( f(t, s) \) as

\[ P(t, T) = \exp \left( -\int_t^T f(t, s) ds \right), \quad 0 \leq t \leq T. \] (17.5)

**Proof.** We check that

\[
\begin{align*}
    \log P(t, T) &= \log P(t, T) - \log P(t, t) \\
    &= \int_t^T \frac{\partial \log P(t, s)}{\partial s} ds \\
    &= -\int_t^T f(t, s) ds.
\end{align*}
\]
Proposition 17.3 also shows that
\[
f(t, t, t) = f(t, t) = \frac{\partial}{\partial T} \int_t^T f(t, s) ds_{|T=t}
\]
\[
= - \frac{\partial}{\partial T} \log P(t, T)_{|T=t}
\]
\[
= - \frac{1}{P(t, T)_{|T=t}} \frac{\partial}{\partial T} P(t, T)_{|T=t}
\]
\[
= - \frac{\partial}{\partial T} \mathbb{E}^* \left[ e^{- \int_t^T \sigma_s ds} \bigg| F_t \right]_{|T=t}
\]
\[
= \mathbb{E}^* \left[ r_T e^{- \int_t^T \sigma_s ds} \bigg| F_t \right]_{|T=t}
\]
\[
= \mathbb{E}^* \left[ r_t \bigg| F_t \right]
\]
\[
= r_t,
\]
\[ i.e. \text{ the short rate } r_t \text{ can be recovered from the instantaneous forward rate as } \]
\[
r_t = f(t, t) = \lim_{T \searrow t} f(t, T).
\]

As a consequence of (17.1) and (17.5) the forward rate \( f(t, T, S) \) can be recovered from (17.2) and the instantaneous forward rate \( f(t, s) \), as:
\[
f(t, T, S) = \frac{\log P(t, T) - \log P(t, S)}{S - T}
\]
\[
= - \frac{1}{S - T} \left( \int_t^T f(t, s) ds - \int_t^S f(t, s) ds \right)
\]
\[
= \frac{1}{S - T} \int_T^S f(t, s) ds, \quad 0 \leq t \leq T < S. \quad (17.6)
\]

Similarly, as a consequence of (17.3) and (17.5) we have the next proposition.

**Proposition 17.4.** The spot forward rate or yield \( f(t, t, T) \) can be written in terms of bond prices as
\[
f(t, t, T) = - \frac{\log P(t, T)}{T - t} = \frac{1}{T - t} \int_t^T f(t, s) ds, \quad 0 \leq t < T. \quad (17.7)
\]

Differentiation with respect to \( T \) of the above relation shows that the yield \( f(t, t, T) \) and the instantaneous forward rate \( f(t, s) \) are linked by the relation
\[
\frac{\partial f}{\partial T} (t, t, T) = -\frac{1}{(T-t)^2} \int_t^T f(t, s) \, ds + \frac{1}{T-t} f(t, T), \quad 0 \leq t < T,
\]

from which it follows that

\[
f(t, T) = \frac{1}{T-t} \int_t^T f(t, s) \, ds + (T-t) \frac{\partial f}{\partial T}(t, t, T) = f(t, t, T) + (T-t) \frac{\partial f}{\partial T}(t, t, T), \quad 0 \leq t < T.
\]

**Forward Vasicek rates**

In this section we consider the Vasicek model, in which the short rate process is the solution (16.2) of (16.1) as illustrated in Figure 16.1.

In the Vasicek model, the forward rate is given by

\[
f(t, T, S) = -\log \frac{P(t, S)}{P(t, T)} = \frac{r_t (C(S-t) - C(T-t)) + A(S-t) - A(T-t))}{S - T}
\]

\[
= -\frac{\sigma^2 - 2ab}{2b^2} - \frac{1}{S - T} \left( \left( \frac{r_t}{b} + \frac{\sigma^2 - ab}{b^2} \right) (e^{-(S-t)b} - e^{-(T-t)b}) - \frac{\sigma^2}{4b^3} (e^{-2(S-t)b} - e^{-2(T-t)b}) \right),
\]

and the spot forward rate, or yield, satisfies

\[
f(t, T) = -\log \frac{P(t, T)}{T-t} = -\frac{r_t C(T-t) + A(T-t)}{T-t}
\]

\[
= -\frac{\sigma^2 - 2ab}{2b^2} + \frac{1}{T-t} \left( \left( \frac{r_t}{b} + \frac{\sigma^2 - ab}{b^3} \right) (1 - e^{-(T-t)b}) - \frac{\sigma^2}{4b^3} (1 - e^{-2(T-t)b}) \right),
\]

with the mean

\[
\mathbb{E}[f(t, t, T)]
\]

\[
= -\frac{\sigma^2 - 2ab}{2b^2} + \frac{1}{T-t} \left( \left( \mathbb{E}\left[ \frac{r_t}{b} \right] + \frac{\sigma^2 - ab}{b^3} \right) (1 - e^{-(T-t)b}) - \frac{\sigma^2}{4b^3} (1 - e^{-2(T-t)b}) \right)
\]

\[
= -\frac{\sigma^2 - 2ab}{2b^2} + \frac{1}{T-t} \left( \frac{r_0}{b} e^{-bt} + \frac{a}{b^2} (1 - e^{-bt}) + \frac{\sigma^2 - ab}{b^3} \right) (1 - e^{-(T-t)b})
\]

\[
- \frac{\sigma^2}{4b^3(T-t)} (1 - e^{-2(T-t)b}).
\]
In this model, the forward rate \( t \mapsto f(t, T, S) \) can be represented as in Figure 17.3, with \( a = 0.06, b = 0.1, \sigma = 0.1 \) and \( r_0 = \%1 \).

Note that the forward rate curve \( t \mapsto f(t, T, S) \) appears flat for small values of \( t \), i.e. longer rates are more stable, while shorter rates show higher volatility or risk. Similar features can be observed in Figure 17.4 for the instantaneous short rate given by

\[
f(t, T) := -\frac{\partial \log P(t, T)}{\partial T} = r_t e^{-(T-t)b} + \frac{a}{b} \left( 1 - e^{-(T-t)b} \right) - \frac{\sigma^2}{2b^2} \left( 1 - e^{-(T-t)b} \right)^2,
\]

from which the relation \( \lim_{T \searrow t} f(t, T) = r_t \) can be easily recovered. We can also evaluate the mean

\[
\mathbb{E}[f(t, T)] = \mathbb{E}[r_t] e^{-(T-t)b} + \frac{a}{b} \left( 1 - e^{-(T-t)b} \right) - \frac{\sigma^2}{2b^2} \left( 1 - e^{-(T-t)b} \right)^2
\]

\[(17.9)\]

The instantaneous forward rate \( t \mapsto f(t, T) \) can be represented as in Figure 17.4, with \( a = 0.06, b = 0.1, \sigma = 0.1 \) and \( r_0 = \%1 \):
Yield curve data

We refer to Chapter III-12 of Charpentier (2014) on the R package “Yield-Curve” Guirreri (2015) for the following code and further details on yield curve and interest rate modeling using R.

```r
install.packages("YieldCurve")
require(YieldCurve)
data(FedYieldCurve)
first(FedYieldCurve,'3 month')
last(FedYieldCurve,'3 month')
mat.Fed=c(0.25,0.5,1,2,3,5,7,10)
n=50
plot(mat.Fed, FedYieldCurve[n,], type="o",xlab="Maturities structure in years", ylab="Interest rates values")
title(main=paste("Federal Reserve yield curve observed at",time(FedYieldCurve[n], sep=" ")))
grid()
```

The next Figure 17.5 is plotted using this code* which is adapted from https://www.quantmod.com/examples/chartSeries3d/chartSeries3d.alpha.R

* Click to open or download.
Fig. 17.5: Federal Reserve yield curves from 1982 to 2012.

