Chapter 1

Exercise 1.1 The payoff $C$ is that of a *put* option with strike price $K = \$3$.

Exercise 1.2 Each of the two possible scenarios yields one equation:

$$
\begin{align*}
5\alpha + \beta &= 0 \\
2\alpha + \beta &= 6,
\end{align*}
$$

with solution

$$
\begin{align*}
\alpha &= -2 \\
\beta &= +10.
\end{align*}
$$

The hedging strategy at $t = 0$ is to **shortsell** $-\alpha = +2$ units of the asset $S$ priced $S_0 = 4$, and to put $\beta = \$10$ in savings. The price $V_0 = \alpha S_0 + \beta$ of the initial portfolio at time $t = 0$ is

$$
V_0 = \alpha S_0 + \beta = -2 \times 4 + 10 = \$2,
$$

which yields the price of the claim at time $t = 0$. In order to hedge then option, one should:

i) At time $t = 0$,

a. Charge the $\$2$ option price.
b. Shortsell $-\alpha = +2$ units of the stock priced $S_0 = 4$, which yields $\$8$.
c. Put $\beta = \$8 + \$2 = \$10$ in savings.

ii) At time $t = 1$,

a. If $S_1 = \$5$, spend $\$10$ from savings to buy back $-\alpha = 2$ stocks.
b. If $S_1 = \$2$, spend $\$4$ from savings to buy back $-\alpha = 2$ stocks, and deliver a $\$10 - \$4 = \$6$ payoff.

Pricing the option by the expected value $E^*[C]$ yields the equality

$$
\$2 = E^*[C]
$$
N. Privault

\[= 0 \times \mathbb{P}^*(C = 0) + 6 \times \mathbb{P}^*(C = 6)\]
\[= 0 \times \mathbb{P}^*(S_1 = 2) + 6 \times \mathbb{P}^*(S_1 = 5)\]
\[= 6 \times q^*,\]

hence the risk-neutral probability measure \(\mathbb{P}^*\) is given by

\[p^* = \mathbb{P}^*(S_1 = 5) = \frac{2}{3} \quad \text{and} \quad q^* = \mathbb{P}^*(S_1 = 2) = \frac{1}{3}.\]

**Exercise 1.3**

a) i) Does this model allow for arbitrage? Yes \(\checkmark\) No

ii) If this model allows for arbitrage opportunities, how can they be realized?

   By shortselling | By borrowing in savings \(\checkmark\) N.A.

b) i) Does this model allow for arbitrage? Yes \(\checkmark\) No

ii) If this model allows for arbitrage opportunities, how can they be realized?

   By shortselling | By borrowing in savings | N.A. \(\checkmark\)

c) i) Does this model allow for arbitrage? Yes \(\checkmark\) No

ii) If this model allows for arbitrage opportunities, how can they be realized?

   By shortselling \(\checkmark\) | By borrowing in savings | N.A.

**Exercise 1.4**

a) We need to search for possible risk-neutral probability measure(s) \(\mathbb{P}^*\) such that \(\mathbb{E}^*[S^{(1)}] = (1 + r)\pi^{(1)}\). Letting

\[
\begin{align*}
  p^* &= \mathbb{P}^*(S^{(1)} = \pi^{(1)}(1 + a)) = \mathbb{P}^* \left( \frac{S^{(1)} - \pi^{(1)}}{\pi^{(1)}} = a \right), \\
  \theta^* &= \mathbb{P}^*(S^{(1)} = \pi^{(1)}(1 + b)) = \mathbb{P}^* \left( \frac{S^{(1)} - \pi^{(1)}}{\pi^{(1)}} = b \right), \\
  q^* &= \mathbb{P}^*(S^{(1)} = \pi^{(1)}(1 + c)) = \mathbb{P}^* \left( \frac{S^{(1)} - \pi^{(1)}}{\pi^{(1)}} = c \right),
\end{align*}
\]

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[https://www.ntu.edu.sg/home/nprivault/index.html](https://www.ntu.edu.sg/home/nprivault/index.html)
We have
\[
\begin{align*}
&\begin{cases}
p^* \pi^{(1)}(1 + a) + \theta^* \pi^{(1)}(1 + b) + q^* \pi^{(1)}(1 + c) = (1 + r) \pi^{(1)} \\
p^* + \theta^* + q^* = 1,
\end{cases}
\end{align*}
\]
from which we obtain
\[
\begin{align*}
&\begin{cases}
p^* a + \theta^* b + q^* c = r, \\
p^* + \theta^* + q^* = 1.
\end{cases} \implies \begin{cases}
p^* = \frac{(1 - \theta)c + \theta^* b - r}{c - a} \in (0, 1), \\
q^* = \frac{r - (1 - \theta)a - \theta^* b}{c - a} \in (0, 1),
\end{cases}
\end{align*}
\]
for any \(\theta \in (0, 1)\) such that
\[
(1 - \theta^*) a - \theta^* b < r < (1 - \theta^*) c + \theta^* b,
\]
or \((1 - \theta^*) a < r < (1 - \theta^*) c\) in case \(b = 0\). Therefore there exists an infinity of risk-neutral probability measures depending on the value of \(\theta^* \in (0, 1)\), and the market is without arbitrage but not complete.

b) Hedging a claim with possible payoff values \(C_a, C_b, C_c\) would require to solve
\[
\begin{align*}
&\begin{cases}
\alpha \pi^{(1)}(1 + a) + \beta \pi^{(0)}(1 + r) = C_a \\
\alpha \pi^{(1)}(1 + b) + \beta \pi^{(0)}(1 + r) = C_b \\
\alpha \pi^{(1)}(1 + c) + \beta \pi^{(0)}(1 + r) = C_c,
\end{cases}
\end{align*}
\]
for \(\alpha\) and \(\beta\), which is generally not possible due to the existence of three conditions with only two unknowns.

Exercise 1.5

a) The risk-neutral condition \(\mathbb{E}^*[R_1] = 0\) reads
\[
b \mathbb{P}^*(R_1 = b) + 0 \times \mathbb{P}^*(R_1 = 0) + (-b) \times (R_1 = -b) = bp^* - bq^* = 0,
\]
hence
\[
p^* = q^* = \frac{1 - \theta^*}{2},
\]
since \(p^* + q^* + \theta^* = 1\).

b) We have
\[
\text{Var}^* \left[ \frac{S_1^{(1)} - S_0^{(1)}}{S_0^{(1)}} \right] = \mathbb{E}^* \left[ R_1^2 \right] - (\mathbb{E}^*[R_1])^2
\]
\[ = \mathbb{E}^*[R_1^2] \]
\[ = b^2 \mathbb{P}^*(R_1 = b) + 0^2 \times \mathbb{P}^*(R_1 = 0) + (-b)^2 \times (R_1 = -b) \]
\[ = b^2(p^* + q^*) \]
\[ = b^2(1 - \theta^*) \]
\[ = \sigma^2, \]

hence
\[ \theta^* = 1 - \frac{\sigma^2}{2b^2} \]

and hence
\[ p^* = q^* = \frac{1 - \theta^*}{2} = \frac{\sigma^2}{2b^2}. \]

Exercise 1.6

a) The possible values of \( R \) are \( a \) and \( b \).

b) We have
\[ \mathbb{E}^*[R] = a \mathbb{P}^*(R = a) + b \mathbb{P}^*(R = b) \]
\[ = \frac{b - r}{b - a} + \frac{r - a}{b - a} \]
\[ = r. \]

c) By Theorem 1.5, there do not exist arbitrage opportunities in this market since from Question (b) there exists a risk-neutral probability measure \( \mathbb{P}^* \) whenever \( a < r < b \).

d) The risk-neutral probability measure is unique hence the market model is complete by Theorem 1.11.

e) Taking
\[ \eta = \frac{\alpha(1 + b) - \beta(1 + a)}{\pi_1(b - a)} \quad \text{and} \quad \xi = \frac{\beta - \alpha}{S_0(b - a)}, \]
we check that
\[ \begin{cases} 
\eta \pi_1 + \xi S_0(1 + a) = \alpha & \text{if } R = a, \\
\eta \pi_1 + \xi S_0(1 + b) = \beta & \text{if } R = b, 
\end{cases} \]
which shows that
\[ \eta \pi_1 + \xi S_1 = C \]
in both cases \( R = a \) and \( R = b \).

f) We have
\[ \pi_0(C) = \eta \pi_0 + \xi S_0 \]
\[ = \frac{\alpha(1 + b) - \beta(1 + a)}{(1 + r)(b - a)} + \frac{\beta - \alpha}{b - a} \]
\[ \alpha(1 + b) - \beta(1 + a) - (1 + r)(\alpha - \beta) \]
\[ = \frac{\alpha b - \beta a - r(\alpha - \beta)}{(1 + r)(b - a)}. \quad (A.1) \]

\( g \) We have
\[ \mathbb{E}^*[C] = \alpha \mathbb{P}^*(R = a) + \beta \mathbb{P}^*(R = b) \]
\[ = \frac{\alpha b - r}{b - a} + \beta \frac{r - a}{b - a}. \quad (A.2) \]

\( h \) Comparing (A.1) and (A.2) above we do obtain
\[ \pi_0(C) = \frac{1}{1 + r} \mathbb{E}^*[C] \]

\( i \) The initial value \( \pi_0(C) \) of the portfolio is interpreted as the arbitrage price of the option contract and it equals the expected value of the discounted payoff.

\( j \) We have
\[ C = (K - S_1)^+ = (11 - S_1)^+ = \begin{cases} 11 - S_1 & \text{if } K > S_1, \\ 0 & \text{if } K \leq S_1. \end{cases} \]

\( k \) We have \( S_0 = 1, a = 8, b = 11, \alpha = 2, \beta = 0 \), hence
\[ \xi = \frac{\beta - \alpha}{S_0(b - a)} = \frac{0 - 2}{11 - 8} = \frac{-2}{3}, \]
\[ \eta = \frac{\alpha(1 + b) - \beta(1 + a)}{\pi_1(b - a)} = \frac{24}{3 \times 1.05}. \]

\( l \) The arbitrage price \( \pi_0(C) \) of the contingent claim payoff \( C \) is
\[ \pi_0(C) = \eta \pi_0 + \xi S_0 = 6.952. \]

Exercise 1.7 Let \( a := -\frac{(152 - 180)}{180} = \frac{7}{45} \) and \( b := \frac{(203 - 180)}{180} = \frac{23}{180} \) denote the potential market returns, with \( r = 0.03 \). From the strike price \( K \) and the risk-neutral probabilities
\[ p_r^* = \frac{r - a}{b - a} = 0.6549 \quad \text{and} \quad q_r^* = \frac{b - r}{b - a} = 0.3451, \]
the price of the option at the beginning of the year is given from Proposition 1.13 as the discounted expected value

\[ \star \]
\[
\frac{1}{1+r} \mathbb{E}^*[(K - S_1)^+] = \frac{1}{1+r} \left( p_r^* (K - 203)^+ + q_r^* (K - 152)^+ \right).
\]

Equating this price with the intrinsic value \((K - 180)^+\) of the put option yields the equation

\[
(K - 180)^+ = \frac{1}{1+r} \left( p_r^* (K - 203)^+ + q_r^* (K - 152)^+ \right)
\]

which requires \(K > 180\) (the case \(K \leq 152\) is not considered because both the option price and option payoff vanish in this case). Hence we consider the equation

\[
K - 180 = \frac{1}{1+r} \left( p_r^* (K - 203)^+ + q_r^* (K - 152)^+ \right),
\]

with the following cases.

i) If \(K \in [180, 203]\) we get

\[
(1 + r)(K - 180) = q_r^* (K - 152),
\]

hence

\[
K = \frac{(1 + r)180 - q_r^* 152}{1 + r - q_r^*} = \frac{(1 + r)180 - q_r^* 152}{p_r^* + r} = 194.11.
\]

ii) If \(K \geq 203\) we find

\[
K = \frac{180(1 + r) - 203p_r^* - 152q_r^*}{r} < 203,
\]

which is out of range and leads to a contradiction.

We note that the above formula

\[
K = \frac{(1 + r)180 - q_r^* 152}{p_r^* + r} = \frac{28b - 180a + r(180(b - a) + 152)}{(b + 1 - a)r - a}
\]

yields a decreasing function \(K(r)\) of \(r\) in the interval \([0, 100\%]\), although the function is not monotone over \(\mathbb{R}_+\).
Chapter 2

Exercise 2.1 Let $m := 2,550$ denote the amount invested each year. By (2.1), the value of the plan after $N = 10$ years becomes

$$m \sum_{k=1}^{N} (1+r)^k = m(1+r)\frac{(1+r)^N - 1}{r},$$

which in turns becomes

$$(1+r)^N m \sum_{k=1}^{N} (1+r)^k = m(1+r)^{N+1} \frac{(1+r)^N - 1}{r},$$

after $N$ additional years without further contributions to the plan. Equating

$$A = 30835 = m(1+r)^{N+1} \frac{(1+r)^N - 1}{r}$$

shows that

$$\frac{(1+r)^{2N+1} - (1+r)^{N+1}}{r} = \frac{A}{m},$$

with $m = 2550$, or

$$\frac{(1+r)^{21} - (1+r)^{11}}{r} = \frac{30835}{2550} \approx 12.09215,$$

hence $r \approx 1.23\%$ according to Figure S.2.
Fig. S.2: Graph of $r \mapsto ((1 + r)^{21} - (1 + r)^{11})/r$.

In the hypothesis $r = 3.25\%$ we would find

$$A = m(1 + r)^{N+1} \frac{(1 + r)^N - 1}{r} = 42040.42.$$

Exercise 2.2 Let $m := 3,581$ denote the amount invested each year. After multiplying (2.1) by $(1 + r)^N$ in order to account for the compounded interests from year 11 until year 20, we get the equality

$$A = m(1 + r)^{N+1} \frac{(1 + r)^N - 1}{r}$$

shows that

$$(1 + r)^{21} - (1 + r)^{11} = r \frac{50862}{3581} \simeq 14.2033r,$$

showing that $r \simeq 2.28\%$ according to Figure S.3.

Fig. S.3: Graph of $r \mapsto ((1 + r)^{21} - (1 + r)^{11})/r$.

Exercise 2.3 We check that for any $P^*$ of the form $P^*(R_t = -1) := p^*$, $P^*(R_t = 0) := 1 - 2p^*$, $P^*(R_t = 1) := p^*$, we have
\[ \mathbb{E}^*[S_1] = S_0(2p^* + 1 - 2p^*) = S_0, \]

and similarly
\[ \mathbb{E}^*[S_2 \mid S_1] = S_1(2p^* + (1 - 2p^*)) = S_1, \]

hence the probability measure \( \mathbb{P}^* \) is risk-neutral.

**Exercise 2.4**

a) In order to check for arbitrage opportunities we look for a risk-neutral probability measure \( \mathbb{P}^* \) which should satisfy
\[ \mathbb{E}^*[S_{k+1} \mid \mathcal{F}_k] = (1 + r)S_k, \quad k = 0, 1, \ldots, N - 1, \]

with \( r = 0 \). Rewriting \( \mathbb{E}^*[S_{k+1} \mid \mathcal{F}_k] \) as
\[
\mathbb{E}^*[S_{k+1}^{(1)} \mid \mathcal{F}_k] = (1 - b)S_k^{(1)} \mathbb{P}^*(R_{k+1} = a \mid \mathcal{F}_k) + S_k^{(1)} \mathbb{P}^*(R_{k+1} = 0 \mid \mathcal{F}_k) + (1 + b)S_k^{(1)} \mathbb{P}^*(R_{k+1} = b),
\]

\( k = 0, 1, \ldots, N - 1 \), it follows that any risk-neutral probability measure \( \mathbb{P}^* \) should satisfy the equations
\[
\begin{cases}
(1 + b)S_k^{(1)} \mathbb{P}^*(R_{k+1} = b) + S_k^{(1)} \mathbb{P}^*(R_{k+1} = 0) + (1 - b)S_k^{(1)} \mathbb{P}^*(R_{k+1} = a) = S_k^{(1)} \\
\mathbb{P}^*(R_{k+1} = b) + \mathbb{P}^*(R_{k+1} = 0) + \mathbb{P}^*(R_{k+1} = -b) = 1,
\end{cases}
\]

\( k = 0, 1, \ldots, N - 1, \ i.e. \)
\[
\begin{cases}
b\mathbb{P}^*(R_k = b) - b\mathbb{P}^*(R_k = -b) = 0, \\
\mathbb{P}^*(R_k = b) + \mathbb{P}^*(R_k = -b) = 1 - \mathbb{P}^*(R_k = 0),
\end{cases}
\]

\( k = 1, 2, \ldots, N \), with solution
\[ \mathbb{P}^*(R_k = b) = \mathbb{P}^*(R_k = -b) = \frac{1 - \theta^*}{2}, \]

\( k = 1, 2, \ldots, N. \)

b) We have
\[
\text{Var}^* \left[ \frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}} \mid \mathcal{F}_k \right]
\]
\[
\begin{align*}
&= \mathbb{E}^* \left[ \left( \frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}} \right)^2 \mid \mathcal{F}_k \right] - \left( \mathbb{E}^* \left[ \frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}} \mid \mathcal{F}_k \right] \right)^2 \\
&= \mathbb{E}^* \left[ \left( \frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}} \right)^2 \mid \mathcal{F}_k \right]
\end{align*}
\]

\[
\begin{align*}
&= b^2 \mathbb{P}_\sigma^*(R_{k+1} = -b \mid \mathcal{F}_k) + b^2 \mathbb{P}_\sigma^*(R_{k+1} = b \mid \mathcal{F}_k) \\
&= b^2 \frac{1 - \mathbb{P}_\sigma^*(R_{k+1} = 0)}{2} + b^2 \frac{1 - \mathbb{P}_\sigma^*(R_{k+1} = 0)}{2} \\
&= b^2 (1 - \theta) \\
&= \sigma^2,
\end{align*}
\]

\(k = 0, 1, \ldots, N - 1\), hence
\[
\mathbb{P}_\sigma^*(R_k = 0) = \theta = 1 - \frac{\sigma^2}{b^2},
\]

and therefore
\[
\mathbb{P}_\sigma^*(R_k = b) = \mathbb{P}_\sigma^*(R_k = -b) = \frac{1 - \mathbb{P}_\sigma^*(R_k = 0)}{2} = \frac{\sigma^2}{2b^2};
\]

\(k = 0, 1, \ldots, N - 1\), under the condition \(0 < \sigma^2 < b^2\).

Exercise 2.5

a) The possible values of \(R_t\) are \(a\) and \(b\).

b) We have
\[
\begin{align*}
\mathbb{E}^*[R_{t+1} \mid \mathcal{F}_t] &= a \mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) + b \mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) \\
&= a \frac{b - r}{b - a} + b \frac{r - a}{b - a} = r.
\end{align*}
\]

c) Letting \(p^* = (r - a) / (b - a)\) and \(q^* = (b - r) / (b - a)\) we have
\[
\begin{align*}
\mathbb{E}^*[S_{t+k} \mid \mathcal{F}_t] &= \sum_{i=0}^{k} (p^*)^i (q^*)^{k-i} \binom{k}{i} (1 + b)^i (1 + a)^{k-i} S_t \\
&= S_t \sum_{i=0}^{k} \binom{k}{i} (p^* (1 + b))^i (q^* (1 + a))^{k-i} \\
&= S_t (p^* (1 + b) + q^* (1 + a))^k \\
&= S_t \left( \frac{r-a}{b-a} (1+b) + \frac{b-r}{b-a} (1+a) \right)^k \\
&= (1+r)^k S_t.
\end{align*}
\]
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Assuming that the formula holds for $k = 1$, its extension to $k \geq 2$ can also be proved recursively from the “tower property” (22.38) of conditional expectations, as follows:

\[
\mathbb{E}^*[S_{t+k} \mid \mathcal{F}_t] = \mathbb{E}^*[\mathbb{E}^*[S_{t+k} \mid \mathcal{F}_{t+k-1}] \mid \mathcal{F}_t]
\]
\[
= (1 + r) \mathbb{E}^*[S_{t+k-1} \mid \mathcal{F}_t]
\]
\[
= (1 + r) \mathbb{E}^*[\mathbb{E}^*[S_{t+k-2} \mid \mathcal{F}_{t+k-1}] \mid \mathcal{F}_t]
\]
\[
= (1 + r)^2 \mathbb{E}^*[S_{t+k-2} \mid \mathcal{F}_t]
\]
\[
= (1 + r)^2 \mathbb{E}^*[\mathbb{E}^*[S_{t+k-3} \mid \mathcal{F}_{t+k-2}] \mid \mathcal{F}_t]
\]
\[
= (1 + r)^3 \mathbb{E}^*[S_{t+k-3} \mid \mathcal{F}_t]
\]
\[
= \cdots
\]
\[
= (1 + r)^{k-2} \mathbb{E}^*[S_{t+2} \mid \mathcal{F}_t]
\]
\[
= (1 + r)^{k-2} \mathbb{E}^*[\mathbb{E}^*[S_{t+2} \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t]
\]
\[
= (1 + r)^{k-1} \mathbb{E}^*[S_{t+1} \mid \mathcal{F}_t]
\]
\[
= (1 + r)^k S_t.
\]

Exercise 2.6

a) We check that

\[
\mathbb{E}^*[R_{t+1} \mid \mathcal{F}_t] = a \mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) + b \mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t)
\]
\[
= a \frac{b-r}{b-a} + b \frac{r-a}{b-a} = r.
\]

b) We have

\[
\mathbb{E}^*[(S_N)\beta] = S_0 \mathbb{E}^* \left[ \left( \prod_{k=1}^{N} (1 + R_k) \right)^\beta \right]
\]
\[
= S_0 \mathbb{E}^* \left[ \prod_{k=1}^{N} (1 + R_k)^\beta \right]
\]
\[
= S_0 \prod_{k=1}^{N} \mathbb{E}^* [(1 + R_k)^\beta],
\]

after using the independence of the returns $(R_k)_{k=1,2,\ldots,N}$, with

\[
\mathbb{E}^* [(1 + R_k)^\beta] = (1 + a)^\beta \frac{b-r}{b-a} + (1 + b)^\beta \frac{r-a}{b-a}, \quad k = 0, 1, \ldots, N,
\]

hence we find
\[ \mathbb{E}^*[(S_N)^\beta] = S_0^\beta \left( (1 + a)^\beta \frac{b - r}{b - a} + (1 + b)^\beta \frac{r - a}{b - a} \right)^N. \]

c) We have
\[
\mathbb{P}^*\left(S_t \geq \alpha \pi_t \text{ for some } t \in \{0, 1, \ldots, N\}\right) = \mathbb{P}^*\left(\max_{t=0,1,\ldots,N} S_t \geq x\right)
\leq \frac{\mathbb{E}\left[(M_N)^\beta\right]}{x^\beta}
\leq \left(\frac{S_0}{(1 + r)x\pi_0}\right)^\beta \left( (1 + a)^\beta \frac{b - r}{b - a} + (1 + b)^\beta \frac{r - a}{b - a} \right)^N,
\]
since the discounted price process
\[
(M_t)_{t=0,1,\ldots,N} := \left(\frac{S_t}{\pi_t}\right)_{t=0,1,\ldots,N}
\]
is a nonnegative martingale.

d) We have
\[
\mathbb{P}^*\left(\max_{t=0,1,\ldots,N} S_t \geq x\right) \leq \frac{\mathbb{E}\left[(M_N)^\beta\right]}{x^\beta}
\leq \left(\frac{S_0}{x}\right)^\beta \left( (1 + a)^\beta \frac{b - r}{b - a} + (1 + b)^\beta \frac{r - a}{b - a} \right)^N,
\]
since the price process \((M_t)_{t=0,1,\ldots,N} := (S_t)_{t=0,1,\ldots,N}\) is a nonnegative submartingale.

**Chapter 3**

Exercise 3.1 (Exercise 2.3 continued). We consider the following trinomial tree.
At time $t = 0$, we find
\[
\pi_0(C) = \frac{1}{(1 + r)^2} \mathbb{E}^*[\{(K - S_2)^+\}]
\]
\[
= p^*(p^* + (1 - 2p^*) + p^*) + (1 - 2p^*)p^* + (p^*)^2
\]
\[
= p^* + (1 - 2p^*)p^* + (p^*)^2
\]
\[
= 2p^* - (p^*)^2.
\]
At time $t = 1$, we find
\[
\pi_1(C) = \frac{1}{1 + r} \mathbb{E}^*[\{(K - S_2)^+ \mid S_1\}]
\]
\[
= \begin{cases} 
  p^* & \text{if } S_1 = 2S_0, \\
  p^* & \text{if } S_1 = S_0, \\
  1 & \text{if } S_1 = 0.
\end{cases}
\]
Exercise 3.2 We have $p^* = (r - a) / (b - a) = 1/2$ and $q^* = (b - r) / (b - a) = 1/2$, and the following underlying asset price tree:
We first price, and then hedge. At time $t = 1$, by Theorem 3.5 we have

$$
\pi_1(C) = V_1 = \begin{cases} 
\frac{3p^* + q^*}{1+r} = \frac{4}{3} & \text{if } S_1 = 2 \\
\frac{p^* + 3q^*}{1+r} = \frac{4}{3} & \text{if } S_1 = 1,
\end{cases}
$$

and $V_0 = \frac{4p^* + q^*}{3(1+r)} = \frac{8}{9}$.

This leads to the following option pricing tree:

Regarding hedging, if $S_1 = 2$ the condition $\xi_2 \cdot S_2 = 0$ reads

$$
S_1 = 2 \implies \begin{cases} 
4\xi_2 + \eta_2(1+r)^2 = 3 \\
2\xi_2 + \eta_2(1+r)^2 = 1,
\end{cases}
$$

hence $(\xi_2, \eta_2) = (1, -4/9)$. On the other hand, if $S_1 = 1$ we have
Notes on Stochastic Finance

\[ S_1 = 1 \implies \begin{cases} 2\xi_2 + \eta_2(1 + r)^2 = 1 \\ \xi_2 + \eta_2(1 + r)^2 = 3, \end{cases} \]

hence \((\xi_2, \eta_2) = (-2, 20/9)\). Finally, at time \(t = 0\) with \(S_0 = 1\) we have

\[ \begin{cases} 2\xi_1 + \eta_1(1 + r) = \frac{4}{3} \\ \xi_1 + \eta_1(1 + r) = \frac{4}{3}, \end{cases} \]

hence \((\xi_1, \eta_1) = (0, 8/9)\). The results can be summarized in the following table:

| \(S_0 = 1\) | \(S_1 = 2, \ V_1 = 4/3\) | \(S_2 = 4\) |
| \(V_0 = 8/9\) | \(\xi_2 = 1, \ \eta_2 = -4/9\) | \(V_2 = 3\) |
| \(\xi_1 = 0\) | \(S_1 = 1, \ V_1 = 4/3\) | \(\xi_2 = -2, \ \eta_2 = 20/9\) |
| \(\eta_1 = 8/9\) | \(V_2 = 3\) | \(V_2 = 3\) |

Table 23.1: CRR pricing and hedging table.

In addition, it can be checked that the portfolio strategy \((\xi_k, \eta_k)_{k=1,2}\) is self-financing as we have

\[ \xi_1 S_1 + \eta_1 A_1 = \frac{8}{9} \times \frac{3}{2} = \begin{cases} 2 - \frac{4}{9} \times \frac{3}{2} \\ -2 + \frac{20}{9} \times \frac{3}{2} \end{cases} = \xi_2 S_1 + \eta_2 A_1. \]

Exercise 3.3

a) We have

\[ \mathbb{E}^*[S_{t+1} \mid \mathcal{F}_t] = \mathbb{E}^*[S_{t+1} \mid S_t] = \frac{S_t}{2} \mathbb{P}^*(R_t = -0.5) + S_t \mathbb{P}^*(R_t = 0) + 2S_t \mathbb{P}^*(R_t = 1) = S_t \left( \frac{r^*}{2} + q^* + 2p^* \right) = S_t, \quad t = 0, 1, \]
with \( r = 0 \).