European Central Bank (ECB) data can be similarly obtained.

```r
data(ECBYieldCurve)
first(ECBYieldCurve, '3 month')
last(ECBYieldCurve, '3 month')
mat.ECB <- c(3/12, 0.5, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30)
for (n in 200:400) {
  plot(mat.ECB, ECBYieldCurve[n, ], type="o", xlab="Maturities structure in years", ylab="Interest rates values", ylim=c(3.1, 5.1))
  title(main=paste("European Central Bank yield curve observed at", time(ECBYieldCurve[n], sep=" ")))
  grid()
  Sys.sleep(0.5)
}
```

The next Figure 17.6 represents the output of the above script.

Fig. 17.6: European Central Bank yield curves.*

Increasing yield curves are typical of economic expansion phases. Decreasing yield curves can occur when central banks attempt to limit inflation by tight-

* The animation works in Acrobat Reader on the entire pdf file.
ening interest rates, such as in the case of an economic recession. In the next section we turn to the modeling of the market curves observed in Figure 17.6.

**LIBOR (London Interbank Offered) Rates**

Recall that the forward rate $f(t, T, S)$, $0 \leq t \leq T \leq S$, is defined using exponential compounding, from the relation

$$f(t, T, S) = -\frac{\log P(t, S) - \log P(t, T)}{S - T}.$$  

(17.10)

In order to compute swaption prices one prefers to use forward rates as defined on the London InterBank Offered Rates (LIBOR) market instead of the standard forward rates given by (17.10). Other types of LIBOR rates include EURIBOR (European Interbank Offered Rates), HIBOR (Hong Kong Interbank Offered Rates), SHIBOR (Shanghai Interbank Offered Rates), SIBOR (Singapore Interbank Offered Rates), TIBOR (Tokyo Interbank Offered Rates), etc.

The forward LIBOR $L(t, T, S)$ for a loan on $[T, S]$ is defined using linear compounding, i.e. by replacing (17.10) with the relation

$$1 + (S - T)L(t, T, S) = \frac{P(t, T)}{P(t, S)}, \quad 0 \leq t \leq T,$$

which yields the following definition.

**Definition 17.5.** The forward LIBOR rate $L(t, T, S)$ at time $t$ for a loan on $[T, S]$ is given by

$$L(t, T, S) = \frac{1}{S - T} \left( \frac{P(t, T)}{P(t, S)} - 1 \right), \quad 0 \leq t \leq T < S.$$  

(17.11)

Note that (17.11) above yields the same formula for the (LIBOR) instantaneous forward rate

$$L(t, T) : = \lim_{S \searrow T} L(t, T, S)$$

$$= \lim_{S \searrow T} \frac{P(t, T) - P(t, S)}{(S - T)P(t, S)}$$

$$= \lim_{\varepsilon \searrow 0} \frac{P(t, T) - P(t, T + \varepsilon)}{\varepsilon P(t, T + \varepsilon)}$$

$$= \frac{1}{P(t, T)} \lim_{\varepsilon \searrow 0} \frac{P(t, T) - P(t, T + \varepsilon)}{\varepsilon}$$

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\[ \frac{1}{P(t,T)} \frac{\partial P(t,T)}{\partial T} = - \frac{\partial \log P(t,T)}{\partial T} = f(t,T), \]

as in (17.4).

In addition, Relation (17.11) shows that the LIBOR rate can be viewed as a forward price \( X_t = X_t / N_t \) with numéraire \( N_t = (S - T)P(t,S) \) and \( X_t = P(t,T) - P(t,S) \), according to Relation (15.7) of Chapter 15. As a consequence, from Proposition 15.4, the LIBOR rate \( (L(t,T,S))_{t \in [T,S]} \) is a martingale under the forward measure \( \hat{P} \) defined by

\[ \frac{d\hat{P}}{dP^*} = \frac{1}{P(0,S)} e^{-\int_0^S r_t dt}. \]

17.2 Forward Swap Rates

The first interest rate swap occurred in 1981 between the World Bank, which was interested in borrowing German Marks and Swiss Francs, and IBM, which already had large amounts of those currencies but needed to borrow U.S. dollars.

The vanilla interest rate swap makes it possible to exchange a sequence of variable forward rates \( f(T,T_k,T_{k+1}) \), \( k = 1, 2, \ldots, n - 1 \), against a fixed rate \( \kappa \) over a time interval \( [T_1, T_n] \). Over the succession of time intervals \( [T_1, T_2], [T_2, T_3], \ldots, [T_{n-1}, T_n] \) defining a tenor structure, see Section 18.1 for details, the accumulation of such exchanges will generate a cumulative discount cash flow

\[
\left( \sum_{k=1}^{n-1} e^{-\int_T^{T_{k+1}} r_s ds} \left( e^{(T_{k+1} - T_k)f(T,T_k,T_{k+1})} - 1 \right) \right) - \left( \sum_{k=1}^{n-1} (e^{(T_{k+1} - T_k)\kappa} - 1) e^{-\int_T^{T_{k+1}} r_s ds} \right)
\]

\[
= \sum_{k=1}^{n-1} e^{-\int_T^{T_{k+1}} r_s ds} \left( e^{(T_{k+1} - T_k)f(T,T_k,T_{k+1})} - e^{(T_{k+1} - T_k)\kappa} \right),
\]

at time \( T = T_0 \), in which we used simple (or linear) interest rate compounding.
This corresponds to a payer swap in which the swap holder receives the floating leg and pays the fixed leg $\kappa$, whereas the holder of a seller swap receives the fixed leg $\kappa$ and pays the floating leg.

The above cash flow is used to make the contract fair, and it can be priced at time $T$ as

$$
\mathbb{E}^* \left[ \sum_{k=1}^{n-1} e^{-\int_T^{T_{k+1}} r_s ds} \left( e^{(T_{k+1} - T_k)(T, T_k, T_{k+1})} - e^{(T_{k+1} - T_k)T} \right) \bigg| \mathcal{F}_T \right] = \sum_{k=1}^{n-1} \left( e^{(T_{k+1} - T_k)(T, T_k, T_{k+1})} - e^{(T_{k+1} - T_k)T} \right) \mathbb{E}^* \left[ e^{-\int_T^{T_{k+1}} r_s ds} \bigg| \mathcal{F}_T \right].
$$

The swap rate $S(T, T_1, T_n)$ is by definition the value of the rate $\kappa$ that makes the contract fair by making this cash flow vanish.