b) We have the following graph:

```
c) The down-an-out barrier call option is priced at time \( t = 0 \) as

\[
V_0 = \mathbb{E}^*[C] = 2.5 \times (p^*)^2 + 0.5 \times p^* q^* = \frac{3}{16}.
\]

At time \( t = 1 \) we have

\[
V_1 = 2.5 \times p^* + 0.5 \times q^* = 2.5 \times \frac{1}{4} + 0.5 \times \frac{1}{4} = \frac{3}{4}
\]

if \( S_1 = 2 \), and \( V_1 = 0 \) in both cases \( S_1 = 1 \) and \( S_1 = 0.5 \).

d) This market is not complete, and not every contingent claim is attainable, because the risk-neutral probability measure \( \mathbb{P}^* \) is not unique, for example \((r^*, q^*, p^*) = (1/4, 5/8, 1/8)\) and \((r^*, q^*, p^*) = (1/2, 1/4, 1/4)\) are both risk-neutral probability measures.

Exercise 3.4 The CRR model can be described by the following binomial tree.
a) By the formulas

\[
V_1 = \frac{1}{1+r} \mathbb{E}^*[V_2 \mid \mathcal{F}_1] = \frac{1}{1+r} \mathbb{E}^*[V_2 \mid S_1]
\]

\[
= \frac{S_0(1+b)^2 - 8}{1+r} \mathbb{P}^*(S_2 = S_0(1+b) \mid S_1)
\]

\[
= \frac{p^*(S_0(1+b)^2 - 8)}{1+r} \mathbb{1}_{\{S_1=S_0(1+b)\}},
\]

and

\[
V_0 = \frac{1}{1+r} \mathbb{E}^*[V_1 \mid \mathcal{F}_0]
\]

\[
= \frac{1}{1+r} \left( p^* \frac{S_0(1+b)^2 - 8}{1+r} \times \mathbb{P}^*(S_1 = S_0(1+b)) + 0 \times \mathbb{P}^*(S_1 = S_0(1+a)) \right)
\]

\[
= (p^*)^2 \frac{(S_0(1+b)^2 - 8)}{(1+r)^2},
\]

we find the table

<table>
<thead>
<tr>
<th>( S_0 = 1 )</th>
<th>( S_1 = 3, V_1 = 1/4 )</th>
<th>( S_2 = 9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_0 = 1/16 )</td>
<td>( S_2 = 3 )</td>
<td>( V_2 = 1 )</td>
</tr>
</tbody>
</table>

| \( S_1 = 1, V_1 = 0 \) | \( S_2 = 1 \)  | \( V_2 = 0 \)  |

Note that we could also directly compute \( V_0 \) from
\[ V_0 = \frac{1}{(1+r)^2} \mathbb{E}^* [V_2 \mid \mathcal{F}_0]. \]

b) When \( S_1 = S_0(1+b) \), the equation \( \xi_2 S_2 + \eta_2 A_2 = V_2 \) reads
\[
\begin{cases}
\xi_2 S_0(1+b)^2 + \eta_2 A_0(1+r)^2 = S_0(1+b)^2 - 8 \\
\xi_2 S_0(1+b)(1+a) + \eta_2 A_0(1+r)^2 = 0,
\end{cases}
\]
which yields
\[
\xi_2 = \frac{S_0(1+b)^2 - 8}{S_0(b-a)(1+b)} \quad \text{and} \quad \eta_2 = -\frac{(S_0(1+b)^2 - 8)(1+a)}{(b-a)A_0(1+r)^2}.
\]

On the other hand, when \( S_1 = S_0(1+a) \) the equation \( \xi_2 S_2 + \eta_2 A_2 = V_2 \) reads
\[
\begin{cases}
\xi_2 S_0(1+a)^2 + \eta_2 A_0(1+r)^2 = 0 \\
\xi_2 S_0(1+b)^2 + \eta_2 A_0(1+r)^2 = 0,
\end{cases}
\]
which has the unique solution \( (\xi_2, \eta_2) = (0,0) \). Next, the equation \( \xi_1 S_1 + \eta_1 A_1 = V_1 \) reads
\[
\begin{cases}
\xi_1 S_0(1+b) + \eta_1 A_0(1+r) = \frac{p^*(S_0(1+b)^2 - 8)}{1+r} \\
\xi_1 S_0(1+a) + \eta_1 A_0(1+r) = 0
\end{cases}
\]
which yields
\[
\xi_1 = p^* \frac{S_0(1+b)^2 - 8}{S_0(b-a)(1+b)} \quad \text{and} \quad \eta_1 = -p^* \frac{(1+a)(S_0(1+b)^2 - 8)}{(b-a)A_0(1+r)^2}.
\]

This can be summarized in the following table:

| \( S_0 = 1 \) | \( S_1 = 3, V_1 = 1/4 \) | \( S_2 = 9 \) |
| \( V_0 = 1/16 \) | \( \xi_2 = 1/6, \eta_2 = -1/8 \) | \( S_1 = 1, V_1 = 0 \) |
| \( \xi_1 = 1/8 \) | \( S_1 = 1, V_1 = 0 \) | \( S_2 = 3 \) |
| \( \eta_1 = -1/16 \) | \( S_1 = 0, \eta_2 = 0 \) | \( S_2 = 1 \) |

Table 23.3: CRR pricing and hedging tree.

When \( S_1 = S_0(1+a) \) at time \( t = 1 \) the option price is \( V_1 = 0 \) and the hedging strategy is to cut all positions: \( \xi_2 = \eta_2 = 0 \). On the other hand,
if \( S_1 = S_0(1 + b) \) then there is a chance of being in the money at maturity and we need to increase our position in the underlying asset from \( \xi_1 = 1/8 \) to \( \xi_2 = 1/6 \).

Note that the self-financing condition

\[
\xi_1 S_1 + \eta_1 A_1 = \xi_2 S_1 + \eta_2 A_1, \tag{A.5}
\]

is verified. For example when \( S_1 = S_0(1 + a) \) we have

\[
\frac{1}{8} S_1 - \frac{1}{16} A_1 = 0 \times S_1 + 0 \times A_1 = 0,
\]

while when \( S_1 = S_0(1 + b) \) we find

\[
\frac{1}{8} S_1 - \frac{1}{16} A_1 = \frac{1}{6} S_1 - \frac{1}{8} A_1 = \frac{1}{4}.
\]

On the other hand, we can also use the self-financing condition (A.5) to recover (A.4) by rewriting the system of equations as

\[
\begin{aligned}
\xi_1 S_0(1 + b) + \eta_1 A_0(1 + r) &= \xi_2 S_0(1 + b) + \eta_2 A_0(1 + r) \\
\xi_1 S_0(1 + a) + \eta_1 A_0(1 + r) &= 0,
\end{aligned}
\]

with \((\xi_2, \eta_2)\) given by (A.3), which recovers

\[
V_1 = \xi_1 S_1 + \eta_1 A_1 = \begin{cases} 
\frac{3}{8} - \frac{2}{16} = \frac{1}{4} & \text{if } S_1 = 3, \\
\frac{1}{8} - \frac{2}{16} = 0 & \text{if } S_1 = 1.
\end{cases}
\]

Exercise 3.5

a) We build a portfolio based on \( \alpha \) units of stock and \$\beta \) in cash. When \( S_1 = 2 \), we should have

\[
\begin{aligned}
4\alpha_2 + \beta_2 &= 0 \\
2\alpha_2 + \beta_2 &= 1,
\end{aligned}
\]

hence \((\alpha_2, \beta_2) = (-1/2, 2)\). On the other hand, when \( S_1 = 1 \) we should have

\[
\begin{aligned}
2\alpha_1 + \beta_1 &= 1 \\
\alpha_1 + \beta_1 &= 0,
\end{aligned}
\]

hence \((\alpha_1, \beta_1) = (1, -1)\).

b) When \( S_1 = 2 \), the price of the claim at \( t = 1 \) is

\[
\alpha_2 S_2 + \beta_2 = 2\alpha_2 + \beta_2 = 1.
\]
When $S_1 = 1$, the price of the claim at $t = 1$ is $\alpha_1 S_1 + \beta_1 = \alpha_1 + \beta_1 = 0$.

c) We build a portfolio based on $\alpha_0$ units of stock and $\beta_0$ in cash. At time $t = 1$, we should have
\[
\begin{cases}
2\alpha_0 + \beta_2 = 1 \\
\alpha_0 + \beta_2 = 0,
\end{cases}
\]
hence $(\alpha_0, \beta_0) = (1, -1)$.

d) The price of the claim $C$ at time $t = 0$ is $\alpha_0 S_0 + \beta_0 = \alpha_0 + \beta_0 = 0$.

e) The probabilities $(p^*, q^*) = ((r - a)/(b - a), (b - r)/(b - a)) = (0, 1)$ are clearly risk-neutral in the sense of Definition 2.11. However, this does not form an equivalent risk-neutral measure in the sense of Definition 2.13.

f) According to Theorem 2.14 this model allows for arbitrage opportunities as the risk-neutral measure $(p^*, q^*)$ is unique and is not an equivalent probability measure. Here the arbitrage opportunity is clearly observed as it is possible to purchase the option at the price 0 of part (d) while receiving a nonzero payoff. This is also due to the fact that the underlying price may increase and cannot decrease.

Exercise 3.6 We have
\[
V_k = \frac{1}{(1 + r)^{N - k}} \mathbb{E}^*[h(S_N) \mid \mathcal{F}_k] = \frac{1}{(1 + r)^{N - k}} \mathbb{E}^*[a + bS_N \mid \mathcal{F}_k] = \frac{a}{(1 + r)^{N - k}} + \frac{b}{(1 + r)^{N - k}} \mathbb{E}^*[S_N \mid \mathcal{F}_k] = \frac{a}{(1 + r)^{N - k}} + bS_k, \quad k = 0, 1, \ldots, N.
\]

Exercise 3.7

a) Taking $q^* = 1 - p^* = 1/4$, we find the binary tree
We find the binary tree

\[
\begin{align*}
S_0 &= 1 \\
S_1 &= 2.5 \quad \text{and} \quad V_1 = 0 \\
S_2 &= 6.25 \quad \text{and} \quad V_2 = 0 \\
S_0 &= 1 \quad \text{and} \quad V_0 = 1/64 \\
S_1 &= 0.5 \quad \text{and} \quad V_1 = 1/8 \\
S_2 &= 0.25 \quad \text{and} \quad V_2 = 1
\end{align*}
\]

and the table

<table>
<thead>
<tr>
<th>( S_0 = 1 )</th>
<th>( S_1 = 2.5, \ V_1 = 0 )</th>
<th>( S_2 = 6.25 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_0 = 1/64 )</td>
<td>( V_2 = 0 )</td>
<td>( S_2 = 1.25 )</td>
</tr>
<tr>
<td>( S_1 = 0.5, \ V_1 = 1/8 )</td>
<td>( S_2 = 0.25 )</td>
<td></td>
</tr>
<tr>
<td>( V_2 = 0 )</td>
<td>( V_2 = 1 )</td>
<td></td>
</tr>
</tbody>
</table>

Table 23.4: CRR pricing tree.

c) Here we compute the hedging strategy from the option prices. When \( S_1 = S_0(1 + b) \) we clearly have \( \xi_2 = \eta_2 = 0 \). When \( S_1 = S_0(1 + a) \), the equation
\[ \xi_2 S_2 + \eta_2 A_2 = V_2 \]
reads
\[
\begin{align*}
\xi_2 S_0 (1 + a)^2 + \eta_2 (1 + r)^2 &= S_0 (K - (1 + a)(1 + b)) \\
\xi_2 S_0 (1 + b)(1 + a) + \eta_2 (1 + r)^2 &= 0
\end{align*}
\]
hence
\[
\xi_2 = -\frac{(K - S_0 (1 + a)(1 + b))}{S_0 (b - a)(1 + a)} \quad \text{and} \quad \eta_2 = \frac{(K - S_0 (1 + a)(1 + b))(1 + b)}{S_0 (b - a)(1 + r)^2}.
\]
Next, at time \( t = 1 \) the equation \( \xi_1 S_1 + \eta_1 A_1 = V_1 \) reads
\[
\begin{align*}
\xi_1 S_0 (1 + a) + \eta_1 (1 + r) &= S_0 q^*(K - (1 + a)(1 + b)) \\
\xi_1 S_0 (1 + b) + \eta_1 (1 + r) &= 0
\end{align*}
\]
which yields
\[
\xi_1 = -\frac{q^*(K - S_0 (1 + a)(1 + b))}{S_0 (b - a)(1 + r)} \quad \text{and} \quad \eta_1 = \frac{q^*(K - S_0 (1 + a)(1 + b))(1 + b)}{S_0 (b - a)(1 + r)^2}.
\]
This can be summarized in the following table:

<table>
<thead>
<tr>
<th>( S_0 = 1 )</th>
<th>( S_1 = 2.5, V_1 = 0 )</th>
<th>( S_2 = 6.25 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_0 = 1/64 )</td>
<td>( \xi_2 = 0, \eta_2 = 0 )</td>
<td>( S_2 = 1.25 )</td>
</tr>
<tr>
<td>( \xi_1 = -1/16 )</td>
<td>( S_1 = 0.5, V_1 = 1/8 )</td>
<td>( V_2 = 0 )</td>
</tr>
<tr>
<td>( \eta_1 = 5/64 )</td>
<td>( \xi_2 = -1, \eta_2 = 5/16 )</td>
<td>( S_2 = 0.25 )</td>
</tr>
</tbody>
</table>

Table 23.5: CRR pricing and hedging tree.

If \( S_1 = S_0 (1 + a) \) then there is a chance of being in the money at maturity and we need short sell by decreasing \( \xi_1 \) from \( \xi_1 = -1/16 \) to \( \xi_2 = -1 \). Note that the self-financing condition
\[
\xi_1 S_1 + \eta_1 A_1 = \xi_2 S_1 + \eta_2 A_1,
\]
is satisfied.

Exercise 3.8
a) We have
Notes on Stochastic Finance

\[ V_2 = \xi_2 S_2 + \eta_2 A_2 + \alpha \frac{S_2}{1-\alpha} = \xi_2 \frac{S_2}{1-\alpha} + \eta_2 A_2. \]

b) We have

\[ V_1 = \xi_1 S_1 + \eta_1 A_1 + \alpha \frac{S_1}{1-\alpha} = \xi_1 \frac{S_1}{1-\alpha} + \eta_1 A_1. \]

c) If \( S_1 = 3 \) we have

\[
V_2 = \xi_2 \frac{S_2}{1-\alpha} + \eta_2 A_2 = \begin{cases}
9 \xi_2 \frac{1}{1-\alpha} + \eta_2 2^2 = 1 & \text{if } S_2 = 9, \\
3 \xi_2 \frac{1}{1-\alpha} + \eta_2 2^2 = 0 & \text{if } S_2 = 3,
\end{cases}
\]

hence \( (\xi_2, \eta_2) = ((1-\alpha)/6, -1/8) \).

If \( S_1 = 1 \) we have

\[
V_2 = \xi_2 \frac{S_2}{1-\alpha} + \eta_2 A_2 = \begin{cases}
3 \xi_2 \frac{1}{1-\alpha} + \eta_2 2^2 = 0 & \text{if } S_2 = 3, \\
\xi_2 \frac{1}{1-\alpha} + \eta_2 2^2 = 0 & \text{if } S_2 = 1,
\end{cases}
\]

hence \( (\xi_2, \eta_2) = (0, 0) \).

d) We have

\[
\begin{cases}
V_1 = \xi_2 S_1 + 2\eta_2 = 3 \times \frac{1-\alpha}{6} - 2 \times \frac{1}{8} = 1 - \frac{2\alpha}{4} & \text{if } S_1 = 3, \\
V_1 = \xi_2 S_1 + 2\eta_2 = 0 \times 1 + 0 \times 2 = 0 & \text{if } S_1 = 1.
\end{cases}
\]

e) We have

\[
V_1 = \xi_1 \frac{S_1}{1-\alpha} + \eta_1 A_1 = \begin{cases}
3 \xi_1 \frac{1}{1-\alpha} + 2\eta_1 = 1 - \frac{2\alpha}{4} & \text{if } S_1 = 3, \\
\xi_1 \frac{1}{1-\alpha} + 2\eta_1 = 0 & \text{if } S_1 = 1,
\end{cases}
\]

hence \( (\xi_1, \eta_1) = ((\alpha - 1)(2\alpha - 1)/8, (2\alpha - 1)/16) \).
f) At time \( k = 0 \) we have

\[
V_0 = \xi_1 S_0 + \eta_1 = \frac{(\alpha - 1)(2\alpha - 1)}{8} + \frac{2\alpha - 1}{16} = \frac{(2\alpha - 1)^2}{16}.
\]

g) Multiplying the prices \( (S_k)_{k=1,2} \) of the original tree by

\[
(\alpha - 1)(2\alpha - 1)/8, (2\alpha - 1)/16.
\]
we find the prices \((\overline{S}_k)_{k=1,2} = (S_k/(1-\alpha)^k)_{k=1,2}\) as in the following tree:

![Tree Diagram]

h) The market returns found in Question (g) are \(\bar{a} = \frac{1}{3}\) and \(\bar{b} = 3\), with \(r = 1\%\). Therefore we have

\[
p^* = \frac{r - a}{b - a} = \frac{1 - 1/3}{3 - 1/3} = \frac{1}{4} \quad \text{and} \quad q^* = \frac{3 - 1}{3 - 1/3} = \frac{b - r}{b - a} = \frac{3}{4}.
\]

i) If \(S_1 = 3\) we have

\[
\frac{1}{1+r} \mathbb{E}^* \left[ (S_2 - K)^+ | \overline{S}_1 = 3 \right] = \$1 \times \frac{p^*}{2} = \frac{1}{8},
\]

which coincides with

\[
V_1 = \xi_2 S_1 + 2\eta_2 = \frac{3}{8} - \frac{2}{8} = \frac{1}{8}.
\]

If \(S_1 = 1\) we have

\[
\frac{1}{1+r} \mathbb{E}^* \left[ (S_2 - K)^+ | \overline{S}_1 = 1 \right] = 0,
\]

which coincides with

\[
V_1 = \xi_2 S_1 + 2\eta_2 = 0.
\]

j) At time \(k = 0\) we have

\[
\frac{1}{(1+r)^2} \mathbb{E}^* \left[ (S_2 - K)^+ \right] = \frac{(p^*)^2}{(1+r)^2} = \frac{1}{64},
\]

which coincides with
Notes on Stochastic Finance

\[ V_0 = \xi_1 S_0 + \eta_1 = \frac{3}{64} - \frac{1}{32} = \frac{1}{64}. \]

We also have

\[ \frac{1}{1 + r} E^\ast [V_1] = \frac{p^\ast}{1 + r} \times \frac{1}{8} = \frac{1}{8} \times \frac{1}{8} = \frac{1}{64}. \]

Exercise 3.9

a) Taking the risk-free interest rate \( r \) equal to zero, the binary call option can be priced as

\[ E^\ast [C] = E^\ast [\mathbb{1}_{[K,\infty)} (S_N)] = P^\ast (S_N \geq K) =: p^\ast \]

under the risk-neutral probability measure \( P^\ast \).

b) Investing \( \$p^\ast \) by purchasing one binary call option yields a potential net return of

\[ \begin{cases} \frac{\$1 - p^\ast}{p^\ast} = \frac{\$1}{p^\ast} - 1 & \text{if } S_N \geq K, \\ \frac{\$0 - p^\ast}{p^\ast} = -100\% & \text{if } S_N < K. \end{cases} \]

c) The corresponding expected return is

\[ p^\ast \times \left( \frac{1}{p^\ast} - 1 \right) + (1 - p^\ast) \times (-1) = 0. \]

d) The corresponding expected return is

\[ p^\ast \times 0.86 + (1 - p^\ast) \times (-1) = p^\ast \times 1.86 - 1, \]

which will be negative if

\[ p^\ast < \frac{1}{1.86} \approx 0.538. \]

That means, the expected gain can be negative even if

\[ 0.538 > p^\ast = P^\ast (S_N \geq K) > 0.5. \]

Similarly, the expected gain

\[ (1 - p^\ast) \times 0.86 + p^\ast \times (-1) = 0.86 - p^\ast \times 1.86, \]

on binary put options will be negative if \( 1 - p^\ast > 1/1.86 \), i.e. if

\[ p^\ast > \frac{0.86}{1.86} \approx 0.462. \]
That means, the expected gain can be negative even if $1 - 0.462 > P^* (S_N < K) > 0.5$. In conclusion, the average gains of both call and put options will be negative if $p^* \in (0.462, 0.538)$.

Note that the average of call and put option gains will still be negative, as

$$\frac{p^* \times 1.86 - 1}{2} + \frac{0.86 - p^* \times 1.86}{2} = \frac{0.86 - 1}{2} < 0.$$

Exercise 3.10

a) Based on the price map of the put spread collar option:

Fig. S.4: Put spread collar price graph.

we deduce the following payoff function graph of the put spread collar option in the next Figure S.5.

Fig. S.5: Put spread collar payoff function.

b) The payoff function can be written as

$$-(K_1 - x)^+ + (K_2 - x)^+ - (x - K_3)^+ = -(80 - x)^+ + (90 - x)^+ - (x - 110)^+, $$

see also https://optioncreator.com/stp7xy2.
Put spread collar option as a combination of call and put options.\

Hence this collar payoff can be realized by

1. issuing (or selling) one put option with strike price $K_1 = 80$, and
2. purchasing one put option with strike price $K_2 = 90$, and
3. issuing (or selling) one call option with strike price $K_3 = 110$.

Exercise 3.11

a) Based on the price map of the call spread collar option:

![Call spread collar price map](image)

we deduce the following payoff function graph of the call spread collar option in the next Figure S.8.

b) The payoff function can be written as

$$-(K_1 - x)^+ + (x - K_2)^+ - (x - K_3)^+$$

* The animation works in Acrobat Reader on the entire pdf file.
Fig. S.8: Call spread collar payoff function.

\[ = -(80 - x)^+ + (x - 100)^+ - (x - 110)^+, \]

see also https://optioncreator.com/st3e4cz.

Fig. S.9: Call spread collar option as a combination of call and put options.*

Hence this collar payoff can be realized by

1. issuing (or selling) one \textit{put option} with strike price \( K_1 = 80 \), and
2. purchasing one \textit{call option} with strike price \( K_2 = 100 \), and
3. issuing (or selling) one \textit{call option} with strike price \( K_3 = 110 \).

* The animation works in Acrobat Reader on the entire pdf file.
Exercise 3.12 We have
\[
\mathbb{E}^* \left[ \phi \left( \frac{S_1 + \ldots + S_N}{N} \right) \right] \leq \mathbb{E}^* \left[ \frac{\phi(S_1) + \ldots + \phi(S_N)}{N} \right]
\]
since \( \phi \) is convex,
\[
= \mathbb{E}^*[\phi(S_1)] + \ldots + \mathbb{E}^*[\phi(S_N)]
\]
\[
= \mathbb{E}^*[\mathbb{E}^*[\phi(S_N) | \mathcal{F}_1]] + \ldots + \mathbb{E}^*[\mathbb{E}^*[\phi(S_N) | \mathcal{F}_N]]
\]
\[
\leq \mathbb{E}^*[\mathbb{E}^*[\phi(S_N) | \mathcal{F}_1]] + \ldots + \mathbb{E}^*[\mathbb{E}^*[\phi(S_N) | \mathcal{F}_N]]
\]
because \((S_n)_{n \in \mathbb{N}}\) is a martingale,
\[
= \mathbb{E}^*[\phi(S_N)] + \ldots + \mathbb{E}^*[\phi(S_N)]
\]
by Jensen’s inequality,
\[
= \mathbb{E}^*[\phi(S_N)].
\]

The above argument is implicitly using the fact that a convex function \( \phi(S_n) \) of a martingale \((S_n)_{n \in \mathbb{N}}\) is itself a submartingale, as
\[
\phi(S_k) = \phi(\mathbb{E}^*[S_N | \mathcal{F}_k]) \leq \mathbb{E}^*[\phi(S_N) | \mathcal{F}_k], \quad k = 1, 2, \ldots, N.
\]

Exercise 3.13 (Exercise 2.5 continued).

a) The condition \( V_N = C \) reads
\[
\begin{cases}
\eta_N \pi_N + \xi_N (1 + a) S_{N-1} = (1 + a) S_{N-1} - K \\
\eta_N \pi_N + \xi_N (1 + b) S_{N-1} = (1 + b) S_{N-1} - K,
\end{cases}
\]
from which we deduce the (static) hedging strategy \( \xi_N = 1 \) and \( \eta_N = -K(1 + r)^{-N}/\pi_0 \).

b) We have
\[
\begin{cases}
\eta_{N-1} \pi_{N-1} + \xi_{N-1} (1 + a) S_{N-2} = \eta_N \pi_{N-1} + \xi_N (1 + a) S_{N-2} \\
\eta_{N-1} \pi_{N-1} + \xi_{N-1} (1 + b) S_{N-2} = \eta_N \pi_{N-1} + \xi_N (1 + b) S_{N-2},
\end{cases}
\]
which yields \( \xi_{N-1} = \xi_N = 1 \) and \( \eta_{N-1} = \eta_N = -K(1 + r)^{-N}/\pi_0 \).
Similarly, solving the self-financing condition
\[
\begin{cases}
\eta_t \pi_t + \xi_t (1 + a) S_{t-1} = \eta_{t+1} \pi_t + \xi_{t+1} (1 + a) S_{t-1} \\
\eta_t \pi_t + \xi_t (1 + b) S_{t-1} = \eta_{t+1} \pi_t + \xi_{t+1} (1 + b) S_{t-1}
\end{cases}
\]
at time \( t \) yields
\[
\xi_t = 1 \quad \text{and} \quad \eta_t = -(1 + r)^{-N} \frac{K}{\pi_0}, \quad t = 1, 2, \ldots, N.
\]

c) We have
d) For all $t = 0, 1, \ldots, N$ we have

\[
(1 + r)^{-(N-t)} \mathbb{E}^*[C \mid \mathcal{F}_t] = (1 + r)^{-(N-t)} \mathbb{E}^*[S_N - K \mid \mathcal{F}_t],
\]

\[
= (1 + r)^{-(N-t)} \mathbb{E}^*[S_N \mid \mathcal{F}_t] - (1 + r)^{-(N-t)} \mathbb{E}^*[K \mid \mathcal{F}_t]
\]

\[
= (1 + r)^{-(N-t)} (1 + r)^{N-t} S_t - K (1 + r)^{-(N-t)}
\]

\[
= S_t - K (1 + r)^{-(N-t)}
\]

\[
= V_t = \pi_t(C).
\]

For a future contract expiring at time $N$ we take $K = S_0(1 + r)^N$ and the contract is usually quoted at time $t$ using the forward price $(1 + r)^{N-t}(S_t - K (1 + r)^{N-t}) = (1 + r)^{N-t} S_t - K = (1 + r)^{N-t} S_t - S_0(1 + r)^N$, or simply using $(1 + r)^{N-t} S_t$. Future contracts are “marked to market” at each time step $t = 1, 2, \ldots, N$ via a positive or negative cash flow exchange $(1 + r)^{N-t} S_t - (1 + r)^{N-t+1} S_{t-1}$ from the seller to the buyer, ensuring that the absolute difference $| (1 + r)^{N-t} S_t - K |$ has been credited to the buyer’s account if it is positive, or to the seller’s account if it is negative.

Exercise 3.14

a) We write

\[
V_N = \begin{cases} 
\xi_N S_{N-1} (1+1/2) + \eta_N = (S_{N-1}(1+1/2))^2 \\
\xi_N S_{N-1} (1-1/2) + \eta_N = (S_{N-1}(1-1/2))^2,
\end{cases}
\]

which yields

\[
\begin{cases} 
\xi_N = 2 S_{N-1} \\
\eta_N = -3(S_{N-1})^2/4.
\end{cases}
\]

b) We have

\[
\mathbb{E}^*[\{S_N\}^2 \mid \mathcal{F}_{N-1}] = p^*(S_{N-1})^2 (1+1/2)^2 + (1-p^*)(S_{N-1})^2 (1-1/2)^2 \\
= \frac{1}{2} (S_{N-1})^2 ((1+1/2)^2 + (1-1/2)^2) \\
= 5(S_{N-1})^2/4.
\]
c) We have
\[
\xi_{N-1}S_{N-1} + \eta_{N-1}A_0 = \begin{cases} 
\xi_{N-1}S_{N-2}(1 + 1/2) + \eta_{N-1} \\
\xi_{N-1}S_{N-2}(1 - 1/2) + \eta_{N-1}
\end{cases} = V_{N-1} = 5(S_{N-1})^2/4 \\
= \begin{cases} 
5(S_{N-2}(1 + 1/2))^2/4 \\
5(S_{N-2}(1 - 1/2))^2/4.
\end{cases}
\]

hence
\[
\begin{cases} 
\xi_{N-1} = 5S_{N-2}/2 \\
\eta_{N-1} = -15(S_{N-2})^2/16.
\end{cases}
\]
d) We have
\[
\xi_{N-1}S_{N-1} + \eta_{N-1}A_0 = 5S_{N-2}S_{N-1}/2 - 15(S_{N-2})^2/16 \\
= \begin{cases} 
5(S_{N-2})^2(1 + 1/2)/2 - 15(S_{N-2})^2/16 \\
5(S_{N-2})^2(1 - 1/2)/2 - 15(S_{N-2})^2/16 \\
15(S_{N-2})^2/4 - 15(S_{N-2})^2/16 \\
5(S_{N-2})^2 - 15(S_{N-2})^2/16 \\
45(S_{N-2})^2/16 \\
5(S_{N-2})^2/16.
\end{cases}
\]

and on the other hand,
\[
\xi_N S_{N-1} + \eta_N A_0 = 2(S_{N-1})^2 - 3(S_{N-1})^2/4 \\
= \begin{cases} 
2(S_{N-2})^2(1 + 1/2)^2 - 3(S_{N-2})^2(1 + 1/2)^2/4 \\
2(S_{N-2})^2(1 - 1/2)^2 - 3(S_{N-2})^2(1 - 1/2)^2/4 \\
45(S_{N-2})^2/16 \\
5(S_{N-2})^2/16.
\end{cases}
\]

Remark: We could also determine \((\xi_{N-1}, \eta_{N-1})\) as in Proposition 3.11, from \((\xi_N, \eta_N)\) and the self-financing condition
\[
\xi_{N-1}S_{N-1} + \eta_{N-1}A_0 = \xi_N S_{N-1} + \eta_N A_{N-1},
\]
as
\[
\xi_{N-1}S_{N-1} + \eta_{N-1}A_0 = \begin{cases} 
\xi_{N-1}S_{N-2}(1 + 1/2) + \eta_{N-1} \\
\xi_{N-1}S_{N-2}(1 - 1/2) + \eta_{N-1} \\
= \xi_N S_{N-1} + \eta_N A_0 \\
= 2(S_{N-1})^2 - 3(S_{N-1})^2/4 \\
= \begin{cases} 
2(S_{N-2})^2(1 + 1/2)^2 - 3(S_{N-2})^2(1 + 1/2)^2/4 \\
2(S_{N-2})^2(1 - 1/2)^2 - 3(S_{N-2})^2(1 - 1/2)^2/4, 
\end{cases}
\end{cases}
\]
which recovers \(\xi_{N-1} = 5S_{N-2}/2\) and \(\eta_{N-1} = -15(S_{N-2})^2/16\).

**Exercise 3.15**

a) By Theorem 2.18 this model admits a unique risk-neutral probability measure \(\mathbb{P}^*\) because \(a < r < b\), and from (2.14) we have
\[
\mathbb{P}^*(R_t = a) = \frac{b - r}{b - a} = \frac{0.07 - 0.05}{0.07 - 0.02},
\]
and
\[
\mathbb{P}(R_t = b) = \frac{r - a}{b - a} = \frac{0.05 - 0.05}{0.07 - 0.02},
\]
t = 1, 2, …, \(N\).

b) There are no arbitrage opportunities in this model, due to the existence of a risk-neutral probability measure.

c) This market model is complete because the risk-neutral probability measure is unique.

d) We have
\(C = (S_N)^2\),

hence
\[
\tilde{C} = \frac{(S_N)^2}{(1 + r)^N} = h(X_N),
\]
with
\(h(x) = x^2(1 + r)^N\).

Now we have
\[
\tilde{V}_t = \tilde{v}(t, X_t),
\]
where the function \(v(t, x)\) is given from Proposition 3.8 by
\[
\tilde{v}(t, x) = \sum_{k=0}^{N-t} \frac{(N-t)!}{k!(N-t-k)!}.
\]
\[ \times (p^*)^k (q^*)^{N-t-k} \left( x \left( \frac{1+b}{1+r} \right)^k \left( \frac{1+a}{1+r} \right)^{N-t-k} \right) \]

\[ = x^2 (1 + r)^N \sum_{k=0}^{N-t} \frac{(N-t)!}{k!(N-t-k)!} \times (p^*)^k (q^*)^{N-t-k} \left( \frac{1+b}{1+r} \right)^{2k} \left( \frac{1+a}{1+r} \right)^{2(N-t-k)} \]

\[ = x^2 (1 + r)^N \sum_{k=0}^{N-t} \frac{(N-t)!}{k!(N-t-k)!} \times \left( \frac{(r-a)(1+b)^2}{(b-a)(1+r)^2} \right)^k \left( \frac{(b-r)(1+a)^2}{(b-a)(1+r)^2} \right)^{N-t-k} \]

\[ = x^2 (1 + r)^N \left( \frac{(r-a)(1+b)^2}{(b-a)(1+r)^2} + \frac{(b-r)(1+a)^2}{(b-a)(1+r)^2} \right)^{N-t} \]

\[ = x^2 \left( (r-a)(1+b)^2 + (b-r)(1+a)^2 \right)^{N-t} \]

\[ = x^2 \left( (r-a)(1+2b+b^2) + (b-r)(1+2a+a^2) \right)^{N-t} \]

\[ = x^2 \left( r(1+2b+b^2) - a(1+2b+b^2) + b(1+2a+a^2) - r(1+2a+a^2) \right)^{N-t} \]

\[ = x^2 \frac{(1+r(a+b+2) - ab)^{N-t}}{(1+r)^{N-2t}}. \]

e) We have

\[ \xi^1_t = \frac{v \left( t, \frac{1+b}{1+r}, X_{t-1} \right) - v \left( t, \frac{1+a}{1+r}, X_{t-1} \right)}{X_{t-1}(b-a)/(1+r)} \]

\[ = X_{t-1} \left( \frac{1+b}{1+r} \right)^2 - \frac{(1+a)^2}{(b-a)/(1+r)} \frac{(1+r(a+b+2) - ab)^{N-t}}{(1+r)^{N-2t}} \]

\[ = S_{t-1}(a+b+2) \frac{(1+r(a+b+2) - ab)^{N-t}}{(1+r)^{N-t}}, \quad t = 1, 2, \ldots, N, \]

representing the quantity of the risky asset to be present in the portfolio at time \( t \). On the other hand we have

\[ \xi^0_t = \frac{V_t - \xi^1_t X_t}{X^0_t} \]

\[ = \frac{V_t - \xi^1_t X_t}{\pi_0} \]
b) We have

\[
X_t(1 + r(a + b + 2) - ab)^{N-t} \frac{X_t - X_{t-1}(a + b + 2)/(1 + r)}{\pi_0(1 + r)^{N-2t}}
\]

\[
S_t(1 + r(a + b + 2) - ab)^{N-t} \frac{S_t - S_{t-1}(a + b + 2)}{\pi_0(1 + r)^N}
\]

\[
= -(S_{t-1})^2(1 + r(a + b + 2) - ab)^{N-t} \frac{(1 + a)(1 + b)}{\pi_0(1 + r)^N},
\]

\[ t = 1, 2, \ldots, N. \]

f) Let us check that the portfolio is self-financing. We have

\[
\xi_{t+1} \cdot \bar{S}_t = \xi_{t+1}^0 S_t^0 + \xi_{t+1}^1 S_t^1
\]

\[
= -(S_t)^2(1 + r(a + b + 2) - ab)^{N-t-1} \frac{(1 + a)(1 + b)}{\pi_0(1 + r)^N} S_t^0
\]

\[
+ (S_t)^2(a + b + 2) \frac{(1 + r(a + b + 2) - ab)^{N-t-1}}{(1 + r)^{N-t-1}}
\]

\[
= (S_t)^2 \frac{(1 + r(a + b + 2) - ab)^{N-t-1}}{(1 + r)^{N-t}}
\]

\[
\times ((a + b + 2)(1 + r) - (1 + a)(1 + b))
\]

\[
= \frac{1}{(1 + r)^{N-t}} (X_t)^2(1 + r(a + b + 2) - ab)^{N-t}
\]

\[
= (1 + r)^t V_t
\]

\[
= \xi_t \cdot \bar{S}_t, \quad t = 1, 2, \ldots, N.
\]

Exercise 3.16

a) We have

\[
V_t = \xi_t S_t + \eta_t \pi_t
\]

\[
= \xi_t (1 + R_t) S_{t-1} + \eta_t (1 + r) \pi_{t-1}.
\]

b) We have

\[
\mathbb{E}^*[R_t | \mathcal{F}_{t-1}] = a \mathbb{P}^*(R_t = a | \mathcal{F}_{t-1}) + b \mathbb{P}^*(R_t = b | \mathcal{F}_{t-1})
\]

\[
= \frac{b - r}{b - a} + \frac{r - a}{b - a}
\]

\[
= \frac{b - r}{b - a} - \frac{r}{b - a}
\]

\[
= r.
\]

c) By the result of Question (a), we have

\[
\mathbb{E}^*[V_t | \mathcal{F}_{t-1}] = \mathbb{E}^*[\xi_t (1 + R_t) S_{t-1} | \mathcal{F}_{t-1}] + \mathbb{E}^*[\eta_t (1 + r) \pi_{t-1} | \mathcal{F}_{t-1}]
\]

\[
= \xi_t S_{t-1} \mathbb{E}^*[1 + R_t | \mathcal{F}_{t-1}] + (1 + r) \mathbb{E}^*[\eta_t \pi_{t-1} | \mathcal{F}_{t-1}]
\]
\[
(1 + r)\xi_t S_{t-1} + (1 + r)\eta_t \pi_{t-1} \\
= (1 + r)\xi_{t-1} S_{t-1} + (1 + r)\eta_{t-1} \pi_{t-1} \\
= (1 + r)V_{t-1},
\]
where we used the self-financing condition.

d) We have
\[
V_{t-1} = \frac{1}{1 + r} \mathbb{E}^*[V_t \mid \mathcal{F}_{t-1}] \\
= \frac{3}{1 + r} \mathbb{P}^*(R_t = a \mid \mathcal{F}_{t-1}) + \frac{8}{1 + r} \mathbb{P}^*(R_t = b \mid \mathcal{F}_{t-1}) \\
= \frac{1}{1 + 0.15} \left( \frac{3}{2} \frac{0.25 - 0.15}{0.25 - 0.05} + \frac{8}{2} \frac{0.15 - 0.05}{0.25 - 0.05} \right) \\
= \frac{1}{1.15} \left( \frac{3}{2} + \frac{8}{2} \right) \\
= 4.78.
\]

Problem 3.17

a) We have
\[
S_k^{(1)} = \begin{cases} 
(1 + b)(1 - \alpha)S_{k-1}^{(1)} & \text{if } R_k = b \\
(1 + a)(1 - \alpha)S_{k-1}^{(1)} & \text{if } R_k = a 
\end{cases} \\
= (1 + R_k)(1 - \alpha)S_{k-1}^{(1)}, \quad k = 1, 2, \ldots, N,
\]
and
\[
S_k^{(1)} = S_0^{(1)} \prod_{i=1}^k (1 + R_i), \quad k = 0, 1, \ldots, N,
\]
with the binary tree

b) The asset price before dividend payment is \( S_k^{(1)}/(1 - \alpha) \), hence the dividend amount is

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\[
\frac{S_k^{(1)}}{1-\alpha} - S_k^{(1)} = \frac{\alpha S_k^{(1)}}{1-\alpha},
\]
therefore, the dividend value represents a percentage \(\alpha/(1-\alpha)\) of the ex-dividend price \(S_k^{(1)}\).

c) When reinvesting the dividend amount \(\frac{\alpha}{1-\alpha} S_k^{(1)}\) into the new portfolio allocation, we have

\[
V_k = \xi_{k+1} S_k^{(1)} + \eta_{k+1} S_k^{(0)}
\]
\[
= \xi_k S_k^{(1)} + \eta_k S_k^{(0)} + \frac{\alpha}{1-\alpha} \xi_k S_k^{(1)}
\]
\[
= \xi_k \frac{S_k^{(1)}}{1-\alpha} + \eta_k S_k^{(0)},
\]
at times \(k = 1, 2, \ldots, N-1\). Moreover, at time \(N\) we will similarly have

\[
V_N = \xi_N S_N^{(1)} + \frac{\alpha}{1-\alpha} \xi_N S_N^{(1)} + \eta_N S_N^{(0)} = \xi_N \frac{S_N^{(1)}}{1-\alpha} + \eta_N S_N^{(0)},
\]
therefore the self-financing condition reads

\[
V_k = \xi_k \frac{S_k^{(1)}}{1-\alpha} + \eta_k S_k^{(0)}, \quad k = 1, 2, \ldots, N.
\] (A.6)

d) By the self-financing condition (A.6) we have

\[
\tilde{V}_k - \tilde{V}_{k-1} = \xi_{k+1} \frac{S_k^{(1)}}{S_k^{(0)}} + \eta_{k+1} - \xi_k \frac{S_{k-1}^{(1)}}{S_{k-1}^{(0)}} - \eta_k
\]
\[
= \frac{\xi_k S_k^{(1)}}{S_k^{(0)}(1-\alpha)} + \eta_k - \xi_k \frac{S_{k-1}^{(1)}}{S_{k-1}^{(0)}} - \eta_k
\]
\[
= \xi_k \left( \frac{S_k^{(1)}}{S_k^{(0)}(1-\alpha)} - \frac{S_{k-1}^{(1)}}{S_{k-1}^{(0)}} \right), \quad k = 1, 2, \ldots, N,
\]
which allows us to conclude from Question (b) that

\[
\mathbb{E}^* \left[ \tilde{V}_k \mid \mathcal{F}_{k-1} \right] - \tilde{V}_{k-1} = \mathbb{E}^* \left[ \tilde{V}_k - \tilde{V}_{k-1} \mid \mathcal{F}_{k-1} \right]
\]
\[
= \mathbb{E}^* \left[ \xi_k \times \left( \frac{S_k^{(1)}}{S_k^{(0)}(1-\alpha)} - \frac{S_{k-1}^{(1)}}{S_{k-1}^{(0)}} \right) \mid \mathcal{F}_{k-1} \right]
\]
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\[ = \xi_k \mathbb{E}^* \left[ \frac{S_k^{(1)}}{S_k^{(0)}} (1 - \alpha) \mid \mathcal{F}_{k-1} \right] - \frac{S_k^{(1)}}{S_k^{(0)}} \mid \mathcal{F}_{k-1} \] \[ = \xi_k \mathbb{E}^* \left[ \frac{S_k^{(1)}}{(1 - \alpha)} \mid \mathcal{F}_{k-1} \right] - (1 + r)S_k^{(1)} \]

\[ = 0, \quad k = 1, 2, \ldots, N, \]

therefore \((\bar{V}_k)_{k=0,1,\ldots,N-1}\) is a martingale under \(\mathbb{P}^*\).

e) Assuming that the portfolio strategy attains the claim \(C\) we have \(C = V_N\) and \(\bar{C} = \bar{V}_N\), hence by the martingale property of \((\bar{V}_k)_{k=0,1,\ldots,N-1}\) under \(\mathbb{P}^*\), we find

\[ \bar{V}_k = \mathbb{E}^* [\bar{V}_N \mid \mathcal{F}_k] = \mathbb{E}^* [\bar{C} \mid \mathcal{F}_k], \quad k = 0, 1, \ldots, N, \]

which shows that

\[ V_k = \frac{1}{(1 + r)^{N-k}} \mathbb{E}^* [C \mid \mathcal{F}_k], \quad k = 0, 1, \ldots, N, \]

f) By a binomial probability computation, we have

\[ V_k = \frac{1}{(1 + r)^{N-k}} \mathbb{E}^* [h(S_N) \mid \mathcal{F}_k] \]

\[ = \frac{1}{(1 + r)^{N-k}} \mathbb{E}^* \left[ h \left( x \prod_{l=t+1}^{N} (1 + R_l) \right) \bigg| \mathcal{F}_k \right]_{x=S_k^{(1)}} \]

\[ = \frac{1}{(1 + r)^{N-k}} \sum_{l=0}^{N-k} \binom{N-k}{l} (p^*)^k (q^*)^{N-k-l} h \left( S_k^{(1)} (1 + b)^k (1 + a)^{N-k-l} (1 - \alpha)^{N-k} \right) \]

\[ = C_0(k, S_k^{(1)} (1 - \alpha)^{N-k}, N, a, b, r). \]

g) We “absorb” the dividend rate \(\alpha\) into new market returns by taking \(a_\alpha, b_\alpha, r_\alpha\) such that

\[ 1 + a_\alpha = (1 + a)(1 - \alpha), \quad 1 + b_\alpha = (1 + b)(1 - \alpha), \quad 1 + r_\alpha = (1 + r)(1 - \alpha), \]

i.e.

\[ a_\alpha = -\alpha + a(1 - \alpha), \quad b_\alpha = -\alpha + b(1 - \alpha), \quad r_\alpha = -\alpha + r(1 - \alpha). \]

As a consequence, we have
\[ V_k = \frac{1}{(1+r)^{N-k}} \]
\[
\times \sum_{l=0}^{N-k} \left( \begin{array}{l} N-k \\ l \end{array} \right) \left( p^* \right)^k \left( q^* \right)^{N-k-l} h(S_k^{(1)}) (1+b)^k (1+a)^{N-k-l} (1-\alpha)^{N-k} \\
= (1-\alpha)^{-(N-k)} \frac{(1+r)^{-N-k}}{1+r} \sum_{l=0}^{N-k} \left( \begin{array}{l} N-k \\ l \end{array} \right) \left( p^* \right)^k \left( q^* \right)^{N-k-l} h(S_k^{(1)}) (1+b\alpha)^k (1+a\alpha)^{N-k-l} \\
= (1-\alpha)^{N-k} C_0(k, S_k^{(1)}, N, a\alpha, b\alpha, r\alpha),
\]

where
\[
p^* := P^*(R_k = b) = \frac{r\alpha - a\alpha}{b\alpha - a\alpha} = \frac{r-a}{b-a} > 0,
\]
and
\[
q^* := P^*(R_k = a) = \frac{b\alpha - r\alpha}{b\alpha - a\alpha} = \frac{b-r}{b-a} > 0,
\]
\[ k = 1, 2, \ldots, N. \]

h) We have
\[
\tilde{V}_k = \frac{1}{(1+r)^N} C_\alpha(k, S_k^{(1)}, N, a\alpha, b\alpha, r\alpha), \quad k = 0, 1, \ldots, N,
\]
hence by the martingale property we have
\[
\tilde{V}_k = \frac{1}{(1+r)^k} C_\alpha(k, S_k^{(1)}, N, a\alpha, b\alpha, r\alpha) \\
= \mathbb{E}^*[\tilde{V}_{k+1} \mid \mathcal{F}_k] \\
= \frac{1}{(1+r)^{k+1}} \mathbb{E}^*[C_\alpha(k, S_{k+1}^{(1)}, N, a\alpha, b\alpha, r\alpha) \mid \mathcal{F}_k] \\
= \frac{1}{(1+r)^{k+1}} \left( p^* C_\alpha(k, S_k^{(1)}(1+b\alpha), N, a\alpha, b\alpha, r\alpha) \\
+ q^* C_\alpha(k, S_k^{(1)}(1+a\alpha), N, a\alpha, b\alpha, r\alpha) \right).
\]
This yields
\[
(1+r) C_\alpha(k, S_k^{(1)}, N, a\alpha, b\alpha, r\alpha) \\
= p^* C_\alpha(k, S_k^{(1)}(1+b\alpha), N, a\alpha, b\alpha, r\alpha) + q^* C_\alpha(k, S_k^{(1)}(1+a\alpha), N, a\alpha, b\alpha, r\alpha).
\]

i) We find the equations
\[
\left\{ \begin{array}{l}
\eta_k S_k^{(0)}(0) + \xi_k (1+a\alpha) S_k^{(1)} = C_\alpha(k, (1+a\alpha) S_{k-1}^{(1)}, N, a\alpha, b\alpha, r\alpha) \\
\eta_k S_k^{(0)}(0) + \xi_k (1+b\alpha) S_k^{(1)} = C_\alpha(k, (1+b\alpha) S_{k-1}^{(1)}, N, a\alpha, b\alpha, r\alpha),
\end{array} \right.
\]

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which imply
\[
\xi_k = \frac{C_\alpha(k, (1 + b_\alpha)S_{k-1}^{(1)}), N, a_\alpha, b_\alpha, r_\alpha)}{C_\alpha(k, (1 + a_\alpha)S_{k-1}^{(1)}, N, a_\alpha, b_\alpha, r_\alpha)} \frac{(b_\alpha - a_\alpha)S_{k-1}^{(1)}}{(b_\alpha - a_\alpha)S_{k-1}^{(1)}}
\]
\[
= (1 - \alpha)^{N-k} \frac{C_0(k, (1 + b_\alpha)S_{k-1}^{(1)}, N, a_\alpha, b_\alpha, r_\alpha)}{C_0(k, (1 + a_\alpha)S_{k-1}^{(1)}, N, a_\alpha, b_\alpha, r_\alpha)} \frac{(b_\alpha - a_\alpha)S_{k-1}^{(1)}}{(b_\alpha - a_\alpha)S_{k-1}^{(1)}}
\]
\[
- (1 - \alpha)^{N-k} \frac{C_0(k, (1 + a_\alpha)S_{k-1}^{(1)}, N, a_\alpha, b_\alpha, r_\alpha)}{C_0(k, (1 + a_\alpha)S_{k-1}^{(1)}, N, a_\alpha, b_\alpha, r_\alpha)} \frac{(b_\alpha - a_\alpha)S_{k-1}^{(1)}}{(b_\alpha - a_\alpha)S_{k-1}^{(1)}}
\]
and
\[
\eta_k = \frac{(1 + b_\alpha)C_\alpha(k, (1 + b_\alpha)S_{k-1}^{(1)}, N, a_\alpha, b_\alpha, r_\alpha)}{C_\alpha(k, (1 + b_\alpha)S_{k-1}^{(1)}, N, a_\alpha, b_\alpha, r_\alpha)} \frac{(b_\alpha - a_\alpha)S_{k-1}^{(0)}}{(b_\alpha - a_\alpha)S_{k-1}^{(0)}}
\]
\[
- (1 + a_\alpha)C_\alpha(k, (1 + a_\alpha)S_{k-1}^{(1)}, N, a_\alpha, b_\alpha, r_\alpha)
\]
\[
\frac{(b_\alpha - a_\alpha)S_{k-1}^{(0)}}{(b_\alpha - a_\alpha)S_{k-1}^{(0)}}
\]
\[
= (1 - \alpha)^{N-k} \frac{(1 + b_\alpha)C_0(k, (1 + b_\alpha)S_{k-1}^{(1)}, N, a_\alpha, b_\alpha, r_\alpha)}{C_0(k, (1 + b_\alpha)S_{k-1}^{(1)}, N, a_\alpha, b_\alpha, r_\alpha)} \frac{(b_\alpha - a_\alpha)S_{k-1}^{(0)}}{(b_\alpha - a_\alpha)S_{k-1}^{(0)}}
\]
\[
- (1 + a_\alpha)C_0(k, (1 + a_\alpha)S_{k-1}^{(1)}, N, a_\alpha, b_\alpha, r_\alpha)
\]
\[
\frac{(b_\alpha - a_\alpha)S_{k-1}^{(0)}}{(b_\alpha - a_\alpha)S_{k-1}^{(0)}}
\]
\[
k = 1, 2, \ldots, N.
\]
j) A possible answer: We have
\[
\xi_k = \frac{1}{(b - a)S_{k-1}^{(1)}} \sum_{l=0}^{N-k} \binom{N-k}{l} \left(p^\star\right)^k (q^\star)^{N-k-l} \times \left((1 - \alpha)^{N-k} S_k^{(1)} (1 + b)^{k+1} (1 + a)^{N-k-l} \right)
\]
\[
- h((1 - \alpha)^{N-k} S_k^{(1)} (1 + b)^{k+1} (1 + a)^{N-k-l+1})
\]
and
\[
\eta_k = \frac{1}{(b - a)S_{k-1}^{(1)}} \sum_{l=0}^{N-k} \binom{N-k}{l} \left(p^\star\right)^k (q^\star)^{N-k-l} \times \left((1 + b)h((1 - \alpha)^{N-k} S_k^{(1)} (1 + b)^{k+1} (1 + a)^{N-k-l+1})
\]
\[
- (1 + a)h((1 - \alpha)^{N-k} S_k^{(1)} (1 + b)^{k+1} (1 + a)^{N-k-l})
\]
Differentiation with respect to $\alpha$ of the general term inside the above summations yields respectively

\[(1 + a)y h'((1 + a)y) - (1 + b)y h'((1 + b)y)\] (A.7)

for $\xi_k$, and

\[(1 + b)y h'((1 + b)y) - (1 + a)y h'((1 + a)y),\] (A.8)

for $\eta_k$, with $y := (1 - \alpha)^N - k S_k^{(1)} (1 + b)^k (1 + a)^N - k - l$ and $a < b$.

We note that the sign of the above quantities (A.7)-(A.8) depends on whether the function $x \mapsto x h'(x)$ is non-decreasing, which is the case for example for the payoff functions $h(x) = (x - K)^+$ and $h(x) = (K - x)^+$ of both European call and put options.

In particular, when the function $x \mapsto x h'(x)$ is non-decreasing, the amount invested on the risky (resp. riskless) asset will be lower (resp. higher) in the presence of a higher dividend.

We also note that the expected return

\[p^*(1 + b)(1 - \alpha) + q^*(1 + a)(1 - \alpha) = r(1 - \alpha)\]

and the variance

\[p^*(1 + b)^2(1 - \alpha)^2 + q^*(1 + a)^2(1 - \alpha)^2 - r^2(1 - \alpha)^2 = (1 - \alpha)^2(p^*(1 + b)^2 + q^*(1 + a)^2 - r^2)\]

of returns are lower in the presence of dividends.

**Problem 3.18**

a) In order to check for arbitrage opportunities we look for a risk-neutral probability measure $\mathbb{P}^*$ which should satisfy

\[\mathbb{E}^*[S_{k+1}^{(1)} \mid \mathcal{F}_k] = (1 + r)S_k^{(1)}, \quad k = 0, 1, \ldots, N - 1.\]

Rewriting $\mathbb{E}^*[S_{k+1}^{(1)} \mid \mathcal{F}_k]$ as

\[\mathbb{E}^*[S_{k+1}^{(1)} \mid \mathcal{F}_k] = (1 + a)S_k^{(1)} \mathbb{P}^*(R_{k+1} = a \mid \mathcal{F}_k) + S_k^{(1)} \mathbb{P}^*(R_{k+1} = 0 \mid \mathcal{F}_k)\]

\[+ (1 + b)S_k^{(1)} \mathbb{P}^*(R_{k+1} = b \mid \mathcal{F}_k) = (1 + a)S_k^{(1)} \mathbb{P}^*(R_{k+1} = a) + S_k^{(1)} \mathbb{P}^*(R_{k+1} = 0)\]

\[+ (1 + b)S_k^{(1)} \mathbb{P}^*(R_{k+1} = b),\]

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\( k = 0, 1, \ldots, N - 1 \), it follows that any risk-neutral probability measure \( \mathbb{P}^* \) should satisfy the equations

\[
\begin{aligned}
(1 + r)S_k^{(1)} = & \\
(1 + b)S_k^{(1)} \mathbb{P}^*(R_{k+1} = b) + S_k^{(1)} \mathbb{P}^*(R_{k+1} = 0) + (1 + a)S_k^{(1)} \mathbb{P}^*(R_{k+1} = a), & \\
\mathbb{P}^*(R_{k+1} = b) + \mathbb{P}^*(R_{k+1} = 0) + \mathbb{P}^*(R_{k+1} = a) = 1, & \\
\end{aligned}
\]

\( k = 0, 1, \ldots, N - 1 \), i.e.

\[
\begin{aligned}
& \begin{cases}
  b \mathbb{P}^*(R_k = b) + a \mathbb{P}^*(R_k = a) = r, \\
  \mathbb{P}^*(R_k = b) + \mathbb{P}^*(R_k = a) = 1 - \mathbb{P}^*(R_k = 0),
\end{cases} & \\
& k = 1, 2, \ldots, N,
\end{aligned}
\]

with solution

\[
\mathbb{P}^*(R_k = b) = \frac{r - (1 - \mathbb{P}^*(R_k = 0))a}{b - a} = \frac{r - (1 - \theta^*)a}{b - a},
\]

and

\[
\mathbb{P}^*(R_k = a) = \frac{(1 - \mathbb{P}^*(R_k = 0))b - r}{b - a} = \frac{(1 - \theta^*)b - r}{b - a},
\]

\( k = 1, 2, \ldots, N \). We check that this ternary tree model is without arbitrage if and only if there exists \( \theta^* = \mathbb{P}^*(R_k = 0) \in (0, 1) \) such that

\[
(1 - \theta^*)a < r < (1 - \theta^*)b, \quad \text{(A.9)}
\]

or

\[
0 < \theta^* < \min \left( \frac{r - a}{-a}, \frac{b - r}{b} \right) = \begin{cases}
  1 - \frac{r}{b} & \text{if } r \geq 0, \\
  1 - \frac{r}{a} & \text{if } r \leq 0.
\end{cases}
\]

Condition (A.9) is necessary in order to have

\[
\mathbb{P}^*(R_k = b) > 0 \quad \text{and} \quad \mathbb{P}^*(R_k = a) > 0,
\]

and it is sufficient because it also implies

\[
\mathbb{P}^*(R_k = b) = 1 - \theta^* - \mathbb{P}^*(R_k = a) \leq 1
\]

and

\[
\mathbb{P}^*(R_k = a) = 1 - \theta^* - \mathbb{P}^*(R_k = b) \leq 1.
\]

b) We will show that this ternary tree model is without arbitrage if and only if \( a < r < b \).
(i) Indeed, if the condition $a < r < b$ is satisfied there always exists $\theta \in (0, 1)$ such that

$$a < (1 - \theta)a < r < (1 - \theta)b < b,$$

as can be seen by taking

$$\theta \in \left(0, \min\left(\frac{r - a}{-a}, \frac{b - r}{b}\right)\right),$$

hence there exists a risk-neutral probability measure $P^*_\theta$, and the market model is without arbitrage.

(ii) Conversely, if this ternary tree model is without arbitrage there exists some $\theta = P^*(R_t = 0) \in (0, 1)$ such that

$$(1 - \theta)a < r < (1 - \theta)b.$$

c) When $r \leq a < 0 < b$ the risky asset overperforms the riskless asset, therefore we can realize arbitrage by borrowing from the riskless asset to purchase the risky asset. When $a < 0 < b \leq r$ the riskless asset overperforms the risky asset, therefore we can realize arbitrage by shortselling the risky asset and save the profit of the short sale on the riskless asset.

d) Under the absence of arbitrage condition $a < r < b$, every value of $\theta \in (0, 1)$ such that

$$0 < \theta < \min\left(\frac{r - a}{-a}, \frac{b - r}{b}\right)$$

satisfies

$$(1 - \theta)a < r < (1 - \theta)b,$$

and gives rise to a different risk-neutral probability measure, hence this ternary tree model is not complete.

In particular, every risk-neutral probability measure $P^*_\theta$ will give rise to a different claim price

$$\pi^*_t(\theta)(C) = \frac{1}{(1 + r)^{N-t}} \mathbb{E}^*_{\theta}[C | \mathcal{F}_t], \quad t = 0, 1, \ldots, N.$$

e) We have

$$\text{Var}^* \left[ \frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}} \right] \bigg| \mathcal{F}_k$$
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\begin{align*}
= \mathbb{E}^* \left[ \left( \frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}} \right)^2 | \mathcal{F}_k \right] - \left( \mathbb{E}^* \left[ \frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}} | \mathcal{F}_k \right] \right)^2 \\
= \mathbb{E}^* \left[ \left( \frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}} \right)^2 | \mathcal{F}_k \right] - r^2 \\
= a^2 \mathbb{P}_\sigma^*(R_{k+1} = a | \mathcal{F}_k) + b^2 \mathbb{P}_\sigma^*(R_{k+1} = b | \mathcal{F}_k) - r^2 \\
= a^2 \frac{(1 - \mathbb{P}_\sigma^*(R_{k+1} = 0))b - r}{b - a} + b^2 \frac{r - (1 - \mathbb{P}_\sigma^*(R_{k+1} = 0))a}{b - a} - r^2 \\
= ab(\theta - 1) + r(a + b) - r^2 \\
= \sigma^2,
\end{align*}

\(k = 0, 1, \ldots, N - 1, \) hence

\[ \mathbb{P}_\sigma^*(R_k = 0) = \theta = 1 + \frac{\sigma^2 + r^2 - r(a + b)}{ab}, \]

and therefore

\[ \mathbb{P}_\sigma^*(R_k = b) = \frac{r - (1 - \mathbb{P}_\sigma^*(R_k = 0))a}{b - a} = \frac{\sigma^2 - r(a - r)}{b(b - a)}, \]

and

\[ \mathbb{P}_\sigma^*(R_k = a) = \frac{(1 - \mathbb{P}_\sigma^*(R_k = 0))b - r}{b - a} = \frac{r(b - r) - \sigma^2}{a(b - a)}, \]

\(k = 1, 2, \ldots, N,\) under the condition

\[ \sigma^2 > \text{Max}(-r(r - a), r(b - r)), \]

in addition to the condition \(0 < \theta < 1,\) i.e.

\[ r(b - r) + rb < \sigma^2 < (b - r)(r - a). \]

Finally, we find

\[ -r(r - a) < \sigma^2 < (b - r)(r - a), \]

if \(r \in (a, 0],\) and

\[ r(b - r) < \sigma^2 < (b - r)(r - a), \]

if \(r \in [0, b).\)

f) In this case the ternary tree becomes a trinomial recombining tree, and the expression of the risk-neutral probability measure becomes

\[ \mathbb{P}_\delta^*(R_k = b) = \frac{r(b + 1) + (1 - \theta)b}{b^2 + 2b}, \]
and
\[ P^*_\theta(R_k = a) = (b + 1) \frac{(1 - \theta)b - r}{b^2 + 2b}, \]
\( k = 1, 2, \ldots, N. \) The market model is without arbitrage if and only if there exists \( \theta := P^*_\theta(R_k = 0) \in (0, 1) \) such that
\[ -(1 - \theta) \frac{b}{b + 1} < r < (1 - \theta)b, \]
or
\[ 0 < \theta < 1 - \frac{r}{b}. \]

\( g) \) Using the tower property (22.38) of conditional expectations, we have
\[
f(k, S^{(1)}_k) = \frac{1}{(1 + r)^{N-k}} \mathbb{E}^*[C \mid \mathcal{F}_k]
\]
\[ = \frac{1}{(1 + r)^{N-k}} \mathbb{E}^* \left[ \mathbb{E}^*[C \mid \mathcal{F}_{k+1}] \mid \mathcal{F}_k \right]
\]
\[ = \frac{1}{(1 + r)^{N-k}} \mathbb{E}^* \left[ (1 + r)^{N-(k+1)} f(k + 1, S^{(1)}_{k+1}) \mid \mathcal{F}_k \right]
\]
\[ = \frac{1}{1 + r} \mathbb{E}^* \left[ f(k + 1, S^{(1)}_{k+1}) \mid \mathcal{F}_k \right]
\]
\[ = \frac{1}{1 + r} \left( f(k + 1, S^{(1)}_k (1 + a)) \mathbb{P}^*_\theta(R_k = a) + f(k + 1, S^{(1)}_k) \mathbb{P}^*_\theta(R_k = 0)
\]
\[ + f(k + 1, S^{(1)}_k (1 + b)) \mathbb{P}^*_\theta(R_k = b) \right). \]

\( h) \) In this case we have \( f(N, x) = (K - x)^+. \)

\( i) \) See the attached code.*†

\( j) \) Taking \( \theta = 0.5 \) we find the following graph:

* Download the modified (trinomial) IPython notebook that can be run here.
† Download the corresponding (binomial) IPython notebook. The Anaconda distribution can be installed from https://www.anaconda.com/distribution/ or tried online at https://jupyter.org/try.

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```python
#matplotlib inline
import networkx as nx
import numpy as np
import matplotlib
import matplotlib.pyplot as plt

N=2;S0=1;r = 0.1; b=1.0; a=-b/(1+b);theta = 0.5;
p = (r-(1-theta)*a)/(b-a)
q = ((1-theta)*b-r)/(b-a)
def plot_tree(g):
    pos={}
    lab={}
    for n in g.nodes():
        pos[n]=(n[0],n[1])
        lab[n]=float("{0:.2f}".format(g.node[n]['value']))
    elarge=g.edges(data=True)
    nx.draw_networkx_edges(g,pos,edgelist=elarge)
    nx.draw_networkx_labels(g,pos,lab,font_size=15,font_family='sans-serif')
    plt.autoscale(enable=True)
    plt.show()

def graph_stock():
    S=nx.Graph()
    for k in range(0,N):
        for l in range(-k,k+1,1):
            S.add_edge((k,l),(k+1,l+1))
            S.add_edge((k,l),(k+1,l))
            S.add_edge((k,l),(k+1,l-1))
    for n in S.nodes():
        k=n[0]
        l=n[1]
        S.node[n]['value']=S0*((1.0+b)**((k+l)/2))*((1.0+a)**((k-l)/2))
    return S

plot_tree(graph_stock())
```

Fig. S.10: Put option pricing.
There also exists extensions of the trinomial model to five states (pentanomial model), six states (hexanomial model), etc.

Chapter 4

Exercise 4.1

a) We need to check whether the four properties of the definition of Brownian motion are satisfied. Checking Conditions 1-2-3 does not pose any particular problem since the time changes $t \mapsto c + t$, $t \mapsto t/c^2$ and $t \mapsto ct^2$ are deterministic, continuous, and increasing. As for Condition 4, $B_{c+t} - B_{c+s}$ clearly has a centered Gaussian distribution with variance $c + t - (c - s) = t - s$, and the same property holds for $cB_{t/c^2}$ since

$$\text{Var} \left( c \left( B_{t/c^2} - B_{s/c^2} \right) \right) = c^2 \text{Var} \left( B_{t/c^2} - B_{s/c^2} \right) = (t - s)c^2 / c^2 = t - s.$$ 

As a consequence, (i) and (ii) are standard Brownian motions.

Concerning (iii), we note that $B_{ct^2}$ is a centered Gaussian random variable with variance $ct^2$ - not $t$, hence $(B_{ct^2})_{t \in \mathbb{R}^+}$ is not a standard Brownian motion.

Regarding (iv), this process does not have independent increments, hence it cannot be a Brownian motion. For example, by (4.1) we have

$$\mathbb{E} \left[ \left( B_t + B_{t/2} - (B_s + B_{s/2}) \right) \left( B_s + B_{s/2} \right) \right] = \mathbb{E} \left[ B_t B_s + B_t B_{s/2} + B_{t/2} B_s + B_{t/2} B_{s/2} \right].$$
\[-\mathbb{E}\left[B_s B_s + B_s B_{s/2} + B_{s/2} B_s + B_{s/2} B_{s/2}\right]\]
\[= s + \frac{s}{2} + s + \frac{s}{2} - \frac{s}{2} - \frac{s}{2} - \frac{s}{2} - \frac{s}{2}\]
\[= \frac{s}{2},\]
which differs from 0, hence the two increments are not independent - otherwise we would have
\[
\mathbb{E}\left[\left(B_t + B_{t/2} - (B_s + B_{s/2})\right)\left(B_s + B_{s/2}\right)\right]
= \mathbb{E}\left[B_t + B_{t/2} - (B_s + B_{s/2})\right]\mathbb{E}\left[(B_s + B_{s/2})\right]
= 0.
\]

b) We have
\[
\int_0^T 2dB_t = 2(B_T - B_0) = 2B_T,
\]
which has a Gaussian distribution with mean 0 and variance 4T. On the other hand,
\[
\int_0^T (2 \times 1_{[0,T/2]}(t) + 1_{(T/2,T]}(t))dB_t = 2(B_{T/2} - B_0) + (B_T - B_{T/2})
= B_T + B_{T/2},
\]
which has a Gaussian distribution with mean 0 and variance
\[
\text{Var}[B_T + B_{T/2}] = \text{Var}[(B_T - B_{T/2}) + 2B_{T/2}]
= \text{Var}[B_T - B_{T/2}] + 4\text{Var}[B_{T/2}]
= \frac{T}{2} + 4\frac{T}{2}
= \frac{5T}{2}.
\]
Equivalently, using the Itô isometry (4.7), we have
\[
\text{Var}\left[\left(\int_0^T (2 \times 1_{[0,T/2]}(t) + 1_{(T/2,T]}(t))dB_t\right)\right]
= \mathbb{E}\left[\left(\int_0^T (2 \times 1_{[0,T/2]}(t) + 1_{(T/2,T]}(t))dB_t\right)^2\right]
= \int_0^T (2 \times 1_{[0,T/2]}(t) + 1_{(T/2,T]}(t))^2 dt
= 4\int_0^{T/2} dt + \int_{T/2}^T dt
= \frac{5T}{2}.
\]
c) The stochastic integral \( \int_0^{2\pi} \sin(t) dB_t \) has a Gaussian distribution with mean 0 and variance
\[
\int_0^{2\pi} \sin^2(t) dt = \frac{1}{2} \int_0^{2\pi} (1 - \cos(2t)) dt = \pi.
\]

d) If \( 0 \leq s \leq t \) we have
\[
\mathbb{E}[B_t B_s] = \mathbb{E}[(B_t - B_s)B_s] + \mathbb{E}[B_s^2] \\
= \mathbb{E}[(B_t - B_s)] \mathbb{E}[B_s] + \mathbb{E}[B_s^2] \\
= 0 + s \\
= s,
\]
and similarly we obtain \( \mathbb{E}[B_t B_s] = t \) when \( 0 \leq t \leq s \), hence in general we have
\[
\mathbb{E}[B_t B_s] = \min(s,t), \quad s,t \in \mathbb{R}_+.
\]

e) By the Itô (4.26) we have
\[
d(f(t)B_t) = f(t)dB_t + B_t df(t) + df(t) \cdot dB_t \\
= f(t)dB_t + B_t f'(t) dt + f'(t) dt \cdot dB_t \\
= f(t)dB_t + B_t f'(t) dt,
\]
and by integration on both sides we get
\[
\int_0^T f(t)dB_t + \int_0^T B_t f'(t) dt = \int_0^T d(f(t)B_t) \\
= f(T)B_T - f(0)B_0 \\
= 0,
\]
since \( f(T) = 0 \) and \( B_0 = 0 \), hence the conclusion. Note that this result can also be obtained by integration by parts.

Exercise 4.2

a) The probability distribution of \( X_n \) is Gaussian with mean zero and variance
\[
\text{Var}[X_n] = \mathbb{E} \left[ \left( \int_0^{2\pi} \sin(nt) dB_t \right)^2 \right] \\
= \int_0^{2\pi} \sin^2(nt) dt \\
= \frac{1}{2} \int_0^{2\pi} \cos(0) dt - \frac{1}{2} \int_0^{2\pi} \cos(2nt) dt \\
= \pi, \quad n \geq 1.
\]
b) The random variables \((X_n)_{n \geq 1}\) have same Gaussian distribution, and they are pairwise independent because

\[
\mathbb{E}[X_nX_m] = \mathbb{E} \left[ \int_0^{2\pi} \sin(nt)dB_t \int_0^{2\pi} \sin(mt)dB_t \right]
\]

\[
= \int_0^{2\pi} \sin(nt) \sin(mt) dt
\]

\[
= \frac{1}{2} \int_0^{2\pi} \cos((n-m)t) dt - \frac{1}{2} \int_0^{2\pi} \cos((n+m)t) dt
\]

and the vector \((X_n, X_m)\) is jointly Gaussian, for \(n, m \geq 1\) such that \(n \neq m\). Note that this condition implies independence only when the random variables have a Gaussian distribution.

Exercise 4.3 We have \(X_t = f(B_t)\) with \(f(x) = \sin^2 x, f'(x) = 2 \sin x \cos x = \sin(2x),\) and \(f''(x) = 2 \cos(2x)\), hence

\[
dX_t = d\sin^2(B_t)
= df(B_t)
= f'(B_t)dB_t + \frac{1}{2} f''(B_t) dt
= \sin(2B_t)dB_t + \cos(2B_t) dt.
\]

Exercise 4.4

a) We have

\[
\mathbb{E}[B_T^3] = \mathbb{E} \left[ \int_0^T dB_t \left( T + 2 \int_0^T B_t dB_t \right) \right]
\]

\[
= T \mathbb{E} \left[ \int_0^T dB_t \right] + 2 \mathbb{E} \left[ \int_0^T dB_t \int_0^T B_t dB_t \right]
\]

\[
= 2 \mathbb{E} \left[ \int_0^T B_t dt \right]
\]

\[
= 2 \int_0^T \mathbb{E}[B_t] dt
\]

\[
= 0.
\]

We also have

\[
\mathbb{E}[B_T^4] = \mathbb{E} \left[ \left( T + 2 \int_0^T B_t dB_t \right)^2 \right]
\]
\begin{align*}
&= \mathbb{E} \left[ T^2 + 2T \int_0^T B_t dB_t + 4 \left( \int_0^T B_t dB_t \right)^2 \right] \\
&= T^2 + 2T \mathbb{E} \left[ \int_0^T B_t dB_t \right] + 4 \mathbb{E} \left[ \left( \int_0^T B_t dB_t \right)^2 \right] \\
&= T^2 + 4 \mathbb{E} \left[ \int_0^T |B_t|^2 dt \right] \\
&= T^2 + 4 \int_0^T \mathbb{E} [|B_t|^2] dt \\
&= T^2 + 4 \int_0^T t dt \\
&= T^2 + 4 \frac{T^2}{2} \\
&= 3T^2.
\end{align*}

b) If \( X \approx \mathcal{N}(0, \sigma^2) \), we have \( X \approx B_T \) with \( \sigma^2 = T \), hence the answer to Question (a) yields

\[ \mathbb{E}[X^3] = 0 \quad \text{and} \quad \mathbb{E}[X^4] = 3\sigma^4. \]

We note that those moments can be recovered directly from the Gaussian probability density function as

\[ \mathbb{E}[X^3] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^3 e^{-x^2/(2\sigma^2)} dx = 0 \]

and

\[ \mathbb{E}[X^4] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^4 e^{-x^2/(2\sigma^2)} dx = 3\sigma^4. \]

Exercise 4.5 Taking expectation on both sides of (4.33) shows that \( C = 0 \). Next, applying Itô’s formula to the function \( f(x) = x^3 \) shows that

\[ (B_T)^3 = f(B_T) \]

\[ = f(B_0) + \int_0^T f'(B_t) dB_t + \frac{1}{2} \int_0^T f''(B_t) dt \]

\[ = 3 \int_0^T B_t^2 dB_t + 3 \int_0^T B_t dt. \]

By the integration by parts formula (4.11) applied to \( f(t) = t \), we find

\[ \int_0^T B_t dt = T B_T - \int_0^T t dB_t = \int_0^T (T - t) dB_t, \]

hence

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\[
(B_T)^3 = 3 \int_0^T B_t^2 dB_t + 3 \left( T B_T - \int_0^T t dB_t \right) \\
= 3 \int_0^T (T - t + B_t^2) dB_t,
\]
and we find \( \zeta_{t,T} = 3(T - t + B_t^2), \ t \in [0, T] \). This type of stochastic integral decomposition can be used for option hedging, cf. Section 7.5.

Exercise 4.6 Let \( f \in L^2([0,T]) \). We have
\[
E \left[ e^{\int_0^T f(s) dB_s} \mid F_t \right] = e^{\int_0^t f(s) dB_s} E \left[ e^{\int_0^T f(s) dB_s} \mid F_t \right] \\
= e^{\int_0^t f(s) dB_s} E \left[ e^{\int_0^T f(s) dB_s} \right] \\
= \exp \left( \int_0^t f(s) dB_s + \frac{1}{2} \int_t^T |f(s)|^2 ds \right), \tag{A.10}
\]
0 \( \leq t \leq T \), where we used the Gaussian moment generating function \( \mathbb{E}[e^{X}] = e^{\sigma^2/2} \) for \( X \approx N(0, \sigma^2) \) and the fact that \( \int_t^T f(s) dB_s \approx N \left( 0, \int_t^T f^2(s) ds \right) \) by Proposition 4.10.

Exercise 4.7 We have
\[
\mathbb{E} \left[ \exp \left( \int_0^t f(s) dB_s - \frac{1}{2} \int_0^t f^2(s) ds \right) \mid F_u \right] \\
= \exp \left( -\frac{1}{2} \int_0^t f^2(s) ds \right) \mathbb{E} \left[ \exp \left( \int_0^t f(s) dB_s \right) \mid F_u \right] \\
= \exp \left( -\frac{1}{2} \int_0^t f^2(s) ds \right) \mathbb{E} \left[ \exp \left( \int_0^u f(s) dB_s + \int_u^t f(s) dB_s \right) \mid F_u \right] \\
= \exp \left( \int_0^u f(s) dB_s - \frac{1}{2} \int_0^t f^2(s) ds \right) \mathbb{E} \left[ \exp \left( \int_0^u f(s) dB_s \right) \mid F_u \right] \\
= \exp \left( \int_0^u f(s) dB_s - \frac{1}{2} \int_0^t f^2(s) ds \right) \mathbb{E} \left[ \exp \left( \int_0^t f(s) dB_s \right) \right] \\
= \exp \left( \int_0^u f(s) dB_s - \frac{1}{2} \int_0^t f^2(s) ds + \frac{1}{2} \int_u^t f^2(s) ds \right) \\
= \exp \left( \int_0^u f(s) dB_s - \frac{1}{2} \int_0^u f^2(s) ds \right), \quad 0 \leq u \leq t.
\]
This result can also be obtained by directly applying (A.10).

Exercise 4.8 We have
\[
E \left[ \exp \left( \beta \int_0^T B_t dB_t \right) \right] = E \left[ \exp \left( \beta (B_T^2 - T) / 2 \right) \right]
= e^{-\beta T/2} E \left[ e^{\beta (B_T)^2 / 2} \right]
= \frac{e^{-\beta T/2}}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{\beta x^2 / 2} e^{-x^2 / (2T)} dx
= \frac{e^{-\beta T/2}}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{(\beta - 1/T)x^2 / 2} dx
= \frac{e^{-\beta T/2}}{\sqrt{1 - \beta T}} \int_{-\infty}^{\infty} \frac{e^{-x^2 / (2(1/T - 1/\beta))}}{\sqrt{2\pi / (1/T - 1/\beta)}} dx
= \frac{e^{-\beta T/2}}{\sqrt{1 - \beta T}},
\]
for all \( \beta < 1/T \), where we applied Relation (22.43) to \( \phi(x) = e^{\beta x^2 / 2} \), knowing that \( B_T \approx \mathcal{N}(0, T) \).

Exercise 4.9

a) Letting \( Y_t = e^{bt} X_t \), we have
\[
dY_t = d(e^{bt} X_t)
= b e^{bt} X_t dt + e^{bt} dX_t
= b e^{bt} X_t dt + e^{bt} (-b X_t dt + \sigma e^{-bt} dB_t)
= \sigma dB_t,
\]

hence
\[
Y_t = Y_0 + \int_0^t dY_s = Y_0 + \sigma \int_0^t dB_s = Y_0 + \sigma B_t,
\]
and
\[
X_t = e^{-bt} Y_t = e^{-bt} Y_0 + \sigma e^{-bt} B_t = e^{-bt} X_0 + \sigma e^{-bt} B_t.
\]

b) Letting \( Y_t = e^{bt} X_t \), we have
\[
dY_t = d(e^{bt} X_t)
= b e^{bt} X_t dt + e^{bt} dX_t
= b e^{bt} X_t dt + e^{bt} (-b X_t dt + \sigma e^{-at} dB_t)
= \sigma e^{(b-a)t} dB_t,
\]

hence we can solve for \( Y_t \) by integrating on both sides as
\[
Y_t = Y_0 + \int_0^t dY_s = Y_0 + \sigma \int_0^t e^{(b-a)s} dB_s, \quad t \in \mathbb{R}_+.
\]

This yields the solution
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\[ X_t = e^{-bt} Y_t = e^{-bt} X_0 + \sigma e^{-bt} \int_0^t e^{(b-a)s} dB_s, \quad t \in \mathbb{R}_+. \]

Comments:

(i) This type of computation appears anywhere discounting by the factor \( e^{-bt} \) is involved.

(ii) The stochastic integral \( \int_0^t e^{(b-a)s} dB_s \) cannot be computed in closed form. It is a centered Gaussian random variable with variance

\[
\int_0^t e^{2(b-a)s} ds = \frac{e^{2(b-a)t} - 1}{2(b-a)}
\]

if \( b \neq a \), and variance \( t \) if \( a = b \).

Exercise 4.10

a) Note that the stochastic integral

\[
\int_0^T \frac{1}{T-s} dB_s
\]

is not defined in \( L^2(\Omega) \) as the function \( s \mapsto \frac{1}{T-s} \) is not in \( L^2([0,T]) \) and by the Itô isometry we have

\[
\mathbb{E} \left[ \left( \int_0^T \frac{1}{T-s} dB_s \right)^2 \right] = \int_0^T \frac{1}{(T-s)^2} ds = \left[ \frac{1}{T-s} \right]_0^\infty = +\infty.
\]

By (4.34) we have

\[
d \left( \frac{X_T^T}{T-t} \right) = \frac{dX_T^T}{T-t} + \frac{X_T^T}{(T-t)^2} dt = \sigma \frac{dB_t}{T-t},
\]

hence by integration using the initial condition \( X_0 = 0 \) we have

\[
\frac{X_T^T}{T-t} = \sigma \int_0^t \frac{1}{T-s} dB_s, \quad t \in [0,T).
\]

b) We have

\[
\mathbb{E}[X_T^T] = (T-t) \sigma \mathbb{E} \left[ \int_0^t \frac{1}{T-s} dB_s \right] = 0, \quad t \in [0,T).
\]

c) By the Itô isometry we have

\[
\text{Var}[X_T^T] = (T-t)^2 \sigma^2 \mathbb{E} \left[ \int_0^t \frac{1}{T-s} dB_s \right]^2
\]
\[
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\]

\[
= (T - t)^2 \sigma^2 \mathbb{E} \left[ \left( \int_0^t \frac{1}{T - s} dB_s \right)^2 \right]
\]

\[
= (T - t)^2 \sigma^2 \int_0^t \frac{1}{(T - s)^2} ds
\]

\[
= (T - t)^2 \sigma^2 \left( \frac{1}{T - t} - \frac{1}{T} \right)
\]

\[
= \sigma^2 \left( 1 - \frac{t}{T} \right), \quad t \in [0, T).
\]

d) We have
\[
\lim_{t \to 0} \left\| X_t^T \right\|_{L^2(\Omega)} = \lim_{t \to 0} \text{Var}[X_t^T] = 0.
\]

Exercise 4.11 Exponential Vasicek model (1). Applying the Itô formula to \( X_t = e^{rt} = f(r_t) \) with \( f(x) = e^x \), we have

\[
dX_t = de^{rt}
\]

\[
= e^{rt} dr_t + \frac{1}{2} e^{rt} |dr_t|^2
\]

\[
= e^{rt} ((a - br_t) dt + \sigma dB_t) + \frac{1}{2} e^{rt} ((a - br_t) dt + \sigma dB_t)^2
\]

\[
= e^{rt} ((a - br_t) dt + \sigma dB_t) + \frac{\sigma^2}{2} e^{rt} dt
\]

\[
= X_t \left( a + \frac{\sigma^2}{2} - b \log(X_t) \right) dt + \sigma X_t dB_t
\]

\[
= X_t (\tilde{a} - \tilde{b}f(X_t)) dt + \sigma g(X_t) dB_t,
\]

hence

\[
\tilde{a} = a + \frac{\sigma^2}{2} \quad \text{and} \quad \tilde{b} = b
\]

the functions \( f(x) \) and \( g(x) \) are given by \( f(x) = \log x \) and \( g(x) = x \). Note that this stochastic differential equation is that of the exponential Vasicek model.

Exercise 4.12 Exponential Vasicek model (2).

a) We have \( Z_t = e^{-at} Z_0 + \sigma \int_0^t e^{-(t-s)a} dB_s \).

b) We have \( Y_t = e^{-at} Y_0 + \frac{\theta}{a} (1 - e^{-at}) + \sigma \int_0^t e^{-(t-s)a} dB_s \).

c) We have \( dX_t = X_t \left( \theta + \frac{\sigma^2}{2} - a \log X_t \right) dt + \sigma X_t dB_t \).

d) We have \( r_t = \exp \left( e^{-at} \log r_0 + \frac{\theta}{a} (1 - e^{-at}) + \sigma \int_0^t e^{-(t-s)a} dB_s \right) \).

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e) Using the Gaussian moment generating function identity $\mathbb{E}[e^X] = e^{\alpha^2/2}$ for $X \sim \mathcal{N}(0, \alpha^2)$, we have

\[
\mathbb{E}[r_t \mid \mathcal{F}_u] = \mathbb{E}\left[ \exp\left( e^{-at} \log r_0 + \frac{\theta}{a} (1 - e^{-at}) + \sigma \int_0^t e^{-(t-s)a} dB_s \right) \mid \mathcal{F}_u \right] \\
= e^{-at} \log r_0 + \frac{\theta}{a} (1 - e^{-at}) + \sigma \int_0^u e^{-(t-s)a} dB_s \mathbb{E}\left[ \exp\left( \sigma \int_u^t e^{-(t-s)a} dB_s \right) \mid \mathcal{F}_u \right] \\
= e^{-at} \log r_0 + \frac{\theta}{a} (1 - e^{-at}) + \sigma \int_0^u e^{-(t-s)a} dB_s \mathbb{E}\left[ \exp\left( \sigma \int_u^t e^{-(t-s)a} dB_s \right) \mid \mathcal{F}_u \right] \\
= \exp\left( e^{-at} \log r_0 + \frac{\theta}{a} (1 - e^{-at}) + \sigma \int_0^u e^{-(t-s)a} dB_s + \frac{\sigma^2}{2} \int_u^t e^{-2(t-s)a} ds \right) \\
= \exp\left( e^{-at} \log r_0 + \frac{\theta}{a} (1 - e^{-at}) + \sigma \int_0^u e^{-(t-s)a} dB_s + \frac{\sigma^2}{4a} (1 - e^{-2(t-u)a}) \right) \\
= \exp\left( e^{-(t-u)a} \log r_0 + \frac{\theta}{a} (1 - e^{-(t-u)a}) + \sigma \int_0^u e^{-(u-s)a} dB_s \right) \\
\quad + \frac{\sigma^2}{4a} (1 - e^{-2(t-u)a}) \\
= \exp\left( e^{-(t-u)a} \log r_u + \frac{\theta}{a} (1 - e^{-(t-u)a}) + \sigma \int_0^u e^{-(u-s)a} dB_s \right) \\
\quad + \frac{\sigma^2}{4a} (1 - e^{-2(t-u)a}) \\
= r_u e^{-(t-u)a} \exp\left( \frac{\theta}{a} (1 - e^{-(t-u)a}) + \frac{\sigma^2}{4a} (1 - e^{-2(t-u)a}) \right).
\]

In particular, for $u = 0$ we find

\[
\mathbb{E}[r_t] = r_0 e^{-at} \exp\left( + \frac{\theta}{a} (1 - e^{-at}) + \frac{\sigma^2}{4a} (1 - e^{-2at}) \right).
\]

f) Similarly, we have

\[
\mathbb{E}[r_t^2 \mid \mathcal{F}_u] = \mathbb{E}\left[ \exp\left( 2 e^{-at} \log r_0 + \frac{2\theta}{a} (1 - e^{-at}) + 2\sigma \int_0^t e^{-(t-s)a} dB_s \right) \mid \mathcal{F}_u \right] \\
= e^{2e^{-at} \log r_0 + \frac{2\theta}{a} (1 - e^{-at}) + 2\sigma \int_0^u e^{-(t-s)a} dB_s} \mathbb{E}\left[ \exp\left( 2\sigma \int_u^t e^{-(t-s)a} dB_s \right) \mid \mathcal{F}_u \right] \\
= e^{2e^{-at} \log r_0 + \frac{2\theta}{a} (1 - e^{-at}) + 2\sigma \int_0^u e^{-(t-s)a} dB_s} \mathbb{E}\left[ \exp\left( 2\sigma \int_u^t e^{-(t-s)a} dB_s \right) \mid \mathcal{F}_u \right] \\
= \exp\left( 2 e^{-at} \log r_0 + \frac{2\theta}{a} (1 - e^{-at}) + 2\sigma \int_0^u e^{-(t-s)a} dB_s + 2\sigma^2 \int_u^t e^{-2(t-s)a} ds \right) \\
= \exp\left( 2 e^{-at} \log r_0 + \frac{2\theta}{a} (1 - e^{-at}) + 2\sigma \int_0^u e^{-(t-s)a} dB_s + \frac{\sigma^2}{a} (1 - e^{-2(t-u)a}) \right) \\
= \exp\left( 2 e^{-(t-u)a} \left( 2 e^{-au} \log r_0 + \frac{2\theta}{a} (1 - e^{-au}) + 2\sigma \int_0^u e^{-(u-s)a} dB_s \right) \right).
\]

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\[
\frac{2\theta}{a} \left(1 - e^{-(t-u)a}\right) + \frac{\sigma^2}{a} \left(1 - e^{-2(t-u)a}\right)
\] 

\[
= \exp \left(2e^{-(t-u)a} \log r_u + \frac{2\theta}{a} \left(1 - e^{-(t-u)a}\right) + \frac{\sigma^2}{a} \left(1 - e^{-2(t-u)a}\right)\right)
\]

\[
= r_u^2 e^{-(t-u)a} \exp \left(\frac{2\theta}{a} \left(1 - e^{-(t-u)a}\right) + \frac{\sigma^2}{a} \left(1 - e^{-2(t-u)a}\right)\right),
\]

hence

\[
\text{Var}[r_t | F_u] = \mathbb{E}[r_t^2 | F_u] - (\mathbb{E}[r_t | F_u])^2
\]

\[
= r_u^2 e^{-(t-u)a} \exp \left(\frac{2\theta}{a} \left(1 - e^{-(t-u)a}\right) + \frac{\sigma^2}{a} \left(1 - e^{-2(t-u)a}\right)\right)
\]

\[
- r_u^2 e^{-(t-u)a} \exp \left(\frac{2\theta}{a} \left(1 - e^{-(t-u)a}\right) + \frac{\sigma^2}{2a} \left(1 - e^{-2(t-u)a}\right)\right)
\]

\[
= r_u^2 e^{-(t-u)a} \exp \left(\frac{2\theta}{a} \left(1 - e^{-(t-u)a}\right) + \frac{\sigma^2}{a} \left(1 - e^{-2(t-u)a}\right)\right)
\]

\[
\times \left(1 - \exp \left(-\frac{\sigma^2}{2a} \left(1 - e^{-2(t-u)a}\right)\right)\right).
\]

\[
g) \text{ We find } \lim_{t \to \infty} \mathbb{E}[r_t] = r_0 \exp \left(\frac{\theta}{a} + \frac{\sigma^2}{4a}\right) \text{ and }
\]

\[
\lim_{t \to \infty} \text{Var}[r_t] = \exp \left(\frac{2\theta}{a} \left(1 - \exp \left(-\frac{\sigma^2}{2a}\right)\right)\right)
\]

\[
= \exp \left(\frac{2\theta}{a} \left(\exp \left(\frac{\sigma^2}{a}\right) - 1\right)\right).
\]

Exercise 4.13 Cox-Ingersoll-Ross (CIR) model.

a) We have

\[
r_t = r_0 + \int_0^t (\alpha - \beta r_s) ds + \sigma \int_0^t \sqrt{r_s} dB_s, \quad t \in \mathbb{R}_+.
\]

b) Taking expectations on both sides of (A.11) and using the fact that the expectation of the stochastic integral with respect to Brownian motion is zero, we find

\[
u(t) = \mathbb{E}[r_t]
\]

\[
= \mathbb{E} \left[ r_0 + \int_0^t (\alpha - \beta r_s) ds + \sigma \int_0^t \sqrt{r_s} dB_s \right]
\]

\[
= \mathbb{E} \left[ r_0 + \int_0^t (\alpha - \beta r_s) ds \right]
\]

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which yields the differential equation \( u'(t) = \alpha - \beta u(t) \). Letting \( w(t) : e^{\beta t}u(t) \) we have
\[
w'(t) = \beta e^{\beta t}u(t) + e^{\beta t}u'(t) = \alpha e^{\beta t},
\]
hence
\[
\mathbb{E}[r_t] = u(t)
\]
\[
= e^{-\beta t}w(t)
\]
\[
= e^{-\beta t} \left( w(0) + \alpha \int_0^t e^{\beta s} ds \right)
\]
\[
= e^{-\beta t} \left( u(0) + \frac{\alpha}{\beta} (e^{\beta t} - 1) \right)
\]
\[
= e^{-\beta t}r_0 + \frac{\alpha}{\beta} (1 - e^{-\beta t}), \quad t \in \mathbb{R}_+.
\]
\[\text{(A.12)}\]

c) By applying Itô’s formula to
\[
r_t^2 = f \left( r_0 + \int_0^t (\alpha - \beta r_s) ds + \sigma \int_0^t \sqrt{r_s} dB_s \right),
\]
with \( f(x) = x^2 \), we find
\[
d(r_t)^2 = r_t (\sigma^2 + 2\alpha - 2\beta r_t) dt + 2\sigma r_t^{5/2} dB_t
\]
or, in integral form,
\[
r_t^2 = r_0^2 + \int_0^t r_s (\sigma^2 + 2\alpha - 2\beta r_s) ds + 2\sigma \int_0^t s^{3/2} dB_s, \quad t \in \mathbb{R}_+.
\]
\[\text{(A.13)}\]
d) Taking again the expectation on both sides of (A.13), we find
\[
v(t) = \mathbb{E}[r_t^2]
\]
\[
= \mathbb{E} \left[ r_0^2 + \int_0^t r_s (\sigma^2 + 2\alpha - 2\beta r_s) ds + 2\sigma \int_0^t s^{3/2} dB_s \right]
\]
\[
= r_0^2 + \mathbb{E} \left[ \int_0^t r_s (\sigma^2 + 2\alpha - 2\beta r_s) ds \right]
\]
\[
= r_0^2 + \int_0^t (\sigma^2 \mathbb{E}[r_s] + 2\alpha \mathbb{E}[r_s] - 2\beta \mathbb{E}[r_s^2]) ds
\]
\[ v(t) = c_0 + c_1 e^{-\beta t} + c_2 e^{-2\beta t}, \quad t \in \mathbb{R}_+. \]

By (A.12) we find
\[
v'(t) = -\beta c_1 e^{-\beta t} = (\sigma^2 + 2\alpha) \left( \frac{\alpha}{\beta} + \left( r_0 - \frac{\alpha}{\beta} \right) e^{-\beta t} \right) - 2\beta (c_0 + c_1 e^{-\beta t}) \]
\[
= \frac{\alpha}{\beta} (\sigma^2 + 2\alpha) + (\sigma^2 + 2\alpha) \left( r_0 - \frac{\alpha}{\beta} \right) e^{-\beta t} - 2\beta c_0 - 2\beta c_1 e^{-\beta t},
\]
t \in \mathbb{R}_+, \text{ hence}
\[
\begin{cases}
0 = \frac{\alpha}{\beta} (\sigma^2 + 2\alpha) - 2\beta c_0, \\
-\beta c_1 = (\sigma^2 + 2\alpha) \left( r_0 - \frac{\alpha}{\beta} \right) - 2\beta c_1,
\end{cases}
\]
and
\[
c_0 = \frac{\alpha}{2\beta^2} (\sigma^2 + 2\alpha), \quad c_1 = \frac{\sigma^2 + 2\alpha}{\beta} \left( r_0 - \frac{\alpha}{\beta} \right),
\]
with
\[
r_0^2 = v(0) = c_0 + c_1 + c_2,
\]
which yields
\[
c_2 = r_0^2 - c_0 - c_1
= r_0^2 - \frac{\alpha}{2\beta^2} (\sigma^2 + 2\alpha) - \frac{\sigma^2 + 2\alpha}{\beta} \left( r_0 - \frac{\alpha}{\beta} \right)
= r_0^2 - (\sigma^2 + 2\alpha) \left( \frac{r_0}{\beta} - \frac{\alpha}{2\beta^2} \right),
\]
and

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\[ \mathbb{E}[r_t^2] = v(t) \]
\[ = c_0 + c_1 e^{-\beta t} + c_2 e^{-2\beta t} \]
\[ = \frac{\alpha}{2\beta^2} (\sigma^2 + 2\alpha) + \frac{\sigma^2 + 2\alpha}{\beta} \left( r_0 - \frac{\alpha}{\beta} \right) e^{-\beta t} \]
\[ + \left( r_0^2 - (\sigma^2 + 2\alpha) \left( \frac{r_0}{\beta} - \frac{\alpha}{2\beta^2} \right) \right) e^{-2\beta t}, \quad t \in \mathbb{R}_+. \]

e) We have
\[ \text{Var}[r_t^2] = \mathbb{E}[r_t^2] - (\mathbb{E}[r_t])^2 \]
\[ = \frac{\alpha}{2\beta^2} (\sigma^2 + 2\alpha) + \frac{\sigma^2 + 2\alpha}{\beta} \left( r_0 - \frac{\alpha}{\beta} \right) e^{-\beta t} \]
\[ + \left( r_0^2 - (\sigma^2 + 2\alpha) \left( \frac{r_0}{\beta} - \frac{\alpha}{2\beta^2} \right) \right) e^{-2\beta t} \]
\[ - \left( \frac{\alpha}{\beta} + (r_0 - \frac{\alpha}{\beta}) e^{-\beta t} \right)^2 \]
\[ = \frac{\alpha}{2\beta^2} (\sigma^2 + 2\alpha) + \frac{\sigma^2 + 2\alpha}{\beta} \left( r_0 - \frac{\alpha}{\beta} \right) e^{-\beta t} \]
\[ + \left( r_0^2 - (\sigma^2 + 2\alpha) \left( \frac{r_0}{\beta} - \frac{\alpha}{2\beta^2} \right) \right) e^{-2\beta t} \]
\[ - \beta \left( r_0 - \frac{\alpha}{\beta} \right) e^{-\beta t} - \left( r_0^2 - 2r_0 \frac{\alpha}{\beta} + \left( \frac{\alpha}{\beta} \right)^2 \right) e^{-2\beta t} \]
\[ = r_0 \frac{\sigma^2}{\beta} (e^{-\beta t} - e^{-2\beta t}) + \frac{\alpha \sigma^2}{2\beta^2} e^{-\beta t} + \frac{\alpha \sigma^2}{2\beta^2} e^{-2\beta t} \]
\[ = r_0 \frac{\sigma^2}{\beta} (e^{-\beta t} - e^{-2\beta t}) + \frac{\alpha \sigma^2}{2\beta^2} (1 - e^{-\beta t})^2, \quad t \in \mathbb{R}_+. \]

Problem 4.14

a) The Itô formula cannot be applied to the function \( f(x) := (x - K)^+ \) because it is not (twice) differentiable.

b) The function \( x \mapsto f_\varepsilon(x) \) can be plotted as follows with \( K = 1 \).

We note that \( f_\varepsilon \) converges uniformly on \( \mathbb{R} \) to the function \( x \mapsto (x - K)^+ \) as we have
\[ 0 \leq f_\varepsilon(x) - (x - K)^+ \leq \frac{\varepsilon}{4}, \quad x \in \mathbb{R}. \] (A.14)

c) Applying the Itô formula to the function \( f_\varepsilon \) we find
\[ f_\varepsilon(B_T) = f_\varepsilon(B_0) + \int_0^T f_\varepsilon'(B_t) dB_t + \frac{1}{2} \int_0^T f_\varepsilon''(B_t) dt \]
\[ = f_\varepsilon(B_0) + \int_0^T f_\varepsilon'(B_t) dB_t + \frac{1}{4\varepsilon} \int_0^T 1_{(K-\varepsilon,K+\varepsilon)}(B_t) dt, \]
and to conclude it suffices to note that
\[ \ell \left( \{ t \in [0,T] : K - \varepsilon < B_t < K + \varepsilon \} \right) = \int_0^T \mathbb{1}_{(K-\varepsilon,K+\varepsilon)}(B_t)dt. \]

d) The derivative \( f'_\varepsilon(x) \) of \( f_\varepsilon(x) \) is given by
\[
f'_\varepsilon(x) := \begin{cases} 
1 & \text{if } x > K + \varepsilon, \\
\frac{1}{2\varepsilon} (x - K + \varepsilon) & \text{if } K - \varepsilon < x < K + \varepsilon, \\
0 & \text{if } x < K - \varepsilon.
\end{cases}
\]

Hence we have
\[
\| \mathbb{1}_{[K,\infty)}(\cdot) - f'_\varepsilon(\cdot) \|_{L^2(\mathbb{R}_+)}^2 = \int_0^\infty \left( \mathbb{1}_{[K,\infty)}(x) - f'_\varepsilon(x) \right)^2 dx \\
= \int_{K-\varepsilon}^{K+\varepsilon} \left( 1 + |f'_\varepsilon(x)|^2 \right) dx
\]
e) i) We have

\[
\mathbb{E} \left[ \int_0^T (\mathbb{1}_{[K,\infty)}(B_t) - f'_\varepsilon(B_t))^2 dt \right] = \int_0^T \mathbb{E} \left[ (\mathbb{1}_{[K,\infty)}(B_t) - f'_\varepsilon(B_t))^2 \right] dt \\
\leq \int_0^T \int_{-\infty}^\infty (\mathbb{1}_{[K,\infty)}(x) - f'_\varepsilon(x))^2 e^{-x^2/(2t)} dx \frac{1}{\sqrt{2\pi t}} dt \\
\leq \int_0^T \int_{K-\varepsilon}^{K+\varepsilon} (\mathbb{1}_{[K,\infty)}(x) - f'_\varepsilon(x))^2 e^{-(K-\varepsilon)^2/(2t)} dx \frac{1}{\sqrt{2\pi t}} dt \\
\leq \|\mathbb{1}_{[K,\infty)}(\cdot) - f'_\varepsilon(\cdot)\|_2^2 \int_0^T e^{-(K-\varepsilon)^2/(2t)} \frac{1}{\sqrt{2\pi t}} dt \\
\leq \left( 2\varepsilon + \frac{2\varepsilon}{3} \right) \int_0^T e^{-(K-\varepsilon)^2/(2t)} \frac{1}{\sqrt{2\pi t}} dt,
\]

where

\[
\int_0^T e^{-(K-\varepsilon)^2/(2t)} \frac{1}{\sqrt{2\pi t}} dt < \int_0^T e^{-K^2/(8t)} \frac{1}{\sqrt{2\pi t}} dt < \infty,
\]

for \( \varepsilon < K/2 \), hence \( \lim_{\varepsilon \to 0} \mathbb{E} \left[ \int_0^T (\mathbb{1}_{[K,\infty)}(B_t) - f'_\varepsilon(B_t))^2 dt \right] = 0 \), and by the Itô isometry

\[
\mathbb{E} \left[ \left( \int_0^\infty (\mathbb{1}_{[K,\infty)}(B_t) - f_\varepsilon(B_t)) dB_t \right)^2 \right] = \mathbb{E} \left[ \int_0^\infty (\mathbb{1}_{[K,\infty)}(B_t) - f_\varepsilon(B_t))^2 dt \right]
\]

we find that

\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ \left( \int_0^\infty \mathbb{1}_{[K,\infty)}(B_t) dB_t - \int_0^\infty f_\varepsilon(B_t) dB_t \right)^2 \right] = 0,
\]

which shows that \( \int_0^\infty f_\varepsilon(B_t) dB_t \) converges to \( \int_0^\infty \mathbb{1}_{[K,\infty)}(B_t) dB_t \) in \( L^2(\Omega) \) as \( \varepsilon \) tends to zero.

ii) By (A.14) we have

\[
\mathbb{E} \left[ ((B_T - K)^+ - f_\varepsilon(B_T))^2 \right] \leq \frac{\varepsilon}{4},
\]

hence \( f_\varepsilon(B_T) \) converges to \( (B_T - K)^+ \) in \( L^2(\Omega) \).

iii) Similarly, \( f_\varepsilon(B_0) \) converges to \( (B_0 - K)^+ \) for any fixed value of \( B_0 \).
As a consequence of (ei), (eii) and (eiii) above, the equation (4.39) shows that
\[
\frac{1}{2\varepsilon} \ell \left( \{ t \in [0, T] : K - \varepsilon < B_t < K + \varepsilon \} \right)
\]
admits a limit in \( L^2(\Omega) \) as \( \varepsilon \) tends to zero, and this limit is denoted by \( \mathcal{L}^K_{[0,T]} \). The formula (4.40) is known as the Tanaka formula.

Problem 4.15

a) We have
\[
0 \leq \mathbb{E}[(X - \varepsilon)^+]
\]
\[
= \frac{1}{\sqrt{2\pi \sigma^2}} \int_{\varepsilon}^{\infty} (x - \varepsilon) e^{-x^2/(2\sigma^2)} dx
\]
\[
= \frac{1}{\sqrt{2\pi \sigma^2}} \int_{\varepsilon}^{\infty} x e^{-x^2/(2\sigma^2)} dx - \frac{\varepsilon}{\sqrt{2\pi \sigma^2}} \int_{\varepsilon}^{\infty} e^{-x^2/(2\sigma^2)} dx
\]
\[
= -\frac{\sigma^2}{\sqrt{2\pi \sigma^2}} \left[ e^{-x^2/(2\sigma^2)} \right]_{\varepsilon}^{\infty} - \varepsilon \mathbb{P}(X \geq \varepsilon)
\]
\[
= \frac{\sigma^2}{\sqrt{2\pi \sigma^2}} e^{-\varepsilon^2/(2\sigma^2)} - \varepsilon \mathbb{P}(X \geq \varepsilon),
\]
which leads to the conclusion.

b) We have
\[
\mathbb{P}(X \in dx \mid X + Y = z) = \frac{\mathbb{P}(X \in dx \text{ and } X + Y \in dz)}{\mathbb{P}(X + Y \in dz)}
\]
\[
= \frac{\mathbb{P}(X \in dx \text{ and } Y \in (dz) - x)}{\mathbb{P}(X + Y \in dz)}
\]
\[
= \frac{\sqrt{2\pi}(\alpha^2 + \beta^2)}{2\pi \alpha \beta} e^{-x^2/(2\alpha^2) - (z-x)^2/(2\beta^2)} e^{-z^2/(2(\alpha^2 + \beta^2))} dx
\]
\[
= \frac{\sqrt{1/\beta^2 + 1/\alpha^2}}{\sqrt{2\pi}} e^{-(x^2(1/\beta^2/\alpha^2) + (x^2+z^2-2xz)(1+\alpha^2/\beta^2)-z^2)/(2(\alpha^2 + \beta^2))} dx
\]
\[
= \frac{\sqrt{1/\beta^2 + 1/\alpha^2}}{\sqrt{2\pi}} e^{-(x^2(2/\beta^2/\alpha^2 + 2\alpha^2/\beta^2) + z^2\alpha^2/\beta^2 - 2xz(1+\alpha^2/\beta^2))/(2(\alpha^2 + \beta^2))} dx
\]
\[
= \frac{\sqrt{1/\beta^2 + 1/\alpha^2}}{\sqrt{2\pi}} e^{-(x(\beta/\alpha + \alpha/\beta) - z/\beta)^2/(2(\alpha^2 + \beta^2))} dx
\]
\[
= \frac{\sqrt{1/\beta^2 + 1/\alpha^2}}{\sqrt{2\pi}} e^{-(x((\alpha^2/\beta^2)/(\alpha \beta)) - z/\beta)^2/(2(\alpha^2 + \beta^2))} dx
\]
\[
= \frac{\sqrt{1/\beta^2 + 1/\alpha^2}}{\sqrt{2\pi}} e^{-(x-z\alpha^2/(\alpha^2 + \beta^2)^2)/(2/(1/\alpha^2 + 1/\beta^2))} dx.
\]
c) Given that \( B_u = x \) we decompose
\[
B_v = (B_v - B_{(u+v)/2}) + (B_{(u+v)/2} - B_u) + x,
\]
and apply the result of Question (b) by taking
\[
X = B_{(u+v)/2} - B_u \quad \text{and} \quad Y = B_v - B_{(u+v)/2},
\]
i.e.
\[
\alpha^2 = \beta^2 = \frac{v-u}{2} \quad \text{and} \quad z = y - x,
\]
which shows that the distribution of \( B_{(u+v)/2} = x + X \) given that \( B_u = x \) and \( B_v = y \) is Gaussian \( \mathcal{N}\left(\frac{x+y}{2}, \frac{v-u}{4}\right) \) with mean
\[
x + \frac{\alpha^2 z}{\alpha^2 + \beta^2} = x + \frac{y-x}{2} = \frac{x+y}{2}
\]
and variance \( \frac{\alpha^2 \beta^2}{\alpha^2 + \beta^2} = \frac{v-u}{4} \).

d) Four linear interpolations are displayed in Figure S.13.