In the sequel we will replace exponential compounding with simple linear compounding. In this case, the discounted cash flow becomes

$$
\left( \sum_{k=1}^{n-1} (T_{k+1} - T_k) e^{-\int_T^{T_{k+1}} r_s ds} f(T, T_k, T_{k+1}) \right) - \left( \sum_{k=1}^{n-1} \kappa(T_{k+1} - T_k) e^{-\int_T^{T_{k+1}} r_s ds} \right)
= \sum_{k=1}^{n-1} (T_{k+1} - T_k) e^{-\int_T^{T_{k+1}} r_s ds} (f(T, T_k, T_{k+1}) - \kappa),
$$

at time $T = T_0$, in which we used simple (or linear) interest rate compounding. This cash flow is used to make the contract fair, and it can be priced at time $T$ as

$$
\mathbb{E}^* \left[ \sum_{k=1}^{n-1} (T_{k+1} - T_k) e^{-\int_T^{T_{k+1}} r_s ds} (f(T, T_k, T_{k+1}) - \kappa) \bigg| \mathcal{F}_T \right] = \sum_{k=1}^{n-1} (T_{k+1} - T_k)(f(T, T_k, T_{k+1}) - \kappa) \mathbb{E}^* \left[ e^{-\int_T^{T_{k+1}} r_s ds} \bigg| \mathcal{F}_T \right]
= \sum_{k=1}^{n-1} (T_{k+1} - T_k) P(T, T_{k+1})(f(T, T_k, T_{k+1}) - \kappa).
$$

**Definition 17.6.** The swap rate $S(T, T_1, T_n)$ is the value of the break-even rate $\kappa$ that makes the contract fair by making this cash flow vanish, i.e.
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\[
\sum_{k=1}^{n-1} (T_{k+1} - T_k) P(T, T_{k+1}) (f(T, T_k, T_{k+1}) - \kappa) = 0. \tag{17.12}
\]

The next Proposition 17.7 makes use of the annuity numéraire

\[
P(T, T_1, T_n) := \mathbb{E}^* \left[ \sum_{k=1}^{n-1} (T_{k+1} - T_k) e^{-\int_T^{T_{k+1}} r_s \, ds} \, \bigg| \, \mathcal{F}_T \right] \tag{17.13}
\]

\[
= \sum_{k=1}^{n-1} (T_{k+1} - T_k) \mathbb{E}^* \left[ e^{-\int_T^{T_{k+1}} r_s \, ds} \, \bigg| \, \mathcal{F}_T \right]
\]

\[
= \sum_{k=1}^{n-1} (T_{k+1} - T_k) P(T, T_{k+1}), \quad 0 \leq T \leq T_1,
\]

which represents the present value at time \( T \) of future $1 receipts at times \( T_1, T_2, \ldots, T_n \), weighted by the lengths \( T_{k+1} - T_k \) of the time intervals \( (T_k, T_{k+1} \), \( k = 1, 2, \ldots, n - 1 \).

The time intervals \( (T_{k+1} - T_k)_{k=1,2,\ldots,n-1} \) in the definition (17.13) of the annuity numéraire can be replaced by coupon payments \( (c_{k+1})_{k=1,2,\ldots,n-1} \) occurring at times \( (T_{k+1})_{k=1,2,\ldots,n-1} \), in which case the annuity numéraire becomes

\[
P(T, T_1, T_n) := \mathbb{E}^* \left[ \sum_{k=1}^{n-1} c_{k+1} e^{-\int_T^{T_{k+1}} r_s \, ds} \, \bigg| \, \mathcal{F}_T \right] \tag{17.14}
\]

\[
= \sum_{k=1}^{n-1} c_{k+1} \mathbb{E}^* \left[ e^{-\int_T^{T_{k+1}} r_s \, ds} \, \bigg| \, \mathcal{F}_T \right]
\]

\[
= \sum_{k=1}^{n-1} c_{k+1} P(T, T_{k+1}), \quad 0 \leq T \leq T_1,
\]

which represents the value at time \( T \) of the future coupon payments discounted according to the bond prices \( (P(T, T_{k+1}))_{k=1,2,\ldots,n-1} \). This expression can also be used to define amortizing swaps in which the value of the notional decreases over time, or accreting swaps in which the value of the notional increases over time.

**LIBOR swap rates**

The LIBOR swap rate \( S(t, T_1, T_n) \) is defined by the same relation as (17.12) with the forward rate \( f(t, T_k, T_{k+1}) \) replaced with the LIBOR rate \( L(t, T_k, T_{k+1}) \), i.e.
Proposition 17.7. The LIBOR swap rate \( S(T,T_1,T_n) \) is given by

\[
S(T,T_1,T_n) = \frac{1}{P(T,T_1,T_n)} \sum_{k=1}^{n-1} (T_{k+1} - T_k)P(T,T_{k+1})L(T,T_k,T_{k+1}).
\]

(17.16)

Proof. By definition, \( S(T, T_1, T_n) \) is the (fixed) break-even rate over \([T_1, T_n]\) that will be agreed in exchange for the family of forward LIBOR rates \(L(T, T_k, T_{k+1}), k = 1, 2, \ldots, n - 1\), and it solves (17.15), i.e.

\[
\sum_{k=1}^{n-1} (T_{k+1} - T_k)P(T,T_{k+1})L(T,T_k,T_{k+1}) - P(T,T_1,T_n)S(T,T_1,T_n) = 0,
\]

which shows (17.16) by solving the above equation for \( S(T, T_1, T_n) \).

Using the Definition 17.11, of LIBOR rates we obtain the next corollary.

Corollary 17.8. The LIBOR swap rate \( S(t, T_1, T_n) \) is given by

\[
S(t, T_1, T_n) = \frac{P(t, T_1) - P(t, T_n)}{P(t, T_1, T_n)}, \quad 0 \leq t \leq T_1.
\]

(17.17)

Proof. By (17.16), (17.11) and a telescoping summation argument we have

\[
S(t, T_1, T_n) = \frac{1}{P(t, T_1, T_n)} \sum_{k=1}^{n-1} (T_{k+1} - T_k)P(t, T_{k+1})L(t, T_k, T_{k+1})
\]

\[
= \frac{1}{P(t, T_1, T_n)} \sum_{k=1}^{n-1} P(t, T_{k+1}) \left( \frac{P(t, T_k)}{P(t, T_{k+1})} - 1 \right)
\]

\[
= \frac{1}{P(t, T_1, T_n)} \sum_{k=1}^{n-1} (P(t, T_k) - P(t, T_{k+1}))
\]

\[
= \frac{P(t, T_1) - P(t, T_n)}{P(t, T_1, T_n)}.
\]

(17.18)
By (17.17), the bond prices $P(t, T_1)$ can be recovered from the values of the forward swap rates $S(t, T_1, T_n)$.

Clearly, a simple expression for the swap rate such as that of Corollary 17.8 cannot be obtained using the standard (i.e. non-LIBOR) rates defined in (17.10). Similarly, it will not be available for amortizing or accreting swaps because the telescoping summation argument does not apply to the expression (17.14) of the annuity numéraire.

When $n = 2$, the swap rate $S(t, T_1, T_2)$ coincides with the forward rate $L(t, T_1, T_2)$, i.e. we have

$$S(t, T_1, T_2) = \frac{P(t, T_1) - P(t, T_2)}{P(t, T_1, T_2)} = \frac{P(t, T_1) - P(t, T_2)}{(T_2 - T_1)P(t, T_2)} = L(t, T_1, T_2).$$

Similarly to the case of LIBOR rates, Relation (17.17) shows that the LIBOR swap rate can be viewed as a forward price with (annuity) numéraire $N_t = P(t, T_1, T_n)$ and $X_t = P(t, T_1) - P(t, T_n)$. Consequently the LIBOR swap rate $(S(t, T_1, T_n)_{t \in [T, S]}$ is a martingale under the forward measure $\hat{\mathbb{P}}$ defined from (15.1) by

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*} = P(T_1, T_n) P(0, T_1, T_n) e^{-\int_0^{T_1} r_t dt}.$$

### 17.3 The HJM Model

From the beginning of this chapter we have started with the modeling of the short rate $(r_t)_{t \in \mathbb{R}_+}$, followed by its consequences on the pricing of bonds $P(t, T)$ and on the expressions of the forward rates $f(t, T, S)$ and $L(t, T, S)$.