![Fig. S.13: Samples of linear interpolations.](image)

e) Clearly, the statement is true for \( n = 0 \) because \( Z_1^{(0)} \) and \( B_1 \) have the same \( \mathcal{N}(0,1) \) distribution. Next, assuming that it holds at the rank \( n \), we note that the terms appearing in the sequence
\[
Z^{(n+1)} = (0, Z_1^{(n+1)}, Z_2^{(n+1)}, \ldots, Z_{2^n+1}^{(n+1)}),
\]
can be written for any \( k = 0, 1, \ldots, 2^n - 1 \) as
\[
\left(\ldots, Z_{2k/2^n+1}^{(n+1)} + \frac{Z_{2k+2/2^n+1}^{(n+1)}}{2} + \mathcal{N}(0, 1/2^{n+2}), \ldots, Z_{2k+2/2^n+1}^{(n+1)} + \mathcal{N}(0, 1/2^{n+2}), \ldots\right)
\]

\[
= \left(\ldots, Z_{k/2^n}^{(n)} + \frac{Z_{k+2/2^n}^{(n)}}{2} + \mathcal{N}(0, 1/2^{n+2}), \ldots, Z_{(k+1)/2^n}^{(n)} + \mathcal{N}(0, 1/2^{n+2}), \ldots\right)
\]
\[
\left( \ldots, Z_{k/2^n}^{(n)}, N\left( \frac{Z_{2k/2^n+1}^{(n)} + Z_{(2k+2)/2^n+1}^{(n)}}{2}, \frac{1}{2^n+2} \right), Z_{(k+1)/2^n}^{(n)}, \ldots \right).
\]

(A.15)

On the other hand, the result of Question (c) shows that given that \(B_{2k/2^n+1} = x\) and \(B_{(2k+2)/2^n+1} = y\), the distribution of \(B_{(2k+1)/2^n+1}\) is

\[
\mathcal{N}\left( \frac{B_{2k/2^n+1} + B_{(2k+2)/2^n+1}}{2} + \frac{(2k + 2)/2^n+1 - (2k + 2)/2^n+1}{4} \right)
\]

\[
= \mathcal{N}\left( \frac{B_{2k/2^n+1} + B_{(2k+2)/2^n+1}}{2}, \frac{1}{2^n+2} \right).
\]

(A.16)

Given that \(Z^{(n)}\) and \(B^{(n)}\) have same distribution, we conclude by comparing (A.15) and (A.16) that \(Z^{(n+1)}\) and \(B^{(n+1)}\) also have same distribution.

f) We have

\[
\mathbb{P} \left( \sup_{t \in [0,1]} |Z_t^{(n+1)} - Z_t^{(n)}| \geq \varepsilon_n \right)
\]

\[
= \mathbb{P} \left( \max_{k=0,1,\ldots,2^{n-1}} |Z_{(2k+1)/2^n+1}^{(n+1)} - Z_{(2k+1)/2^n+1}^{(n)}| \geq \varepsilon_n \right)
\]

\[
\leq \mathbb{P} \left( \bigcup_{k=0,1,\ldots,2^{n-1}} \{ |Z_{(2k+1)/2^n+1}^{(n+1)} - Z_{(2k+1)/2^n+1}^{(n)}| \geq \varepsilon_n \} \right)
\]

\[
\leq \sum_{k=0}^{2^{n-1}} \mathbb{P} (|Z_{(2k+1)/2^n+1}^{(n+1)} - Z_{(2k+1)/2^n+1}^{(n)}| \geq \varepsilon_n)
\]

\[
= 2^n \mathbb{P} (|Z_{1/2^n+1}^{(n+1)} - Z_{1/2^n+1}^{(n)}| \geq \varepsilon_n)
\]

\[
= 2^n \mathbb{P} \left( \left| Z_{1/2^n+1}^{(n+1)} - \frac{Z_{0/2^n}^{(n)} + Z_{1/2^n}^{(n)}}{2} \right| \geq \varepsilon_n \right) .
\]

g) Since

\[
Z_{1/2^n+1}^{(n+1)} = \frac{Z_{0/2^n}^{(n)} + Z_{1/2^n}^{(n)}}{2} + \mathcal{N}(0, 1/2^{n+2}) = Z_{1/2^n+1}^{(n)} + \mathcal{N}(0, 1/2^{n+2}),
\]

we have

\[
\mathbb{P} \left( \sup_{t \in [0,1]} |Z_t^{(n+1)} - Z_t^{(n)}| \geq \varepsilon_n \right) \leq 2^n \mathbb{P} (|Z_{1/2^n+1}^{(n+1)} - Z_{1/2^n+1}^{(n)}| \geq \varepsilon_n)
\]
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\[ \begin{aligned}
= 2^n \mathbb{P} \left( \left| Z_{1/2^{n+1}}^{(n+1)} - \frac{Z_{0/2^n}^{(n)} + Z_{1/2^n}^{(n)}}{2} \right| \geq \varepsilon_n \right) \\
= 2^n \mathbb{P} \left( |\mathcal{N}(0, 1/2^{n+2})| \geq \varepsilon_n \right) \\
\leq \frac{2^n}{\varepsilon_n \sqrt{2\pi}} e^{-\varepsilon_n^2 2^{n+1}},
\end{aligned} \]

where we applied the bound of Question (a) to the Gaussian random variable

\[ Z_{1/2^{n+1}}^{(n+1)} - \frac{Z_{0/2^n}^{(n)} + Z_{1/2^n}^{(n)}}{2} \sim \mathcal{N}(0, 1/2^{n+2}). \]

h) We have

\[ \sum_{n \geq 0} \mathbb{P} \left( \| Z^{(n+1)} - Z^{(n)} \|_\infty \geq 2^{-n/4} \right) = \sum_{n \geq 0} \mathbb{P} \left( \sup_{t \in [0,1]} |Z_{t}^{(n+1)} - Z_{t}^{(n)}| \geq \varepsilon_n \right) \]

\[ \leq \sum_{n \geq 0} \frac{2^n}{\varepsilon_n \sqrt{2\pi}} e^{-\varepsilon_n^2 2^{n+1}} \]

\[ = \frac{1}{\sqrt{2\pi}} \sum_{n \geq 0} 2^{3n/4} e^{-1+n/2} < \infty, \]

since

\[ \lim_{n \to \infty} \frac{2^{3(n+1)/4} e^{-2^{1+n+1}/2}}{2^{3n/4} e^{-2^{1+n}/2}} = 2^{3/4} \lim_{n \to \infty} \frac{e^{-2^{1+n}/2(\sqrt{2}-1)}}{e^{-2^{1+n}/2}} = 0. \]

Hence the Borel-Cantelli lemma shows that

\[ \mathbb{P} \left( \| Z^{(n+1)} - Z^{(n)} \|_\infty \geq 2^{-n/4} \text{ for infinitely many } n \right) = 0, \]

therefore we have

\[ \mathbb{P} \left( \| Z^{(n+1)} - Z^{(n)} \|_\infty < 2^{-n/4} \text{ except for finitely many } n \right) = 1. \]

i) The result of Question (h) shows that with probability one we have

\[ \lim_{p,q \to \infty} \| Z^{(p)} - Z^{(q)} \|_\infty = \lim_{p,q \to \infty} \left\| \sum_{n=q}^{p-1} Z^{(n+1)} - Z^{(n)} \right\|_\infty \]

\[ \leq \lim_{p,q \to \infty} \sum_{n=q}^{p-1} \| Z^{(n+1)} - Z^{(n)} \|_\infty \]

\[ \leq \lim_{p \to \infty} \sum_{n>q} \| Z^{(n+1)} - Z^{(n)} \|_\infty \]
hence the sequence \((Z^{(n)})_{n \geq 0}\) is Cauchy in \(C_0([0,1])\) for the \(\| \cdot \|_{\infty}\) norm. Since \(C_0([0,1])\) is a complete space for the \(\| \cdot \|_{\infty}\) norm, this implies that, with probability one, the sequence \((Z^{(n)}_{k})_{n \geq 0}\) admits a limit in \(C_0([0,1])\).

j) 1. By construction we have \(Z^{(n)}_0 = 0\) for all \(n \in \mathbb{N}\), hence \(Z_0 = \lim_{n \to \infty} Z^{(n)}_0 = 0\), almost surely.

2. The sample trajectories \(t \mapsto Z_t\) are continuous, because the limit \(Z\) belongs to \(C_0([0,1])\) with probability 1.

3. The result of Question (e) shows that for any fixed \(m \geq 1\), the sequences

\[
Z_{t_1} - Z_{t_0}, Z_{t_2} - Z_{t_1}, \ldots, Z_{t_m} - Z_{t_{m-1}}
\]

and

\[
B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}}
\]

have same distribution when the \(t_k\)'s are dyadic rationals of the form \(t_k = i_n/2^n\), \(k = 0, 1, \ldots, n\). This property extends to any sequence \(t_0, t_1, \ldots, t_m\) of real numbers by approximation of each \(t_k > 0\) by a sequence \((i_n)_{n \in \mathbb{N}}\) such that \(t_k = \lim_{n \to \infty} i_n/2^n\) and taking the limit as \(n\) tends to infinity.

4. By a similar argument as in the above point 3, one can show that for any \(0 \leq s < t\), \(Z_t - Z_s\) has the Gaussian distribution \(\mathcal{N}(0, t - s)\).

Problem 4.16

a) We have

\[
\mathbb{E} \left[ Q^{(n)}_T \right] = \sum_{k=1}^{n} \mathbb{E} \left[ (B_{kT/n} - B_{(k-1)T/n})^2 \right]
\]

\[
= \sum_{k=1}^{n} \left( \frac{T}{k} - \frac{T}{n} \right)
\]

\[
= T, \quad n \geq 1.
\]

b) We have

\[
\mathbb{E} \left[ (Q^{(n)}_T)^2 \right] = \mathbb{E} \left[ \left( \sum_{k=1}^{n} (B_{kT/n} - B_{(k-1)T/n})^2 \right)^2 \right]
\]

\[
= \mathbb{E} \left[ \sum_{k,l=1}^{n} (B_{kT/n} - B_{(k-1)T/n})^2 (B_{lT/n} - B_{(l-1)T/n})^2 \right]
\]

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\[= \sum_{k=1}^{n} \mathbb{E} \left[ (B_{kT/n} - B_{(k-1)T/n})^4 \right] + 2 \sum_{1 \leq k < l \leq n} \mathbb{E} \left[ (B_{kT/n} - B_{(k-1)T/n})^2 \right] \mathbb{E} \left[ (B_{lT/n} - B_{(l-1)T/n})^2 \right] \]

\[= 3 \sum_{k=1}^{n} \frac{(kT/n - (k-1)T/n)^2}{n} + 2 \sum_{1 \leq k < l \leq n} \frac{(kT/n - (k-1)T/n)(lT/n - (l-1)T/n)}{n^2} \]

\[= \frac{T^2}{n} + \frac{n(n-1)T^2}{n^2} = T^2 + \frac{2T^2}{n}, \quad n \geq 1, \]

hence

\[\text{Var} \left[ Q_T^{(n)} \right] = \mathbb{E} \left[ \left( Q_T^{(n)} \right)^2 \right] - \left( \mathbb{E} \left[ Q_T^{(n)} \right] \right)^2 = \frac{2T^2}{n}, \quad n \geq 1.\]

c) We have

\[\|Q_T^{(n)} - T\|_{L^2(\Omega)}^2 = \mathbb{E} \left[ (Q_T^{(n)} - \mathbb{E}[Q_T^{(n)}])^2 \right] = \text{Var} \left[ Q_T^{(n)} \right] = \frac{n(n+2)T^2}{n^2} - T^2 = \frac{2T^2}{n},\]

hence

\[\lim_{n \to \infty} \|Q_T^{(n)} - T\|_{L^2(\Omega)}^2 = \lim_{n \to \infty} \frac{2T^2}{n} = 0,\]

showing that

\[\lim_{n \to \infty} Q_T^{(n)} = T\]

in \(L^2(\Omega)\).

d) We have

\[\sum_{k=1}^{n} \left( B_{kT/n} - B_{(k-1)T/n} \right)B_{(k-1)T/n} = \frac{1}{2} \sum_{k=1}^{n} B_{kT/n}^2 - B_{(k-1)T/n}^2 - \frac{1}{2} \sum_{k=1}^{n} (B_{kT/n} - B_{(k-1)T/n})(B_{kT/n} - B_{(k-1)T/n})\]
\[
\begin{align*}
&= \frac{1}{2}((B_T)^2 - (B_0)^2) \\
&- \frac{1}{2} \sum_{k=1}^{n} (B_{kT/n} - B_{(k-1)T/n})(B_{kT/n} - B_{(k-1)T/n}) \\
&= \frac{1}{2}((B_T)^2 - Q_T^{(n)}),
\end{align*}
\]

which converges to \(((B_T)^2 - T)/2\) in \(L^2(\Omega)\) as \(n\) tends to infinity, hence

\[
\int_0^T B_t dB_t = \lim_{n \to \infty} \sum_{k=1}^{n} (B_{kT/n} - B_{(k-1)T/n})B_{(k-1)T/n} = \frac{(B_T)^2 - T}{2}.
\]

e) We have

\[
\mathbb{E} \left[ \tilde{Q}_T^{(n)} \right] = \sum_{k=1}^{n} \mathbb{E} \left[ (B_{(k-1/2)T/n} - B_{(k-1)T/n})^2 \right]
\]

\[
= \sum_{k=1}^{n} ((k - 1/2)T/n - (k - 1)T/n)
\]

\[
= \frac{T}{2}, \quad n \geq 1.
\]

Next, we have

\[
\mathbb{E} \left[ (\tilde{Q}_T^{(n)})^2 \right] = \mathbb{E} \left[ \left( \sum_{k=1}^{n} (B_{(k-1/2)T/n} - B_{(k-1)T/n})^2 \right)^2 \right]
\]

\[
= \mathbb{E} \left[ \sum_{k,l=1}^{n} (B_{(k-1/2)T/n} - B_{(k-1)T/n})^2(B_{lT/n} - B_{(l-1)T/n})^2 \right]
\]

\[
= \sum_{k=1}^{n} \mathbb{E} \left[ (B_{(k-1/2)T/n} - B_{(k-1)T/n})^4 \right]
\]

\[
+ 2 \sum_{1 \leq k < l \leq n} \mathbb{E} \left[ (B_{(k-1/2)T/n} - B_{(k-1)T/n})^2 \right] \mathbb{E} \left[ (B_{(l-1/2)T/n} - B_{(l-1)T/n})^2 \right]
\]

\[
= 3 \sum_{k=1}^{n} ((k - 1/2)T/n - (k - 1)T/n)^2
\]

\[
+ 2 \sum_{1 \leq k < l \leq n} ((k - 1/2)T/n - (k - 1)T/n)((l - 1/2)T/n - (l - 1)T/n)
\]

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\[= 3 \frac{T^2}{4n} + \frac{n(n-1)T^2}{4n^2}\]
\[= \frac{n(n+2)T^2}{4n^2}, \quad n \geq 1.\]

Finally, we find
\[
\|\tilde{Q}_T^{(n)} - T/2\|^2_{L^2(\Omega)} = \mathbb{E} [(\tilde{Q}_T^{(n)} - \mathbb{E}[\tilde{Q}_T^{(n)}])^2]
\]
\[= \text{Var} [\tilde{Q}_T^{(n)}]
\]
\[= \frac{n(n+2)T^2}{4n^2} - \frac{T^2}{4}
\]
\[= \frac{T^2}{2n},\]

hence
\[
\lim_{n \to \infty} \|\tilde{Q}_T^{(n)} - T/2\|^2_{L^2(\Omega)} = \lim_{n \to \infty} \frac{T^2}{2n} = 0,
\]

showing that
\[
\lim_{n \to \infty} \tilde{Q}_T^{(n)} = \frac{T}{2}
\]
in \(L^2(\Omega)\).

f) We have
\[
\sum_{k=1}^{n} \left( B_{kT/n} - B_{(k-1)T/n} \right) B_{(k-1/2)T/n}
\]
\[= \sum_{k=1}^{n} \left( B_{kT/n} - B_{(k-1/2)T/n} \right) B_{(k-1/2)T/n}
\]
\[+ \sum_{k=1}^{n} \left( B_{(k-1/2)T/n} - B_{(k-1)T/n} \right) B_{(k-1/2)T/n}
\]
\[= \frac{1}{2} \sum_{k=1}^{n} B_{kT/n}^2 - B_{(k-1/2)T/n}^2
\]
\[- \frac{1}{2} \sum_{k=1}^{n} \left( B_{kT/n} - B_{(k-1/2)T/n} \right) \left( B_{kT/n} - B_{(k-1/2)T/n} \right)
\]
\[+ \frac{1}{2} \sum_{k=1}^{n} B_{(k-1/2)T/n}^2 - B_{(k-1)T/n}^2
\]
\[+ \frac{1}{2} \sum_{k=1}^{n} \left( B_{(k-1/2)T/n} - B_{(k-1)T/n} \right) \left( B_{(k-1/2)T/n} - B_{(k-1)T/n} \right)
\]
\[\frac{1}{2}(B_T)^2 - \frac{1}{2} \sum_{k=1}^{n} (B_{kT/n} - B_{(k-1/2)T/n})(B_{kT/n} - B_{(k-1/2)T/n}) + \frac{1}{2} \sum_{k=1}^{n} (B_{(k-1/2)T/n} - B_{(k-1)T/n})(B_{(k-1/2)T/n} - B_{(k-1)T/n}),\]

which converges to \(((B_T)^2 - T + T)/2 = (B_T)^2/2\) in \(L^2(\Omega)\) as \(n\) tends to infinity, hence

\[\int_{0}^{T} B_t \circ dB_t = \lim_{n \to \infty} \sum_{k=1}^{n} (B_{kT/n} - B_{(k-1)T/n})B_{(k-1)T/n} = \frac{(B_T)^2}{2},\]

see Section 2.4 of Mikosch (1998) for further details on the Stratonovich integral.

\(g)\) We have

\[\mathbb{E}[(\tilde{Q}_T^{(n)})^2] = \sum_{k=1}^{n} \mathbb{E}[(B_{(k-\alpha)T/n} - B_{(k-1)T/n})^2] = \sum_{k=1}^{n} ((k - \alpha)T/n - (k - 1)T/n) = (1 - \alpha)\frac{T}{2}, \quad n \geq 1.\]

Next, we have

\[\mathbb{E}[(\tilde{Q}_T^{(n)})^2] = \mathbb{E}\left[\left(\sum_{k=1}^{n} (B_{(k-\alpha)T/n} - B_{(k-1)T/n})^2\right)^2\right]\]

\[= \mathbb{E} \left[\sum_{k,l=1}^{n} (B_{(k-\alpha)T/n} - B_{(k-1)T/n})^2(B_{lT/n} - B_{(l-1)T/n})^2\right]\]

\[= \sum_{k=1}^{n} \mathbb{E}[(B_{(k-\alpha)T/n} - B_{(k-1)T/n})^4]\]

\[+ 2 \sum_{1 \leq k < l \leq n} \mathbb{E}[(B_{(k-\alpha)T/n} - B_{(k-1)T/n})^2] \mathbb{E}[(B_{(l-\alpha)T/n} - B_{(l-1)T/n})^2]\]

\[= 3 \sum_{k=1}^{n} ((k - \alpha)T/n - (k - 1)T/n)^2\]

\[+ 2 \sum_{1 \leq k < l \leq n} ((k - \alpha)T/n - (k - 1)T/n)((l - \alpha)T/n - (l - 1)T/n)\]

\[= 3(1 - \alpha)^2\frac{T^2}{n} + (1 - \alpha)^2\frac{n(n - 1)T^2}{n^2}\]
\[ = (1 - \alpha)^2 \frac{n(n + 2)T^2}{n^2}, \quad n \geq 1. \]

Finally we find
\[
\|\tilde{Q}_T^{(n)} - (1 - \alpha)T/2\|_{L^2(\Omega)}^2 = \mathbb{E} \left[ (\tilde{Q}_T^{(n)} - \mathbb{E} [\tilde{Q}_T^{(n)}])^2 \right] \\
= \text{Var} [\tilde{Q}_T^{(n)}] \\
= (1 - \alpha)^2 \frac{n(n + 2)T^2}{n^2} - (1 - \alpha)^2T^2 \\
= 2(1 - \alpha)^2 \frac{T^2}{n},
\]

hence
\[
\lim_{n \to \infty} \|\tilde{Q}_T^{(n)} - (1 - \alpha)T\|_{L^2(\Omega)}^2 = (1 - \alpha)^2 \lim_{n \to \infty} \frac{T^2}{n} = 0.
\]

Next, we have
\[
\sum_{k=1}^{n} (B_{kT}/n - B_{(k-1)T}/n)B_{(k-\alpha)T}/n \\
= \sum_{k=1}^{n} (B_{kT}/n - B_{(k-\alpha)T}/n)B_{(k-\alpha)T}/n + \sum_{k=1}^{n} (B_{(k-\alpha)T}/n - B_{(k-1)T}/n)B_{(k-\alpha)T}/n \\
= \frac{1}{2} \sum_{k=1}^{n} B_{kT}^2/n - B_{(k-\alpha)T}^2/n - \frac{1}{2} \sum_{k=1}^{n} (B_{kT}/n - B_{(k-\alpha)T}/n)(B_{kT}/n - B_{(k-\alpha)T}/n) \\
+ \frac{1}{2} \sum_{k=1}^{n} B_{(k-\alpha)T}/n - B_{(k-1)T}/n \\
+ \frac{1}{2} \sum_{k=1}^{n} (B_{(k-\alpha)T}/n - B_{(k-1)T}/n)(B_{(k-\alpha)T}/n - B_{(k-1)T}/n) \\
= \frac{1}{2} (B_T)^2 - \frac{1}{2} \sum_{k=1}^{n} (B_{kT}/n - B_{(k-\alpha)T}/n)(B_{kT}/n - B_{(k-\alpha)T}/n) \\
+ \frac{1}{2} \sum_{k=1}^{n} (B_{(k-\alpha)T}/n - B_{(k-1)T}/n)(B_{(k-\alpha)T}/n - B_{(k-1)T}/n),
\]

which converges to
\[
\frac{(B_T)^2 - \alpha T + (1 - \alpha)T}{2} = \frac{(B_T)^2 + (1 - 2\alpha)T}{2}
\]

in \(L^2(\Omega)\) as \(n\) tends to infinity, hence
\[
\int_0^T B_t \circ d^\alpha B_t = \lim_{n \to \infty} \sum_{k=1}^{n} (B_{kT/n} - B_{(k-1)T/n}) B_{(k-\alpha)T/n} \times \frac{(B_T)^2}{2} + (1 - 2\alpha)T.
\]

In particular we find
\[
\int_0^T B_t \circ d^0 B_t = \lim_{n \to \infty} \sum_{k=1}^{n} (B_{kT/n} - B_{(k-1)T/n}) B_{kT/n} = \frac{(B_T)^2 + T}{2},
\]
and we note that
\[
\int_0^T B_t \circ dB_t = \frac{1}{2} \left( \int_0^T B_t dB_t + \int_0^T B_t \circ d^1 B_t \right).
\]

h) We have
\[
\lim_{n \to \infty} \sum_{k=1}^{n} (k - \alpha) \frac{T}{n} \left( \frac{kT}{n} - (k - 1) \frac{T}{n} \right) = \lim_{n \to \infty} \frac{T}{n} \sum_{k=1}^{n} (k - \alpha) \frac{T}{n} = \lim_{n \to \infty} \frac{T}{n} \sum_{k=1}^{n} \frac{T}{n} - \alpha \lim_{n \to \infty} \frac{T}{n} \sum_{k=1}^{n} \frac{T}{n} = T^2 \lim_{n \to \infty} \frac{n(n+1)}{2n^2} - \alpha \lim_{n \to \infty} \frac{T^2}{n} = \frac{T^2}{2},
\]
which does not depend on \( \alpha \in [0, 1] \) hence the stochastic phenomenon of the previous questions does not occur when approximating the deterministic integral \( \int_0^T t dt = T^2/2 \) by Riemann sums.

In mathematical finance we choose to use the Itô integral (which corresponds to the choice \( \alpha = 1 \)) because it is suitable for the modeling of market returns as
\[
\frac{dS_t}{S_t} \simeq S_{t+\Delta t} - S_t = \mu \Delta t + \sigma \Delta B_t = \mu \Delta t + (B_{t+\Delta t} - B_t) \sigma
\]
or
\[
dS_t \simeq S_{t+\Delta t} - S_t = \mu S_t \Delta t + \sigma S_t \Delta B_t = \mu S_t \Delta t + \sigma S_t (B_{t+\Delta t} - B_t),
\]
based on the value $S_t$ at the left endpoint of the discretized time interval $[t, t + \Delta t]$.

**Chapter 5**

Exercise 5.1 For all $x \in \mathbb{R}$ we have

$$
P(S_T \leq x) = P(S_0 e^{\sigma B_T + (\mu - \sigma^2/2)T} \leq x)
$$

$$
= P\left(\sigma B_T + \left(\mu - \frac{\sigma^2}{2}\right) T \leq \log \frac{x}{S_0}\right)
$$

$$
= P\left(B_T \leq \frac{1}{\sigma} \left(\log \frac{x}{S_0} - \left(\mu - \frac{\sigma^2}{2}\right) T\right)\right)
$$

$$
= \int_{-\infty}^{\log(x/S_0) - (\mu - \sigma^2/2)T/\sigma} e^{-y^2/(2T)} dy \sqrt{2\pi T}.
$$

After differentiation with respect to $x$ we find the probability density function

$$
f(x) = \frac{dP(S_T \leq x)}{dx}
$$

$$
= \frac{\partial}{\partial x} \int_{-\infty}^{\log(x/S_0) - (\mu - \sigma^2/2)T/\sigma} e^{-y^2/(2T)} dy \sqrt{2\pi T}
$$

$$
= \frac{1}{x\sigma \sqrt{2\pi T}} e^{-\left(-\left(\mu - \sigma^2/2\right) T + \log(x/S_0)\right)^2/(2\sigma^2 T)}, \quad x > 0.
$$

Exercise 5.2 Taking expectations on both sides of (5.23) shows that

$$
\mathbb{E}[S_T] = C(S_0, r, T) + \mathbb{E} \left[\int_0^T \zeta_{t,T} dB_t\right] = C(S_0, r, T),
$$

hence

$$
C(S_0, r, T) = \mathbb{E}[S_T]
$$

$$
= \mathbb{E}[S_0 e^{\mu T + \sigma B_T - \sigma^2 T/2}]
$$

$$
= S_0 e^{\mu T - \sigma^2 T/2} \mathbb{E}[e^{\sigma B_T}]
$$

$$
= S_0 e^{\mu T - \sigma^2 T/2 + \sigma^2 T/2}
$$

$$
= S_0 e^{\mu T},
$$

where we used the moment generating function

$$
\mathbb{E}[e^{\sigma B_T}] = e^{\sigma^2 T/2}
$$

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https://www.ntu.edu.sg/home/nprivault/index.html
of the Gaussian random variable $B_T \sim N(0, T)$. On the other hand, the
discounted asset price $X_t := e^{-rt}S_t$ satisfies $dX_t = \sigma X_t dB_t$, which shows that

$$X_T = X_0 + \sigma \int_0^T X_t dB_t.$$ 

Multiplying both sides by $e^{rT}$ shows that

$$S_T = e^{rT}S_0 + \sigma \int_0^T e^{rT}X_t dB_t = e^{rT}S_0 + \sigma \int_0^T e^{r(T-t)}S_t dB_t,$$

which recovers the relation $C(S_0, r, T) = S_0 e^{rT}$, and shows that $\zeta_{t,T} = \sigma e^{r(T-t)}S_t$, $t \in [0, T]$.

Exercise 5.3
a) We have $S_t = f(X_t)$, $t \in \mathbb{R}_+$, where $f(x) = S_0 e^x$ and $(X_t)_{t \in \mathbb{R}_+}$ is the
Itô process given by

$$X_t := \int_0^t \sigma_s dB_s + \int_0^t u_s ds, \quad t \in \mathbb{R}_+,$$

or in differential form

$$dX_t := \sigma_t dB_t + u_t dt, \quad t \in \mathbb{R}_+,$$

hence

$$dS_t = df(X_t)$$

$$= f'(X_t) dX_t + \frac{1}{2} f''(X_t) (dX_t)^2$$

$$= u_t f'(X_t) dt + \sigma_t f'(X_t) dB_t + \frac{1}{2} \sigma_t^2 f''(X_t) dt$$

$$= S_0 u_t e^{X_t} dt + S_0 \sigma_t e^{X_t} dB_t + \frac{1}{2} S_0 \sigma_t^2 e^{X_t} dt$$

$$= u_t S_t dt + \sigma_t S_t dB_t + \frac{1}{2} \sigma_t^2 S_t dt.$$ 

b) The process $(S_t)_{t \in \mathbb{R}_+}$ satisfies the stochastic differential equation

$$dS_t = \left( u_t + \frac{1}{2} \sigma_t^2 \right) S_t dt + \sigma_t S_t dB_t.$$ 

Exercise 5.4
a) We have $\mathbb{E}[S_t] = 1$ because the expected value of the Itô stochastic integral is zero. Regarding the variance, using the Itô isometry (4.7) we have
Notes on Stochastic Finance

\[
\text{Var}[S_t] = \sigma^2 \mathbb{E} \left[ \left( \int_0^t e^{\sigma B_s - \sigma^2 s/2} dB_s \right)^2 \right] \\
= \sigma^2 \mathbb{E} \left[ \int_0^t \left( e^{\sigma B_s - \sigma^2 s/2} \right)^2 ds \right] \\
= \sigma^2 \int_0^t \mathbb{E} \left[ e^{2\sigma B_s - \sigma^2 s} \right] ds \\
= \sigma^2 \int_0^t e^{-\sigma^2 s} \mathbb{E} \left[ e^{2\sigma B_s} \right] ds \\
= \sigma^2 \int_0^t e^{\sigma^2 s} ds \\
= \sigma^2 \int_0^t e^{\sigma^2 s} ds \\
= e^{\sigma^2 t} - 1.
\]

b) Taking \( f(x) = \log x \), we have

\[
d\log(S_t) = df(S_t) \\
= \sigma f'(S_t)dS_t + \frac{1}{2} \sigma^2 f''(S_t)(dS_t)^2 \\
= \sigma f'(S_t)e^{\sigma B_t - \sigma^2 t/2}dB_t + \frac{1}{2} \sigma^2 f''(S_t)e^{2\sigma B_t - \sigma^2 t}dt \\
= \frac{\sigma}{S_t}e^{\sigma B_t - \sigma^2 t/2}dB_t - \frac{\sigma^2}{2S_t^2}e^{2\sigma B_t - \sigma^2 t}dt. \tag{A.17}
\]

c) We check that when \( S_t = e^{\sigma B_t - \sigma^2 t/2}, \ t \in \mathbb{R}_+ \), we have

\[
\log S_t = \sigma B_t - \sigma^2 t/2, \quad \text{and} \quad d\log S_t = \sigma dB_t - \frac{\sigma^2}{2} dt.
\]

On the other hand, we also find

\[
\sigma dB_t - \frac{\sigma^2}{2} dt = \frac{\sigma}{S_t}e^{\sigma B_t - \sigma^2 t/2}dB_t - \frac{\sigma^2}{2S_t^2}e^{2\sigma B_t - \sigma^2 t}dt,
\]

showing by (A.17) that the equation

\[
d\log S_t = \frac{\sigma}{S_t}e^{\sigma B_t - \sigma^2 t/2}dB_t - \frac{\sigma^2}{2S_t^2}e^{2\sigma B_t - \sigma^2 t}dt
\]

is satisfied. By uniqueness of solutions, we conclude that \( S_t := e^{\sigma B_t - \sigma^2 t/2} \)

solves

\[
S_t = 1 + \sigma \int_0^t e^{\sigma B_s - \sigma^2 s/2} dB_s, \quad t \in \mathbb{R}_+.
\]
Exercise 5.5

a) We have \( f(t) = f(0) e^{ct} \) (interest rate compounding) and

\[
S_t = S_0 e^{\sigma B_t - \sigma^2 t/2 + rt}, \quad t \in \mathbb{R}_+,
\]

(geometric Brownian motion).

b) Those quantities can be directly computed from the expression of \( S_t \) as a function of the \( N(0, t) \) random variable \( B_t \). Alternatively, taking expectations in the stochastic differential equations

\[
dS_t = rS_t dt + \sigma S_t dB_t
\]

that \( u(t) := \mathbb{E}[S_t] \) satisfies the ordinary differential equation \( u'(t) = ru(t) \) with \( u(0) = S_0 \) and solution \( u(t) = \mathbb{E}[S_t] = S_0 e^{rt} \). On the other hand, taking expectations on both sides of

\[
dS_t^2 = 2S_t dS_t + (dS_t)^2 = 2rS_t^2 dt + \sigma^2 S_t^2 dt + 2\sigma S_t dB_t,
\]

or

\[
S_t^2 = S_0^2 + 2r \int_0^t S_u^2 du + \sigma^2 \int_0^t S_u^2 du + 2\sigma \int_0^t S_u dB_u,
\]

we find

\[
v(t) = \mathbb{E} [S_t^2]
\]

\[
= S_0^2 + (2r + \sigma^2) \mathbb{E} \left[ \int_0^t S_u^2 du \right] + 2\sigma \mathbb{E} \left[ \int_0^t S_u dB_u \right]
\]

\[
= S_0^2 + (2r + \sigma^2) \int_0^t \mathbb{E} [S_u^2] du
\]

\[
= S_0^2 + (2r + \sigma^2) \int_0^t v(u) du,
\]

hence \( v(t) := \mathbb{E} [S_t^2] \) satisfies the ordinary differential equation

\[
v'(t) = (\sigma^2 + 2r)v(t),
\]

with \( v(0) = S_0^2 \) and solution

\[
v(t) = \mathbb{E} [S_t^2] = S_0^2 e^{(\sigma^2 + 2r)t},
\]

hence

\[
\text{Var}[S_t] = \mathbb{E} [S_t^2] - (\mathbb{E}[S_t])^2
\]

\[
= v(t) - u^2(t)
\]

\[
= S_0^2 e^{(\sigma^2 + 2r)t} - S_0^2 e^{2rt}
\]

\[
= S_0^2 e^{2rt} (e^{\sigma^2 t} - 1), \quad t \in \mathbb{R}_+.
\]

c) We have

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\[ d \log S_t = \frac{1}{S_t} dS_t - \frac{1}{2 S_t^2} (dS_t)^2 = r dt + \sigma dB_t - \frac{\sigma^2}{2} dt, \quad t \in \mathbb{R}_+. \]

d) Using the Itô formula \((4.25)\) in two variables we find

\[
df(S_t, Y_t) = \frac{\partial f}{\partial x}(S_t, Y_t) dS_t + \frac{\partial f}{\partial y}(S_t, Y_t) dY_t \\
+ \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(S_t, Y_t) (dS_t)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial y^2}(S_t, Y_t) (dY_t)^2 \\
+ \frac{\partial^2 f}{\partial x \partial y}(S_t, Y_t) dS_t \cdot dY_t \\
= \frac{\partial f}{\partial x}(S_t, Y_t) (r S_t dt + \sigma S_t dB_t) + \frac{\partial f}{\partial y}(S_t, Y_t) (\mu Y_t dt + \eta Y_t dW_t) \\
+ \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 f}{\partial x^2}(S_t, Y_t) dt + \frac{\eta^2 Y_t^2}{2} \frac{\partial^2 f}{\partial y^2}(S_t, Y_t) dt + \rho \sigma Y_t S_t \frac{\partial^2 f}{\partial x \partial y}(S_t, Y_t) dt.
\]

Exercise 5.6

a) We have

\[ F_t = \xi_t S_t + \eta_t A_t = \beta \frac{F_t}{S_t} S_t - (\beta - 1) \frac{F_t}{A_t} A_t, \quad t \in \mathbb{R}_+, \]

with \(\xi_t = \beta F_t / S_t\) and \(\eta_t = - (\beta - 1) F_t / A_t, t \in \mathbb{R}_+.\)

b) We have

\[
dF_t = \xi_t dS_t + \eta_t dA_t \\
= \frac{\beta F_t}{S_t} dS_t - (\beta - 1) \frac{F_t}{A_t} dA_t \\
= \frac{\beta F_t}{S_t} dS_t - (\beta - 1) r F_t dt \\
= \beta F_t (r dt + \sigma dB_t) - (\beta - 1) r F_t dt \\
= r F_t dt + \beta \sigma F_t dB_t, \quad t \in \mathbb{R}_+.
\]

c) We have

\[ F_t = F_0 e^{\beta \sigma B_t + r t - \beta^2 \sigma^2 t/2} \]
\[ = F_0 \left( e^{\sigma B_t + r t/2 - \beta - \beta^2 t/2} \right)^{\beta/2} \]
\[ = F_0 \left( e^{\sigma B_t + r t/2 - \beta - \beta^2 t/2 - r(1-1/\beta) t - (\beta-1)^2 t/2} \right)^{\beta} \]
\[ = F_0 \left( e^{\sigma B_t + r t/2 - \beta - \beta^2 t/2} \right)^{\beta} e^{-(\beta-1)rt - \beta(\beta-1)\sigma^2 t/2} \]
\[ = \left( S_0 e^{\sigma B_t + r t/2 - \beta - \beta^2 t/2} \right)^{\beta} e^{-(\beta-1)rt - \beta(\beta-1)\sigma^2 t/2} \]
\[ = S_t^\beta e^{-(\beta-1)rt - \beta(\beta-1)\sigma^2 t/2}, \quad t \in \mathbb{R}_+.\]
Exercise 5.7 Letting $X_t := f(t) e^{\sigma B_t - \sigma^2 t/2}$, $t \in \mathbb{R}_+$, we have
\[
\begin{align*}
  dX_t &= e^{\sigma B_t - \sigma^2 t/2} f'(t) \, dt + f(t) d\left( e^{\sigma B_t - \sigma^2 t/2} \right) \\
  &= e^{\sigma B_t - \sigma^2 t/2} f'(t) \, dt + f(t) \sigma e^{\sigma B_t - \sigma^2 t/2} \, dB_t \\
  &= \frac{f'(t)}{f(t)} X_t \, dt + \sigma X_t \, dB_t \\
  &= h(t) X_t \, dt + \sigma X_t \, dB_t,
\end{align*}
\]
hence
\[
\frac{d}{dt} \log f(t) = \frac{f'(t)}{f(t)} = h(t),
\]
which shows that
\[
\log f(t) = \log f(0) + \int_0^t h(s) \, ds,
\]
and
\[
X_t = f(t) e^{\sigma B_t - \sigma^2 t/2} \\
  = f(0) \exp \left( \int_0^t h(s) \, ds + \sigma B_t - \frac{\sigma^2}{2} t \right) \\
  = X_0 \exp \left( \int_0^t h(s) \, ds + \sigma B_t - \frac{\sigma^2}{2} t \right), \quad t \in \mathbb{R}_+.
\]

Exercise 5.8
a) We have
\[
S_t = e^{X_t} \\
  = e^{X_0} + \int_0^t u_s e^{X_s} \, dB_s + \int_0^t v_s e^{X_s} \, ds + \frac{1}{2} \int_0^t u_s^2 e^{X_s} \, ds \\
  = e^{X_0} + \sigma \int_0^t e^{X_s} \, dB_s + \nu \int_0^t e^{X_s} \, ds + \frac{\sigma^2}{2} \int_0^t e^{X_s} \, ds \\
  = S_0 + \sigma \int_0^t S_s \, dB_s + \nu \int_0^t S_s \, ds + \frac{\sigma^2}{2} \int_0^t S_s \, ds.
\]
b) Let $r > 0$. The process $(S_t)_{t \in \mathbb{R}_+}$ satisfies the stochastic differential equation
\[
dS_t = r S_t \, dt + \sigma S_t \, dB_t
\]
when $r = \nu + \sigma^2/2$.
c) We have
Var\[X_t\] = Var\[(B_T - B_t)\sigma\] = \sigma^2 \ Var\[B_T - B_t\] = (T - t)\sigma^2, \ t \in [0, T].

d) Let the process \( (S_t)_{t \in \mathbb{R}^+} \) be defined by \( S_t = S_0 e^{\sigma B_t + \nu t}, t \in \mathbb{R}^+ \). Using the time splitting decomposition

\[
S_T = S_t \frac{S_T}{S_t} = S_t e^{(B_T - B_t)\sigma + \nu \tau},
\]

we have

\[
P(S_T > K | S_t = x) = P(S_t e^{(B_T - B_t)\sigma + (T-t)\nu} > K | S_t = x)
= P(x e^{(B_T - B_t)\sigma + (T-t)\nu} > K)
= P(e^{(B_T - B_t)\sigma} > K e^{-(T-t)\nu} / x)
= P\left(\frac{B_T - B_t}{\sqrt{T-t}} > \frac{1}{\sigma \sqrt{T-t}} \log(K e^{-(T-t)\nu} / x)\right)
= 1 - \Phi\left(\frac{\log(K e^{-(T-t)\nu} / x)}{\sigma \sqrt{\tau}}\right)
= \Phi\left(\frac{\log(x/K) + \nu \tau}{\sigma \sqrt{\tau}}\right),
\]

where \( \tau = T - t. \)

Problem 5.9 (Exercise 4.14 continued).

a) The option payoff is \((B_T - K)^+\) at maturity.

b) We can ignore what happens between two crossings as every crossing resets the portfolio to its state right before the previous crossing. Based on this, it is clear that every of the four possible scenarios will lead to a portfolio value \((B_T - K)^+\) at maturity:

i) If \( B_0 < 1 \) and \( B_T < 1 \) we issue the option for free and finish with an empty portfolio and zero payoff.

ii) If \( B_0 < 1 \) and \( B_T > 1 \) we issue the option for free and finish with one AUD and one SGD to refund, which yields the payoff \( B_T - 1 = (B_T - 1)^+ \).

iii) If \( B_0 > 1 \) and \( B_T < 1 \) we purchase one AUD and borrow one SGD at the start, however the AUD will be sold and the SGD refunded before maturity, resulting into an empty portfolio and zero payoff.

iv) If \( B_0 > 1 \) and \( B_T > 1 \) we purchase one AUD and borrow one SGD right before maturity, which yields the payoff \( B_T - 1 = (B_T - 1)^+ \).
Therefore we are hedging the option in all cases. Note that \( P(B_T = K) = 0 \) so the case \( B_T = 1 \) can be ignored with probability one.

c) The portfolio strategy is given by

\[
\xi_t = 1_{[K,\infty)}(B_t) \quad \text{and} \quad \eta_t = -1_{[K,\infty)}(B_t), \quad t \in [0, T].
\]

It is called a \textit{stop-loss/start-gain} strategy.

d) Noting that \( \int_0^t \eta_s dA_s = 0 \) because \( A_t = A_0 \) is constant, \( t \in [0, T] \), we find by (4.40) that

\[
\int_0^t \eta_s dA_s + \int_0^t \xi_s dB_s = \int_0^T 1_{[K,\infty)}(B_t) dB_t
\]

\[
= (B_T - K)^+ - (B_0 - K)^+ - \frac{1}{2} \mathcal{L}^K_{[0,T]}.
\]

e) Question (d) shows that

\[
(B_T - K)^+ = (B_0 - K)^+ + \int_0^t \eta_s dA_s + \int_0^t \xi_s dB_s + \frac{1}{2} \mathcal{L}^K_{[0,T]},
\]

\textit{i.e.} the initial premium \( (B_0 - K)^+ \) plus the sum of portfolio profits and losses is not sufficient to cover the terminal payoff \( (B_T - K)^+ \), and that we fall short of this by the positive amount \( \frac{1}{2} \mathcal{L}^K_{[0,T]} > 0 \). Therefore the portfolio allocation \( (\xi_t, \eta_t)_{t \in [0,T]} \) is \textit{not} self-financing.

\textbf{Additional comments:}
The stop-loss / start-gain strategy described here cannot implemented in practice because it would require infinitely many transactions when Brownian motion crosses the level \( K \), as illustrated in Figure S.14.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{brownian_crossings.png}
\caption{Brownian crossings of level 1.}
\end{figure}
The arbitrage price of the option can in fact be computed as the expected discounted option payoff

\[
\pi_t = e^{-(T-t)r} \mathbb{E}^*[\{(B_T - K)^+ | \mathcal{F}_t\}]
\]

\[
= e^{-(T-t)r} \mathbb{E}^*[\{(B_T - B_t + x - K)^+ | \mathcal{F}_t\}_{x=B_t}]
\]

\[
= e^{-(T-t)r} \mathbb{E}^*[\{(B_T - B_t + x - K)^+\}_{x=B_t}]
\]

\[
= e^{-(T-t)r} \int_{-\infty}^{\infty} (y + B_t - K)^+ e^{-y^2/(2(T-t))} \frac{dy}{\sqrt{2\pi(T-t)}}
\]

\[
= e^{-(T-t)r} \int_{K-B_t}^{\infty} (y + B_t - K)^+ e^{-y^2/(2(T-t))} \frac{dy}{\sqrt{2\pi(T-t)}}
\]

\[
= e^{-(T-t)r} \int_{K-B_t}^{\infty} y e^{-y^2/(2(T-t))} \frac{dy}{\sqrt{2\pi(T-t)}}
\]

\[
+(B_t - K) e^{-(T-t)r} \int_{K-B_t}^{\infty} e^{-y^2/(2(T-t))} \frac{dy}{\sqrt{2\pi(T-t)}}
\]

\[
= e^{-(T-t)r} \int_{(K-B_t)/\sqrt{T-t}}^{\infty} y e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}
\]

\[
+(B_t - K) e^{-(T-t)r} \int_{(K-B_t)/\sqrt{T-t}}^{\infty} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}
\]

\[
= \frac{e^{-(T-t)r}}{\sqrt{2\pi}} \left[ -e^{-y^2/2}\right]_{(K-B_t)/\sqrt{T-t}}^{\infty}
\]

\[
+(B_t - K) e^{-(T-t)r} \left( \Phi \left( \frac{B_t - K}{\sqrt{T-t}} \right) \right)
\]

\[
= \frac{e^{-(T-t)r}}{\sqrt{2\pi}} e^{-(K-B_t)^2/(2(T-t))}
\]

\[
+(B_t - K) e^{-(T-t)r} \left( \Phi \left( \frac{B_t - K}{\sqrt{T-t}} \right) \right)
\]

\[
=: g(t, B_t),
\]

where the function

\[
g(t, x) := \frac{e^{-(T-t)r}}{\sqrt{2\pi}} e^{-(K-x)^2/(2(T-t))} + (x - K) e^{-(T-t)r} \Phi \left( \frac{x - K}{\sqrt{T-t}} \right), \quad t \in [0, T),
\]

solves the Black-Scholes heat equation

\[
\frac{\partial g}{\partial t}(t, x) + r \frac{\partial g}{\partial x}(t, x) + \frac{1}{2} \frac{\partial^2 g}{\partial^2 x}(t, x) = 0
\]

with terminal condition \( g(T, x) = (x - K)^+ \). The Delta gives the amount to be invested in AUD at time \( t \) and is given by
\[ \xi_t = \frac{\partial g}{\partial x}(t, B_t) \]
\[ = (K - B_t) \frac{e^{-(T-t)r}}{\sqrt{2\pi(T-t)}} e^{-(K-B_t)^2/(2(T-t))} \]
\[ + (B_t - K) \frac{e^{-(T-t)r}}{\sqrt{2\pi(T-t)}} e^{-(B_t-K)^2/(2(T-t))} + e^{-(T-t)r} \Phi \left( \frac{B_t - K}{\sqrt{T-t}} \right) \]
\[ = e^{-(T-t)r} \Phi \left( \frac{B_t - K}{\sqrt{T-t}} \right) \]
\[ =: h(t, B_t), \]

with
\[ h(t, x) := e^{-(T-t)r} \Phi \left( \frac{x - K}{\sqrt{T-t}} \right), \quad t \in [0, T), \]

and \( h(T, x) = 1_{[K,\infty)}(x). \)

Fig. S.15: Brownian path started at \( B_0 > 1. \)

Fig. S.16: Risk-neutral pricing of the FX option by \( \pi_t(B_t) = g(t, B_t) \) vs stop-loss / start-gain pricing.
The “one or nothing” stop-loss / start-gain strategy is not self-financing because in practice there is an impossibility to buy/sell the AUD at exactly SGD1.00 to the existence of an order book that generates a gap between bid/ask prices as the sample of Figure S.18 with 383.16964 < 384.07141.

The existence of the order book will force buying and selling within a certain range \([K - \varepsilon, K + \varepsilon]\), typically resulting into selling lower than \(K = 1.00\) and buying higher than \(K = 1.00\). This potentially results into a trading loss that can be proportional to the time

\[
\ell \left( \{ t \in [0, T] : K - \varepsilon < B_t < K + \varepsilon \} \right)
\]

spent by the exchange rate \((B_t)_{t \in [0, T]}\) within the range \([K - \varepsilon, K + \varepsilon]\).

The Itô-Tanaka formula (4.40)

\[
(B_T - K)^+ = (B_0 - K)^+ + \int_0^T 1_{[K, \infty)}(B_t) dB_t + \frac{1}{2} \ell^K_{[0, T]},
\]
precisely shows that the trading loss equals half the \( \text{local time} L^K_{[0,T]} \) spent by \( (B_t)_{t \in [0,T]} \) at the level \( K \). When \( \varepsilon \) is small we have

\[
\frac{1}{2} L^K_{[0,T]} \simeq \frac{1}{4\varepsilon} \ell(\{ t \in [0,T] : K - \varepsilon < B_t < K + \varepsilon \}),
\]

therefore the proportionality coefficient is \( 1/(4\varepsilon) \).

![Fig. S.19: Time spent by Brownian motion in the range \([K - \varepsilon, K + \varepsilon]\).](image)

More generally, we could show that there is no self-financing (buy and hold) portfolio that can remain constant over time intervals, and that the self-financing portfolio has to be constantly re-adjusted in time as illustrated in Figure S.17. This invalidates the stop-loss / start-gain strategy as a self-financing portfolio strategy.

**Chapter 6**

**Exercise 6.1** By the Itô formula we have

\[
dV_t = dg(t, B_t) = \frac{\partial g}{\partial t} (t, B_t) dt + \frac{\partial g}{\partial x} (t, B_t) dB_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} (t, B_t) dt. \quad (A.18)
\]

Consider a hedging portfolio with value \( V_t = \eta_t A_t + \xi_t B_t \), satisfying the self-financing condition

\[
dV_t = \eta_t dA_t + \xi_t dB_t = \xi_t dB_t, \quad t \in \mathbb{R}_+.
\]

(A.19)

By respective identification of the terms in \( dB_t \) and \( dt \) in (A.18) and (A.19) we get

\[
\begin{align*}
0 &= \frac{\partial g}{\partial t} (t, B_t) dt + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} (t, B_t) dt, \\
\xi_t dB_t &= \frac{\partial g}{\partial x} (t, B_t) dB_t,
\end{align*}
\]

hence
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\begin{equation}
0 = \frac{\partial g}{\partial t}(t, B_t) + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, B_t), \quad \xi_t = \frac{\partial g}{\partial x}(t, B_t),
\end{equation}

and

\begin{equation}
0 = \frac{\partial g}{\partial t}(t, B_t) + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, B_t), \quad \xi_t = \frac{\partial g}{\partial x}(t, B_t),
\end{equation}

hence the function \( g(t, x) \) satisfies the heat equation

\begin{equation}
0 = \frac{\partial g}{\partial t}(t, x) + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, x), \quad x > 0, \tag{A.20}
\end{equation}

with terminal condition \( g(T, x) = x^2 \), and \( \xi_t \) is given by the partial derivative

\[ \xi_t = \frac{\partial g}{\partial x}(t, B_t), \quad t \in \mathbb{R}_+. \]

In order to solve (A.20) we substitute a solution of the form \( g(t, x) = x^2 + f(t) \) and find \( 1 + f'(t) = 0 \), which yields \( f(T - t) = T - t \) and \( g(t, x) = x^2 + T - t, t \in [0, T] \).

Exercise 6.2 By the Itô formula we have

\begin{equation}
dV_t = dg(t, S_t) \tag{A.21}
= \frac{\partial g}{\partial t}(t, S_t)dt + \beta(\alpha - S_t) \frac{\partial g}{\partial x}(t, S_t)dt + \frac{1}{2} \sigma^2 S_t \frac{\partial^2 g}{\partial x^2}(t, S_t)dt + \sigma \sqrt{S_t} \frac{\partial g}{\partial x}(t, S_t)dB_t.
\end{equation}

By respective identification of the terms in \( dB_t \) and \( dt \) in (6.29) and (A.21) we get

\begin{equation}
\begin{cases}
rg(t, S_t)dt + \beta(\alpha - S_t) \frac{\partial g}{\partial x}(t, S_t)dt - r \xi_t S_t dt \\
\quad = \frac{\partial g}{\partial t}(t, S_t)dt + \beta(\alpha - S_t) \frac{\partial g}{\partial x}(t, S_t)dt + \frac{1}{2} \sigma^2 S_t \frac{\partial^2 g}{\partial x^2}(t, S_t)dt,
\end{cases}
\end{equation}

\[ \sigma \xi_t \sqrt{S_t} dB_t = \sigma \sqrt{S_t} \frac{\partial g}{\partial x}(t, S_t)dB_t, \]

hence
\[
\begin{aligned}
  r g(t, S_t) + \beta(\alpha - S_t) \xi_t - r \xi_t S_t &= \frac{\partial g}{\partial t}(t, S_t) + \beta(\alpha - S_t) \frac{\partial g}{\partial x}(t, S_t) + \frac{1}{2} \sigma^2 S_t \frac{\partial^2 g}{\partial x^2}(t, S_t), \\
  \xi_t &= \frac{\partial g}{\partial x}(t, S_t),
\end{aligned}
\]

and
\[
\begin{aligned}
  r g(t, S_t) + \beta(\alpha - S_t) \xi_t - r \xi_t S_t &= \frac{\partial g}{\partial t}(t, S_t) + \frac{1}{2} \sigma^2 S_t \frac{\partial^2 g}{\partial x^2}(t, S_t), \\
  \xi_t &= \frac{\partial g}{\partial x}(t, S_t),
\end{aligned}
\]

hence the function \( g(t, x) \) satisfies the PDE
\[
rg(t, x) = \frac{\partial g}{\partial t}(t, x) + rx \frac{\partial g}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x \frac{\partial^2 g}{\partial x^2}(t, x), \quad x > 0,
\]

and \( \xi_t \) is given by the partial derivative
\[
\xi_t = \frac{\partial g}{\partial x}(t, S_t), \quad t \in \mathbb{R}^+.
\]

Exercise 6.3

a) Let \( V_t := \xi_t S_t + \eta_t A_t \) denote the hedging portfolio value at time \( t \in [0, T] \). Since the dividend yield \( \delta S_t \) per share is continuously reinvested in the portfolio, the portfolio change \( dV_t \) decomposes as
\[
\begin{aligned}
  dV_t &= \underbrace{\eta_t dA_t + \xi_t dS_t}_{\text{trading profit and loss}} + \underbrace{\delta \xi_t S_t dt}_{\text{dividend payout}} \\
  &= r \eta_t A_t dt + \xi_t ((\mu - \delta) S_t dt + \sigma S_t dB_t) + \delta \xi_t S_t dt \\
  &= r \eta_t A_t dt + \xi_t (\mu S_t dt + \sigma S_t dB_t) \\
  &= r V_t dt + (\mu - r) \xi_t S_t dt + \sigma \xi_t S_t dB_t, \quad t \in \mathbb{R}^+.
\end{aligned}
\]

b) By Itô’s formula we have
\[
\begin{aligned}
  d g(t, S_t) &= \frac{\partial g}{\partial t}(t, S_t) dt + (\mu - \delta) S_t \frac{\partial g}{\partial x}(t, S_t) dt \\
  &\quad + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 g}{\partial x^2}(t, S_t) dt + \sigma S_t \frac{\partial g}{\partial x}(t, S_t) dB_t,
\end{aligned}
\]

hence by identification of the terms in \( dB_t \) and \( dt \) in the expressions of \( dV_t \) and \( d g(t, S_t) \), we get
\[
\xi_t = \frac{\partial g}{\partial x}(t, S_t),
\]

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and we derive the Black-Scholes PDE with dividend

\[ rg(t, x) = \frac{\partial g}{\partial t}(t, x) + (r - \delta)x \frac{\partial g}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 g}{\partial x^2}(t, x). \quad (A.22) \]

c) In order to solve (A.22) we note that, letting \( f(t, x) := e^{(T-t)\delta} g(t, x) \), the PDE (A.22) reads

\[ rf(t, x) = \delta f(t, x) + \frac{\partial f}{\partial t}(t, x) + (r - \delta)x \frac{\partial f}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}(t, x), \]

hence \( f(t, x) := e^{(T-t)\delta} g(t, x) \), satisfies the standard Black-Scholes PDE with interest rate \( r - \delta \), i.e. we have

\[ (r - \delta)f(t, x) = \frac{\partial f}{\partial t}(t, x) + (r - \delta)x \frac{\partial f}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}(t, x), \]

with same terminal condition \( f(T, x) = g(T, x) = (x - K)^+ \), hence we have

\[ f(t, x) = Bl(K, x, \sigma, r - \delta, T - t) \]

\[ = x \Phi(d^\delta_+(T - t)) - K e^{-(r-\delta)(T-t)} \Phi(d^\delta_-(T - t)), \]

where

\[ d^\delta_{\pm}(T - t) := \frac{\log(x/K) + (r - \delta \pm \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}. \]

Consequently, the pricing function of the European call option with dividend rate \( \delta \) is

\[ g(t, x) = e^{-(T-t)\delta} f(t, x) \]

\[ = e^{-(T-t)\delta} Bl(K, x, \sigma, r - \delta, T - t) \]

\[ = x e^{-(T-t)\delta} \Phi(d^\delta_+(T - t)) - K e^{-(T-t)r} \Phi(d^\delta_-(T - t)), \quad 0 \leq t \leq T. \]

We also have

\[ g(t, x) = Bl(x e^{-(T-t)\delta}, K, \sigma, r, T - t), \quad 0 \leq t \leq T. \]

Exercise 6.4

a) We easily check that \( g_c(t, 0) = 0 \), as when \( x = 0 \) we have \( d_+(T - t) = d_-(T - t) = -\infty \) for all \( t \in [0, T) \). On the other hand, we have
\[
\lim_{t \to T} d_+(T - t) = \lim_{t \to T} d_-(T - t) = \begin{cases} +\infty, & x > K, \\ 0, & x = K, \\ -\infty, & x < K, \end{cases}
\]

which allows us to recover the boundary condition

\[
g_c(T, x) = \lim_{T \to T} g_c(t, x) = \begin{cases} x\Phi(+\infty) - K\Phi(+\infty) = x - K, & x > K \\ \frac{x}{2} - \frac{K}{2} = 0, & x = K \\ x\Phi(-\infty) - K\Phi(-\infty) = 0, & x < K \end{cases} = (x - K)^+
\]
at \( t = T \). Similarly, we can check that

\[
\lim_{T \to \infty} d_+(T - t) = \begin{cases} +\infty, & r > \frac{\sigma^2}{2}, \\ 0, & r = \frac{\sigma^2}{2}, \\ -\infty, & r < \frac{\sigma^2}{2} \end{cases}
\]

and \( \lim_{T \to \infty} d_-(T - t) = +\infty \), hence

\[
\lim_{T \to \infty} B_0(K, x, \sigma, r, T - t) = x, \quad t \in \mathbb{R}_+.
\]

b) We check that \( g_p(t, 0) = K e^{-(T-t)r} \) and \( g_p(t, \infty) = 0 \) as when \( x = 0 \) we have \( d_+(T - t) = d_-(T - t) = -\infty \) and as \( x \) tends to infinity we have \( d_+(T - t) = d_-(T - t) = +\infty \) for all \( t \in [0, T) \). On the other hand, we have

\[
g_p(T, x) = \begin{cases} K\Phi(+\infty) - x\Phi(+\infty) = K - x, & x < K \\ \frac{K}{2} - \frac{x}{2} = 0, & x = K \\ K\Phi(-\infty) - x\Phi(-\infty) = 0, & x > K \end{cases} = (K - x)^+
\]
at \( t = T \). Similarly, we can check that
Notes on Stochastic Finance

\[
\lim_{T \to \infty} d_{-}(T - t) = \begin{cases} 
+\infty, & r > \frac{\sigma^2}{2}, \\
0, & r = \frac{\sigma^2}{2}, \\
-\infty, & r < \frac{\sigma^2}{2},
\end{cases}
\]

and \(\lim_{T \to \infty} d_{+}(T - t) = +\infty\), hence

\[
\lim_{T \to \infty} \text{Bl}_{p}(K, x, \sigma, r, T - t) = 0, \quad t \in \mathbb{R}_{+}.
\]

Exercise 6.5 (Exercise 3.14 continued).

a) Substituting \(g(x, t) = x^2 f(t)\) in (6.30), we find \(f'(t) = -(r + \sigma^2)f(t)\), hence

\[
f(t) = f(0) e^{-(r+\sigma^2)t} = f(T) e^{(r+\sigma^2)(T-t)},
\]

hence \(g(x, t) = f(T)x^2 e^{(r+\sigma^2)(T-t)} = x^2 e^{(r+\sigma^2)(T-t)}\) due to the terminal condition \(g(x, T) = x^2\).

b) We have \(\xi_t = \frac{\partial g}{\partial x} g(S_t, t) = 2S_t e^{(r+\sigma^2)(T-t)},\) and

\[
\eta_t = \frac{1}{A_t} (g(S_t, t) - \xi_t S_t)
= \frac{1}{A_0 e^{rt}} \left( S_t^2 e^{(r+\sigma^2)(T-t)} - 2S_t^2 e^{(r+\sigma^2)(T-t)} \right)
= -\frac{S_t^2}{A_0} e^{(T-t)r+(T-t)\sigma^2}, \quad t \in [0, T].
\]

Exercise 6.6

a) Counting approximately 46 days to maturity, we have

\[
d_{-}(T - t) = \frac{(r - \sigma^2/2)(T - t) + \log(S_t/K)}{\sigma \sqrt{T - t}}
= \frac{(0.04377 - (0.9)^2/2)(46/365) + \log(17.2/36.08)}{0.9 \sqrt{46/365}}
= -2.461179058,
\]

and

\[
d_{+}(T - t) = d_{-}(T - t) + 0.9 \sqrt{46/365} = -2.14167602.
\]

From the standard Gaussian cumulative distribution table we get

\[
\Phi(d_{+}(T - t)) = \Phi(-2.14) = 0.0161098
\]
and
\[ \Phi(d_-(T-t)) = \Phi(-2.46) = 0.00692406, \]
hence
\[ f(t, S_t) = S_t \Phi(d_+(T-t)) - K e^{-(T-t)r} \Phi(d_-(T-t)) 
= 17.2 \times 0.0161098 - 36.08 \times e^{-0.04377\times 46/365} \times 0.00692406 
= HK$ 0.028642744. \]

For comparison, running the corresponding Black-Scholes R script yields
\[ \text{BSCall}(17.2, 36.08, 0.04377, 46/365, 0.9) = 0.02864235. \]

b) We have
\[ \eta_t = \frac{\partial f}{\partial x} (t, S_t) = \Phi(d_+(T-t)) = \Phi(-2.14) = 0.0161098, \quad (A.23) \]
hence one should only hold a fractional quantity equal to 16.10 units in
the risky asset in order to hedge 1000 such call options when \( \sigma = 0.90 \).
c) From the curve it turns out that when \( f(t, S_t) = 10 \times 0.023 = HK$ 0.23 \),
the volatility \( \sigma \) is approximately equal to \( \sigma = 122\% \).

This approximate value of implied volatility can be found under the column “Implied Volatility (IV.)” on this set of market data from the Hong
Kong Stock Exchange:

**Updated: 6 November 2008**

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<th>DW Code</th>
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<th>UL</th>
<th>Call Type</th>
<th>DW Listing</th>
<th>Maturity (D-M-Y)</th>
<th>Strike</th>
<th>Entitlement Ratio</th>
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<td>FB</td>
<td>0066</td>
<td>Call</td>
<td>Standard</td>
<td>18-12-2007</td>
<td>36.08</td>
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**Market Data**

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<th>O/S (%)</th>
<th>Delta (%)</th>
<th>IV. (%)</th>
<th>Day High</th>
<th>Day Low</th>
<th>Closing Price #</th>
<th>T/O ('000)</th>
<th>UL Price ($)</th>
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<td>0.780</td>
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<td>0</td>
<td>17.200</td>
</tr>
</tbody>
</table>

Fig. S.20: Market data for the warrant #01897 on the MTR Corporation.

**Remark**: a typical value for the volatility in standard market conditions
would be around 20\%. The observed volatility value \( \sigma = 1.22 \) per year is
actually quite high.

Exercise 6.7

a) We find $h(x) = x - K$.

b) Letting $g(t, x)$, the PDE rewrites as

$$(x - \alpha(t))r = -\alpha'(t) + rx,$$

hence $\alpha(t) = \alpha(0) e^{rt}$ and $g(t, x) = x - \alpha(0) e^{rt}$. The final condition

$$g(T, x) = h(x) = x - K$$

yields $\alpha(0) = K e^{-rT}$ and $g(t, x) = x - K e^{-(T-t)r}$.

c) We have

$$\xi_t = \frac{\partial g}{\partial x}(t, S_t) = 1,$$

hence

$$\eta_t = \frac{V_t - \xi_t S_t}{A_t} = \frac{g(t, S_t) - S_t}{A_t} = \frac{S_t - K e^{-(T-t)r} - S_t}{A_t} = -K e^{-rT}.$$

Note that we could also have directly used the identification

$$V_t = g(S_t, t) = S_t - K e^{-(T-t)r} = S_t - K e^{-rT} A_t = \xi_t S_t + \eta_t A_t,$$

which immediately yields $\xi_t = 1$ and $\eta_t = -K e^{-rT}$.

d) It suffices to take $K = 0$, which shows that $g(t, x) = x, \xi_t = 1$ and $\eta_t = 0$.

Exercise 6.8

a) We develop two approaches.

(i) By financial intuition. We need to replicate a fixed amount of $1 at maturity $T$, without risk. For this there is no need to invest in the stock. Simply invest $g(t, S_t) := e^{-(T-t)r}$ at time $t \in [0, T]$ and at maturity $T$ you will have $g(T, S_T) = e^{(T-t)r} g(t, S_t) = 1$.

(ii) By analysis and the Black-Scholes PDE. Given the hint, we try plugging a solution of the form $g(t, x) = f(t)$, not depending on the variable $x$, into the Black-Scholes PDE (6.31). Given that here we have

$$\frac{\partial g}{\partial x}(t, x) = 0, \quad \frac{\partial^2 g}{\partial x^2}(t, x) = 0, \quad \text{and} \quad \frac{\partial g}{\partial t}(t, x) = f'(t),$$

we find that the Black-Scholes PDE reduces to $rf(t) = f'(t)$ with the terminal condition $f(T) = g(T, x) = 1$. This equation has
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for solution \( f(t) = e^{-(T-t)r} \) and this is also the unique solution
\( g(t, x) = f(t) = e^{-(T-t)r} \) of the Black-Scholes PDE (6.31) with
terminal condition \( g(T, x) = 1 \).

b) We develop two approaches.

(i) By financial intuition. Since the terminal payoff $1 is risk-free we do
not need to invest in the risky asset, hence we should keep \( \xi_t = 0 \).
Our portfolio value at time \( t \) becomes
\[
V_t = g(t, S_t) = e^{-(T-t)r} = \xi_t S_t + \eta_t A_t = \eta_t A_t
\]
with \( A_t = e^{rt} \), so that we find \( \eta_t = e^{-rT}, t \in [0, T] \). This portfolio
strategy remains constant over time, hence it is clearly self-financing.

(ii) By analysis. The Black-Scholes theory of Proposition 6.1 tells us that
\[
\xi_t = \frac{\partial g}{\partial x}(t, x) = 0,
\]
and
\[
\eta_t = \frac{V_t - \xi_t S_t}{A_t} = \frac{V_t}{A_t} = \frac{e^{-(T-t)r}}{e^{rt}} = e^{-rT}.
\]

Exercise 6.9 Log-contracts.

a) Substituting the function \( g(x, t) := f(t) + \log x \) in the PDE (6.30) we
have
\[
0 = f'(t) + r - \frac{\sigma^2}{2},
\]
hence
\[
f(t) = f(0) - \left( r - \frac{\sigma^2}{2} \right) t,
\]
with \( f(0) = \left( r - \frac{\sigma^2}{2} \right) T \) in order to match the terminal condition
\( g(x, T) := \log x \), hence we have
\[
g(x, t) = \left( r - \frac{\sigma^2}{2} \right) (T - t) + \log x, \quad x > 0.
\]

b) Substituting the function
\[
h(x, t) := u(t)g(x, t) = u(t) \left( \left( r - \frac{\sigma^2}{2} \right) (T - t) + \log x \right)
\]
in the PDE (6.32), we find \( u'(t) = ru(t) \), hence \( u(t) = u(0) e^{rt} = e^{-(T-t)r} \), with \( u(T) = 1 \), and we conclude to
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\[ h(x, t) = u(t)g(x, t) = e^{-(T-t)r} \left( r - \frac{\sigma^2}{2} \right) (T - t) + \log x, \]

\[ x > 0, \ t \in [0, T]. \]
c) We have

\[ \xi_t = \frac{\partial h}{\partial x} (t, S_t) = \frac{e^{-(T-t)r}}{S_t}, \quad 0 \leq t \leq T, \]

and

\[ \eta_t = \frac{1}{A_t} (h(t, S_t) - \xi_t S_t) \]

\[ = \frac{e^{-r(T-t)}}{A_0} \left( \left( r - \frac{\sigma^2}{2} \right) (T - t) + \log x - 1 \right), \]

\[ 0 \leq t \leq T. \]

Exercise 6.10 Binary options.

a) From Proposition 6.1, the function \( C_d(t, x) \) solves the Black-Scholes PDE

\[
\begin{aligned}
&\left\{ \begin{array}{l}
\frac{\partial C}{\partial t} (t, x) + r x \frac{\partial C}{\partial x} (t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 C}{\partial x^2} (t, x), \\
C(T, x) = 1_{[K, \infty)}(x).
\end{array} \right.
\end{aligned}
\]

b) We check by direct differentiation that the Black-Scholes PDE is satisfied by the function \( C(t, x) \), together with the terminal condition \( C(T, x) = 1_{[K, \infty)}(x) \) as \( t \) tends to \( T \).

Exercise 6.11

a) By (4.29) we have

\[ S_t = S_0 e^{\alpha t} + \sigma \int_0^t e^{(t-s)\alpha} dB_s. \]

b) By the self-financing condition (5.8) we have

\[
\begin{aligned}
dV_t &= \eta_t dA_t + \xi_t dS_t \\
&= r \eta_t dt + \alpha \xi_t S_t dt + \sigma \xi_t dB_t \\
&= r V_t dt + (\alpha - r) \xi_t S_t dt + \sigma \xi_t dB_t,
\end{aligned}
\]

\( t \in \mathbb{R}_+ \). Rewriting (6.33) under the form of an Itô process

\[ S_t = S_0 + \int_0^t v_s ds + \int_0^t u_s dB_s, \quad t \in \mathbb{R}_+, \]

\( \diamond \)
with  
\[ u_t = \sigma, \quad \text{and} \quad v_t = \alpha S_t, \quad t \in \mathbb{R}_+, \]
the application of Itô's formula Theorem 4.19 to \( V_t = C(t, S_t) \) shows that
\[
dC(t, S_t) = v_t \partial C \partial x (t, S_t) dt + u_t \partial C \partial x (t, S_t) dB_t \\
+ \frac{1}{2} |u_t|^2 \partial^2 C \partial x^2 (t, S_t) dt \\
= \frac{\partial C}{\partial t} (t, S_t) dt + \alpha S_t \frac{\partial C}{\partial x} (t, S_t) dt + \frac{1}{2} \sigma^2 \partial^2 C \partial x^2 (t, S_t) dt + \sigma \frac{\partial C}{\partial x} (t, S_t) dB_t.
\]
(A.25)

Identifying the terms in \( dB_t \) and \( dt \) in (A.24) and (A.25) above, we get
\[
\begin{aligned}
\{rC(t, S_t) &= \frac{\partial C}{\partial t} (t, S_t) + rS_t \frac{\partial C}{\partial x} (t, S_t) + \frac{1}{2} \sigma^2 \partial^2 C \partial x^2 (t, S_t), \\
\} \xi_t &= \frac{\partial C}{\partial x} (t, S_t),
\end{aligned}
\]
hence the function \( C(t, x) \) satisfies the usual Black-Scholes PDE
\[
rC(t, x) = \frac{\partial C}{\partial t} (t, x) + rx \frac{\partial C}{\partial x} (t, x) + \frac{1}{2} \sigma^2 \partial^2 C \partial x^2 (t, x), \quad x > 0, \quad t \in [0, T],
\]
(A.26)
with the terminal condition \( C(T, x) = e^x, x \in \mathbb{R} \).

c) By substituting (6.34) into the Black-Scholes PDE (A.26) we find the ordinary differential equation
\[
xh'(t) + \frac{\sigma^2}{2r} h'(t) h(t) + rh(t) + \frac{\sigma^2}{2} (h(t))^2 = 0, \quad x > 0, \quad t \in [0, T],
\]
which reduces to the ordinary differential equation \( h'(t) + rh(t) = 0 \) with terminal condition \( h(T) = 1 \) and solution \( h(t) = e^{(T-t)r}, t \in [0, T] \), which yields
\[
C(t, x) = \exp \left( -(T-t)r + xe^{(T-t)r} + \frac{\sigma^2}{4r} (e^{2(T-t)r} - 1) \right).
\]
d) We have
\[
\xi_t = \frac{\partial C}{\partial x} (t, S_t) = \exp \left( S_t e^{(T-t)r} + \frac{\sigma^2}{4r} (e^{2(T-t)r} - 1) \right).
\]

Exercise 6.12

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a) Noting that \( \varphi(x) = \Phi'(x) = (2\pi)^{-1/2} e^{-x^2/2} \), we have the

\[
\frac{\partial h}{\partial d}(S, d) = S \varphi(d + \sigma \sqrt{T}) - K e^{-rT} \varphi(d)
\]

\[
= \frac{S}{\sqrt{2\pi}} e^{-(d+\sigma \sqrt{T})^2/2} - \frac{K}{\sqrt{2\pi}} e^{-rT} e^{-d^2/2}
\]

\[
= \frac{S}{\sqrt{2\pi}} e^{-d^2/2-\sigma\sqrt{T}d-\sigma^2T/2} - \frac{K}{\sqrt{2\pi}} e^{-rT} e^{-d^2/2},
\]

hence the vanishing of \( \frac{\partial h}{\partial d}(S, d_*(S)) \) at \( d = d_*(S) \) yields

\[
\frac{S}{\sqrt{2\pi}} e^{-d_*^2(S)/2-\sigma\sqrt{T}d_*(S)-\sigma^2T/2} - \frac{K}{\sqrt{2\pi}} e^{-rT} e^{-d_*^2(S)/2} = 0,
\]

i.e. \( d_*(S) = \frac{\log(S/K) + rT - \sigma^2T/2}{\sigma \sqrt{T}} \). We can also check that

\[
\frac{\partial^2 h}{\partial d^2}(S, d_*(S)) = \frac{\partial}{\partial d} \left( \frac{S}{\sqrt{2\pi}} e^{-(d_*(S)+\sigma \sqrt{T})^2/2} - \frac{K}{\sqrt{2\pi}} e^{-rT} e^{-d_*^2(S)/2} \right)
\]

\[
= -(d_*(S) + \sigma \sqrt{T}) \frac{S}{\sqrt{2\pi}} e^{-(d_*(S)+\sigma \sqrt{T})^2/2} + \frac{K}{\sqrt{2\pi}} d_*(S) e^{-rT} e^{-d_*^2(S)/2}
\]

\[
= -(d_*(S) + \sigma \sqrt{T}) \frac{K}{\sqrt{2\pi}} e^{-rT} e^{-d_*^2(S)/2} + \frac{K}{\sqrt{2\pi}} d_*(S) e^{-rT} e^{-d_*^2(S)/2}
\]

\[
= -\sigma \sqrt{T} \frac{K}{\sqrt{2\pi}} e^{-rT} e^{-d_*^2(S)/2} < 0,
\]

hence the function \( d \mapsto h(S, d) := S \Phi(d + \sigma \sqrt{T}) - K e^{-rT} \Phi(d) \) admits a maximum at \( d = d_*(S) \), and

\[
h(S, d_*(S)) = S \Phi(d_*(S) + \sigma \sqrt{T}) - K e^{-rT} \Phi(d_*(S))
\]

\[
= S \Phi \left( \frac{\log(S/K) + (r + \sigma^2/2)T}{\sigma \sqrt{T}} \right) - K e^{-rT} \Phi \left( \frac{\log(S/K) + (r - \sigma^2/2)T}{\sigma \sqrt{T}} \right)
\]

is the Black-Scholes call option price.

b) Since \( \frac{\partial h}{\partial d}(S, d_*(S)) = 0 \), we find

\[
\Delta = \frac{d}{dS} h(S, d_*(S)) = \frac{\partial h}{\partial S}(S, d_*(S)) + d_*(S) \frac{\partial h}{\partial d}(S, d_*(S))
\]

\[
= \Phi(d_*(S) + \sigma \sqrt{T}) = \Phi \left( \frac{\log(S/K) + rT + \sigma^2T/2}{\sigma \sqrt{T}} \right).
\]

Exercise 6.13 When \( \sigma > 0 \) we have
\[
\frac{\partial g_c}{\partial \sigma} = x \Phi'(d_+(T-t)) \frac{\partial}{\partial \sigma} d_+(T-t) - K e^{-(T-t)r} \Phi'(d_-(T-t)) \frac{\partial}{\partial \sigma} d_-(T-t) \\
= x \Phi'(d_+(T-t)) \frac{\partial}{\partial \sigma} d_+(T-t) - K e^{-(T-t)r} \Phi'(d_+(T-t)) e^{(T-t)r + \log(x/K)} \frac{\partial}{\partial \sigma} d_-(T-t) \\
= x \Phi'(d_+(T-t)) \frac{\partial}{\partial \sigma} (d_+(T-t) - d_-(T-t)) \\
= x \Phi'(d_+(T-t)) \frac{\partial}{\partial \sigma} (\sigma \sqrt{T-t}) \\
= x \sqrt{T-t} \Phi'(d_+(T-t)),
\]

where we used the fact that
\[
\Phi'(d_-(T-t)) = \frac{1}{\sqrt{2\pi}} e^{-(d_-(T-t))^2/2} \\
= \frac{1}{\sqrt{2\pi}} e^{-(d_-(T-t))^2/2 + (T-t)r + \log(x/K)} \\
= \Phi'(d_+(T-t)) e^{(T-t)r + \log(x/K)}.
\]

We note that the Black-Scholes European call price is an increasing function of the volatility parameter \(\sigma > 0\). Relation (6.35) can be obtained from
\[
(d_+(T-t))^2 - (d_-(T-t))^2 \\
= ((d_+(T-t) + d_-(T-t))(d_+(T-t) - d_-(T-t)) \\
= 2r(T-t) + 2 \log \frac{x}{K}.
\]

**Exercise 6.14**

a) Given that
\[
p^* = \frac{r_N - a_N}{b_N - a_N} = \frac{1}{2} \quad \text{and} \quad q^* = \frac{b_N - r_N}{b_N - a_N} = \frac{1}{2},
\]

Relation (3.13) reads
\[
\tilde{v}(t, x) = \frac{1}{2} \tilde{v}(t + T/N, x(1 + rT/N)(1 - \sigma \sqrt{T/N})) \\
+ \frac{1}{2} \tilde{v}(t + T/N, x(1 + rT/N)(1 + \sigma \sqrt{T/N})).
\]

After letting \(\Delta T := T/N\) and applying Taylor’s formula at the second order we obtain
\[
0 = \frac{1}{2} (\tilde{v}(t + \Delta T, x(1 + r\Delta T - \sigma \sqrt{\Delta T})) - \tilde{v}(t, x))
\]
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\[ + \frac{1}{2} (\tilde{v}(t + \Delta T, x(1 + r\Delta T + \sigma\sqrt{\Delta T})) - \tilde{v}(t, x)) + o(\Delta T) \]
\[ = \frac{1}{2} \left( \Delta T \frac{\partial \tilde{v}}{\partial t} (t, x) + x(r\Delta T - \sigma\sqrt{\Delta T}) \frac{\partial \tilde{v}}{\partial x} (t, x) \right. \]
\[ + \frac{x^2}{2} (r\Delta T - \sigma\sqrt{\Delta T})^2 \frac{\partial^2 \tilde{v}}{\partial x^2} (t, x) + o(\Delta T) \left. \right) \]
\[ + \frac{1}{2} \left( \Delta T \frac{\partial \tilde{v}}{\partial t} (t, x) + x(r\Delta T + \sigma\sqrt{\Delta T}) \frac{\partial \tilde{v}}{\partial x} (t, x) \right. \]
\[ + \frac{x^2}{2} (r\Delta T + \sigma\sqrt{\Delta T})^2 \frac{\partial^2 \tilde{v}}{\partial x^2} (t, x) + o(\Delta T) \left. \right) + o(\Delta T) \]
\[ = \Delta T \frac{\partial \tilde{v}}{\partial t} (t, x) + rx\Delta T \frac{\partial \tilde{v}}{\partial x} (t, x) + \frac{x^2}{2} (\sigma\sqrt{\Delta T}) \frac{\partial^2 \tilde{v}}{\partial x^2} (t, x) + o(\Delta T), \]

which shows that

\[ \frac{\partial \tilde{v}}{\partial t} (t, x) + rx \frac{\partial \tilde{v}}{\partial x} (t, x) + \frac{x^2}{2} \frac{\partial^2 \tilde{v}}{\partial x^2} (t, x) = -\frac{o(\Delta T)}{\Delta T}, \]

hence as \( N \) tends to infinity (or as \( \Delta T \) tends to 0) we find*

\[ 0 = \frac{\partial \tilde{v}}{\partial t} (t, x) + rx \frac{\partial \tilde{v}}{\partial x} (t, x) + \frac{x^2}{2} \frac{\partial^2 \tilde{v}}{\partial x^2} (t, x), \]

showing that the function \( v(t, x) := e^{(T-t)r}\tilde{v}(t, x) \) solves the classical Black-Scholes PDE

\[ rv(t, x) = \frac{\partial v}{\partial t} (t, x) + rx \frac{\partial v}{\partial x} (t, x) + \frac{x^2}{2} \frac{\partial^2 v}{\partial x^2} (t, x). \]

b) Similarly, we have

\[ \xi_t^{(1)} (x) = \frac{v(t, (1 + b_N)x) - v(t, (1 + a_N)x)}{x(b_N - a_N)} \]
\[ = \frac{v(t, (1 + r/N)(1 + \sigma\sqrt{T/N})x) - v(t, (1 + r/N)(1 - \sigma\sqrt{T/N})x)}{2x(1 + r/N)\sigma\sqrt{T/N}} \]
\[ \rightarrow \frac{\partial v}{\partial x} (t, x), \]

as \( N \) tends to infinity.

Problem 6.15

a) When the risk-free rate is \( r = 0 \) the two possible returns are \((5 - 4) / 4 = 25\%\) and \((2 - 4) / 4 = -50\%\). Under the risk-neutral probability measure

* The notation \( o(\Delta T) \) denotes any function of \( \Delta T \) such that \( \lim_{\Delta T \to 0} o(\Delta T) / \Delta T = 0. \)
given by \( \mathbb{P}^*(S_1 = 5) = (4 - 2)/(5 - 2) = 2/3 \) and \( \mathbb{P}^*(S_1 = 2) = (5 - 4)/(5 - 2) = 1/3 \) the expected return is \( 2 \times 25%/3 - 50%/3 = 0\% \).
In general the expected return can be shown to be equal to the risk-free rate \( r \).

b) The two possible returns become \((3 \times 5 - 4 - 2 \times 4)/4 = 75\% \) and \((3 \times 2 - 4 - 2 \times 4)/4 = -150\% \). Under the risk-neutral probability measure given by \( \mathbb{P}^*(S_1 = 5) = (4 - 2)/(5 - 2) = 2/3 \) and \( \mathbb{P}^*(S_1 = 2) = (5 - 4)/(5 - 2) = 1/3 \) the expected return is \( 2 \times 75%/3 - 150%/3 = 0\% \). Similarly to Question (a), the expected return can be shown to be equal to the risk-free rate \( r \) when \( r \neq 0 \).

c) We decompose the amount \( F_t \) invested in one unit of the fund as

\[
F_t = \beta F_t \quad \text{purchased/sold} - (\beta - 1)F_t \quad \text{borrowed/saved}
\]

meaning that we invest the amount \( \beta F_t \) in the risky asset \( S_t \), and borrow/save the amount \( -(\beta - 1)F_t \) from/on the saving account.

d) We have

\[
F_t = \xi_t S_t + \eta_t A_t = \beta F_t S_t - (\beta - 1) \frac{F_t}{A_t} A_t, \quad t \in \mathbb{R}_+,
\]

with \( \xi_t = \beta F_t / S_t \) and \( \eta_t = -(\beta - 1)F_t / A_t, \quad t \in \mathbb{R}_+ \).

e) We have

\[
dF_t = \xi_t dS_t + \eta_t dA_t \\
= \beta \frac{F_t}{S_t} dS_t - (\beta - 1) \frac{F_t}{A_t} dA_t \\
= \beta \frac{F_t}{S_t} dS_t - (\beta - 1)rF_t dt \\
= \beta F_t (r dt + \sigma dB_t) - (\beta - 1)rF_t dt \\
= rF_t dt + \beta \sigma F_t dB_t, \quad t \in \mathbb{R}_+.
\]

By (A.27), the return of the fund \( F_t \) is \( \beta \) times the return of the risky asset \( S_t \), up to the cost of borrowing \((\beta - 1)r\) per unit of time.

f) The discounted fund value \( (e^{-rt}F_t)_{t \in \mathbb{R}_+} \) is a martingale under the risk-neutral probability measure \( \mathbb{P}^* \) as we have

\[
d(e^{-rt}F_t) = \beta \sigma e^{-rt}F_t dB_t, \quad t \in \mathbb{R}_+.
\]

g) We have

\[
F_t = F_0 e^{\beta \sigma B_t + r t - \beta^2 \sigma^2 t/2}
\]

and

\[
S_t^\beta = \left( S_0 e^{\sigma B_t + r t - \sigma^2 t/2} \right)^\beta = F_0 e^{\beta \sigma B_t + \beta r t - \beta^2 \sigma^2 t/2},
\]

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h) We have
\[
F_t = S_t^{\beta} e^{-(\beta-1)rt-\beta(\beta-1)\sigma^2 t/2}, \quad t \in \mathbb{R}_+.
\]
Note that when \( \beta = 0 \) we have \( F_t = e^{rt} \), i.e. in this case the fund \( F_t \) coincides with the money market account.

i) We have
\[
e^{-r(T-t)} \mathbb{E}^* \left[ (F_T - K)^+ | F_t \right] = F_t \Phi \left( \frac{\log(F_t/K) + (r + \beta^2\sigma^2/2)(T-t)}{|\beta|\sigma\sqrt{T-t}} \right) - Ke^{-r(T-t)} \Phi \left( \frac{\log(F_t/K) + (r - \beta^2\sigma^2/2)(T-t)}{|\beta|\sigma\sqrt{T-t}} \right),
\]
\( t \in [0, T) \).

j) We have
\[
\Phi \left( \frac{\log(F_t/K) + (r + \beta^2\sigma^2/2)(T-t)}{|\beta|\sigma\sqrt{T-t}} \right)
\]
\[
= \Phi \left( \frac{\log(S_t^{\beta} e^{-(\beta-1)rt-\beta(\beta-1)\sigma^2 t/2} / K) + (r + \beta^2\sigma^2/2)(T-t)}{|\beta|\sigma\sqrt{T-t}} \right)
\]
\[
= \Phi \left( \frac{\log(S_t^{\beta}/K) - (\beta-1)rt - \beta(\beta-1)\sigma^2 t/2 + (r + \beta^2\sigma^2/2)(T-t)}{|\beta|\sigma\sqrt{T-t}} \right)
\]
\[
= \Phi \left( \frac{\log(S_t^{\beta}/K) + (\beta - \beta(\beta-1)\beta^2) + \beta r(T-t) + (T-t)\beta\sigma^2/2}{|\beta|\sigma\sqrt{T-t}} \right)
\]
\[
= \Phi \left( \frac{\log(S_t/K_{\beta}(t)) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \right), \quad t \in [0, T),
\]
if \( \beta > 0 \), with \( K_{\beta}(t) := K^{1/\beta} e^{(\beta-1)(rT/\beta -(T/2-t)\sigma^2)} \).

j) When \( \beta < 0 \) we find that the Delta of the call option on \( F_T \) with strike price \( K \) is
\[
\Phi \left( \frac{\log(F_t/K) + (r + \beta^2\sigma^2/2)(T-t)}{|\beta|\sigma\sqrt{T-t}} \right)
\]
\[
= \Phi \left( \frac{\log(S_t^{\beta}/K_{\beta}) + \beta r(T-t) + (T-t)\beta\sigma^2/2}{|\beta|\sigma\sqrt{T-t}} \right)
\]
\[
= \Phi \left( \frac{-\log(S_t/K_{\beta}(t)) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \right), \quad t \in [0, T),
\]
which coincides, up to a negative sign, with the Delta of the put option on \( S_T \) with strike price \( K_\beta(t) := K^{1/\beta} e^{(\beta-1)(rT/\beta-(T/2-t))\sigma^2)} \).