In this section we choose a different starting point and consider the problem of directly modeling the instantaneous forward rate $f(t, T)$. The graph given in Figure 17.7 presents a possible random evolution of a forward interest rate curve using the Musiela convention, i.e. we will write

$$g(x) = f(t, t + x) = f(t, T),$$

under the substitution $x = T - t$, $x \geq 0$, and represent a sample of the instantaneous forward curve $x \mapsto f(t, t + x)$ for each $t \in \mathbb{R}_+$.
In the Heath-Jarrow-Morton (HJM) model, the instantaneous forward rate \( f(t, T) \) is modeled under \( \mathbb{P}^* \) by a stochastic differential equation of the form

\[
d_t f(t, T) = \alpha(t, T)dt + \sigma(t, T)dB_t, \quad 0 \leq t \leq T, \tag{17.20}
\]

where \( t \mapsto \alpha(t, T) \) and \( t \mapsto \sigma(t, T) \), \( 0 \leq t \leq T \), are allowed to be random (adapted) processes. In the above equation, the date \( T \) is fixed and the differential \( dt \) is with respect to \( t \).

Under basic Markovianity assumptions, a HJM model with deterministic coefficients \( \alpha(t, T) \) and \( \sigma(t, T) \) will yield a short rate process \((r_t)_{t \in \mathbb{R}_+}\) of the form

\[
dr_t = (a(t) - b(t)r_t)dt + \sigma(t)dB_t,
\]

cf. § 6.6 of Privault (2012), which is the Hull and White (1990) model, with explicit solution

\[
r_t = r_s e^{-\int_s^t b(\tau)d\tau} + \int_s^t e^{-\int_s^\tau b(\tau)d\tau}a(u)du + \int_s^t \sigma(u)e^{-\int_u^\tau b(\tau)d\tau}dB_u,
\]

\( 0 \leq s \leq t \).

**The HJM condition**

How to “encode” absence of arbitrage in the defining HJM Equation (17.20) is an important question. Recall that under absence of arbitrage, the bond price \( P(t, T) \) has been constructed as

\[
P(t, T) = \mathbb{E}^* \left[ \exp \left( -\int_t^T r_s ds \right) \mid \mathcal{F}_t \right] = \exp \left( -\int_t^T f(t, s)ds \right), \tag{17.21}
\]

cf. Proposition 17.3, hence the discounted bond price process is given by

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Forward Rate Modeling

\[ t \mapsto \exp \left( - \int_0^t r_s ds \right) P(t, T) = \exp \left( - \int_0^t r_s ds - \int_t^T f(t, s) ds \right) \quad (17.22) \]

is a martingale under \( \mathbb{P}^* \) by Proposition 16.1 and Relation (17.5) in Proposition 17.3. This shows that \( \mathbb{P}^* \) is a risk-neutral probability measure, and by the first fundamental theorem of asset pricing Theorem 5.7 we conclude that the market is without arbitrage opportunities.

**Proposition 17.9.** (*HJM Condition* Heath et al. (1992)). Under the condition

\[ \alpha(t, T) = \sigma(t, T) \int_t^T \sigma(t, s) ds, \quad t \in [0, T], \quad (17.23) \]

which is known as the HJM absence of arbitrage condition, the discounted bond price process (17.22) is a martingale, and the probability measure \( \mathbb{P}^* \) is risk-neutral.

**Proof.** Consider the spot forward rate, or yield, given from (17.7) as

\[ f(t, t, T) = \frac{1}{T-t} \int_t^T f(t, s) ds, \]

and consider the process \((X_t)_{t \in [0, T]}\) defined as

\[ X_t := \int_t^T f(t, s) ds = - \log P(t, T), \quad 0 \leq t \leq T, \]

such that \( P(t, T) = e^{-X_t} \), with the relation

\[ f(t, t, T) = \frac{1}{T-t} \int_t^T f(t, s) ds = \frac{X_t}{T-t}, \quad 0 \leq t \leq T, \quad (17.24) \]

where the dynamics of \( t \mapsto f(t, s) \) is given by (17.20). We note that when \( f(t, s) = g(t)h(s) \) is a smooth function which satisfies the separation of variables property we have the relation

\[ dt \left( \int_t^T g(t)h(s) ds \right) = dt \left( g(t) \int_t^T h(s) ds \right) \]

\[ = \int_t^T h(s) ds dg(t) + g(t) dt \int_t^T h(s) ds \]

\[ = g'(t) \left( \int_t^T h(s) ds \right) dt - g(t)h(t) dt, \]

which extends to \( f(t, s) \) as

\[ dt \int_t^T f(t, s) ds = - f(t, t) dt + \int_t^T dt f(t, s) ds = -r_t dt + \int_t^T dt f(t, s) ds, \]

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which can be seen as a form of the Leibniz integral rule. Therefore we have

\[ d_t X_t = d_t \int_t^T f(t, s) \, ds \]

\[ = -f(t, t) \, dt + \int_t^T d_t f(t, s) \, ds \]

\[ = -f(t, t) \, dt + \int_t^T \alpha(t, s) \, ds \, dt + \int_t^T \sigma(t, s) \, d s d B_t \]

\[ = -r_t \, dt + \left( \int_t^T \alpha(t, s) \, ds \right) \, dt + \left( \int_t^T \sigma(t, s) \, d s d B_t \right) \]

hence we have

\[ |d_t X_t|^2 = \left( \int_t^T \sigma(t, s) \, ds \right)^2 \, dt. \]

Hence by Itô’s calculus we have

\[ d_t P(t, T) = d_t e^{-X_t} \]

\[ = -e^{-X_t} d_t X_t + \frac{1}{2} e^{-X_t} (d_t X_t)^2 \]

\[ = -e^{-X_t} d_t X_t + \frac{1}{2} e^{-X_t} \left( \int_t^T \sigma(t, s) \, ds \right)^2 \, dt \]

\[ = -e^{-X_t} \left( -r_t \, dt + \int_t^T \alpha(t, s) \, ds \, dt + \int_t^T \sigma(t, s) \, d s d B_t \right) \]

\[ + \frac{1}{2} e^{-X_t} \left( \int_t^T \sigma(t, s) \, ds \right)^2 \, dt , \]

and the discounted bond price satisfies

\[ d_t \left( \exp \left( -\int_0^t r_s ds \right) P(t, T) \right) \]

\[ = -r_t \exp \left( -\int_0^t r_s ds - X_t \right) \, dt + \exp \left( -\int_0^t r_s ds \right) d_t P(t, T) \]

\[ = -r_t \exp \left( -\int_0^t r_s ds - X_t \right) \, dt - \exp \left( -\int_0^t r_s ds - X_t \right) \, d_t X_t \]

\[ + \frac{1}{2} \exp \left( -\int_0^t r_s ds - X_t \right) \left( \int_t^T \sigma(t, s) \, ds \right)^2 \, dt \]

\[ = -r_t \exp \left( -\int_0^t r_s ds - X_t \right) \, dt \]

\[ - \exp \left( -\int_0^t r_s ds - X_t \right) \left( -r_t \, dt + \int_t^T \alpha(t, s) \, ds \, dt + \int_t^T \sigma(t, s) \, d s d B_t \right) \]

\[ + \frac{1}{2} \exp \left( -\int_0^t r_s ds - X_t \right) \left( \int_t^T \sigma(t, s) \, ds \right)^2 \, dt \]
Forward Rate Modeling

\[
\begin{align*}
&= -\exp\left( -\int_{0}^{t} r_s ds - X_t \right) \int_{t}^{T} \sigma(t, s) ds dB_t \\
&\quad - \exp\left( -\int_{0}^{t} r_s ds - X_t \right) \left( \int_{t}^{T} \alpha(t, s) ds dt - \frac{1}{2} \left( \int_{t}^{T} \sigma(t, s) ds \right)^2 \right) dt.
\end{align*}
\]

Thus, the discounted bond price process

\[
t \mapsto \exp \left( -\int_{0}^{t} r_s ds \right) P(t, T)
\]

will be a martingale provided that

\[
\int_{t}^{T} \alpha(t, s) ds - \frac{1}{2} \left( \int_{t}^{T} \sigma(t, s) ds \right)^2 = 0, \quad 0 \leq t \leq T. \quad (17.25)
\]

Differentiating the above relation with respect to \( T \) yields

\[
\alpha(t, T) = \sigma(t, T) \int_{t}^{T} \sigma(t, s) ds,
\]

which is in fact equivalent to (17.25).