Chapter 7

Exercise 7.1 (Exercise 6.1 continued). Since \( r = 0 \) we have \( \mathbb{P} = \mathbb{P}^* \) and

\[
g(t, B_t) = \mathbb{E}^* [B_T^2 | \mathcal{F}_t] = \mathbb{E}^* [(B_T - B_t + B_t)^2 | \mathcal{F}_t] = \mathbb{E}^* [(B_T - B_t + x)^2]_{x = B_t} = \mathbb{E}^* [(B_T - B_t)^2 + 2x(B_T - B_t) + x^2]_{x = B_t} = \mathbb{E}^* [(B_T - B_t)^2] + 2x \mathbb{E}^*[B_T - B_t] + B_t^2 = B_t^2 + T - t, \quad 0 \leq t \leq T,
\]

hence \( \xi_t \) is given by the partial derivative

\[
\xi_t = \frac{\partial g}{\partial x}(t, B_t) = 2B_t, \quad 0 \leq t \leq T,
\]

with

\[
\eta_t = \frac{g(t, B_t) - \xi_t B_t}{A_0} = \frac{B_t^2 + (T - t) - 2B_t^2}{A_0} = \frac{(T - t) - B_t^2}{A_0}, \quad 0 \leq t \leq T.
\]

Exercise 7.2 Since \( B_T \sim \mathcal{N}(0, T) \), we have

\[
\mathbb{E}[\phi(S_T)] = \mathbb{E} [\phi(S_0 e^{\sigma B_T + (r - \sigma^2/2)T})] = \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^\infty \phi(S_0 e^{\sigma y + (r - \sigma^2/2)T}) e^{-y^2/(2T)} dy = \frac{1}{\sqrt{2\pi \sigma^2 T}} \int_{-\infty}^\infty \phi(x) e^{-(\sigma^2/2-r)T + \log(x)} e^{\log(x) \sigma^2 T} \frac{dx}{x} = \int_{-\infty}^\infty \phi(x) g(x) dx,
\]

under the change of variable

\[
x = S_0 e^{\sigma y + (r - \sigma^2/2)T}, \quad \text{with } dx = \sigma S_0 e^{\sigma y + (r - \sigma^2/2)T} dy = \sigma x dy,
\]

i.e.

\[
y = \frac{(\sigma^2/2 - r)T + \log(x/S_0)}{\sigma} \quad \text{and } \quad dy = \frac{dx}{\sigma x},
\]

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https://www.ntu.edu.sg/home/nprivault/index.html
where

\[ g(x) := \frac{1}{x\sqrt{2\pi\sigma^2T}} e^{-((\sigma^2/2-r)T+\log(x/S_0))^2/(2\sigma^2T)} \]

is the lognormal probability density function with location parameter \((r - \sigma^2/2)T + \log S_0\) and scale parameter \(\sigma\sqrt{T}\).

Exercise 7.3 We have

\[
\mathbb{E}^*[\phi(pS_{T_1} + qS_{T_2})] \leq \mathbb{E}^*[p\phi(S_{T_1}) + q\phi(S_{T_2})]
\]

since \(\phi\) is convex,

\[
= p\mathbb{E}^*[\phi(S_{T_1})] + q\mathbb{E}^*[\phi(S_{T_2})] = p\mathbb{E}^*[\phi(S_{T_2})] + q\mathbb{E}^*[\phi(S_{T_2})]
\]

because \((S_t)_{t \in \mathbb{R}_+}\) is a martingale,

\[
\leq p\mathbb{E}^*[\mathbb{E}^*[\phi(S_{T_2}) | \mathcal{F}_{T_1}]] + q\mathbb{E}^*[\phi(S_{T_2})]
\]

by Jensen’s inequality,

\[
= p\mathbb{E}^*[\phi(S_{T_2})] + q\mathbb{E}^*[\phi(S_{T_2})]
\]

by the tower property,

\[
= \mathbb{E}^*[\phi(S_{T_2})],
\]

see Exercise 13.6 for an extension to arbitrary summations.

Remark: This type of technique can be useful in order to get an upper price estimate from Black-Scholes when the actual option price is difficult to compute: here the closed-form computation would involve a double integration of the form

\[
\mathbb{E}^*[\phi(pS_{T_1} + qS_{T_2})] = \mathbb{E}^* \left[ \phi \left( pS_0 e^{\sigma B_{T_1} - \sigma^2 T_1/2} + qS_0 e^{\sigma B_{T_2} - \sigma^2 T_2/2} \right) \right]
\]

\[
= \mathbb{E}^* \left[ \phi \left( S_0 e^{\sigma B_{T_1} - \sigma^2 T_1/2} + q e^{(B_{T_2} - B_{T_1})\sigma - (T_2 - T_1)\sigma^2/2} \right) \right]
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi \left( S_0 e^{\sigma x - \sigma^2 T_1/2} + q e^{e^{\sigma y - (T_2 - T_1)\sigma^2/2}} \right) \times e^{-x^2/(2T_1)} - y^2(2(T_2 - T_1)) \frac{dxdy}{\sqrt{T_1(T_2 - T_1)}}
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( S_0 e^{\sigma x - \sigma^2 T_1/2} + q e^{e^{\sigma y - (T_2 - T_1)\sigma^2/2}} - K \right)^+ \times e^{-x^2/(2T_1)} - y^2(2(T_2 - T_1)) \frac{dxdy}{\sqrt{T_1(T_2 - T_1)}}
\]

\[
= \frac{1}{2\pi} \int_{\{(x,y) \in \mathbb{R}^2 : S_0 e^{\sigma x(x+q e^{\sigma y - (T_2 - T_1)\sigma^2/2})} e^{\sigma^2 T_1/2} \geq K e^{\sigma^2 T_1/2} \}} \times e^{-x^2/(2T_1)} - y^2(2(T_2 - T_1)) \frac{dxdy}{\sqrt{T_1(T_2 - T_1)}}
\]

\[
= \ldots
\]
Exercise 7.4

a) Using Jensen’s inequality and the martingale property of the discounted asset price process \( (e^{-rt} S_t)_{t \in \mathbb{R}^+} \) under the risk-neutral probability measure \( \mathbb{P}^* \), we have

\[
e^{-(T-t)r} \mathbb{E}^*[(S_T - K)^+ | \mathcal{F}_t] \geq e^{-(T-t)r} \left( \mathbb{E}^*[S_T - K | \mathcal{F}_t] \right)^+
= e^{-(T-t)r} \left( e^{(T-t)r} S_t - K \right)^+
= (S_t - K e^{-(T-t)r})^+, \quad t \in [0, T].
\]

b) Similarly, by Jensen’s inequality and the martingale property we find

\[
e^{-(T-t)r} \mathbb{E}^*[(K - S_T)^+ | \mathcal{F}_t] \geq e^{-(T-t)r} \left( \mathbb{E}^*[K - S_T | \mathcal{F}_t] \right)^+
= e^{-(T-t)r} \left( K - e^{(T-t)r} S_t \right)^+
= (K e^{-(T-t)r} - S_t)^+, \quad t \in [0, T].
\]
Exercise 7.5

a) (i) The bull spread option can be realized by purchasing one European call option with strike price \( K_1 \) and by short selling (or issuing) one European call option with strike price \( K_2 \), because the bull spread payoff function can be written as

\[
x \mapsto (x - K_1)^+ - (x - K_2)^+.
\]

see https://optioncreator.com/st3ce7z.

(ii) The bear spread option can be realized by purchasing one European put option with strike price \( K_2 \) and by short selling (or issuing) one European put option with strike price \( K_1 \), because the bear spread payoff function can be written as

\[
x \mapsto -(K_1 - x)^+ + (K_2 - x)^+,
\]

see https://optioncreator.com/stmomsb.

* The animation works in Acrobat Reader on the entire pdf file.
Fig. S.24: Bear spread option as a combination of call and put options.\

b) (i) The bull spread option can be priced at time $t \in [0, T)$ using the Black-Scholes formula as

$$Bl(K_1, S_t, \sigma, r, T - t) - Bl(K_2, S_t, \sigma, r, T - t).$$

(ii) The bear spread option can be priced at time $t \in [0, T)$ using the Black-Scholes formula as

$$Bl(K_2, S_t, \sigma, r, T - t) - Bl(K_1, S_t, \sigma, r, T - t).$$

Exercise 7.6

a) We have

$$C_t = e^{-(T-t)r} \mathbb{E}^*[S_T - K \mid \mathcal{F}_t]$$

$$= e^{-(T-t)r} \mathbb{E}^*[S_T \mid \mathcal{F}_t] - K e^{-(T-t)r}$$

$$= e^{rt} \mathbb{E}^*[e^{-rT}S_T \mid \mathcal{F}_t] - K e^{-(T-t)r}$$

$$= e^{rt} e^{-rT} S_T - K e^{-(T-t)r}$$

$$= S_t - K e^{-(T-t)r}.$$ 

We can check that the function $g(x, t) = x - K e^{-(T-t)r}$ satisfies the Black-Scholes PDE

$$rg(x, t) = \frac{\partial g}{\partial t} (x, t) + rx \frac{\partial g}{\partial x} (x, t) + \frac{\sigma^2}{2} x^2 \frac{\partial^2 g}{\partial x^2} (x, t)$$

with terminal condition $g(x, T) = x - K$, since $\partial g(x, t) / \partial t = -rK e^{-(T-t)r}$ and $\partial g(x, t) / \partial x = 1$. 

* The animation works in Acrobat Reader on the entire pdf file.
b) We simply take $\xi_t = 1$ and $\eta_t = -Ke^{-rT}$ in order to have
$$C_t = \xi_t S_t + \eta_t e^{rt} = S_t - K e^{-(T-t)r}, \quad t \in [0, T].$$

Note again that this hedging strategy is constant over time, and the relation $\xi_t = \partial g(S_t, t)/\partial x$ for the option Delta, cf. (A.23), is satisfied.

Exercise 7.7 Option pricing with dividends (Exercise 6.3 continued).

a) Let $\widehat{P}$ denote the probability measure under which the process $(\widehat{B}_t)_{t \in \mathbb{R}_+}$ defined by
$$d\widehat{B}_t = \frac{\mu - r}{\sigma} dt + dB_t$$
is a standard Brownian motion. Under absence of arbitrage the asset price process $(S_t)_{t \in \mathbb{R}_+}$ has the dynamics
$$dS_t = (\mu - \delta)S_t dt + \sigma S_t dB_t$$
$$= (r - \delta)S_t dt + \sigma S_t d\widehat{B}_t,$$
and the discounted asset price process $(\tilde{S}_t)_{t \in \mathbb{R}_+} = (e^{-r t} S_t)_{t \in \mathbb{R}_+}$ satisfies
$$d\tilde{S}_t = -\delta \tilde{S}_t dt + \sigma \tilde{S}_t d\widehat{B}_t.$$

Assuming that the dividend yield $\delta S_t$ per share is continuously reinvested in the portfolio, the self-financing portfolio condition
$$dV_t = \underbrace{\eta_t dA_t + \xi_t dS_t}_{\text{trading profit and loss}} + \underbrace{\delta \xi_t S_t dt}_{\text{dividend payout}}$$
$$= r \eta_t A_t dt + \xi_t ((r - \delta)S_t dt + \sigma S_t d\widehat{B}_t) + \delta \xi_t S_t dt$$
$$= r \eta_t A_t dt + \xi_t (r S_t dt + \sigma S_t d\widehat{B}_t)$$
$$= r V_t dt + \sigma \xi_t S_t d\widehat{B}_t, \quad t \in \mathbb{R}_+,$$

which yields
$$d\tilde{V}_t = d (e^{-r t} V_t)$$
$$= -r e^{-r t} V_t dt + e^{-r t} dV_t$$
$$= \sigma \xi_t e^{-r t} S_t d\widehat{B}_t$$
$$= \sigma \xi_t \tilde{S}_t d\widehat{B}_t$$
$$= \xi_t (d\tilde{S}_t + \delta \tilde{S}_t dt), \quad t \in \mathbb{R}_+.$$

Therefore, we have
$$\tilde{V}_t - \tilde{V}_0 = \int_0^t d\tilde{V}_u.$$
b) We have
\[
  \mathbf{V}_t = \xi u S_u \mathbf{\tilde{B}}_u = \xi u S_u + \delta \mathcal{F}_u, \quad t \in \mathbb{R}_+.
\]
Here, the asset price process \((e^{\delta t} S_t)_{t \in \mathbb{R}_+}\) with added dividend yield satisfies the equation
\[
d(e^{\delta t} S_t) = r e^{\delta t} S_t dt + \sigma (e^{\delta t} S_t) d\mathbf{\tilde{B}}_t,
\]
and after discount, the process \((e^{-r t} e^{\delta t} S_t)_{t \in \mathbb{R}_+} = (e^{-(r - \delta) t} S_t)_{t \in \mathbb{R}_+}\) is a martingale under \(\hat{\mathbb{P}}\).

b) We have
\[
  \mathbf{V}_t = \mathbf{V}_0 + \sigma \int_0^t \xi u S_u d\mathbf{\tilde{B}}_u, \quad t \in \mathbb{R}_+,
\]
which is a martingale under \(\hat{\mathbb{P}}\) from Proposition 7.1, hence
\[
  \mathbf{V}_t = \mathbb{E}[\mathbf{V}_t | \mathcal{F}_t] = e^{-r T} \mathbb{E}[\mathbf{V}_T | \mathcal{F}_t] = e^{-r T} \mathbb{E}[(S_T - K)^+ | \mathcal{F}_t],
\]
which implies
\[
  \mathbf{V}_t = e^{r t} \mathbf{V}_t = e^{-(T - t) r} \mathbb{E}[(S_T - K)^+ | \mathcal{F}_t], \quad 0 \leq t \leq T.
\]

c) After discounting the payoff \((S_T - K)^+\) at the continuously compounded interest rate \(r\), we obtain
\[
  \mathbf{V}_t = e^{-(T - t) r} \mathbb{E}[(S_T - K)^+ | \mathcal{F}_t] = e^{-(T - t) r} \mathbb{E}[(S_T e^{\sigma \mathbf{\tilde{B}}_T + (r - \delta - \sigma^2/2) T} - K)^+ | \mathcal{F}_t] = e^{-(T - t) r} Bl(K, x, \sigma, r - \delta, T - t) = e^{-(T - t) r} S_t \Phi(d_+^\delta(T - t)) - K e^{-(T - t) r} \Phi(d_-^\delta(T - t)), \quad t \in [0, T),
\]
where
\[
  d_+^\delta(T - t) := \frac{\log(S_t/K) + (r - \delta + \sigma^2/2)(T - t)}{|\sigma|\sqrt{T - t}}
\]
and
\[
  d_-^\delta(T - t) := \frac{\log(S_t/K) + (r - \delta - \sigma^2/2)(T - t)}{|\sigma|\sqrt{T - t}}.
\]
We also have
Exercise 7.8 We start by pricing the “inner” at-the-money option with payoff 
$$(S_{T_2} - S_{T_1})^+$$ and strike price $K = S_{T_1}$ at time $T_1$ as
\[
ge^{-\frac{(T_2-T_1)}{r}} \mathbb{E}^* \left[ (S_{T_2} - S_{T_1})^+ | \mathcal{F}_{T_1} \right] = S_{T_1} \Phi \left( \frac{r + \sigma^2/2}{\sigma \sqrt{T_2 - T_1}} \right)
- S_{T_1} e^{-\frac{(T_2-T_1)}{r}} \Phi \left( \frac{r - \sigma^2/2}{\sigma \sqrt{T_2 - T_1}} \right) = S_{T_1} \Phi \left( \frac{r + \sigma^2/2}{\sigma \sqrt{T_2 - T_1}} \right) - S_{T_1} e^{-\frac{(T_2-T_1)}{r}} \Phi \left( \frac{r - \sigma^2/2}{\sigma \sqrt{T_2 - T_1}} \right),
\]
where we applied (7.18) with $T = T_2$, $t = T_1$, and $K = S_{T_1}$. As a consequence, the forward start option can be priced as
\[
e^{-\frac{(T_1-t)}{r}} \mathbb{E}^* \left[ e^{-\frac{(T_2-T_1)}{r}} \mathbb{E}^* \left[ (S_{T_2} - S_{T_1})^+ | \mathcal{F}_{T_1} \right] | \mathcal{F}_t \right] = e^{-\frac{(T_1-t)}{r}} \times \mathbb{E}^* \left[ S_{T_1} \Phi \left( \frac{r + \sigma^2/2}{\sigma \sqrt{T_2 - T_1}} \right) - S_{T_1} e^{-\frac{(T_2-T_1)}{r}} \Phi \left( \frac{r - \sigma^2/2}{\sigma \sqrt{T_2 - T_1}} \right) | \mathcal{F}_t \right] = e^{-\frac{(T_1-t)}{r}} \times \left( \Phi \left( \frac{r + \sigma^2/2}{\sigma \sqrt{T_2 - T_1}} \right) - e^{-\frac{(T_2-T_1)}{r}} \Phi \left( \frac{r - \sigma^2/2}{\sigma \sqrt{T_2 - T_1}} \right) \right) \mathbb{E}^*[S_{T_1} | \mathcal{F}_t] = S_t \left( \Phi \left( \frac{r + \sigma^2/2}{\sigma \sqrt{T_2 - T_1}} \right) - e^{-\frac{(T_2-T_1)}{r}} \Phi \left( \frac{r - \sigma^2/2}{\sigma \sqrt{T_2 - T_1}} \right) \right),
\]
$0 \leq t \leq T_1$.

Exercise 7.9 (Exercise 6.9 continued). We have
\[
C(t, S_t) = e^{-\frac{(T-t)}{r}} \mathbb{E}^* \left[ \log S_T | \mathcal{F}_t \right] = e^{-\frac{(T-t)}{r}} \mathbb{E}^* \left[ (\log S_t) + (\hat{B}_T - \hat{B}_t) \sigma + \left( r - \frac{\sigma^2}{2} \right) (T - t) | \mathcal{F}_t \right] = e^{-\frac{(T-t)}{r}} \log S_t + e^{-\frac{(T-t)}{r}} \left( r - \frac{\sigma^2}{2} \right) (T - t),
\]
t $\in [0, T]$.

Exercise 7.10 (Exercise 6.5 continued).
a) For all $t \in [0, T]$ we have

$$C(t, S_t) = e^{-(T-t)r} S_t^2 \mathbb{E}\left[ \frac{S_T^2}{S_t^2} \right] = e^{-(T-t)r} S_t^2 \mathbb{E}\left[ e^{2(B_T-B_t)\sigma-\sigma^2(T-t)+2(T-t)r} \right] = S_t^2 e^{(r+\sigma^2)(T-t)}.$$ 

b) For all $t \in [0, T]$ we have

$$\xi_t = \frac{\partial C(t, x)|_{x=S_t}}{\partial x} = 2S_t e^{(r+\sigma^2)(T-t)},$$

i.e.

$$\xi_t S_t = 2S_t^2 e^{(r+\sigma^2)(T-t)} = 2C(t, S_t),$$

and

$$\eta_t = \frac{C(t, S_t) - \xi_t S_t}{A_t} = \frac{e^{-rt}}{A_0} \left( S_t^2 e^{(r+\sigma^2)(T-t)} - 2S_t^2 e^{(r+\sigma^2)(T-t)} \right) = -\frac{S_t^2}{A_0} e^{\sigma^2(T-t)+(T-2t)r},$$

i.e.

$$\eta_t A_t = -S_t^2 \frac{A_t}{A_0} e^{\sigma^2(T-t)+(T-2t)r} = -S_t^2 e^{\sigma^2(T-t)+(T-t)r} = -C(t, S_t).$$

As for the self-financing condition, we have

$$dC(t, S_t) = d(S_t^2 e^{(r+\sigma^2)(T-t)})$$

$$= -(r+\sigma^2) e^{(r+\sigma^2)(T-t)} S_t^2 dt + e^{(r+\sigma^2)(T-t)} d(S_t^2)$$

$$= -(r+\sigma^2) e^{(r+\sigma^2)(T-t)} S_t^2 dt + e^{(r+\sigma^2)(T-t)} (2S_t dS_t + \sigma^2 S_t^2 dt)$$

$$= -r e^{(r+\sigma^2)(T-t)} S_t^2 dt + 2S_t e^{(r+\sigma^2)(T-t)} dS_t,$$

and

$$\xi_t dS_t + \eta_t dA_t = 2S_t e^{(r+\sigma^2)(T-t)} dS_t - r \frac{S_t^2}{A_0} e^{\sigma^2(T-t)+(T-2t)r} A_t dt$$

$$= 2S_t e^{(r+\sigma^2)(T-t)} dS_t - r S_t^2 e^{\sigma^2(T-t)+(T-t)r} dt,$$

which recovers $dC(t, S_t) = \xi_t dS_t + \eta_t dA_t$, i.e. the portfolio strategy is self-financing.
a) The discounted process $X_t := e^{-rt}S_t$ satisfies

$$dX_t = (\alpha - r)X_t dt + \sigma e^{-rs}dB_s,$$

which is a martingale when $\alpha = r$ by Proposition 7.1, as in this case it becomes a stochastic integral with respect to a standard Brownian motion. This fact can be recovered by directly computing the conditional expectation $\mathbb{E}[X_t \mid \mathcal{F}_s]$ and showing it is equal to $X_s$. By (4.29), see Exercise 6.11, we have

$$S_t = S_0 e^{\alpha t} + \sigma \int_0^t e^{(t-s)\alpha} dB_s,$$

hence

$$X_t = S_0 + \sigma \int_0^t e^{-rs}dB_s, \quad t \in \mathbb{R}_+,$$

and

$$\mathbb{E}[X_t \mid \mathcal{F}_s] = \mathbb{E}\left[ S_0 + \sigma \int_0^t e^{-ru} dB_u \mid \mathcal{F}_s \right]$$

$$= \mathbb{E}[S_0] + \sigma \mathbb{E}\left[ \int_0^t e^{-ru} dB_u \mid \mathcal{F}_s \right]$$

$$= S_0 + \sigma \mathbb{E}\left[ \int_s^t e^{-ru} dB_u \mid \mathcal{F}_s \right] + \sigma \mathbb{E}\left[ \int_s^t e^{-ru} dB_u \mid \mathcal{F}_s \right]$$

$$= S_0 + \sigma \int_0^s e^{-ru} dB_u + \sigma \mathbb{E}\left[ \int_0^t e^{-ru} dB_u \mid \mathcal{F}_s \right]$$

$$= S_0 + \sigma \int_0^s e^{-ru} dB_u$$

$$= X_s, \quad 0 \leq s \leq t.$$

b) We rewrite the stochastic differential equation satisfied by $(S_t)_{t \in \mathbb{R}_+}$ as

$$dS_t = \alpha S_t dt + \sigma dB_t = rS_t dt + \sigma d\hat{B}_t,$$

where

$$d\hat{B}_t := \frac{\alpha - r}{\sigma} S_t dt + dB_t,$$

which allows us to rewrite (4.29) with $\alpha = -r$ as

$$S_t = e^{rt}\left( S_0 + \sigma \int_0^t e^{-rs} d\hat{B}_s \right) = S_0 e^{rt} + \sigma \int_0^t e^{(t-s)r} d\hat{B}_s. \quad (A.28)$$

Taking

$$\psi_t := \frac{\alpha - r}{\sigma} S_t, \quad 0 \leq t \leq T,$$

in the Girsanov Theorem 7.3, the process $(\hat{B}_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion under the probability measure $\mathbb{P}_\alpha$ defined by
d) We have

\[
\frac{d\mathbb{P}_{\alpha}}{d\mathbb{P}} := \exp \left( - \int_0^T \psi_t dB_t - \frac{1}{2} \int_0^T \psi_t^2 dt \right) = \exp \left( - \frac{\alpha - r}{\sigma} \int_0^T S_t dB_t - \frac{1}{2} \left( \frac{\alpha - r}{\sigma} \right)^2 \int_0^T S_t^2 dt \right),
\]

and \((X_t)_{t \in \mathbb{R}_+}^\ast\) is a martingale under \(\mathbb{P}_{\alpha}\).

c) Using (A.28) under the risk-neutral probability measure \(\mathbb{P}^\ast\), we have

\[
C(t, S_t) = e^{-(T-t)r} \mathbb{E}_{\alpha}[\exp(S_T) \mid F_t] = e^{-(T-t)r} \mathbb{E}_{\alpha} \left[ \exp \left( e^{rT} S_0 + \sigma \int_0^T e^{(T-u)r} d\widehat{B}_u \right) \mid F_t \right] = e^{-(T-t)r} \mathbb{E}_{\alpha} \left[ \exp \left( \sigma \int_0^T e^{(T-u)r} d\widehat{B}_u \right) \mid F_t \right] = e^{-(T-t)r} \mathbb{E}_{\alpha} \left[ \exp \left( \sigma \int_0^T e^{(T-u)r} e^{(T-u)r} d\widehat{B}_u \right) \mid F_t \right] = e^{-(T-t)r} \mathbb{E}_{\alpha} \left[ \exp \left( \frac{\sigma^2}{2} \int_0^T (e^{(T-u)r})^2 du \right) \right] = e^{-(T-t)r} \mathbb{E}_{\alpha} \left[ \exp \left( \frac{\sigma^2}{2} e^{2(T-t)r} - 1 \right) \right], \quad 0 \leq t \leq T.
\]

d) We have

\[
\xi_t = \frac{\partial C}{\partial x}(t, S_t) = \exp \left( S_t e^{(T-t)r} + \frac{\sigma^2}{4r} (e^{2(T-t)r} - 1) \right)
\]

and

\[
\eta_t = \frac{C(t, S_t) - \xi_t S_t}{A_t} = e^{-(T-t)r} \frac{\mathbb{E}_{\alpha}}{A_t} \exp \left( S_t e^{(T-t)r} + \frac{\sigma^2}{4r} (e^{2(T-t)r} - 1) \right) - \frac{S_t}{A_t} \exp \left( S_t e^{(T-t)r} + \frac{\sigma^2}{4r} (e^{2(T-t)r} - 1) \right).
\]

e) We have

\[
dC(t, S_t) = r e^{-(T-t)r} \exp \left( S_t e^{(T-t)r} + \frac{\sigma^2}{4r} (e^{2(T-t)r} - 1) \right) dt - r S_t \exp \left( S_t e^{(T-t)r} + \frac{\sigma^2}{4r} (e^{2(T-t)r} - 1) \right) dt
\]
Exercise 7.12

a) Using (A.28) under the risk-neutral probability measure \( \mathbb{P}^* \), we have

\[
C(t, S_t) = e^{-(T-t)r} \mathbb{E}_\alpha [S_T^2 \mid \mathcal{F}_t]
\]

\[
= e^{-(T-t)r} \mathbb{E}_\alpha \left[ \left( e^{rT} S_0 + \sigma \int_0^T e^{(T-u)r} \, d\hat{B}_u \right)^2 \mid \mathcal{F}_t \right]
\]

\[
= e^{-(T-t)r} \mathbb{E}_\alpha \left[ \left( e^{rT} S_0 + \sigma \int_0^t e^{(T-u)r} \, d\hat{B}_u + \sigma \int_t^T e^{(T-u)r} \, d\hat{B}_u \right)^2 \mid \mathcal{F}_t \right]
\]

\[
= e^{-(T-t)r} \mathbb{E}_\alpha \left[ \left( e^{rT} S_0 + \sigma \int_0^t e^{(T-u)r} \, d\hat{B}_u \right)^2 \mid \mathcal{F}_t \right] + 2\sigma e^{-(T-t)r} \left( e^{rT} S_0 + \sigma \int_0^t e^{(T-u)r} \, d\hat{B}_u \right) \mathbb{E}_\alpha \left[ \int_t^T e^{(T-u)r} \, d\hat{B}_u \mid \mathcal{F}_t \right]
\]

\[
+ \sigma^2 e^{-(T-t)r} \mathbb{E}_\alpha \left[ \left( \int_t^T e^{(T-u)r} \, d\hat{B}_u \right)^2 \mid \mathcal{F}_t \right]
\]

On the other hand we have

\[
\xi_t dS_t + \eta_t dA_t = \xi_t dS_t
\]

\[
+ r e^{-(T-t)r} \exp \left( S_t e^{(T-t)r} + \frac{\sigma^2}{4r} \left( e^{2(T-t)r} - 1 \right) \right) dt
\]

\[
- r S_t \exp \left( S_t e^{(T-t)r} + \frac{\sigma^2}{4r} \left( e^{2(T-t)r} - 1 \right) \right) dt,
\]

showing that

\[
dC(t, S_t) = \xi_t dS_t + \eta_t dA_t,
\]

and confirming that the strategy \( (\xi_t, \eta_t)_{t \in \mathbb{R}^+} \) is self-financing.
Exercise 7.13 (Exercise 6.2 continued). If
\[ b) \] We find
\[
\frac{\partial C}{\partial x}(t, S_t) = 2 e^{(T-t)r} S_t, \quad 0 \leq t \leq T.
\]

Exercise 7.14

Exercise 7.14 (Exercise 6.2 continued). If \( C = \phi(S_T) \) such that \((\xi_t, \eta_t)_{t \in [0,T]}\) hedges the claim payoff \( C \), the arbitrage price of the claim payoff \( C \) at time \( t \in [0, T] \) is given by
\[
\pi_t(X) = V_t = e^{-r(T-t)} \mathbb{E}^*[\phi(S_T) \mid \mathcal{F}_t], \quad 0 \leq t \leq T,
\]
where \( \mathbb{E}^* \) denotes expectation under the risk-neutral measure \( \mathbb{P}^* \). Hence, from the noncentral Chi square probability density function
\[
f_{T-t}(x) = \frac{2\beta}{\sigma^2(1 - e^{-\beta(T-t)})} \exp \left( -\frac{2\beta(x + r_t e^{-\beta(T-t)})}{\sigma^2(1 - e^{-\beta(T-t)})} \right) \left( \frac{x}{r_t e^{-\beta(T-t)}} \right)^{\alpha\beta/\sigma^2 - 1/2}
\times I_{2\alpha\beta/\sigma^2 - 1} \left( \frac{4\beta \sqrt{r_t x e^{-\beta(T-t)}}}{\sigma^2(1 - e^{-\beta(T-t)})} \right),
\]
of \( S_T \) given \( S_t, x > 0 \), we find
\[
g(t, S_t) = e^{-r(T-t)} \mathbb{E}^*[\phi(S_T) \mid \mathcal{F}_t]
\]
\[
= \frac{2\beta e^{-r(T-t)}}{\sigma^2(1 - e^{-\beta(T-t)})} \int_0^\infty \phi(x) \exp \left( -\frac{2\beta(x + S_t e^{-\beta(T-t)})}{\sigma^2(1 - e^{-\beta(T-t)})} \right) \left( \frac{x}{S_t e^{-\beta(T-t)}} \right)^{\alpha\beta/\sigma^2 - 1/2}
I_{2\alpha\beta/\sigma^2 - 1} \left( \frac{4\beta \sqrt{S_t x e^{-\beta(T-t)}}}{\sigma^2(1 - e^{-\beta(T-t)})} \right) dx
\]
\( 0 \leq t \leq T \), under the Feller condition \( 2\alpha\beta \geq \sigma^2 \).
Notes on Stochastic Finance

a) We have

\[
\frac{\partial f}{\partial t}(t, x) = (r - \sigma^2/2)f(t, x), \quad \frac{\partial f}{\partial x}(t, x) = \sigma f(t, x),
\]

and

\[
\frac{\partial^2 f}{\partial x^2}(t, x) = \sigma^2 f(t, x),
\]

hence

\[
dS_t = df(t, B_t)
\]

\[
= \frac{\partial f}{\partial t}(t, B_t) dt + \frac{\partial f}{\partial x}(t, B_t) dB_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, B_t) dt
\]

\[
= \left( r - \frac{\sigma^2}{2} \right) f(t, B_t) dt + \sigma f(t, B_t) dB_t + \frac{1}{2} \sigma^2 f(t, B_t) dt
\]

\[
= rf(t, B_t) dt + \sigma f(t, B_t) dB_t
\]

\[
= rS_t dt + \sigma S_t dB_t.
\]

b) We have

\[
\mathbb{E} \left[ e^{\sigma B_T} \mid \mathcal{F}_t \right] = \mathbb{E} \left[ e^{(B_T - B_t + B_t)\sigma} \mid \mathcal{F}_t \right]
\]

\[
= e^{\sigma B_t} \mathbb{E} \left[ e^{(B_T - B_t)\sigma} \mid \mathcal{F}_t \right]
\]

\[
= e^{\sigma B_t} \mathbb{E} \left[ e^{(B_T - B_t)\sigma} \right]
\]

\[
= e^{\sigma B_t + \sigma^2(T-t)/2}.
\]

c) We have

\[
\mathbb{E} [S_T \mid \mathcal{F}_t] = \mathbb{E} \left[ e^{\sigma B_T + r T - \sigma^2 T/2} \mid \mathcal{F}_t \right]
\]

\[
= e^{r T - \sigma^2 T/2} \mathbb{E} \left[ e^{\sigma B_T} \mid \mathcal{F}_t \right]
\]

\[
= e^{r T - \sigma^2 T/2} e^{\sigma B_t + \sigma^2(T-t)/2}
\]

\[
= e^{r T + \sigma B_t - \sigma^2 t/2}
\]

\[
= e^{(T-t)r + \sigma B_t + rt - \sigma^2 t/2}
\]

\[
= e^{(T-t)r} S_t, \quad t \in [0, T].
\]

d) We have

\[
V_t = e^{-(T-t)r} \mathbb{E} [C \mid \mathcal{F}_t]
\]

\[
= e^{-(T-t)r} \mathbb{E} [S_T - K \mid \mathcal{F}_t]
\]

\[
= e^{-(T-t)r} \mathbb{E} [S_T \mid \mathcal{F}_t] - e^{-(T-t)r} \mathbb{E} [K \mid \mathcal{F}_t]
\]

\[
= S_t - e^{-(T-t)r} K, \quad t \in [0, T].
\]
e) We take $\xi_t = 1$ and $\eta_t = -Ke^{-rT}/A_0$, $t \in [0,T]$.

f) We find

$$V_T = \mathbb{E}[C \mid \mathcal{F}_T] = C.$$

Exercise 7.15 Binary options. (Exercise 6.10 continued).

a) By definition of the indicator (or step) functions $\mathbb{1}_{[K,\infty)}$ and $\mathbb{1}_{[0,K]}$ we have

$$\mathbb{1}_{[K,\infty)}(x) = \begin{cases} 1 & \text{if } x \geq K, \\ 0 & \text{if } x < K, \end{cases}$$

resp. $\mathbb{1}_{[0,K]}(x) = \begin{cases} 1 & \text{if } x \leq K, \\ 0 & \text{if } x > K, \end{cases}$

which shows the claimed result by the definition of $C_b$ and $P_b$.

b) We have

$$\pi_t(C_b) = e^{-(T-t)r} \mathbb{E}[C_b \mid \mathcal{F}_t] = e^{-(T-t)r} \mathbb{E}[\mathbb{1}_{[K,\infty)}(S_T) \mid S_t] = e^{-(T-t)r} P(S_T \geq K \mid S_t) = C_b(t, S_t).$$

c) We have $\pi_t(C_b) = C_b(t, S_t)$, where

$$C_b(t, x) = e^{-(T-t)r} P(S_T > K \mid S_t = x) = e^{-(T-t)r} \Phi \left( \frac{(r - \sigma^2/2)(T-t) + \log(S_t/K)}{\sigma \sqrt{T-t}} \right) = e^{-(T-t)r} \Phi (d_-(T-t)),$$

with

$$d_-(T-t) = \frac{(r - \sigma^2/2)(T-t) + \log(S_t/K)}{\sigma \sqrt{T-t}}.$$

d) The price of this modified contract with payoff

$$C_\alpha = \mathbb{1}_{[K,\infty)}(S_T) + \alpha \mathbb{1}_{[0,K]}(S_T)$$

is given by

$$\pi_t(C_\alpha) = e^{-(T-t)r} \mathbb{E}[\mathbb{1}_{[K,\infty)}(S_T) + \alpha \mathbb{1}_{[0,K]}(S_T) \mid S_t] = e^{-(T-t)r} P(S_T \geq K \mid S_t) + \alpha e^{-(T-t)r} P(S_T \leq K \mid S_t) = e^{-(T-t)r} P(S_T \geq K \mid S_t) + \alpha (1 - P(S_T \geq K \mid S_t)) = \alpha e^{-(T-t)r} / e^{-(T-t)r} + (1 - \alpha) P(S_T \geq K \mid S_t)$$
\[
= \alpha e^{-(T-t)r} + (1-\alpha) e^{-(T-t)r} \Phi \left( \frac{(T-t)r - (T-t)\sigma^2/2 + \log(S_t/K)}{\sigma \sqrt{T-t}} \right).
\]

Fig. S.25: Price of a binary call option.

e) We note that
\[
\mathbb{1}_{[K,\infty)}(S_T) + \mathbb{1}_{[0,K]}(S_T) = \mathbb{1}_{[0,\infty)}(S_T),
\]
almost surely since \( \mathbb{P}(S_T = K) = 0 \), hence
\[
\pi_t(C_b) + \pi_t(P_b) = e^{-(T-t)r} \mathbb{E}[C_b | \mathcal{F}_t] + e^{-(T-t)r} \mathbb{E}[P_b | \mathcal{F}_t]
= e^{-(T-t)r} \mathbb{E}[C_b + P_b | \mathcal{F}_t]
= e^{-(T-t)r} \mathbb{E} \left[ \mathbb{1}_{[K,\infty)}(S_T) + \mathbb{1}_{[0,K]}(S_T) | \mathcal{F}_t \right]
= e^{-(T-t)r} \mathbb{E} \left[ \mathbb{1}_{[0,\infty)}(S_T) | \mathcal{F}_t \right]
= e^{-(T-t)r} \mathbb{E}[1 | \mathcal{F}_t]
= e^{-(T-t)r}, \quad 0 \leq t \leq T.
\]

f) We have
\[
\pi_t(P_b) = e^{-(T-t)r} - \pi_t(C_b)
= e^{-(T-t)r} - e^{-(T-t)r} \Phi \left( \frac{(T-t)r - (T-t)\sigma^2/2 + \log(x/K)}{\sigma \sqrt{T-t}} \right)
= e^{-(T-t)r} \left( 1 - \Phi(d_-(T-t)) \right)
= e^{-(T-t)r} \Phi(-d_-(T-t)).
\]

g) We have
\[
\xi_t = \frac{\partial C_b}{\partial x}(t, S_t)
\]

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\[
\begin{align*}
&= e^{-(T-t)r} \frac{\partial}{\partial x} \Phi \left( \frac{(T-t)r - (T-t)\sigma^2/2 + \log(x/K)}{\sigma \sqrt{T-t}} \right)_{x=S_t} \\
&= e^{-(T-t)r} \frac{1}{\sigma S_t \sqrt{2(T-t)\pi}} e^{-\left(\frac{d_- (T-t)}{2}\right)} \\
&> 0.
\end{align*}
\]

The Black-Scholes hedging strategy of such a call option does not involve short selling because \( \xi_t > 0 \) for all \( t \), cf. Figure S.26 which represents the risky investment in the hedging portfolio of a binary call option.

![Fig. S.26: Risky hedging portfolio value for a binary call option.](image)

Figure S.27 presents the risk-free hedging portfolio value for a binary call option.

![Fig. S.27: Risk-free hedging portfolio value for a binary call option.](image)

h) Here we have

\[ \xi_t = \frac{\partial P_b}{\partial x}(t, S_t) \]
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\[ = e^{-\left(T-t\right)r} \frac{\partial}{\partial x} \Phi \left( -\frac{(T-t)r - \left(T-t\right)\sigma^2/2 + \log(x/K)}{\sigma\sqrt{T-t}} \right) \bigg|_{x=S_t} \]

\[ = -e^{-\left(T-t\right)r} \frac{1}{\sigma\sqrt{2\left(T-t\right)\pi S_t}} e^{-\left(d_-(T-t)\right)^2/2} \]

\[ < 0. \]

The Black-Scholes hedging strategy of such a put option does involve short selling because \( \xi_t < 0 \) for all \( t \).

Exercise 7.16 Using Itô’s formula and the fact that the expectation of the stochastic integral with respect to \( (W_t)_{t \in \mathbb{R}_+} \) is zero, cf. Relation (4.16), we have

\[ C(x,T) = e^{-rT} \mathbb{E} \left[ \phi(S_T) \mid S_0 = x \right] \tag{A.29} \]

\[ = \phi(x) - r \mathbb{E} \left[ \int_0^T e^{-rs} \phi(S_t) dt \mid S_0 = x \right] \]

\[ + r \mathbb{E} \left[ S_t \int_0^T e^{-rt} \phi'(S_t) dt \mid S_0 = x \right] \]

\[ + \sigma \mathbb{E} \left[ S_t \int_0^T e^{-rt} \phi'(S_t) dB_t \mid S_0 = x \right] \]

\[ + \frac{1}{2} \mathbb{E} \left[ \int_0^T e^{-rt} \phi''(S_t) \sigma^2(S_t) dt \mid S_0 = x \right] \]

\[ = \phi(x) - r \mathbb{E} \left[ \int_0^T e^{-rs} \phi(S_t) dt \mid S_0 = x \right] \]

\[ + r \mathbb{E} \left[ S_t \int_0^T e^{-rt} \phi'(S_t) dt \mid S_0 = x \right] \]

\[ + \frac{1}{2} \mathbb{E} \left[ \int_0^T e^{-rt} \phi''(S_t) \sigma^2(S_t) dt \mid S_0 = x \right] \]

\[ = \phi(x) - \int_0^T r e^{-rt} \mathbb{E} \left[ \phi(S_t) \mid S_0 = x \right] dt \]

\[ + r \int_0^T e^{-rt} \mathbb{E} \left[ S_t \phi'(S_t) \mid S_0 = x \right] dt \]

\[ + \frac{1}{2} \int_0^T e^{-rt} \mathbb{E} \left[ \phi''(S_t) \sigma^2(S_t) \mid S_0 = x \right] dt, \]

hence by differentiation with respect to \( T \) we find

\[ \Theta_T = \frac{\partial}{\partial T} \left( e^{-rT} \mathbb{E} \left[ \phi(S_T) \mid S_0 = x \right] \right) \]

\[ = -re^{-rT} \mathbb{E} \left[ \phi(S_T) \mid S_0 = x \right] + re^{-rT} \mathbb{E} \left[ S_T \phi'(S_T) \mid S_0 = x \right] \]

\[ + \frac{1}{2} e^{-rT} \mathbb{E} \left[ \phi''(S_T) \sigma^2(S_T) \mid S_0 = x \right]. \]
Problem 7.17 Choose options.

a) We take conditional expectations in the equality
\[(S_T - K)^+ - (K - S_T)^+ = S_T - K\]
to find
\[
\begin{align*}
C(t, S_t, K, T) - P(t, S_t, K, T) &= e^{-(T-t)r} \mathbb{E}^*[\mathbb{F}_t] - e^{-(T-t)r} \mathbb{E}^*[\mathbb{F}_t] - e^{-(T-t)r} \mathbb{E}^*[\mathbb{F}_t] \\
&= e^{-(T-t)r} \mathbb{E}^*[S^T - K | \mathbb{F}_t] \\
&= e^{-(T-t)r} \mathbb{E}^*[S^T | \mathbb{F}_t] - Ke^{-(T-t)r} \\
&= S_t - Ke^{-(T-t)r}, \quad t \in [0, T].
\end{align*}
\]

b) The price this contract at time \(t \in [0, T]\) can be written as
\[
e^{-(T-t)r} \mathbb{E}^*[P(T, S_T, K, U) | \mathbb{F}_t] = e^{-(T-t)r} \mathbb{E}^*[e^{-(U-T)r} \mathbb{E}^*[\mathbb{F}_T] | \mathbb{F}_t] = e^{-(T-t)r} \mathbb{E}^*[\mathbb{F}_t] = P(t, S_t, K, U).
\]

c) From the call-put parity (7.38) the payoff of this contract can be written as
\[
\begin{align*}
&\text{Max}(P(T, S_T, K, U), C(T, S_T, K, U)) \\
&= \text{Max}(P(T, S_T, K, U), P(T, S_T, K, U) + S_T - Ke^{-(U-T)r}) \\
&= P(T, S_T, K, U) + \text{Max}(S_T - Ke^{-(U-T)r}, 0).
\end{align*}
\]

d) The contract of Question (c) is priced at any time \(t \in [0, T]\) as
\[
e^{-(T-t)r} \mathbb{E}^*[\text{Max}(P(T, S_T, K, U), C(T, S_T, K, U)) | \mathbb{F}_t] \\
&= e^{-(T-t)r} \mathbb{E}^*[P(T, S_T, K, U) | \mathbb{F}_t] \\
&\quad + e^{-(T-t)r} \mathbb{E}^*[\text{Max}(S_T - Ke^{-(U-T)r}, 0) | \mathbb{F}_t] \\
&= e^{-(T-t)r} \mathbb{E}^*[e^{-(U-T)r} \mathbb{E}^*[\mathbb{F}_T] | \mathbb{F}_t] \\
&\quad + e^{-(T-t)r} \mathbb{E}^*[\text{Max}(S_T - Ke^{-(U-T)r}, 0) | \mathbb{F}_t] \\
&= e^{-(U-t)r} \mathbb{E}^*[\mathbb{F}_T] \\
&\quad + e^{-(T-t)r} \mathbb{E}^*[\text{Max}(S_T - Ke^{-(U-T)r}, 0) | \mathbb{F}_t] \\
&= P(t, S_t, K, U) + C(t, S_t, K e^{-(U-T)r}, T). \quad \text{(A.30)}
\]
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Fig. S.28: Black-Scholes price of the maximum chooser option.

e) By (A.30) and Relation (6.3) in Proposition 6.1 we have

\[
\xi_t = \frac{\partial C}{\partial x}(t, S_t, K, e^{-(U-T)r}, T) + \frac{\partial P}{\partial x}(t, S_t, K, U) \\
= \Phi \left( \frac{\log(e^{(U-T)T}S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right) \\
- \Phi \left( -\frac{\log(S_t/K) + (r + \sigma^2/2)(U-t)}{\sigma \sqrt{U-t}} \right) \\
= \Phi \left( \frac{\log(S_t/K) + (U-t)r + (T-t)\sigma^2/2}{\sigma \sqrt{T-t}} \right) \\
- \Phi \left( -\frac{\log(S_t/K) + (r + \sigma^2/2)(U-t)}{\sigma \sqrt{U-t}} \right),
\]

Fig. S.29: Delta of the maximum chooser option.

f) From the call-put parity (7.38) the payoff of this contract can be written as

\[
\min(P(T, S_T, K, U), C(T, S_T, K, U))
\]
= \min(C(T, S_T, K, U) - S_T + K e^{-(U-T)r}, C(T, S_T, K, U)) \\
= C(T, S_T, K, U) + \min(-S_T + K e^{-(U-T)r}, 0) \\
= C(T, S_T, K, U) - \text{Max}(S_T - K e^{-(U-T)r}, 0).