**Forward HJM rates**

The HJM coefficients in the Vasicek model are in fact deterministic, for example, taking \( a = 0 \) we have

\[
d_t f(t, T) = \sigma^2 e^{-(T-t)b} \int_{t}^{T} e^{(t-s)b} ds dt + \sigma e^{-(T-t)b} dB_t,
\]

i.e.

\[
\alpha(t, T) = \sigma^2 e^{-(T-t)b} \int_{t}^{T} e^{(t-s)b} ds = \sigma^2 e^{-(T-t)b} \frac{1 - e^{-(T-t)b}}{b},
\]

and \( \sigma(t, T) = \sigma e^{-(T-t)b} \), and the HJM condition reads

\[
\alpha(t, T) = \sigma^2 e^{-(T-t)b} \int_{t}^{T} e^{(t-s)b} ds = \sigma(t, T) \int_{t}^{T} \sigma(t, s) ds. \quad (17.26)
\]

Random simulations of the Vasicek instantaneous forward rates are provided in Figures 17.8 and 17.9.
Fig. 17.8: Forward instantaneous curve \((t, x) \mapsto f(t, t + x)\) in the Vasicek model.*

Fig. 17.9: Forward instantaneous curve \(x \mapsto f(0, x)\) in the Vasicek model.†

For \(x = 0\) the first “slice” of this surface is actually the short rate Vasicek process \(r_t = f(t, t) = f(t, t + 0)\) which is represented in Figure 17.10 using another discretization.

† The animation works in Acrobat Reader on the entire pdf file.
17.4 Yield Curve Modeling

Nelson-Siegel parametrization of instantaneous forward rates

In the Nelson-Siegel parametrization the instantaneous forward rate curves are parametrized by 4 coefficients $z_1, z_2, z_3, z_4$, as

$$g(x) = z_1 + (z_2 + z_3x)e^{-xz_4}, \quad x \geq 0.$$  

An example of a graph obtained by the Nelson-Siegel parametrization is given in Figure 17.11, for $z_1 = 1$, $z_2 = -10$, $z_3 = 100$, $z_4 = 10$.

Svensson parametrization of instantaneous forward rates

The Svensson parametrization has the advantage to reproduce two humps instead of one, the location and height of which can be chosen via 6 parameters $z_1, z_2, z_3, z_4, z_5, z_6$ as

$$g(x) = z_1 + (z_2 + z_3x)e^{-xz_4} + z_5xe^{-xz_6}, \quad x \geq 0.$$
A typical graph of a Svensson parametrization is given in Figure 17.12, for $z_1 = 6.6$, $z_2 = -5$, $z_3 = -100$, $z_4 = 10$, $z_5 = -1/2$, $z_6 = 1$.

![Graph of $x \mapsto g(x)$ in the Svensson model.](image)

**Fig. 17.12:** Graph of $x \mapsto g(x)$ in the Svensson model.

Figure 17.13 presents a fit of the market data of Figure 17.1 using a Svensson curve.

![Comparison of market rates vs a Svensson curve.](image)

**Fig. 17.13:** Comparison of market rates vs a Svensson curve.

The attached IPython notebook can be run here to fit a Svensson curve to market data.

**Vasicek parametrization**

In the Vasicek model, the instantaneous forward rate process is given by

$$f(t, T) = \frac{a}{b} - \frac{\sigma^2}{2b^2} + \left( r_t - \frac{a}{b} + \frac{\sigma^2}{b^2} \right) e^{-bx} - \frac{\sigma^2}{2b^2} e^{-2bx},$$

in the Musiela notation ($x = T - t$), hence we have

$$\frac{\partial f}{\partial T}(t, T) = \left( -br_t + a - \frac{\sigma^2}{b} + \frac{\sigma^2}{b} e^{-(T-t)b} \right) e^{-(T-t)b},$$

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and one can check that the sign of the derivatives of $f$ can only change once at most. As a consequence, the possible forward curves in the Vasicek model are limited to one change of “regime” per curve, as illustrated in Figure 17.14 for various values of $r_t$, and in Figure 17.15.

Fig. 17.14: Graphs of forward rates with $b = 0.16$, $a/b = 0.04$, $r_0 = 2\%$, $\sigma = 4.5\%$.

The next figure is also using the parameters $b = 0.16$, $a/b = 0.04$, $r_0 = 2\%$, and $\sigma = 4.5\%$.

Fig. 17.15: Forward instantaneous curve $(t, x) \mapsto f(t, t + x)$ in the Vasicek model.

One may think of constructing an instantaneous rate process taking values in the Svensson space, however this type of modeling is not consistent with absence of arbitrage, and it can be proved that the HJM curves cannot live in the Nelson-Siegel or Svensson spaces, cf. §3.5 of Björk (2004b). In other words, it can be shown that the forward yield curves produced by the Vasicek model are included neither in the Nelson-Siegel space, nor in the Svensson space. In addition, the Vasicek yield curves do not appear to correctly model the market forward curves cf. also Figure 17.1 above.
Another way to deal with the curve fitting problem is to use deterministic shifts for the fitting of one forward curve, such as the initial curve at $t = 0$, cf. e.g. § 8.2 of Privault (2012).

**Fitting the Nelson-Siegel and Svensson models to yield curve data**

Recall that in the Nelson-Siegel parametrization the instantaneous forward rate curves are parametrized by four coefficients $z_1, z_2, z_3, z_4$, as

$$f(t, t + x) = z_1 + (z_2 + z_3 x) e^{-x z_4}, \quad x \geq 0. \quad (17.28)$$

Taking $x = T - t$, the yield $f(t, t, T)$ is given as

$$f(t, t, T) = \frac{1}{T - t} \int_t^T f(t, s)ds$$

$$= \frac{1}{x} \int_0^x f(t, t + y)dy$$

$$= z_1 + \frac{z_2}{x} \int_0^x e^{-y z_4}dy + \frac{z_3}{x} \int_0^x y e^{-y z_4}dy$$

$$= z_1 + z_2 \frac{1 - e^{-x z_4}}{xz_4} + z_3 \frac{1 - e^{-x z_4} + x e^{-x z_4}}{xz_4}.$$

The expression (17.28) can be represented in the parametrization

$$f(t, t + x) = z_1 + (z_2 + z_3 x) e^{-x z_4} = \beta_0 + \beta_1 e^{-x/\lambda} + \frac{\beta_2}{\lambda} x e^{-x/\lambda}, \quad x \geq 0,$$

cf. Charpentier (2014), with $\beta_0 = z_1, \beta_1 = z_2, \beta_2 = z_3 / z_4, \lambda = 1 / z_4$. 

```
require(YieldCurve)
data(ECBYieldCurve)
mat.ECB<-c(3/12, 0.5, 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30)
first(ECBYieldCurve, '1 month')
Nelson.Siegel(first(ECBYieldCurve, '1 month'), mat.ECB)
```
for (n in seq(from=70, to=290, by=10)) {
  ECB.NS <- Nelson.Siegel(ECBYieldCurve[,], mat.ECB)
  ECB.S <- Svensson(ECBYieldCurve[,], mat.ECB)
  ECB.NS.yield.curve <- NSrates(ECB.NS, mat.ECB)
  ECB.S.yield.curve <- Srates(ECB.S, mat.ECB, "Spot")
  plot(mat.ECB, as.numeric(ECBYieldCurve[,]), type="o", lty=1, col=1, ylab="Interest rates", xlab="Maturity in years", ylim=c(3.2,4.8))
  lines(mat.ECB, as.numeric(ECB.NS.yield.curve), type="l", lty=3, col=2, lwd=2)
  lines(mat.ECB, as.numeric(ECB.S.yield.curve), type="l", lty=2, col=6, lwd=2)
  title(main=paste("ECB yield curve observed at", time(ECBYieldCurve[,], sep=" "), "vs fitted yield curve"))
  legend("bottomright", legend=c("ECB data", "Nelson-Siegel", "Svensson"), col=c(1,2,6), lty=1, bg="gray90")
  grid()
  Sys.sleep(2.5)}