\( g \) The contract of Question (f) is priced at any time \( t \in [0, T] \) as
\[
e^{-(T-t)r} E^* \left[ \min \left( P(T, S_T, K, U), C(T, S_T, K, U) \right) \mid \mathcal{F}_t \right] \\
= e^{-(T-t)r} E^* \left[ C(T, S_T, K, U) \mid \mathcal{F}_t \right] \\
- e^{-(T-t)r} E^* \left[ \text{Max} (S_T - K e^{-(U-T)r}, 0) \mid \mathcal{F}_t \right] \\
= e^{-(T-t)r} E^* \left[ e^{-(U-T)r} E^* \left[ (S_U - K)^+ \mid \mathcal{F}_T \right] \mid \mathcal{F}_t \right] \\
- e^{-(T-t)r} E^* \left[ \text{Max} (S_T - K e^{-(U-T)r}, 0) \mid \mathcal{F}_t \right] \\
= e^{-(U-t)r} E^* \left[ (S_U - K)^+ \mid \mathcal{F}_t \right] \\
- e^{-(T-t)r} E^* \left[ \text{Max} (S_T - K e^{-(U-T)r}, 0) \mid \mathcal{F}_t \right] \\
= C(t, S_t, K, U) - C(t, S_t, K e^{-(U-T)r}, T). \tag{A.31}
\]

Fig. S.30: Black-Scholes price of the minimum chooser option.

\( h \) By (A.31) and Relation (6.3) in Proposition 6.1 we have
\[
\xi_t = \frac{\partial C}{\partial x}(t, S_t, K, U) - \frac{\partial C}{\partial x}(t, S_t, K e^{-(U-T)r}, T) \\
= \Phi \left( \frac{\log(S_t/K) + (r + \sigma^2/2)(U - t)}{\sigma \sqrt{U - t}} \right) \\
- \Phi \left( \frac{\log(e^{(U-T)r}S_t/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} \right) \\
= \Phi \left( \frac{\log(S_t/K) + (r + \sigma^2/2)(U - t)}{\sigma \sqrt{U - t}} \right)
\]
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\[ -\Phi \left( \frac{\log(S_t/K) + (U - t)r + (T - t)\sigma^2/2}{\sigma \sqrt{T - t}} \right). \]

Fig. S.31: Delta of the minimum chooser option.

i) Such a contract is priced as the sum of a European call and a European put option with maturity \( U \), and is priced at time \( t \in [0, T] \) as \( P(t, S_t, K, U) + C(t, S_t, K, U) \). Its hedging strategy is the sum of the hedging strategies of Questions (e) and (h), i.e.

\[
\xi_t = \Phi \left( \frac{\log(S_t/K) + (r + \sigma^2/2)(U - t)}{\sigma \sqrt{T - t}} \right) - \Phi \left( -\frac{\log(S_t/K) + (r + \sigma^2/2)(U - t)}{\sigma \sqrt{T - t}} \right) = 2\Phi \left( \frac{\log(S_t/K) + (r + \sigma^2/2)(U - t)}{\sigma \sqrt{T - t}} \right) - 1.
\]

j) When \( U = T \), the contracts of Questions (e), (f) and (i) have respective payoffs

- \( \text{Max}((S_T - K)^+, (K - S_T)^+) = |S_T - K| \),
- \( \text{min}((S_T - K)^+, (K - S_T)^+) = 0 \), and
- \( (S_T - K)^+ + (K - S_T)^+ = |S_T - K| \),

where \( |S_T - K| \) is known as the payoff of a \textit{straddle option}.

Problem 7.18

a) The self-financing condition reads

\[
dV_t = \eta_t dA_t + \xi_t dS_t = r\eta_t A_t dt + \mu\xi_t S_t dt + \sigma\xi_t S_t dB_t
\]
\[ V_t = V_0 + \int_0^T (rV_t + (\mu - r)\xi_t S_t)dt + \sigma \xi_t S_t dB_t, \]

hence

\[ V_T = V_0 + \int_0^T (rV_t + (\mu - r)\xi_t S_t)dt + \sigma \xi_t S_t dB_t. \]

b) The portfolio value \( V_t \) rewrites as

\[ V_t = V_T - \int_t^T \left( rV_s + \frac{\mu - r}{\sigma} \pi_s \right) ds - \int_t^T \pi_s dB_s \]

\[ = V_T - r \int_t^T V_s ds - \int_t^T \pi_s d\hat{B}_s. \]

c) We have

\[ V_t = V_T - r \int_t^T V_s ds - \int_t^T \pi_s d\hat{B}_s, \]

hence

\[ dV_t = rV_t dt + \pi_t d\hat{B}_t, \]

and after discounting we find

\[ d\tilde{V}_t = -r e^{-rt} V_t dt + e^{-rt} dV_t \]

\[ = -r e^{-rt} V_t dt + e^{-rt} (rV_t dt + \pi_t d\hat{B}_t) \]

\[ = e^{-rt} \pi_t d\hat{B}_t, \]

which shows that

\[ \tilde{V}_T = V_0 + \int_0^T e^{-rt} \pi_t d\hat{B}_t, \]

after integration in \( t \in [0, T] \).

d) We have

\[ dV_t = du(t, S_t) \]

\[ = \frac{\partial u}{\partial t}(t, S_t) dt + \mu S_t \frac{\partial u}{\partial x}(t, S_t) dt + \sigma S_t \frac{\partial^2 u}{\partial x^2}(t, S_t) dB_t \]

\[ + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 u}{\partial x^2}(t, S_t) dt. \]

(A.32)

e) By matching the Itô formula (A.32) term by term to the BSDE (7.41) we find that \( V_t = u(t, S_t) \) satisfies the PDE

\[ \frac{\partial u}{\partial t}(t, x) + \mu \frac{\partial u}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 u}{\partial x^2}(t, x) + f \left( t, x, u(t, x), \sigma x \frac{\partial u}{\partial x}(t, x) \right) = 0. \]

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https://www.ntu.edu.sg/home/nprivault/index.html
f) In this case we have
\[ \frac{\partial u}{\partial t}(t, x) + \mu x \frac{\partial u}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 u}{\partial x^2}(t, x) - ru(t, x) - (\mu - r)x \frac{\partial u}{\partial x}(t, x) = 0, \]
which recovers the Black-Scholes PDE
\[ ru(t, x) = \frac{\partial u}{\partial t}(t, x) + r x \frac{\partial u}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 u}{\partial x^2}(t, x). \]

g) In the Black-Scholes model the Delta of the European call option is given by
\[ \xi_t = \Phi \left( \frac{(r + \sigma^2/2)(T - t) + \log(S_t/K)}{\sigma \sqrt{T - t}} \right), \]
hence
\[ \pi_t = \sigma \xi_t S_t = \sigma S_t \Phi \left( \frac{(r + \sigma^2/2)(T - t) + \log(S_t/K)}{\sigma \sqrt{T - t}} \right), \quad t \in [0, T]. \]
h) Replacing the self-financing condition with
\[ dV_t = \eta_t dA_t + \xi_t dS_t - \gamma S_t (\xi_t^-)^{-} dt \]
\[ = r \eta_t A_t dt + \mu \xi_t S_t dt + \sigma \xi_t S_t dB_t - \gamma S_t (\xi_t^-)^{-} dt \]
\[ = r V_t dt + (\mu - r) \xi_t S_t dt - \gamma S_t (\xi_t^-)(^-) dt + \sigma \xi_t S_t dB_t, \]
we get the BSDE
\[ V_t = V_T - \int_t^T (r V_s + (\mu - r) \pi_s + \gamma (\pi_s^-)^) ds - \int_t^T \pi_s dB_s. \]
i) In this case we have
\[ f(t, x, u, z) = -ru - \frac{\mu - r}{\sigma} z - \gamma z^- \]
and the BSDE reads
\[ dV_t = ru(t, S_t) dt + (\mu - r) \xi_t S_t dt - \gamma S_t (\xi_t^-)^{-} dt + \sigma \xi_t S_t dB_t. \]
j) We find the nonlinear PDE
\[ \frac{\partial u}{\partial t}(t, x) + \mu x \frac{\partial u}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 u}{\partial x^2}(t, x) - \gamma \sigma x \left( \frac{\partial u}{\partial x}(t, x) \right)^{-} = ru(t, x), \]
with the terminal condition \( u(T, x) = g(x) \).
k) The self-financing condition reads
\[ dV_t = r \mathbb{1}_{\{\eta_t > 0\}} A_t \eta_t dt + R \mathbb{1}_{\{\eta_t < 0\}} A_t \eta_t dt + \xi_t dS_t \]
which yields the BSDE
\[ V_t = V_T - \int_t^T (r V_s + (\mu - r) \pi_s - (R - r) (V_s - \xi_s S_s)^-) \, ds - \int_t^T \pi_s dB_s, \]
hence we have
\[ f(t, x, u, z) = -ru - \frac{(\mu - r)}{\sigma} z + (R - r) \left( u - \frac{z}{\sigma} \right)^- \]
and the nonlinear PDE
\[ \frac{\partial u}{\partial t} (t, x) + \mu \frac{\partial u}{\partial x} (t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} (t, x) + f \left( t, x, u(t, x), \sigma x \frac{\partial u}{\partial x} (t, x) \right) = 0 \]
rewrites as
\[ \frac{\partial u}{\partial t} (t, x) + r \frac{\partial u}{\partial x} (t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} (t, x) = ru(t, x) + (r - R) \left( u(t, x) - x \frac{\partial u}{\partial x} (t, x) \right)^-. \]

1) The sum of profits and losses of the portfolio \((\xi_t, \eta_t)_{t \in \mathbb{R}_+}\)
\[ V_0 + \int_0^T \eta_t dA_t + \int_0^T \xi_t dS_t = V_0 + \int_0^T dV_t + \int_0^T dU_t \]
\[ = V_T + U_T - U_0 \]
\[ > V_T = C, \]
hence the corresponding portfolio strategy superhedging the claim payoff \(V_T = C\).

Exercise 7.19 Girsanov Theorem. For all \(n \geq 1\), let
\[ \psi_t^{(n)} := \mathbb{1}_{\{\psi_t \in [-n,n]\}} \psi_t, \quad 0 \leq t \leq T. \]
Since \((\psi_t^{(n)})_{t \in [0,T]}\) is a bounded process it satisfies the Novikov integrability condition (7.7), hence for all \(n \geq 1\) and random variable \(F \in L^1(\Omega)\) we have
\[ \mathbb{E}[F] = \mathbb{E} \left[ F \left( B + \int_0^T \psi_s^{(n)} dS_s \right) \exp \left( - \int_0^T \psi_s^{(n)} dB_s - \frac{1}{2} \int_0^T (\psi_s^{(n)})^2 \, ds \right) \right], \]
which yields
\[ \mathbb{E}[F] = \lim_{n \to \infty} \mathbb{E} \left[ F \left( B_t + \int_0^T \psi^n_s \, ds \right) \exp \left( - \int_0^T \psi^n_s \, dB_s - \frac{1}{2} \int_0^T (\psi^n_s)^2 \, ds \right) \right] \]

\[ \geq \mathbb{E} \left[ \liminf_{n \to \infty} F \left( B_t + \int_0^T \psi^n_s \, ds \right) \exp \left( - \int_0^T \psi^n_s \, dB_s - \frac{1}{2} \int_0^T (\psi^n_s)^2 \, ds \right) \right] \]

\[ = \mathbb{E} \left[ F \left( B_t + \int_0^T \psi_s \, ds \right) \exp \left( - \int_0^T \psi_s \, dB_s - \frac{1}{2} \int_0^T (\psi_s)^2 \, ds \right) \right], \]

where we applied Fatou’s Lemma 22.3.

**Problem 7.20**

a) We have

\[
\frac{\text{Cov}(dS_t / S_t, dM_t / M_t)}{\text{Var}[dM_t / M_t]} = \frac{\text{Cov}((r + \alpha)dt + \beta(dM_t / M_t - rdt) + \sigma_S dB_t, \mu dt + \sigma_M dB_t)}{\text{Var}[\mu dt + \sigma_M dB_t]}
\]

\[ = \frac{\text{Cov}((r + \alpha)dt + \beta(\mu dt + \sigma_M dB_t - rdt) + \sigma_S dB_t, \mu dt + \sigma_M dB_t)}{\text{Var}[\mu dt + \sigma_M dB_t]}
\]

\[ = \frac{\text{Cov}(\beta \sigma_M dB_t + \sigma_S dB_t, \sigma_M dB_t)}{\text{Var}[\sigma_M dB_t]}
\]

\[ = \frac{\text{Cov}(\beta \sigma_M dB_t, \sigma_M dB_t) + \text{Cov}(\sigma_S dB_t, \sigma_M dB_t)}{\text{Var}[\sigma_M dB_t]}
\]

\[ = \frac{\text{Cov}(\beta \sigma_M dB_t, \sigma_M dB_t)}{\text{Var}[\sigma_M dB_t]}
\]

\[ = \beta \frac{\text{Var}[\sigma_M dB_t]}{\text{Var}[\sigma_M dB_t]}
\]

\[ = \beta.
\]

b) We have

\[ dS_t = (r + \alpha)S_t dt + \beta \left( \frac{dM_t}{M_t} - r \right) S_t dt + \sigma_S S_t dB_t
\]

\[ = (r + \alpha)S_t dt + \beta S_t (\mu dt + \sigma_M dB_t - rdt) + \sigma_S S_t dB_t
\]

\[ = (r + \alpha + \beta(\mu - r))S_t dt + S_t (\beta \sigma_M dB_t + \sigma_S dB_t)
\]

\[ = (r + \alpha + \beta(\mu - r))S_t dt + S_t \sqrt{\beta^2 \sigma^2_M + \sigma^2_S} \frac{\beta \sigma_M dB_t + \sigma_S dB_t}{\sqrt{\beta^2 \sigma^2_M + \sigma^2_S}}.
\]

Now, we have
\[
\left( \frac{\beta \sigma_M dB_t + \sigma_S dW_t}{\sqrt{\beta^2 \sigma_M^2 + \sigma_S^2}} \right)^2 = \frac{(\beta \sigma_M dB_t)^2 + \beta \sigma_M \sigma_S dB_t \cdot dW_t + (\sigma_S dW_t)^2}{\beta^2 \sigma_M^2 + \sigma_S^2}
\]

By the characterization of Brownian motion as the only continuous martingale whose quadratic variation is \(dt\), it follows that the process \((Z_t)_{t \in \mathbb{R}_+}\) defined by

\[
dZ_t = \frac{\beta \sigma_M dB_t + \sigma_S dW_t}{\sqrt{\beta^2 \sigma_M^2 + \sigma_S^2}}
\]

is a standard Brownian motion, see e.g. Theorem 7.36 page 203 of Klebaner (2005). Hence, we have

\[
dS_t = (r + \alpha + \beta (\mu - r)) S_t dt + S_t (\beta \sigma_M dB_t + \sigma_S dW_t)
\]

\[
= (r + \alpha + \beta (\mu - r)) S_t dt + S_t \sqrt{\beta^2 \sigma_M^2 + \sigma_S^2} dZ_t.
\]

In the sequel we assume that \(\beta\) is allowed to depend locally on the state of the benchmark market index on \(M_t\), as \(\beta(M_t), t \in \mathbb{R}_+\).

c) We take

\[
\begin{align*}
\frac{dB_t^*}{\sigma_M} &= dB_t + \frac{\mu - r}{\sigma_M} dt \quad (A.34) \\
\frac{dW_t^*}{\sigma_S} &= dW_t + \frac{\alpha}{\sigma_S} dt \quad (A.35)
\end{align*}
\]

in order to have

\[
\begin{cases}
\frac{dM_t}{M_t} = \mu dt + \sigma_M dB_t = r dt + \sigma_M dB_t^*, \\
\frac{dS_t}{S_t} = (r + \alpha) dt + \beta(M_t) \times \left( \frac{dM_t}{M_t} - rd dt \right) + \sigma_S dW_t \\
= (r + \alpha) dt + \sigma_M \beta(M_t) dB_t^* + \sigma_S dW_t^* \quad (A.36)
\end{cases}
\]

d) By the Girsanov theorem, \((B_t^*)_{t \in [0,T]}\) is a standard Brownian motion under the probability measure \(\mathbb{P}^*\) defined by its density

\[
\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp \left( -\frac{\mu - r}{\sigma_M} B_T - \frac{(\mu - r)^2}{2\sigma_M^2} T \right),
\]
and \((W^*_t)_{t \in [0,T]}\) is a standard Brownian motion under the probability measure \(\mathbb{P}^*_W\) defined by its density

\[
\frac{d\mathbb{P}^*_W}{d\mathbb{P}} = \exp \left( -\frac{\alpha}{\sigma_S} W_T - \frac{\alpha^2}{2\sigma_S^2} T \right).
\]

We conclude that \((B^*_t)_{t \in [0,T]}\) and \((W^*_t)_{t \in [0,T]}\) are independent standard Brownian motions under the probability measure \(\mathbb{P}^*\) defined by its density

\[
\frac{d\mathbb{P}^*}{d\mathbb{P}} = \frac{d\mathbb{P}_B}{d\mathbb{P}} \times \frac{d\mathbb{P}_W}{d\mathbb{P}} = \exp \left( -\frac{\mu - r}{\sigma_M} B_T - \frac{\alpha}{\sigma_S} W_T - \frac{(\mu - r)^2}{2\sigma_M^2} T - \frac{\alpha^2}{2\sigma_S^2} T \right).
\]

Indeed, for any sequence \(t_0 = 0 < t_1 < \cdots < t_{n-1} < t_n = T\) we have

\[
\mathbb{E}^* \left[ f(B^*_{t_1} - B^*_{t_0}, \ldots, B^*_{t_n} - B^*_{t_{n-1}}) \right] = \mathbb{E} \left[ \frac{d\mathbb{P}^*}{d\mathbb{P}} f(B^*_{t_1} - B^*_{t_0}, \ldots, B^*_{t_n} - B^*_{t_{n-1}}) \right]
\]

\[
= \mathbb{E} \left[ f(B^*_{t_1} - B^*_{t_0}, \ldots, B^*_{t_n} - B^*_{t_{n-1}}) \times \exp \left( -\frac{\mu - r}{\sigma_M} B_T - \frac{(\mu - r)^2}{2\sigma_M^2} T - \frac{\alpha}{\sigma_S} W_T - \frac{\alpha^2}{2\sigma_S^2} T \right) \right]
\]

\[
= \mathbb{E} \left[ f(B^*_{t_1} - B^*_{t_0}, \ldots, B^*_{t_n} - B^*_{t_{n-1}}) \exp \left( -\frac{\mu - r}{\sigma_M} B_T - \frac{(\mu - r)^2}{2\sigma_M^2} T \right) \right]
\]

\[
\times \mathbb{E} \left[ \exp \left( -\frac{\alpha}{\sigma_S} W_T - \frac{\alpha^2}{2\sigma_S^2} T \right) \right]
\]

\[
= \mathbb{E} \left[ f(B^*_{t_1} - B^*_{t_0}, \ldots, B^*_{t_n} - B^*_{t_{n-1}}) \exp \left( -\frac{\mu - r}{\sigma_M} B_T - \frac{(\mu - r)^2}{2\sigma_M^2} T \right) \right]
\]

\[
= \mathbb{E} \left[ f(B_{t_1} - B_{t_0}, \ldots, B_{t_n} - B_{t_{n-1}}) \right],
\]

and similarly for \((W^*_t)_{t \in [0,T]}\).

e) By (A.36), the discounted price processes

\[
(\tilde{S}_t)_{t \in \mathbb{R}_+} := (e^{-rt}S_t)_{t \in \mathbb{R}_+} \quad \text{and} \quad (\tilde{M}_t)_{t \in \mathbb{R}_+} := (e^{-rt}M_t)_{t \in \mathbb{R}_+}
\]

satisfy

\[
\begin{cases}
\frac{d\tilde{M}_t}{d\tilde{t}} = \sigma_M \tilde{M}_t dB^*_t, \\
\frac{d\tilde{S}_t}{d\tilde{t}} = \sigma_M \beta(M_t) \tilde{S}_t dB^*_t + \sigma_S \tilde{S}_t dW^*_t,
\end{cases}
\]

hence by the Girsanov theorem of Question (d) the discounted two-dimensional process \((\tilde{S}_t, \tilde{M}_t)_{t \in \mathbb{R}_+}\) is a martingale under the probability measure \(\mathbb{P}^*\), showing that \(\mathbb{P}^*\) is a risk-neutral probability measure. Therefore, by Theorem 6.8 the market made of \(S_t\) and \(M_t\) is without arbitrage opportunities due to the existence of a risk-neutral probability measure \(\mathbb{P}^*\).
f) The self-financing condition for the portfolio strategy \((\xi_t, \zeta_t, \eta_t)_{t \in [0,T]}\) reads
\[
\eta_{t+dt} A_{t+dt} + \xi_{t+dt} S_{t+dt} + \zeta_{t+dt} M_{t+dt} = \eta_t A_{t+dt} + \xi_t S_{t+dt} + \zeta_t M_{t+dt}
\]
which yields
\[
A_{t+dt} d\eta_t + S_{t+dt} d\xi_t + M_{t+dt} d\zeta_t = 0,
\]
i.e.
\[
dA_t \cdot d\eta_t + dS_t \cdot d\xi_t + dM_t \cdot d\zeta_t + A_t d\eta_t + S_t d\xi_t + M_t d\zeta_t = 0,
\]
hence
\[
dV_t = \eta_t dA_t + \xi_t dS_t + \zeta_t dM_t
\]
\[
= \xi_t \left(r S_t dt + \sigma_M \beta(M_t) S_t dB_t^* + \sigma_S S_t dW_t^* \right) + \zeta_t \left(\sigma_M M_t dt + \sigma_M M_t dB_t^* \right) + \eta_t A_t dt
\]
\[
= r \xi_t S_t dt + \sigma_M \beta(M_t) \xi_t S_t dB_t^* + \sigma_S \xi_t S_t dW_t^* + r \zeta_t M_t dt + \sigma_M \zeta_t M_t dB_t^* + \eta_t A_t dt
\]
\[
= r \xi_t S_t dt + \sigma_M \beta(M_t) \xi_t S_t dB_t^* + \sigma_S \xi_t S_t dW_t^* + r \zeta_t M_t dt + \left(\sigma_M \beta(M_t) \xi_t S_t + \sigma_M \zeta_t M_t \right) dB_t^*
\]
\[
= r V_t d\eta_t + \sigma_S \xi_t S_t dW_t^* + \left(\sigma_M \beta(M_t) \xi_t S_t + \sigma_M \zeta_t M_t \right) dB_t^*. \tag{A.37}
\]

On the other hand, by the Itô formula for two state variables, we have
\[
df(t, S_t, M_t) = \frac{\partial f}{\partial t}(t, S_t, M_t) dt + \frac{\partial f}{\partial x}(t, S_t, M_t) dS_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, S_t, M_t)(dS_t)^2
\]
\[
+ \frac{\partial f}{\partial y}(t, S_t, M_t) dM_t + \frac{1}{2} \frac{\partial^2 f}{\partial x \partial y}(t, S_t, M_t)(dM_t)^2 + \frac{\partial^2 f}{\partial y^2}(t, S_t, M_t) dS_t \cdot dM_t
\]
\[
= \frac{\partial f}{\partial t}(t, S_t, M_t) dt + \frac{\partial f}{\partial x}(t, S_t, M_t) \left(r S_t dt + \sigma_M \beta(M_t) S_t dB_t^* + \sigma_S S_t dW_t^* \right)
\]
\[
+ \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, S_t, M_t) \left(\sigma_M^2 S_t^2 + \sigma_S^2 S_t^2 \right) dt
\]
\[
+ \frac{\partial f}{\partial y}(t, S_t, M_t) \left(\sigma_M M_t dB_t^* \right) + \frac{1}{2} \frac{\partial^2 f}{\partial y^2}(t, S_t, M_t) \left(\sigma_M^2 M_t^2 \right) dt
\]
\[
+ \sigma_M^2 S_t M_t \beta(M_t) \frac{\partial^2 f}{\partial x \partial y}(t, S_t, M_t) dt
\]
\[
= \frac{\partial f}{\partial t}(t, S_t, M_t) dt + \frac{\partial f}{\partial x}(t, S_t, M_t) \left(r S_t + \sigma_M \beta(M_t) \right) dt + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, S_t, M_t) \sigma_M S_t dB_t^*
\]
\[
= \frac{\partial f}{\partial t}(t, S_t, M_t) dt + \frac{\partial f}{\partial x}(t, S_t, M_t) \left(r S_t + \sigma_M \beta(M_t) \right) dt + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, S_t, M_t) \sigma_M S_t dB_t^*
\]
h) By identification of terms in (A.37) and (A.38), we find

\[ rf(t, S_t, M_t) = \frac{\partial f}{\partial t}(t, S_t, M_t) + rS_t \frac{\partial f}{\partial x}(t, S_t, M_t) \]

\[ + \frac{1}{2} (\sigma_S^2 + \sigma_M^2 \beta^2(M_t)) S_t^2 \frac{\partial^2 f}{\partial x^2}(t, S_t, M_t) \]

\[ + rM_t \frac{\partial f}{\partial y}(t, S_t, M_t) + \frac{1}{2} \sigma_M^2 M_t^2 \frac{\partial^2 f}{\partial y^2}(t, S_t, M_t) + \sigma_M S_t M_t \beta(M_t) \frac{\partial^2 f}{\partial x \partial y}(t, S_t, M_t) dt, \]

which yields the PDE

\[ rf(t, x, y) = \frac{\partial f}{\partial t}(t, x, y) + r x \frac{\partial f}{\partial x}(t, x, y) + \frac{1}{2} x^2 (\sigma_S^2 + \sigma_M^2 \beta^2(y)) \frac{\partial^2 f}{\partial x^2}(t, x, y) \]

\[ + r y \frac{\partial f}{\partial y}(t, x, y) + \frac{1}{2} \sigma_M^2 y^2 \frac{\partial^2 f}{\partial y^2}(t, x, y) + \sigma_M x y \beta(y) \frac{\partial^2 f}{\partial x \partial y}(t, x, y), \]

with the terminal condition

\[ f(T, x, y) = h(x, y), \quad x, y > 0. \]

By identification of terms in \( dB_t^* \) and \( dW_t^* \) in (A.37) and (A.38), we find

\[ \xi_t = \frac{\partial f}{\partial x}(t, S_t, M_t) \]

and
\[ \sigma M \beta(M_t) S_t \frac{\partial f}{\partial x}(t, S_t, M_t) + \sigma M M_t \frac{\partial f}{\partial y}(t, S_t, M_t) = \sigma M \beta(M_t) \xi_t S_t + \sigma M \zeta_t M_t, \]

hence
\[ \zeta_t = \frac{\partial f}{\partial y}(t, S_t, M_t), \]

and by the relation \( V_t = \xi_t S_t + \zeta_t M_t + \eta_t \) we find
\[ \eta_t = \frac{V_t - \xi_t S_t - \zeta_t M_t}{A_t}, \]
\[ f(t, S_t, M_t) - S_t \frac{\partial f}{\partial x}(t, S_t, M_t) - M_t \frac{\partial f}{\partial y}(t, S_t, M_t) = \frac{1}{A_0 e^{rt}}, \quad t \in [0, T]. \]

i) When the option payoff depends only on \( S_T \) we can look for a solution of (A.39) of the form \( f(t, x) \), in which case (A.39) simplifies to
\[ rf(t, x, y) = \frac{\partial f}{\partial t}(t, x, y) + r x \frac{\partial f}{\partial x}(t, x, y) + \frac{1}{2} x^2 (\sigma_S^2 + \sigma_M^2 \beta^2(y)) \frac{\partial^2 f}{\partial x^2}(t, x, y), \]

When \( \beta(M_t) = \beta \) is a constant, (A.40) becomes the Black-Scholes PDE with squared volatility parameter
\[ \sigma^2 := \sigma_S^2 + \sigma_M^2 \beta^2. \]

When the option is a European call option with strike price \( K \) on \( S_T \), its solution is given by the Black-Scholes function
\[
\begin{align*}
f(t, x) &= Bl(K, x, \sigma, r, T-t) = x \Phi(d_+(T-t)) - Ke^{-(T-t)r} \Phi(d_-(T-t)), \\
\text{with} \quad d_+(T-t) &= \log(x/K) + \left(r + \frac{\sigma_S^2 + \sigma_M^2 \beta^2}{2}\right)(T-t) / |\sigma| \sqrt{T-t}, \\
\text{d_-} &\quad (T-t) := \log(x/K) + \left(r - \frac{\sigma_S^2 + \sigma_M^2 \beta^2}{2}\right)(T-t) / |\sigma| \sqrt{T-t}, \\
\text{and} \quad \xi_t &= \frac{\partial f}{\partial x}(t, S_t, M_t) = \Phi(d_+(T-t)), \\
\text{with} \quad \eta_t &= -K A_t e^{-(T-t)r} \Phi(d_-(T-t)) = -K A_0 e^{-Tr} \Phi(d_-(T-t)), \quad t \in [0, T].
\end{align*}
\]
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j) Similarly to Question (i), when the option is a European put option with strike price \( K \) on \( S_T \), its solution is given by the Black-Scholes put price function

\[
 f(t, x) = K e^{-(T-t)r} \Phi(-d_-(T-t)) - x \Phi(-d_+(T-t)),
\]

with

\[
 \zeta_t = \frac{\partial f}{\partial y}(t, S_t, M_t) = -\Phi(-d_+(T-t)), \quad t \in [0, T).
\]

and

\[
 \eta_t = \frac{K}{A_0} e^{-Tr} \Phi(-d_-(T-t)), \quad t \in [0, T).
\]

Remark. By the answer to Question (b) we have

\[
dS_t = (r + \alpha + \beta(\mu - r))S_t dt + \sqrt{\beta^2 \sigma^2_M + \sigma^2} S_t dB_t
\]

where \((Z_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion, hence the answers to Questions (i) and j can be recovered from the pricing relation

\[
f(t, S_t) = e^{-(T-t)r} \mathbb{E}^*[\Phi(S_T) \mid \mathcal{F}_t], \quad 0 \leq t \leq T.
\]

Chapter 8

Exercise 8.1 We need to compute the average

\[
\frac{1}{T} \mathbb{E}\left[\int_0^T v_t dt\right] = \frac{1}{T} \int_0^T \mathbb{E}[v_t] dt = \frac{1}{T} \int_0^T u(t) dt,
\]

where \(u(t) := \mathbb{E}[v_t]\). Taking expectation on both sides of the equation

\[
v_t = v_0 - \lambda \int_0^t (v_s - m) ds + \eta \int_0^t \sqrt{v_s} dB_s,
\]

we find

\[
u(t) = \mathbb{E}[v_t]
= \mathbb{E}\left[v_0 - \lambda \int_0^t (v_s - m) ds + \eta \int_0^t \sqrt{v_s} dB_s\right]
= v_0 - \lambda \mathbb{E}\left[\int_0^t (v_s - m) ds\right]
\]
\begin{align*}
&= v_0 - \lambda \int_0^t (\mathbb{E}[v_s] - m) ds \\
&= v_0 - \lambda \int_0^t (u(s) - m) ds, \quad t \geq 0,
\end{align*}

hence by differentiation with respect to \( t \in \mathbb{R} \) we find the ordinary differential equation

\[
u'(t) = \lambda m - \lambda u(t),
\]

cf. e.g. Exercise 4.13-(b). This equation can be rewritten as

\[
(e^{\lambda t} u(t))' = \lambda e^{\lambda t} u(t) + e^{\lambda t} u'(t) = \lambda m e^{\lambda t},
\]

which can be integrated as

\[
e^{\lambda t} u(t) = \left( u(0) + \lambda m \int_0^t e^{\lambda s} ds \right) \\
= \mathbb{E}[v_0] + m(e^{\lambda t} - 1) \\
= m e^{\lambda t} + \mathbb{E}[v_0] - m \quad t \in \mathbb{R}_+,
\]

from which we conclude that

\[
u(t) = m + (\mathbb{E}[v_0] - m) e^{-\lambda t}, \quad t \in \mathbb{R}_+,
\]

and

\[
\frac{1}{T} \mathbb{E} \left[ \int_0^T v_t dt \right] = \frac{1}{T} \int_0^T u(t) dt \\
= \frac{1}{T} \int_0^T (m + (\mathbb{E}[v_0] - m) e^{-\lambda t}) dt \\
= m + \frac{\mathbb{E}[v_0] - m}{T} \int_0^T e^{-\lambda t} dt \\
= m + (\mathbb{E}[v_0] - m) \frac{1 - e^{-\lambda T}}{\lambda T}.
\]

Exercise 8.2

a) By e.g. Exercise 4.13-(b), we have

\[
\mathbb{E}[v_t] = \mathbb{E}[v_0] e^{-\lambda t} + m(1 - e^{-\lambda t}), \quad t \in \mathbb{R}_+,
\]

hence

\[
\text{VST}_T = \frac{1}{T} \mathbb{E} \left[ \int_0^T \frac{1}{S_t^2} (dS_t)^2 \right] \\
= \frac{1}{T} \mathbb{E} \left[ \int_0^T \frac{1}{S_t^2} \left( (r - \alpha v_t) S_t dt + S_t \sqrt{\beta + v_t dB_t^{(1)})} \right)^2 \right]
\]

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\[
\frac{1}{T} \mathbb{E} \left[ \int_0^T (\beta + v_t) dt \right] = \beta + \frac{1}{T} \int_0^T \mathbb{E}[v_t] dt,
\]

which yields

\[
VS_T = \beta + \frac{1}{T} \int_0^T (\mathbb{E}[v_0] e^{-\lambda t} + m(1 - e^{-\lambda t})) dt
= \beta + \frac{1}{T} \int_0^T (\mathbb{E}[v_0] e^{-\lambda t} + m(1 - e^{-\lambda t})) dt
= \beta + \frac{1}{T} (\mathbb{E}[v_0] - m) \int_0^T e^{-\lambda t} dt
= \beta + m + \frac{1}{T} (\mathbb{E}[v_0] - m) \frac{e^{\lambda T} - 1}{\lambda T}.
\]

Note that if the process \((v_t)_t \in \mathbb{R}_+\) is started in the gamma stationary distribution then we have \(\mathbb{E}[v_0] = \mathbb{E}[v_t] = m, t \in \mathbb{R}_+\), and the variance swap rate \(VS_T = \beta + m\) becomes independent of the time \(T\).

b) The stochastic differential equation \(d\sigma_t = \alpha \sigma_t dB_t^{(2)}\) is solved as

\[
\sigma_t = \sigma_0 e^{\alpha B_t^{(2)} - \alpha^2 t/2}, \quad t \in \mathbb{R}_+,
\]

hence we have

\[
VS_T = \frac{1}{T} \mathbb{E} \left[ \int_0^T \frac{1}{S_t^2} (dS_t)^2 \right]
= \frac{1}{T} \mathbb{E} \left[ \int_0^T \frac{1}{S_t^2} (\sigma_t S_t dB_t^{(1)})^2 \right]
= \frac{1}{T} \mathbb{E} \left[ \int_0^T \sigma_t^2 dt \right]
= \frac{\sigma_0^2}{T} \int_0^T \mathbb{E} \left[ e^{2\alpha B_t^{(2)} - \alpha^2 t} \right] dt
= \frac{\sigma_0^2}{T} \int_0^T e^{-\alpha^2 t} \mathbb{E} \left[ e^{2\alpha B_t^{(2)}} \right] dt
= \frac{\sigma_0^2}{T} \int_0^T e^{-\alpha^2 t + 2\alpha^2 t} dt
= \frac{\sigma_0^2}{T} \int_0^T e^{\alpha^2 t} dt
= \frac{\sigma_0^2}{\alpha^2 T} (e^{\alpha^2 T} - 1).
\]

Exercise 8.3
a) By the Itô formula we have
\[ \log \frac{S_T}{S_0} = \log S_T - \log S_0 = \int_0^T \frac{dS_t}{S_t} - \frac{1}{2} \int_0^T \frac{\sigma_t^2}{S_t^2} dt. \]

b) By (8.34) we have
\[ \mathbb{E}^* \left[ \left. \int_0^T \sigma_t^2 dt \right| \mathcal{F}_t \right] = 2 \mathbb{E}^* \left[ \left. \int_0^T \frac{dS_t}{S_t} \right| \mathcal{F}_t \right] - 2 \mathbb{E}^* \left[ \log \frac{S_T}{S_0} \bigg| \mathcal{F}_t \right]. \]

\[ = 2 \int_0^t \frac{dS_u}{S_u} + 2r(T-t) - 2 \mathbb{E}^* \left[ \log \frac{S_T}{S_0} \bigg| \mathcal{F}_t \right]. \]

c) At time \( t \in [0,T] \) we check that
\[ L_t + e^{-(T-t)r} \frac{2}{S_t} S_t + 2 e^{-rT} \left( \int_0^t \frac{dS_u}{S_u} + (T-t)r - 1 \right) A_t \]
\[ = L_t + 2r(T-t) e^{-(T-t)r} + 2 e^{-(T-t)r} \int_0^t \frac{dS_u}{S_u} \]
\[ = V_t. \]

d) By (8.35) we have
\[ dV_t = d \left( L_t + 2r(T-t) e^{-(T-t)r} + 2 e^{-(T-t)r} \int_0^t \frac{dS_u}{S_u} \right) \]
\[ = dL_t - 2r e^{-(T-t)r} dt + 2r^2 (T-t) e^{-(T-t)r} dt \]
\[ + 2r e^{-(T-t)r} \int_0^t \frac{dS_u}{S_u} dt + 2 e^{-(T-t)r} \frac{dS_t}{S_t} \]
\[ = dL_t + e^{-(T-t)r} \frac{2}{S_t} dS_t + 2 e^{-rT} \left( \int_0^t \frac{dS_u}{S_u} + (T-t)r - 1 \right) dA_t, \]

with \( dA_t = r e^{rt} dt \), hence the portfolio is self-financing.

Chapter 9

Exercise 9.1

a) We have \( \frac{\partial C}{\partial x} (T-t, x, K) = \frac{\partial f}{\partial x} \left( T-t, \frac{x}{K} \right) \) and
\[ \frac{\partial C}{\partial K} (T-t, x, K) = f \left( T-t, \frac{x}{K} \right) - \frac{x}{K} \frac{\partial f}{\partial K} \left( T-t, \frac{x}{K} \right) \]
\[ = \frac{1}{K} C(T-t, x, K) - \frac{x}{K} \frac{\partial C}{\partial x} (T-t, x, K), \]

hence
\[
\frac{\partial C}{\partial x}(T-t, x, K) = \frac{1}{x} C(T-t, x, K) - \frac{K}{x} \frac{\partial C}{\partial K}(T-t, x, K).
\]

b) We have \(\frac{\partial^2 C}{\partial x^2}(T-t, x, K) = \frac{1}{K^2} \frac{\partial^2 f}{\partial z^2} \left( T-t, \frac{x}{K} \right) \) and

\[
\frac{\partial^2 C}{\partial K^2}(T-t, x, K)
= - \frac{x}{K^2} \frac{\partial f}{\partial z} \left( T-t, \frac{x}{K} \right) + \frac{x^2}{K^2} \frac{\partial f}{\partial z} \left( T-t, \frac{x}{K} \right)
= \frac{x^2}{K^2} \frac{\partial^2 C}{\partial x^2}(T-t, x, K),
\]

hence

\[
\frac{\partial^2 C}{\partial x^2}(T-t, x, K) = \frac{K^2}{x^2} \frac{\partial^2 C}{\partial K^2}(T-t, x, K).
\]

c) Noting that

\[
\frac{\partial C}{\partial t}(T-t, x, K) = - \frac{\partial C}{\partial T}(T-t, x, K),
\]

we can rewrite the Black-Scholes PDE as

\[
rC(T-t, x, K) = - \frac{\partial C}{\partial T}(T-t, x, K)
+ rx \left( \frac{1}{x} C(T-t, x, K) - \frac{K}{x} \frac{\partial C}{\partial K}(T-t, x, K) \right)
+ \frac{\sigma^2 x^2 K^2}{2} \frac{\partial^2 C}{\partial x^2}(T-t, x, K),
\]

i.e.

\[
\frac{\partial C}{\partial T}(T-t, x, K) = -rK \frac{\partial C}{\partial K}(T-t, x, K) + \frac{\sigma^2 x^2 K^2}{2} \frac{\partial^2 C}{\partial x^2}(T-t, x, K),
\]

Remarks:

1. Using the Black-Scholes Greek \textbf{Gamma} expression

\[
\frac{\partial^2 C}{\partial x^2}(T-t, x, K) = \frac{1}{\sigma x \sqrt{T-t}} \Phi'(d_+(T-t))
= \frac{1}{\sigma x \sqrt{2\pi(T-t)}} e^{-(d_+(T-t))^2/2},
\]
we can recover the lognormal probability density function \( \varphi_T(y) \) of geometric Brownian motion \( S_T \) as follows:

\[
\varphi_T(K) = e^{(T-t)r} \frac{\partial^2 C}{\partial K^2} (T-t, x, K) \\
= e^{(T-t)r} \frac{x^2}{K^2} \frac{\partial^2 C}{\partial x^2} (T-t, x, K) \\
= \frac{e^{(T-t)r}x}{\sigma K^2 \sqrt{2\pi (T-t)}} e^{-(d_+(T-t))^2/2} \\
= \frac{1}{\sigma K \sqrt{2\pi (T-t)}} e^{-(d_-(T-t))^2/2} \\
= \frac{1}{\sigma K \sqrt{2\pi (T-t)}} \exp \left( -\frac{(r-\sigma^2/2)(T-t) + \log(x/K))^2}{2(T-t)\sigma^2} \right),
\]

knowing that

\[
-\frac{1}{2}(d_-(T-t))^2 = -\frac{1}{2} \left( \frac{\log(x/K) + (r-\sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right)^2 \\
= -\frac{1}{2} \left( \frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right)^2 + (T-t)r + \log \frac{x}{K} \\
= -\frac{1}{2}(d_+(T-t))^2 + (T-t)r + \log \frac{x}{K},
\]

which can be obtained from the relation

\[
(d_+(T-t))^2 - (d_-(T-t))^2 \\
= \left( (d_+(T-t) + d_-(T-t))((d_+(T-t) - d_-(T-t)) \right) \\
= 2r(T-t) + 2\log \frac{x}{K}.
\]

2. Using the expressions of the Black-Scholes Greeks Delta and Theta we can also recover

\[
\frac{\partial C}{\partial T} (T-t, x, K) + rK \frac{\partial C}{\partial K} (T-t, x, K) \\
= \frac{K^2 \frac{\partial^2 C}{\partial K^2} (T-t, x, K)}{x^2 \frac{\partial^2 C}{\partial x^2} (T-t, x, K)} \\
- \frac{\partial C}{\partial t} (T-t, x, K) + rK \left( \frac{1}{K} C(T-t, x, K) - \frac{x}{K} \frac{\partial C}{\partial x} (T-t, x, K) \right) \\
= 2 \frac{x^2 \frac{\partial^2 C}{\partial x^2} (T-t, x, K)}{x^2 \frac{\partial^2 C}{\partial x^2} (T-t, x, K)}
\]

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\[ = 2 \frac{x \sigma \Phi'(d_+(T-t))/(2\sqrt{T-t})}{x^2 \Phi'(d_+(T-t))/(x \sigma \sqrt{T-t})} \]

\[ + 2 \frac{r \mathcal{C}(T-t,x,K) - r x \Phi(d_+(T-t))}{x^2 \Phi'(d_+(T-t))/(x \sigma \sqrt{T-t})} \]

\[ = \sigma^2. \]

Exercise 9.2

a) We have

\[ \frac{\partial M_C}{\partial K}(K,S,r,\tau) = \frac{\partial C}{\partial K}(K,S,\sigma_{\text{imp}}(K),r,\tau) + \sigma'_{\text{imp}}(K) \frac{\partial C}{\partial \sigma}(K,S,\sigma_{\text{imp}}(K),r,\tau). \]

b) We have

\[ \frac{\partial C}{\partial K}(K,S,\sigma_{\text{imp}}(K),r,\tau) + \sigma'_{\text{imp}}(K) \frac{\partial C}{\partial \sigma}(K,S,\sigma_{\text{imp}}(K),r,\tau) \leq 0, \]

which shows that

\[ \sigma'_{\text{imp}}(K) \leq - \frac{\partial C}{\partial \sigma}(K,S,\sigma_{\text{imp}}(K),r,\tau) \frac{\partial C}{\partial K}(K,S,\sigma_{\text{imp}}(K),r,\tau) \]

c) We have

\[ \frac{\partial P}{\partial K}(K,S,\sigma_{\text{imp}}(K),r,\tau) + \sigma'_{\text{imp}}(K) \frac{\partial P}{\partial \sigma}(K,S,\sigma_{\text{imp}}(K),r,\tau) \geq 0, \]

which shows that

\[ \sigma'_{\text{imp}}(K) \geq - \frac{\partial P}{\partial \sigma}(K,S,\sigma_{\text{imp}}(K),r,\tau) \frac{\partial P}{\partial K}(K,S,\sigma_{\text{imp}}(K),r,\tau) \]

Exercise 9.3

a) We have

\[ \sigma_{\text{imp}}(K,S) \simeq \sigma_{\text{loc}}((K+S)/2) \]

\[ = \sigma_0 + \beta((K+S)/2 - S_0)^2 \]

\[ = \sigma_0 + \frac{\beta}{4}(K - (2S_0 - S))^2. \]

b) We find

\[ \frac{\partial}{\partial S} \left( (S,K,T,\sigma_{\text{imp}}(K,S),r) \right) = \frac{\partial \mathcal{B}_l}{\partial x}(x,K,T,\sigma_{\text{imp}}(K,S),r)_{x=S} \]
(a) At the money $K = S_0$.