Fig. 17.16: ECB data vs fitted yield curve.*

Two-Factor Model

The correlation problem is another issue of concern when using the affine models considered so far. Let us compare three bond price simulations with maturity $T_1 = 10$, $T_2 = 20$, and $T_3 = 30$ based on the same Brownian path, as given in Figure 17.17. Clearly, the bond prices $F(r_t, T_1) = P(t, T_1)$ and $F(r_t, T_2) = P(t, T_2)$ with maturities $T_1$ and $T_2$ are linked by the relation

$$P(t, T_2) = P(t, T_1) \exp \left( A(t, T_2) - A(t, T_1) + r_t (C(t, T_2) - C(t, T_1)) \right),$$

(17.29)

meaning that bond prices with different maturities could be deduced from each other, which is unrealistic.

* The animation works in Acrobat Reader on the entire pdf file.
In affine short rates models, by \((17.29)\), \(\log P(t, T_1)\) and \(\log P(t, T_2)\) are linked by the linear relationship

\[
\log P(t, T_2) = \log P(t, T_1) + A(t, T_2) - A(t, T_1) + r_t(C(t, T_2) - C(t, T_1))
\]

\[
= \log P(t, T_1) + A(t, T_2) - A(t, T_1) + (C(t, T_2) - C(t, T_1)) \frac{\log P(t, T_1) - C(t, T_1)}{A(t, T_1)}
\]

\[
= \left(1 + \frac{C(t, T_2) - C(t, T_1)}{A(t, T_1)}\right) \log P(t, T_1) + A(t, T_2) - A(t, T_1) - (C(t, T_2) - C(t, T_1)) \frac{C(t, T_1)}{A(t, T_1)}
\]

with constant coefficients, which yields the perfect (positive or negative) correlation

\[
\text{Cor}(\log P(t, T_1), \log P(t, T_2)) = \pm 1,
\]

depending on the sign of the coefficient \(1 + (C(t, T_2) - C(t, T_1))/A(t, T_1)\), cf. § 8.3 of Privault (2012),

A solution to the correlation problem is to consider a two-factor model based on two control processes \((X_t)_{t \in \mathbb{R}_+}\), \((Y_t)_{t \in \mathbb{R}_+}\) which are solution of

\[
\begin{align*}
    dX_t &= \mu_1(t, X_t) dt + \sigma_1(t, X_t) dB^{(1)}_t, \\
    dY_t &= \mu_2(t, Y_t) dt + \sigma_2(t, Y_t) dB^{(2)}_t,
\end{align*}
\]

(17.30)

where \((B^{(1)}_t)_{t \in \mathbb{R}_+}, (B^{(2)}_t)_{t \in \mathbb{R}_+}\) have correlated Brownian motion with

\[
\text{Cov}(B^{(1)}_s, B^{(2)}_t) = \rho \min(s, t), \quad s, t \in \mathbb{R}_+,
\]

(17.31)

and

\[
 dB^{(1)}_t \cdot dB^{(2)}_t = \rho dt,
\]

(17.32)
for some correlation parameter $\rho \in [-1,1]$. In practice, $(B^{(1)})_{t \in \mathbb{R}_+}$ and $(B^{(2)})_{t \in \mathbb{R}_+}$ can be constructed from two independent Brownian motions $(W^{(1)})_{t \in \mathbb{R}_+}$ and $(W^{(2)})_{t \in \mathbb{R}_+}$, by letting
\[
\begin{cases}
B^{(1)}_t = W^{(1)}_t, \\
B^{(2)}_t = \rho W^{(1)}_t + \sqrt{1 - \rho^2} W^{(2)}_t,
\end{cases}
\quad t \in \mathbb{R}_+,
\]
and Relations (17.31) and (17.32) are easily satisfied from this construction. In two-factor models one chooses to build the short-term interest rate $r_t$ via
\[
r_t := X_t + Y_t, \quad t \in \mathbb{R}_+.
\]
By the previous standard arbitrage arguments we define the price of a bond with maturity $T$ as
\[
P(t, T) := \mathbb{E}^* \left[ \exp \left( -\int_t^T r_s ds \right) \bigg| \mathcal{F}_t \right] \\
= \mathbb{E}^* \left[ \exp \left( -\int_t^T r_s ds \right) \bigg| X_t, Y_t \right] \\
= \mathbb{E}^* \left[ \exp \left( -\int_t^T (X_s + Y_s) ds \right) \bigg| X_t, Y_t \right] \\
= F(t, X_t, Y_t), \tag{17.33}
\]
since the couple $(X_t, Y_t)_{t \in \mathbb{R}_+}$ is Markovian. Applying the Itô formula with two variables to
\[
F(t, X_t, Y_t) = P(t, T) = \mathbb{E}^* \left[ \exp \left( -\int_t^T r_s ds \right) \bigg| \mathcal{F}_t \right],
\]
and using the fact that the discounted process
\[
t \mapsto e^{-\int_0^t r_s ds} P(t, T) = \mathbb{E}^* \left[ \exp \left( -\int_0^T r_s ds \right) \bigg| \mathcal{F}_t \right]
\]
is an $\mathcal{F}_t$-martingale under $\mathbb{P}^*$, we can derive a PDE
\[
-(x + y)F(t, x, y) + \mu_1(t, x) \frac{\partial F}{\partial x}(t, x, y) + \mu_2(t, y) \frac{\partial F}{\partial y}(t, x, y) \\
+ \frac{1}{2} \sigma_1^2(t, x) \frac{\partial^2 F}{\partial x^2}(t, x, y) + \frac{1}{2} \sigma_2^2(t, y) \frac{\partial^2 F}{\partial y^2}(t, x, y) \\
+ \rho \sigma_1(t, x) \sigma_2(t, y) \frac{\partial^2 F}{\partial x \partial y}(t, x, y) + \frac{\partial F}{\partial t}(t, X_t, Y_t) = 0, \tag{17.34}
\]
on $\mathbb{R}^2$ for the bond price $P(t,T)$. In the Vasicek model

\[
\begin{align*}
\begin{cases}
dX_t &= -aX_t dt + \sigma dB_t^{(1)}, \\
dY_t &= -bY_t dt + \eta dB_t^{(2)},
\end{cases}
\end{align*}
\]

this yields the solution $F(t,x,y)$ of (17.34) as

\[
P(t,T) = F(t,X_t,Y_t) = F_1(t,X_t)F_2(t,Y_t) \exp(\rho U(t,T)),
\]

where $F_1(t,X_t)$ and $F_2(t,Y_t)$ are the bond prices associated to $X_t$ and $Y_t$ in the Vasicek model, and

\[
U(t,T) := \frac{\sigma \eta}{ab} \left( T - t + \frac{e^{-(T-t)a} - 1}{a} + \frac{e^{-(T-t)b} - 1}{b} - \frac{e^{-(a+b)(T-t)} - 1}{a+b} \right)
\]

is a correlation term which vanishes when $(B_t^{(1)})_{t \in \mathbb{R}_+}$ and $(B_t^{(2)})_{t \in \mathbb{R}_+}$ are independent, i.e. when $\rho = 0$, cf Brigo and Mercurio (2006), Chapter 4, Appendix A, and § 8.4 of Privault (2012).

Partial differentiation of $\log P(t,T)$ with respect to $T$ leads to the instantaneous forward rate

\[
f(t,T) = f_1(t,T) + f_2(t,T) - \rho \frac{\sigma \eta}{ab} (1 - e^{-(T-t)a})(1 - e^{-(T-t)b}),
\]

where $f_1(t,T)$, $f_2(t,T)$ are the instantaneous forward rates corresponding to $X_t$ and $Y_t$ respectively, cf. § 8.4 of Privault (2012).

An example of a forward rate curve obtained in this way is given in Figure 17.18.

![Graph of forward rates in a two-factor model](https://www.ntu.edu.sg/home/nprivault/indext.html)

Fig. 17.18: Graph of forward rates in a two-factor model.
Forward Rate Modeling

Next, in Figure 17.19 we present a graph of the evolution of forward curves in a two factor model.

Fig. 17.19: Random evolution of instantaneous forward rates in a two-factor model.

17.5 The BGM Model

The models (HJM, affine, etc.) considered in the previous chapter suffer from various drawbacks such as nonpositivity of interest rates in Vasicek model, and lack of closed-form solutions in more complex models. The Brace et al. (1997) (BGM) model has the advantage of yielding positive interest rates, and to permit to derive explicit formulas for the computation of prices for interest rate derivatives such as caps and swaptions on the LIBOR market.

In the BGM model we consider two bond prices \( P(t, T_1) \), \( P(t, T_2) \) with maturities \( T_1, T_2 \) and the forward measure

\[
\frac{d\mathbb{P}_2}{d\mathbb{P}_2^*} = \frac{e^{-\int_0^{T_2} r_s ds}}{P(0, T_2)},
\]

with numéraire \( P(t, T_2) \), cf. (15.6). The forward LIBOR rate \( L(t, T_1, T_2) \) is modeled as a driftless geometric Brownian motion under \( \mathbb{P}_2 \), i.e.

\[
\frac{dL(t, T_1, T_2)}{L(t, T_1, T_2)} = \gamma_1(t) dB_t^{(2)},
\]

(17.37)

\( 0 \leq t \leq T_1, \) \( i = 1, 2, \ldots, n - 1 \), for some deterministic volatility function of time \( \gamma_1(t) \), with solution

\[
L(u, T_1, T_2) = L(t, T_1, T_2) \exp \left( \int_t^u \gamma_1(s) dB_s^{(2)} - \frac{1}{2} \int_t^u |\gamma_1|^2(s) ds \right),
\]

i.e. for \( u = T_1 \),
\[ L(T_1, T_1, T_2) = L(t, T_1, T_2) \exp \left( \int_t^{T_1} \gamma_1(s) dB_s^{(2)} - \frac{1}{2} \int_t^{T_1} |\gamma_1|^2(s) ds \right). \]

Since \( L(t, T_1, T_2) \) is a geometric Brownian motion under \( \mathbb{P}_2 \), standard caplets can be priced at time \( t \in [0, T_1] \) from the Black-Scholes formula.

In the next Table 17.1 we summarize some stochastic models used for interest rates.

<table>
<thead>
<tr>
<th>Model</th>
<th>Mean reverting SDE</th>
<th>BGM model</th>
<th>HJM model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Short rate ( r_t )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Forward rate ( f(t, T, S) )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Instantaneous forward rate ( f(t, s) )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 17.1: Stochastic interest rate models.

The following Graph 17.20 summarizes the notions introduced in this chapter.
Forward Rate Modeling

**Bond price**

\[ P(t, T) = e^{\int_t^T r_s ds} \]

**Short rate**

\[ r_t = f(t, t) \]

**LIBOR rate**

\[ L(t, T, S) = \frac{P(t, T) - P(t, S)}{(T - S) P(t, S)} \]

**Forward rate**

\[ f(t, T, S) = \log \frac{P(t, T)}{P(t, S)} - \frac{S - T}{T - t} \]

**Instantaneous forward rate**

\[ f(t, T) = L(t, T) = \lim_{S \to T} f(t, T, S) = \lim_{S \to T} L(t, T, S) \]

**Spot forward rate (yield)**

\[ f(t, t, T) = \int_t^T f(t, s) ds / (T - t) \]

1. Can be modeled by Vasicek and other short rate models
2. Can be modeled from \( dP(t, T)/P(t, T) \).
3. Can be modeled in the BGM model
4. Can be modeled in the HJM model

Fig. 17.20: Roadmap of stochastic interest rate modeling.
Exercises

Exercise 17.1 Consider a tenor structure \( \{T_1, T_2\} \) and a bond with maturity \( T_2 \) and price given at time \( t \in [0, T_2] \) by
\[
P(t, T_2) = \exp \left( -\int_t^{T_2} f(t, s) ds \right), \quad t \in [0, T_2],
\]
where the instantaneous yield curve \( f(t, s) \) is parametrized as
\[
f(t, s) = r_1 \mathbb{1}_{[0, T_1]}(s) + r_2 \mathbb{1}_{[T_1, T_2]}(s), \quad s \in [t, T_2].
\]
Find a formula to estimate the values of \( r_1 \) and \( r_2 \) from the data of \( P(0, T_2) \) and \( P(T_2, T_2) \).

Same question for when \( f(t, s) \) is parametrized as
\[
f(t, s) = r_1 s \mathbb{1}_{[0, T_1]}(s) + (r_1 T_1 + r_2 (s - T_1)) \mathbb{1}_{[T_1, T_2]}(s), \quad s \in [t, T_2].
\]

Exercise 17.2 (Exercise 4.10 continued). Bridge model. Assume that the price \( P(t, T) \) of a zero-coupon bond is modeled as
\[
P(t, T) = e^{-\mu (T-t) + X_t^T}, \quad t \in [0, T],
\]
where \( \mu > 0 \).

a) Show that the terminal condition \( P(T, T) = 1 \) is satisfied.

b) Compute the forward rate
\[
f(t, T, S) = -\frac{1}{S-T} (\log P(t, S) - \log P(t, T)).
\]

c) Compute the instantaneous forward rate
\[
f(t, T) = -\lim_{S \downarrow T} \frac{1}{S-T} (\log P(t, S) - \log P(t, T)).
\]

d) Show that the limit \( \lim_{T \downarrow t} f(t, T) \) does not exist in \( L^2(\Omega) \).

e) Show that \( P(t, T) \) satisfies the stochastic differential equation
\[
\frac{dP(t, T)}{P(t, T)} = \sigma dB_t + \frac{\sigma^2}{2} dt - \frac{\log P(t, T)}{T-t} dt, \quad t \in [0, T].
\]

f) Rewrite the equation of Question (e) as
Forward Rate Modeling

\[
\frac{dP(t, T)}{P(t, T)} = \sigma dB_t + r_t^T dt, \quad t \in [0, T],
\]

where \((r_t^T)_{t \in [0, T]}\) is a process to be determined.

g) Show that we have the expression

\[
P(t, T) = \mathbb{E}^* \left[ e^{-\int_t^T r_s^T ds} \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T.
\]

h) Compute the conditional density

\[
\mathbb{E}^* \left[ \frac{dP_T}{dP^*} \mid \mathcal{F}_t \right] = \frac{P(t, T)}{P(0, T)} e^{-\int_0^t r_s^T ds}
\]

of the forward measure \(P_T\) with respect to \(P^*\).

i) Show that the process \(\hat{B}_t := B_t - \sigma t, \quad 0 \leq t \leq T\),

is a standard Brownian motion under \(P_T\).

j) Compute the dynamics of \(X_t^S\) and \(P(t, S)\) under \(P_T\).

Hint: Show that

\[
-\mu(S - T) + \sigma(S - T) \int_0^t \frac{1}{S - s} dB_s = \frac{S - T}{S - t} \log P(t, S).
\]

k) Compute the bond option price

\[
\mathbb{E}^* \left[ e^{-\int_t^T r_s^T ds} (P(T, S) - K)^+ \mid \mathcal{F}_t \right] = P(t, T) \mathbb{E}_T [(P(T, S) - K)^+ \mid \mathcal{F}_t],
\]

\(0 \leq t < T < S\).

Hint: Given \(X\) a centered Gaussian random variable with mean \(m\) and variance \(v^2\) given \(\mathcal{F}_t\), we have:

\[
\mathbb{E} \left[ (e^X - \kappa)^+ \mid \mathcal{F}_t \right] = e^{m + v^2/2} \Phi \left( \frac{v}{2} + \frac{1}{v} (m + v^2/2 - \log \kappa) \right) - \kappa \Phi \left( -\frac{v}{2} + \frac{1}{v} (m + v^2/2 - \log \kappa) \right).
\]

Exercise 17.3 Consider a short rate process \((r_t)_{t \in \mathbb{R}_+}\) of the form \(r_t = h(t) + X_t\), where \(h(t)\) is a deterministic function of time and \((X_t)_{t \in \mathbb{R}_+}\) is a Vasicek process started at \(X_0 = 0\).

a) Compute the price \(P(0, T)\) at time \(t = 0\) of a bond with maturity \(T\), using \(h(t)\) and the function \(A(T)\) defined in (16.29) for the pricing of Vasicek bonds.
b) Show how the function $h(t)$ can be estimated from the market data of the initial instantaneous forward rate curve $f(0,t)$.

**Exercise 17.4**

a) Given two LIBOR spot rates $L(t,t,T)$ and $L(t,t,S)$, compute the corresponding LIBOR forward rate $L(t,T,S)$.

b) Assuming that $L(t,t,T) = 2\%$, $L(t,t,S) = 2.5\%$ and $t = 0$, $T = 1$, $S = 2T = 2$, would you buy a LIBOR forward contract over $[T,2T]$ with rate $L(0,T,2T)$ if $L(T,T,2T)$ remained at $L(T,T,2T) = L(0,0,T) = 2\%$?

**Exercise 17.5** (Exercise 16.3 continued).

a) Compute the forward rate $f(t,T,S)$ in this model.

b) Compute the instantaneous forward rate $f(t,T)$ in this model.

c) Derive the stochastic equation satisfied by the instantaneous forward rate $f(t,T)$.

d) Check that the HJM absence of arbitrage condition is satisfied in this equation.

**Exercise 17.6**  Stochastic string model (Santa-Clara and Sornette (2001)).

Consider an instantaneous forward rate $f(t,x)$ solution of

$$
\frac{df(t,x)}{dt} = \alpha x^2 dt + \sigma dx B(t,x),
$$

(17.38)

with a flat initial curve $f(0,x) = r$, where $x$ represents the time to maturity, and $(B(t,x))_{(t,x) \in \mathbb{R}_+^2}$ is a standard Brownian sheet with covariance

$$
\mathbb{E}[B(s,x)B(t,y)] = \min(s,t) \times \min(x,y), \quad s,t,x,y \in \mathbb{R}_+,
$$

and initial conditions $B(t,0) = B(0,x) = 0$ for all $t,x \in \mathbb{R}_+$.

a) Solve the equation (17.38) for $f(t,x)$.

b) Compute the short-term interest rate $r_t = f(t,0)$.

c) Compute the value at time $t \in [0,T]$ of the bond price

$$
P(t,T) = \exp \left( - \int_0^{T-t} f(t,x)dx \right)
$$

with maturity $T$.  

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d) Compute the variance $\mathbb{E} \left[ \left( \int_0^{T-t} B(t,x)dx \right)^2 \right]$ of the centered Gaussian random variable $\int_0^{T-t} B(t,x)dx$.

e) Compute the expected value $\mathbb{E}^* [P(t,T)]$.

f) Find the value of $\alpha$ such that the discounted bond price 

$$e^{-rt} P(t,T) = \exp \left( -rT - \frac{\alpha}{3} t(T-t)^3 - \sigma \int_0^{T-t} B(t,x)dx \right), \quad t \in [0,T].$$

satisfies $e^{-rt} \mathbb{E}^* [P(t,T)] = e^{-rT}$.

g) Compute the bond option price $\mathbb{E}^* \left[ \exp \left( -\int_0^T r_s ds \right) (P(T,S) - K)^+ \right]$ by the Black-Scholes formula, knowing that for any centered Gaussian random variable $X \simeq N(0,v^2)$ with variance $v^2$ we have

$$\mathbb{E}[(xe^{m+X} - K)^+] = xe^{m+v^2/2} \Phi \left( (m+\log(x/K))/v \right) - K \Phi \left( (m+\log(x/K))/v \right).$$

Exercise 17.7  (Exercise 17.2 continued).

a) Compute the forward rate

$$f(t,T,S) = -\frac{1}{S-T} (\log P(t,S) - \log P(t,T)).$$

b) Compute the instantaneous forward rate

$$f(t,T) = -\lim_{S \searrow T} \frac{1}{S-T} (\log P(t,S) - \log P(t,T)).$$

c) Show that the limit $\lim_{T \searrow t} f(t,T)$ does not exist in $L^2(\Omega)$.

d) Show that $P(t,T)$ satisfies the stochastic differential equation

$$\frac{dP(t,T)}{P(t,T)} = \sigma dB_t + \frac{1}{2} \sigma^2 dt - \frac{\log P(t,T)}{T-t} dt, \quad t \in [0,T].$$

e) Show, using the results of Exercise 16.9-(c), that

$$P(t,T) = \mathbb{E}^* \left[ e^{-\int_t^T r_s^T ds} | \mathcal{F}_t \right],$$

where $(r_s^T)_{t \in [0,T]}$ is a stochastic process to be determined.

f) Compute the conditional density

$$\mathbb{E}^* \left[ \frac{dP_T}{dP} | \mathcal{F}_t \right] = \frac{P(t,T)}{P(0,T)} e^{-\int_0^t r_s^T ds}.$$
of the forward measure $\mathbb{P}_T$ with respect to $\mathbb{P}^*$.

\textbf{g)} Show that the process

$$\hat{B}_t := B_t - \sigma t, \quad 0 \leq t \leq T,$$

is a standard Brownian motion under $\mathbb{P}_T$.

\textbf{h)} Compute the dynamics of $X^S_t$ and $P(t, S)$ under $\mathbb{P}_T$.

\textit{Hint:} Show that

$$-\mu(S - T) + \sigma(S - T) \int_0^t \frac{1}{S - s} dB_s = \frac{S - T}{S - t} \log P(t, S).$$

\textbf{i)} Compute the bond option price

$$\mathbb{E}^* \left[ e^{-\int_t^T \frac{1}{S - S'} dS} (P(T, S) - K)^+ \bigg| \mathcal{F}_t \right] = P(t, T) \mathbb{E}_T \left[ (P(T, S) - K)^+ \bigg| \mathcal{F}_t \right],$$

$$0 \leq t < T < S.$$