(b) Out of the money $K > S_0$.

Fig. S.32: Implied vs local volatility.

\[
+ \frac{\partial \sigma_{\text{imp}}}{\partial S} \frac{\partial B}{\partial \sigma}(x, K, T, \sigma, r)_{\sigma = \sigma_{\text{imp}}(K, S)}
= \Delta + \nu \frac{\beta}{2} (K - (2S_0 - S)),
\]

where

\[
\Delta = \frac{\partial B}{\partial x} (x, K, T, \sigma_{\text{imp}}(K, S), r)_{x = S}
\]

is the Black-Scholes Delta and

\[
\nu = \frac{\partial B}{\partial \sigma} (S, K, T, \sigma, r)_{\sigma = \sigma_{\text{imp}}(K, S)}
\]

is the Black-Scholes Vega, cf. §2.2 of Hagan et al. (2002).

Chapter 10

Exercise 10.1

a) We have $S_t = S_0 e^{\sigma W_t}$, $t \in \mathbb{R}_+$.

b) We have

\[
\mathbb{E}[S_T] = S_0 \mathbb{E} [e^{\sigma W_T}] = S_0 e^{\sigma^2 T/2}.
\]

c) We have

\[
P \left( \max_{t \in [0, T]} W_t \geq a \right) = 2 \int_{-\infty}^{a} e^{-x^2/(2T)} \frac{dx}{\sqrt{2\pi T}}, \quad a \geq 0,
\]

and

\[
P \left( \max_{t \in [0, T]} W_t \leq a \right) = 2 \int_{0}^{a} e^{-x^2/(2T)} \frac{dx}{\sqrt{2\pi T}}, \quad a \geq 0,
\]
hence the probability density function $\varphi$ of $\max_{t \in [0,T]} W_t$ is given by

$$\varphi(a) = \sqrt{\frac{2}{\pi T}} e^{-a^2/(2T)} \mathbf{1}_{(0,\infty)}(a), \quad a \in \mathbb{R}.$$

d) We have

$$\mathbb{E}[M_0^T] = S_0 \mathbb{E}\left[ \exp \left( \sigma \max_{t \in [0,T]} W_t \right) \right] = S_0 \int_0^\infty e^{\sigma x} \varphi(x) dx = \frac{2S_0}{\sqrt{2\pi T}} \int_0^\infty e^{-(x-\sigma T)^2/(2T)+\sigma^2 T/2} dx = \frac{2S_0 \sigma^2 T/2}{\sqrt{2\pi T}} \int_{-\sigma T}^{\infty} e^{-x^2/(2T)} dx = \frac{2S_0 \sigma^2 T/2}{\sqrt{2\pi T}} \int_{-\infty}^{\sigma \sqrt{T}} e^{-x^2/2} dx = 2S_0 e^{\sigma^2 T/2} \Phi(\sigma \sqrt{T}) = 2 \mathbb{E}[S_T] \Phi(\sigma \sqrt{T}).$$

Remarks:

(i) From the inequality

$$0 \leq \mathbb{E}[(W_T - \sigma T)^+] = \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} (x - \sigma T)^+ e^{-x^2/(2T)} dx = -\frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} (x - \sigma T) e^{-x^2/(2T)} dx = \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} x e^{-x^2/(2T)} dx - \frac{\sigma T}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{-x^2/(2T)} dx = \sqrt{\frac{T}{2\pi}} \int_{\sigma \sqrt{T}}^{\infty} e^{-x^2/2} dx - \frac{\sigma T}{\sqrt{2\pi T}} \int_{\sigma \sqrt{T}}^{\infty} e^{-x^2/2} dx = \sqrt{\frac{T}{2\pi}} \left[ e^{-x^2/2} \right]_{\sigma \sqrt{T}}^{\infty} - \sigma T \Phi(-\sigma \sqrt{T}) = \sqrt{\frac{T}{2\pi}} e^{-\sigma^2 T/2} - \sigma T(1 - \Phi(\sigma \sqrt{T})),$$

we get

$$\Phi(\sigma \sqrt{T}) \geq 1 - \frac{e^{-\sigma^2 T/2}}{\sigma \sqrt{2\pi T}},$$

hence

$$\mathbb{E}[M_0^T] = 2S_0 e^{\sigma^2 T/2} \Phi(\sigma \sqrt{T}).$$
\[
\geq 2S_0 e^{\sigma^2 T/2} \left( 1 - \frac{e^{-\sigma^2 T/2}}{\sqrt{2\pi T}} \right)
\]
\[
= 2 \mathbb{E}[S_T] \left( 1 - \frac{e^{-\sigma^2 T/2}}{\sqrt{2\pi T}} \right)
\]
\[
= 2S_0 \left( e^{\sigma^2 T/2} - \frac{1}{\sqrt{2\pi T}} \right).
\]

(ii) We observe that the ratio between the expected gains by selling at the maximum and selling at time \( T \) is given by \( 2\Phi(\sigma\sqrt{T}) \), which cannot be greater than 2.

![Fig. S.33: Average return by selling at the maximum vs selling at maturity.](image)

By a symmetry argument, we have

\[
\mathbb{P} \left( \min_{t \in [0,T]} W_t \leq a \right) = \mathbb{P} \left( - \max_{t \in [0,T]} (-W_t) \leq a \right)
\]
\[
= \mathbb{P} \left( - \max_{t \in [0,T]} W_t \leq a \right)
\]
\[
= \mathbb{P} \left( \max_{t \in [0,T]} W_t \geq -a \right)
\]
\[
= 2 \int_{-a}^{\infty} e^{-x^2/(2T)} \frac{dx}{\sqrt{2\pi T}}, \quad a < 0,
\]

i.e. the probability density function \( \varphi \) of \( \min_{t \in [0,T]} W_t \) is given by

\[
\varphi(a) = \sqrt{\frac{2}{\pi T}} e^{-a^2/(2T)} \mathbb{I}_{(-\infty,0]}(a), \quad a \in \mathbb{R}.
\]

f) We have
\[ \mathbb{E} [m_0^T] = S_0 \mathbb{E} \left[ \exp \left( \sigma \min_{t \in [0,T]} W_t \right) \right] \]

\[ = S_0 \int_{-\infty}^{0} e^{\sigma x} \varphi(x) \, dx \]

\[ = \frac{2S_0}{\sqrt{2\pi T}} \int_{-\infty}^{0} e^{\sigma^2 x^2 / (2T)} \, dx \]

\[ = \frac{2S_0}{\sqrt{2\pi T}} e^{\sigma^2 T / 2} \int_{-\infty}^{-\sigma T} e^{-x^2 / (2T)} \, dx \]

\[ = \frac{2S_0}{\sqrt{2\pi}} e^{\sigma^2 T / 2} \int_{-\infty}^{-\sigma \sqrt{T}} e^{-x^2 / 2} \, dx \]

\[ = 2S_0 e^{\sigma^2 T / 2} \Phi(-\sigma \sqrt{T}) \]

\[ = 2 \mathbb{E}[S_T] \Phi(-\sigma \sqrt{T}). \]

**Remarks:**

(i) From the inequality

\[ 0 \leq \mathbb{E} [(-\sigma T - W_T)^+] \]

\[ = \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} (-\sigma T - x)^+ e^{-x^2 / (2T)} \, dx \]

\[ = -\frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{-\sigma T} (\sigma T + x) e^{-x^2 / (2T)} \, dx \]

\[ = -\frac{\sigma T}{\sqrt{2\pi T}} \int_{-\infty}^{-\sigma T} e^{-x^2 / (2T)} \, dx - \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{-\sigma T} xe^{-x^2 / (2T)} \, dx \]

\[ = -\frac{\sigma T}{\sqrt{2\pi}} \int_{-\infty}^{-\sigma \sqrt{T}} e^{-x^2 / 2} \, dx - \sqrt{\frac{T}{2\pi}} \int_{-\infty}^{-\sigma \sqrt{T}} xe^{-x^2 / 2} \, dx \]

\[ = -\sigma T \Phi(-\sigma \sqrt{T}) + \sqrt{\frac{T}{2\pi}} \left[ e^{-x^2 / 2} \right]_{-\infty}^{-\sigma \sqrt{T}} \]

\[ = -\sigma T \Phi(-\sigma \sqrt{T}) + \sqrt{\frac{T}{2\pi}} e^{-\sigma^2 T / 2}, \]

we get

\[ e^{\sigma^2 T / 2} \Phi(-\sigma \sqrt{T}) \leq \frac{1}{\sigma \sqrt{2\pi T}}, \] hence

\[ \mathbb{E} [m_0^T] \leq \frac{2S_0}{\sigma \sqrt{2\pi T}}. \]

(ii) The ratio between the expected gains by maturity \( T \) vs selling at the minimum is given by \( 2 \Phi(-\sigma \sqrt{T}) \), which is at most 1 and tends to 0 as \( \sigma \) and \( T \) tend to infinity.
(iii) Given that $\mathbb{E}[M_T^0] = 2 \mathbb{E}[S_T]\Phi(\sigma\sqrt{T})$, we find the bound

$$2 \mathbb{E}[S_T]\Phi(-\sigma\sqrt{T}) \leq \mathbb{E}[S_T] \leq 2 \mathbb{E}[S_T]\Phi(\sigma\sqrt{T}),$$

with equality if $\sigma = 0$ or $T = 0$. We also have

$$2 \mathbb{E}[S_T] - \mathbb{E}[M_T^0] = 2e^{\sigma^2T/2}(1 - \Phi(\sigma\sqrt{T}))$$

$$= 2e^{\sigma^2T/2}\Phi(-\sigma\sqrt{T})$$

$$= \mathbb{E}[m_T^0],$$

hence we have

$$\mathbb{E}[m_T^0] + \mathbb{E}[M_T^0] = 2\mathbb{E}[S_T], \quad \text{or} \quad \mathbb{E}[S_T] - \mathbb{E}[m_T^0] = \mathbb{E}[M_T^0] - \mathbb{E}[S_T],$$

and

$$2\mathbb{E}[S_T] - \frac{2S_0}{\sigma\sqrt{2\pi T}} \leq \mathbb{E}[M_T^0] \leq 2\mathbb{E}[S_T].$$

Exercise 10.2 (Exercise 10.1 continued).

a) Regarding call option prices we have, assuming $K \geq S_0$,

$$\mathbb{E}[(M_T^0 - K)^+] = S_0 \mathbb{E} \left[ \left( \exp \left( \sigma \max_{t \in [0,T]} W_t \right) - K \right)^+ \right]$$

$$= \int_0^\infty (S_0 e^{\sigma x} - K)^+ \varphi(x) dx$$

$$= \frac{2}{\sqrt{2\pi T}} \int_0^\infty (S_0 e^{\sigma x} - K)^+ e^{-x^2/(2T)} dx$$

$$= \frac{2}{\sqrt{2\pi T}} \int_{\sigma^{-1} \log(K/S_0)}^\infty (S_0 e^{\sigma x} - K) e^{-x^2/(2T)} dx$$

$$= \frac{2S_0}{\sqrt{2\pi T}} \int_{\sigma^{-1} \log(K/S_0)}^\infty e^{\sigma x - x^2/(2T)} dx$$
\[-\frac{2K}{\sqrt{2\pi T}} \int_0^\infty e^{-x^2/(2T)} \, dx = 2S_0 \frac{\sqrt{2\pi T}}{\sqrt{2\pi T}} \int_0^\infty e^{-(x-\sigma T)^2/(2T)+\sigma^2 T/2} \, dx \]
\[= 2S_0 \frac{\sqrt{2\pi T}}{\sqrt{2\pi T}} \int_{\sigma T}^\infty e^{-x^2/(2T)} \, dx \]
\[= 2S_0 e^{\sigma^2 T/2} \int_{-\sigma T}^\infty e^{-x^2/(2T)} \, dx \]
\[= 2S_0 e^{\sigma^2 T/2} \Phi(\sigma T/\sqrt{T}) - K \]
\[-2K \Phi(\sigma T) - K = 2S_0 e^{\sigma^2 T/2} \Phi(\sigma T) - K - 2K \Phi(\sigma T) - K = 2S_0 e^{\sigma^2 T/2} \Phi(\sigma T) - K - 2K \Phi(\sigma T) - K = 2S_0 e^{\sigma^2 T/2} \Phi(\sigma T) - K - 2K \Phi(\sigma T) - K = 2S_0 e^{\sigma^2 T/2} \Phi(\sigma T) - K - 2K \Phi(\sigma T) - K.
\]

When \( K \leq S_0 \), by “completion of the square” and use of the Gaussian cumulative distribution function \( \Phi(\cdot) \) we have

\[\mathbb{E} \left[ \left( \max_{t \in [0,T]} S_t - K \right)^+ \right] = \mathbb{E} \left[ \max_{t \in [0,T]} S_t - K \right] = \mathbb{E} \left[ \max_{t \in [0,T]} S_t \right] - \mathbb{E}[K] = \mathbb{E} \left[ \max_{t \in [0,T]} S_t \right] - K = S_0 \mathbb{E} \left[ \exp \left( \sigma \max_{t \in [0,T]} W_t \right) \right] - K = S_0 \int_0^\infty e^{\sigma x} \varphi(x) \, dx - K = \frac{2S_0}{\sqrt{2\pi T}} \int_0^\infty e^{\sigma x - x^2/(2T)} \, dx = \frac{2S_0}{\sqrt{2\pi T}} \int_{\sigma T}^\infty e^{-x^2/(2T)} \, dx = \frac{2S_0}{\sqrt{2\pi T}} e^{\sigma^2 T/2} \int_{-\sigma T}^\infty e^{-x^2/(2T)} \, dx = 2S_0 e^{\sigma^2 T/2} \Phi(\sigma T) - K = 2S_0 e^{\sigma^2 T/2} \left( 1 - \Phi(-\sigma T) \right) - K = 2 \mathbb{E}[S_T] \Phi(\sigma T) - K,
\]
hence

\[e^{-\sigma^2 T/2} \mathbb{E} \left[ \left( M_0^T - K \right)^+ \right] = 2S_0 \Phi(\sigma \sqrt{T}) - K e^{-\sigma^2 T/2}.
\]
Recall that when \( r = \sigma^2 / 2 \) the price of the finite expiration American call option price is the Black-Scholes price with maturity \( T \), with

\[
\text{Bl}_{\text{Call}}(S_0, K, \sigma, r, T) = S_0 \Phi((\sigma T + \sigma^{-1} \log(S_0/K))/\sqrt{T}) - K e^{-\sigma^2 T/2} \Phi(\sigma^{-1} \log(S_0/K)/\sqrt{T})
\]

\[
\leq \begin{cases} 
2S_0 \Phi((\sigma T + \sigma^{-1} \log(S_0/K))/\sqrt{T}) - 2K e^{-\sigma^2 T/2} \Phi(\sigma^{-1} \log(S_0/K)/\sqrt{T}) & \text{if } K \geq S_0, \\
2S_0 \Phi(\sigma \sqrt{T}) - K e^{-\sigma^2 T/2} & \text{if } K \leq S_0.
\end{cases}
\]

\[
= \begin{cases} 
2 \times \text{Bl}_{\text{Call}}(S_0, K, \sigma, r, T) & \text{if } K \geq S_0, \\
2S_0 \Phi(\sigma \sqrt{T}) - K e^{-\sigma^2 T/2} & \text{if } K \leq S_0,
\end{cases}
\]

\[
= \text{Max} \left( 2 \times \text{Bl}_{\text{Call}}(S_0, K, \sigma, r, T), 2S_0 \Phi(\sigma \sqrt{T}) - K e^{-\sigma^2 T/2} \right).
\]

Fig. S.35: Black-Scholes call price upper bound with \( S_0 = 1 \).

b) Regarding put option prices we have, assuming \( S_0 \geq K \),

\[
\mathbb{E} \left[ (K - m_0^T)^+ \right] = S_0 \mathbb{E} \left[ \left( K - \exp \left( \sigma \min_{t \in [0,T]} W_t \right) \right)^+ \right]
\]

\[
= \int_0^{\infty} (K - S_0 e^{\sigma x})^+ \varphi(x) dx
\]

\[
= \frac{2}{\sqrt{2\pi T}} \int_{-\infty}^{0} (K - S_0 e^{\sigma x})^+ e^{-x^2/(2T)} dx
\]

\[
= \frac{2}{\sqrt{2\pi T}} \int_{-\infty}^{-\sigma^{-1} \log(K/S_0)} (K - S_0 e^{\sigma x}) e^{-x^2/(2T)} dx
\]

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\begin{align*}
&= \frac{2K}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{-x^2/(2T)} dx \\
&= \frac{2S_0}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{\sigma x^2/(2T)} dx \\
&= \frac{2K}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{-x^2/(2T)} dx \\
&= \frac{2S_0}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{-(x-\sigma T)^2/(2T)+\sigma^2T/2} dx \\
&= \frac{2K}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{-x^2/(2T)} dx \\
&= 2K \Phi(-\sigma^{-1} \log(S_0/K) / \sqrt{T}) \\
&\quad - 2S_0 e^{\sigma^2T/2} \Phi(-\sigma T + \sigma^{-1} \log(S_0/K) / \sqrt{T}),
\end{align*}

with

\[ e^{-\sigma^2T/2} \mathbb{E}\left[ (K - m_0^T)^+ \right] = K e^{-\sigma^2T/2} - 2S_0 \Phi(-\sigma\sqrt{T}) \]

if \( S_0 \leq K \). Therefore we deduce the bounds

\[ \text{Bl}_{\text{put}}(S_0, K, \sigma, r, T) \]
\[ = K e^{-\sigma^2T/2} \Phi(-\sigma^{-1} \log(S_0/K) / \sqrt{T}) - S_0 \Phi(-\sigma T + \sigma^{-1} \log(S_0/K) / \sqrt{T}) \]

\( \leq \) American put option price

\[ \begin{cases} 
2K e^{-\sigma^2T/2} \Phi(-\sigma^{-1} \log(S_0/K) / \sqrt{T}) - 2S_0 \Phi(-\sigma T + \sigma^{-1} \log(S_0/K) / \sqrt{T}) & \text{if } S_0 \geq K, \\
K e^{-\sigma^2T/2} - 2S_0 \Phi(-\sigma\sqrt{T}) & \text{if } S_0 \leq K,
\end{cases} \]

\[ = \begin{cases} 
2 \times \text{Bl}_{\text{put}}(S_0, K, \sigma, r, T) & \text{if } S_0 \geq K, \\
K e^{-\sigma^2T/2} - 2S_0 \Phi(-\sigma\sqrt{T}) & \text{if } S_0 \leq K,
\end{cases} \]

\[ = \max \left( 2 \times \text{Bl}_{\text{put}}(S_0, K, \sigma, r, T), K e^{-\sigma^2T/2} - 2S_0 \Phi(-\sigma\sqrt{T}) \right) \]

for the finite expiration American put option price when \( r = \sigma^2/2 \).
Exercise 10.3

a) We have

\[ P(\tau_a \geq t) = P(X_t > a) = \int_a^\infty \varphi_{X_t}(x)dx \]
\[ = \sqrt{\frac{2}{\pi t}} \int_y^\infty e^{-x^2/(2t)}dx, \quad y > 0. \]

b) We have

\[ \varphi_{\tau_a}(t) = \frac{d}{dt} P(\tau_a \leq t) = \frac{d}{dt} \int_a^\infty \varphi_{X_t}(x)dx \]
\[ = -\frac{1}{2} \sqrt{\frac{2}{\pi t}} t^{-3/2} \int_a^\infty e^{-x^2/(2t)}dx + \frac{1}{2} \sqrt{\frac{2}{\pi t}} t^{-3/2} \int_a^\infty \frac{x^2}{t} e^{-x^2/(2t)}dx \]
\[ = \frac{1}{2} \sqrt{\frac{2}{\pi t}} t^{-3/2} \left( -\int_a^\infty e^{-x^2/(2t)}dx + a e^{-a^2/(2t)} + \int_a^\infty e^{-x^2/(2t)}dx \right) \]
\[ = \frac{a}{\sqrt{2\pi t^3}} e^{-a^2/(2t)}, \quad t > 0. \]

c) We have

\[ \mathbb{E} [(\tau_a)^{-2}] = \int_0^\infty t^{-2} \varphi_{\tau_a}(t)dt \]
\[ = \frac{a}{\sqrt{2\pi}} \int_0^\infty t^{-7/2} e^{-a^2/(2t)}dt \]
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\[ \frac{2a}{\sqrt{2\pi}} \int_0^\infty x^4 e^{-a^2 x^2/2} dx = \frac{3}{a^2}, \]

by the change of variable \( x = t^{-1/2}, \) i.e. \( x^2 = 1/t, \ t = x^{-2}, \) and \( dt = -2x^{-3} dx. \)

Remark: We have

\[ \mathbb{E}[\tau_a] = \frac{a}{\sqrt{2\pi}} \int_0^\infty t^{-1/2} e^{-a^2/(2t)} dt = +\infty. \]

Exercise 10.4

a) Using the expression

\[ \varphi_{X_0} (x) = \sqrt{\frac{2}{\pi T}} e^{-(x-\mu T)^2/(2T)} + 2\mu e^{2\mu x} \Phi \left( \frac{x+\mu T}{\sqrt{T}} \right), \ x \leq 0. \]  \hfill (A.41)

of the probability density function of the minimum

\[ \bar{X}_0^T := \min_{t \in [0,T]} \tilde{W}_t = \min_{t \in [0,T]} (W_t + \mu t) \]

of drifted Brownian motion \( \tilde{W}_t = W_t + \mu t \) over \( t \in [0,T] \) given in Proposition 10.4, we find

\[ \mathbb{E} \left[ \min_{t \in [0,T]} S_t \right] = S_0 \int_{-\infty}^0 e^{\sigma x} \varphi_{\bar{X}_0^T} (x) dx \]

\[ = S_0 \int_{-\infty}^0 e^{\sigma x} \sqrt{\frac{2}{\pi T}} e^{-(x-\mu T)^2/(2T)} dx \]

\[ + 2\mu S_0 \int_{-\infty}^0 e^{\sigma x} e^{2\mu x} \Phi \left( \frac{x+\mu T}{\sqrt{T}} \right) dx \]

\[ = 2S_0 e^{\sigma^2 T/2-\mu \sigma T} \Phi((\mu-\sigma)\sqrt{T}) + \frac{2\mu S_0}{2\mu-\sigma} \Phi(-\mu \sqrt{T}) \]

\[ - \frac{2\mu S_0}{2\mu-\sigma} e^{\sigma^2 T/2-\mu \sigma T} \Phi((\mu-\sigma)\sqrt{T}), \]

with \( \mu := r/\sigma - \sigma/2, \) which yields

\[ \mathbb{E} \left[ \min_{t \in [0,T]} S_t \right] = S_0 \left( 1 - \frac{\sigma^2}{2r} \right) \Phi \left( \frac{r-\sigma^2/2}{\sigma} \sqrt{T} \right) \]

\[ + S_0 \left( 1 + \frac{\sigma^2}{2r} \right) \Phi \left( \frac{-r+\sigma^2/2}{\sigma} \sqrt{T} \right). \]

\[ \diamond \]

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https://www.ntu.edu.sg/home/nprivault/index.html
b) When $S_0 \leq K$ we have

$$\mathbb{E} \left[ \left( K - \min_{t \in [0,T]} S_t \right)^+ \right] = \mathbb{E} \left[ K - \min_{t \in [0,T]} S_t \right]$$

$$= K - \mathbb{E} \left[ \min_{t \in [0,T]} S_t \right]$$

$$= K - S_0 \left( 1 - \frac{\sigma^2}{2r} \right) \Phi \left( \frac{r - \sigma^2/2 \sqrt{T}}{\sigma} \right)$$

$$- S_0 \left( 1 + \frac{\sigma^2}{2r} \right) \Phi \left( - \frac{r + \sigma^2/2 \sqrt{T}}{\sigma} \right).$$

Next, when $S_0 \geq K$ we have, using the probability density function $\varphi_{\tilde{X}_0}(x)$,

$$\mathbb{E} \left[ \left( K - \min_{t \in [0,T]} S_t \right)^+ \right] = \mathbb{E} \left[ \left( K - S_0 \min_{t \in [0,T]} e^{\sigma \tilde{X}_0^T} \right)^+ \right]$$

$$= \int_{-\infty}^{0} (K - S_0 e^{\sigma x})^+ \varphi_{\tilde{X}_0}(x) dx$$

$$= S_0 \sqrt{\frac{2}{\pi T}} \int_{-\infty}^{0} (K - S_0 e^{\sigma x})^+ e^{-(x-\mu T)^2/(2T)} dx$$

$$+ 2\mu S_0 \int_{0}^{\infty} (K - S_0 e^{\sigma x})^+ e^{2\mu x} \Phi \left( \frac{x + \mu T}{\sqrt{T}} \right) dx$$

$$= K \Phi \left( - \frac{(r - \sigma^2/2)T + \log(S_0/K)}{\sigma \sqrt{T}} \right)$$

$$+ K \left( \frac{S_0}{K} \right)^{1-2r/\sigma^2} \Phi \left( \frac{(r - \sigma^2/2)T + \log(K/S_0)}{\sigma \sqrt{T}} \right)$$

$$- 2S_0 e^{rT} \Phi \left( - \frac{(r + \sigma^2/2)T + \log(S_0/K)}{\sigma \sqrt{T}} \right)$$

$$- S_0 \left( 1 - \frac{\sigma^2}{2r} \right) \left( \frac{S_0}{K} \right)^{-2r/\sigma^2} \Phi \left( \frac{(r + \sigma^2/2)T + \log(K/S_0)}{\sigma \sqrt{T}} \right)$$

$$+ S_0 e^{rT} \left( 1 - \frac{\sigma^2}{2r} \right) \Phi \left( - \frac{(r + \sigma^2/2)T + \log(S_0/K)}{\sigma \sqrt{T}} \right).$$

c) When $r = 0$ and $S_0 \leq K$ we find

$$\mathbb{E} \left[ \left( K - \min_{t \in [0,T]} S_t \right)^+ \right] = K - 2S_0 \left( 1 + \frac{\sigma^2 T}{4} \right) \Phi \left( - \frac{\sigma \sqrt{T}}{2} \right) + S_0 \sigma \sqrt{\frac{T}{2\pi}} e^{-\sigma^2 T/8}. \quad (A.42)$$

Next, when $r = 0$ and $S_0 \geq K$ we find
\[ \mathbb{E} \left[ \left( K - \min_{t \in [0,T]} S_t \right)^+ \right] = K \Phi \left( \frac{\sigma^2 T/2 + \log(K/S_0)}{\sigma \sqrt{T}} \right) \] 

(A.43)

\[ -S_0 \left( 1 + \log \frac{S_0}{K} + \frac{\sigma^2 T}{2} \right) \Phi \left( -\frac{\sigma^2 T/2 + \log(S_0/K)}{\sigma \sqrt{T}} \right) \]

\[ + S_0 \sigma \sqrt{\frac{T}{2\pi}} e^{-\frac{(\sigma^2 T/2+\log(S_0/K))^2}{(2\sigma^2 T)}}. \]

In Figure S.37, using a finite expiration American put option pricer from the R fOptions package we plot the graph of American put price vs \((A.42)-(A.43)\), together with the European put price, according to the following R code.

```
install.packages("fOptions")
library(fOptions)
d1 <- function(S,K,r,T,sigma) {return((log(S/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T)))}
d2 <- function(S,K,r,T,sigma) {return(d1(S, K, r, T, sigma) - sigma * sqrt(T))}
BSPut <- function(S, K, r, T, sigma){return(K*exp(-r*T) * pnorm(-d2(S, K, r, T, sigma)) - S*pnorm(-d1(S, K, r, T, sigma)))}
Optimal_Put_Option <- function(S,K,T,sigma){return(K*pnorm(d1(K,S,0,T,sigma),0,1) -S*(1+(sigma*sigma*T/2)+log(S/K))*pnorm(-d1(S,K,0,T,sigma),0,1) +S*sigma*sqrt(T/(2*pi)))*exp(-d1(S,K,0,T,sigma) *d1(S,K,0,T,sigma)/(2*sigma*sigma*T)))}
curve(BSPut(1,x,0.5,1), from=0, to=2 , xlab="K", lwd = 3, ylim=c(0,1),ylab="Price",col="blue")
par(new=TRUE)
curve(Optimal_Put_Option(1,x,1,1), from=0, to=2 , xlab="K", lwd = 3, ylim=c(0,1),ylab="",col="red")
par(new=TRUE)
curve(BSAmericanApproxOption("p",1,x,1, 0, b=0, 1,title = NULL, description = NULL)@price, from=0, to=2 , xlab="K", lwd = 3, ylim=c(0,1),ylab="",col="orange")
grid (lty = 5)
legend(0,0.1,legend=c("Upper bound","American put price","Black-Scholes put price"),col=c("red","orange", "blue"), lty=1:1, cex=1.)
```

Fig. S.37: “Optimal” exercise put price upper bound with \(S_0 = 1\).
Chapter 11

Exercise 11.1  Barrier options.

a) By (12.14) and (11.28) we find
\[
\xi_t = \frac{\partial g}{\partial y}(t, S_t) = \Phi\left(\delta^T-t\left(\frac{S_t}{K}\right)\right) - \Phi\left(\delta^T-t\left(\frac{S_t}{B}\right)\right) \\
+ \frac{K}{B} e^{-(T-t)r} \left(1 - \frac{2r}{\sigma^2}\right) \left(\frac{S_t}{B}\right)^{-2r/\sigma^2} \left(\Phi\left(\delta^T-t\left(\frac{B^2}{KS_t}\right)\right) - \Phi\left(\delta^T-t\left(\frac{B}{S_t}\right)\right)\right) \\
+ \frac{2r}{\sigma^2} \left(\frac{S_t}{B}\right)^{-1-2r/\sigma^2} \left(\Phi\left(\delta^T-t\left(\frac{B^2}{KS_t}\right)\right) - \Phi\left(\delta^T-t\left(\frac{B}{S_t}\right)\right)\right) \\
- \frac{2}{\sigma \sqrt{2\pi(T-t)}} \left(1 - \frac{K}{B}\right) \exp\left(-\frac{1}{2} \left(\delta^T-t\left(\frac{S_t}{B}\right)\right)^2\right),
\]

\[0 < S_t \leq B, \ 0 \leq t \leq T, \ 	ext{cf. also Exercise 7.1-(ix) of Shreve (2004) and Figure 11.16 above.}\]

b) We find
\[
P(Y_T \leq a & W_T \geq b) = P(W_T \leq 2a - b), \quad a < b < 0,
\]
hence
\[
f_{Y_T, W_T}(a, b) = \frac{dP(Y_T \leq a & W_T \leq b)}{dadb} = -\frac{dP(Y_T \leq a & W_T \geq b)}{dadb},
\]
a, b \in \mathbb{R}, satisfies
\[
f_{Y_T, W_T}(a, b) = \sqrt{\frac{2}{\pi T}} 1_{(-\infty, \min(0,b))}(a) \frac{(b - 2a)}{T} e^{-(2a-b)^2/(2T)}
\]
\[
= \begin{cases} 
\sqrt{\frac{2}{\pi T}} \frac{(b - 2a)}{T} e^{-(2a-b)^2/(2T)}, & a < \min(0,b), \\
0, & a > \min(0,b).
\end{cases}
\]

c) We find
\[
f_{Y_T, \widehat{W}_T}(a, b) = 1_{(-\infty, \min(0,b))}(a) \frac{1}{T} \sqrt{\frac{2}{\pi T}} (b - 2a) e^{-\mu^2T/2 + \mu b -(2a-b)^2/(2T)}
\]
d) The function \( g(t, x) \) is given by the Relations (11.12) and (11.13) above.

Exercise 11.2 Barrier forward contracts.

a) Up-and-in barrier long forward contract. We have

\[
e^{-(T-t)r} \mathbb{E}[C \mid \mathcal{F}_t] = e^{-(T-t)r} \mathbb{E} \left[ (S_T - K) \mathbb{1}_{\left\{ \max_{0 \leq u \leq T} S_u > B \right\}} \mid \mathcal{F}_t \right]
\]

\[
= \mathbb{1}_{\left\{ \max_{0 \leq u \leq t} S_u > B \right\}} (S_t - K e^{-(T-t)r}) + \mathbb{1}_{\left\{ \max_{0 \leq u \leq t} S_u \leq B \right\}} \phi(t, S_t),
\]

where the function

\[
\phi(t, x) := x \Phi \left( \delta_+^{T-t} \left( x/B \right) \right) - K e^{-(T-t)r} \Phi \left( \delta_-^{T-t} \left( x/B \right) \right)
\]

\[
+ B \left( B/x \right)^{2r/\sigma^2} \Phi \left( -\delta_+^{T-t} \left( B/x \right) \right)
\]

\[
- K e^{-(T-t)r} \left( B/x \right)^{-1+2r/\sigma^2} \Phi \left( -\delta_-^{T-t} \left( B/x \right) \right)
\]

solves the Black-Scholes PDE with the terminal condition

\[
\phi(T, x) = \left( x - K + \left( B/x \right)^{2r/\sigma^2} \left( B - x K/B \right) \right) \mathbb{1}_{[B, \infty)}(x),
\]

as in the proof of Proposition 11.3. Note that only the values of \( \phi(t, x) \) with \( x \in [0, B] \) are used for pricing.
Fig. S.38: Price of the up-and-in long forward contract with $K = 60 < B = 80$.

As for the hedging strategy, we find

$$\xi_t = \frac{\partial \phi}{\partial x}(t,S_t) = \Phi \left( \delta_{+}^{T-t}(x/B) \right) + \frac{1}{\sqrt{2\pi}} e^{-\left(\delta_{+}^{T-t}(x/B)\right)^2/2}$$

$$- \frac{1}{x\sqrt{2\pi}} K e^{-(T-t)/2}\varphi \left( \delta_{-}^{T-t}(x/B) \right)^2/2 - \frac{2r}{\sigma^2} (B/x)^{1+2r/\sigma^2} \Phi \left( -\delta_{+}^{T-t}(B/x) \right)$$

$$+ \frac{1}{\sqrt{2\pi}} (B/x)^{1+2r/\sigma^2} e^{-\left(\delta_{-}^{T-t}(B/x)\right)^2/2}$$

$$- \frac{K(1-2r/\sigma^2)}{B} e^{-(T-t)r}(B/x)^{2r/\sigma^2} \Phi \left( -\delta_{-}^{T-t}(B/x) \right)$$

$$- \frac{K}{B\sqrt{2\pi}} (B/x)^{2r/\sigma^2} e^{-(T-t)r-\left(\delta_{-}^{T-t}(B/x)\right)^2/2}$$

$$= \Phi \left( \delta_{+}^{T-t}(x/B) \right) - \frac{2r}{\sigma^2} (B/x)^{1+2r/\sigma^2} \Phi \left( -\delta_{+}^{T-t}(B/x) \right)$$

$$+ \frac{1}{\sqrt{2\pi}} (1-K/B) e^{-\left(\delta_{+}^{T-t}(x/B)\right)^2/2} + \frac{B}{x} e^{-(T-t)r-\left(\delta_{+}^{T-t}(x/B)\right)^2/2}$$

$$- \frac{K}{B}(1-2r/\sigma^2) e^{-(T-t)r}(B/x)^{2r/\sigma^2} \Phi \left( -\delta_{-}^{T-t}(B/x) \right),$$

since by (12.20) we have

$$e^{-\left(\delta_{-}^{T-t}(B/x)\right)^2/2} = e^{(T-t)(x/B)^{2r/\sigma^2} e^{-\left(\delta_{-}^{T-t}(x/B)\right)^2/2}}$$

and

$$e^{-\left(\delta_{-}^{T-t}(x/B)\right)^2/2} = e^{(T-t)(B/x)^{2r/\sigma^2} e^{-\left(\delta_{+}^{T-t}(B/x)\right)^2/2}}.$$
b) Up-and-out barrier long forward contract. We have

\[ e^{-(T-t)r} \mathbb{E}[C \mid F_t] = e^{-(T-t)r} \mathbb{E} \left[ (S_T - K) \mathbb{1} \left\{ \max_{0 \leq u \leq T} S_u < B \right\} \right| F_t \]

\[ = \mathbb{1} \left\{ \max_{0 \leq u \leq t} S_u \leq B \right\} \phi(t, S_t), \quad (A.45) \]

where the function

\[ \phi(t, x) := x \Phi \left( -\delta_+^{T-t} \left( x/B \right) \right) - K e^{-(T-t)r} \Phi \left( -\delta_-^{T-t} \left( x/B \right) \right) \]

\[ - B \left( B/x \right)^{2r/\sigma^2} \Phi \left( -\delta_+^{T-t} \left( B/x \right) \right) \]

\[ + K e^{-(T-t)r} \left( B/x \right)^{-1+2r/\sigma^2} \Phi \left( -\delta_-^{T-t} \left( B/x \right) \right) \]

solves the Black-Scholes PDE with the terminal condition

\[ \phi(T, x) = (x - K) \mathbb{1}_{[0, B]}(x) - \left( B/x \right)^{2r/\sigma^2} \left( B - x \frac{K}{B} \right) \mathbb{1}_{[B, \infty)}(x). \]

Note that only the values of \( \phi(t, x) \) with \( x \in [B, \infty) \) are used for pricing.
Fig. S.40: Price of the up-and-out long forward contract with \( K = 60 < B = 80 \).

As for the hedging strategy, we find

\[
\xi_t = \frac{\partial \phi}{\partial x}(t, S_t) = \Phi(-\delta^T_t(x/B)) - \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{\delta^T_t(x/B)}{2}\right)^2/2} \\
+ \frac{1}{x\sqrt{2\pi}} K e^{-(T-t)r-\left(\delta^T_t(x/B)\right)^2/2} + \frac{2r}{\sigma^2}(B/x)^{1+2r/\sigma^2} \Phi(-\delta^T_t(B/x)) \\
- \frac{1}{\sqrt{2\pi}} (B/x)^{1+2r/\sigma^2} e^{-\left(\delta^T_t(B/x)\right)^2/2} \\
+ \frac{K(1-2r/\sigma^2)}{B} e^{-(T-t)r(B/x)^{2r/\sigma^2} \Phi(-\delta^T_t(B/x))} \\
+ \frac{K}{B\sqrt{2\pi}} (B/x)^{2r/\sigma^2} e^{-\left(\delta^T_t(B/x)\right)^2/2} \\
= \Phi\left(-\delta^T_t(x/B)\right) + \frac{2r}{\sigma^2}(B/x)^{1+2r/\sigma^2} \Phi\left(-\delta^T_t(B/x)\right) \\
- \frac{1}{\sqrt{2\pi}} e^{-\left(\delta^T_t(x/B)\right)^2/2} - \frac{1}{\sqrt{2\pi}} \frac{B}{x} e^{-(T-t)r-\left(\delta^T_t(x/B)\right)^2/2} \\
+ \frac{K}{B\sqrt{2\pi}} e^{-\left(\delta^T_t(x/B)\right)^2/2} + \frac{1}{\sqrt{2\pi}} \frac{K}{x} e^{-(T-t)r-\left(\delta^T_t(x/B)\right)^2/2} \\
+ \frac{K}{B} (1-2r/\sigma^2) e^{-(T-t)r(B/x)^{2r/\sigma^2} \Phi(-\delta^T_t(B/x))} \\
= \Phi\left(-\delta^T_t(x/B)\right) + \frac{2r}{\sigma^2}(B/x)^{1+2r/\sigma^2} \Phi\left(-\delta^T_t(B/x)\right) \\
- \frac{1}{\sqrt{2\pi}} (1-K/B) \left(e^{-\left(\delta^T_t(x/B)\right)^2/2} + \frac{B}{x} e^{-(T-t)r-\left(\delta^T_t(x/B)\right)^2/2}\right) \\
+ \frac{K}{B} \left(1-\frac{2r}{\sigma^2}\right) e^{-(T-t)r(B/x)^{2r/\sigma^2} \Phi\left(-\delta^T_t\left(\frac{B}{x}\right)\right)},
\]

by (12.20).
c) **Down-and-in barrier long forward contract.** We have

\[
e^{-r(T-t)} \mathbb{E}[C \mid F_t] = e^{-r(T-t)} \mathbb{E} \left[ (S_T - K) \mathbb{I} \left\{ \min_{0 \leq u \leq T} S_u < B \right\} \mid F_t \right]
\]

\[
= \mathbb{I} \left\{ \min_{0 \leq u \leq t} S_u < B \right\} (S_t - K e^{-r(T-t)}) + \mathbb{I} \left\{ \min_{0 \leq u \leq t} S_u \geq B \right\} \phi(t, S_t)
\]

where the function

\[
\phi(t, x) := x \Phi \left( -\delta^T-t(x/B) \right) - K e^{-r(T-t)} \Phi \left( -\delta^T-t(x/B) \right)
\]

\[
+B \left( B/x \right)^{2r/\sigma^2} \Phi \left( \delta^T-t(B/x) \right)
\]

\[
-K e^{-r(T-t)} \left( B/x \right)^{-1+2r/\sigma^2} \Phi \left( \delta^T-t(B/x) \right)
\]

solves the Black-Scholes PDE with the terminal condition

\[
\phi(T, x) = \left( x - K + \left( B/x \right)^{2r/\sigma^2} \left( B - x \frac{K}{B} \right) \right) \mathbb{I}_{[0,B]}(x).
\]
Fig. S.42: Price of the down-and-in long forward contract with \( K = 60 < B = 80 \).

As for the hedging strategy, we find

\[
\xi_t = \frac{\partial \phi}{\partial x}(t, S_t)
\]

\[
= \Phi \left( -\delta^T_t (x/B) \right) + \frac{2r}{\sigma^2} (B/x)^{1+2r/\sigma^2} \Phi \left( \delta^T_t (B/x) \right)
\]

\[
- \frac{1}{\sqrt{2\pi}} \frac{1}{(1-K/B)} \left( e^{-(\delta^T_t (x/B))^2/2} + \frac{B}{x} e^{-(T-t) r - (\delta^T_t (x/B))^2/2} \right)
\]

\[
+ \frac{K}{B} \left( 1 - 2r/\sigma^2 \right) e^{-(T-t) r} (B/x)^{2r/\sigma^2} \Phi \left( \delta^T_t (B/x) \right).
\]

Fig. S.43: Delta of the down-and-in long forward contract with \( K = 60 < B = 80 \).

d) Down-and-out barrier long forward contract. We have

\[
e^{-(T-t)r} \mathbb{E}[C \mid F_t] = e^{-(T-t)r} \mathbb{E} \left[ (S_T - K) \mathbb{1} \left\{ \min_{0 \leq u \leq T} S_u > B \right\} \right]_{F_t}
\]

\[
= \mathbb{1} \left\{ \min_{0 \leq u \leq t} S_u \geq B \right\} \phi(t, S_t) \tag{A.47}
\]

where the function

where the function
φ(t, x) := xΦ\left(\delta^T_t \left(\frac{x}{B}\right)\right) - Ke^{-(T-t)r} Φ\left(\delta^-_t \left(\frac{x}{B}\right)\right)
- B\left(\frac{B}{x}\right)^2r/\sigma^2 \Phi\left(\delta^+_t \left(\frac{B}{x}\right)\right)
+ Ke^{-(T-t)r} \left(\frac{B}{x}\right)^{-1+2r/\sigma^2} \Phi\left(\delta^-_t \left(\frac{B}{x}\right)\right)

solves the Black-Scholes PDE with the terminal condition

φ(T, x) = (x - K)1_{[B,\infty)}(x) - \left(B - x\frac{K}{B}\right)\left(\frac{B}{x}\right)^{2r/\sigma^2} 1_{[0,B]}(x).

Note that φ(t, x) above coincides with the price of (11.13) of the standard down-and-out barrier call option in the case K < B, cf. Exercise 11.1-(d).

Fig. S.44: Price of the down-and-out long forward contract with K = 60 < B = 80.

As for the hedging strategy, we find

ξ_t = \frac{\partial φ}{\partial x}(t, S_t)
= Φ\left(\delta^+_t \left(\frac{x}{B}\right)\right) - \frac{2r}{\sigma^2} \left(\frac{B}{x}\right)^{1+2r/\sigma^2} \Phi\left(\delta^+_t \left(\frac{B}{x}\right)\right)
+ \frac{1}{\sqrt{2\pi}} \left(1 - \frac{K}{B}\right) \left(e^{-\left(\delta^+_t \left(\frac{x}{B}\right)\right)^2/2} + \frac{B}{x} e^{-\left((T-t)r - (\delta^-_t \left(\frac{x}{B}\right)\right)^2/2}\right)
- \frac{K}{B} \left(1 - \frac{2r}{\sigma^2}\right) e^{-\left((T-t)r\right)} \left(\frac{B}{x}\right)^{2r/\sigma^2} \Phi\left(\delta^-_t \left(\frac{B}{x}\right)\right).
Fig. S.45: Delta of the down-and-out long forward contract with $K = 60 < B = 80$.

e) Up-and-in barrier short forward contract. The price of the up-and-in barrier short forward contract is identical to (A.44) with a negative sign.

f) Up-and-out barrier short forward contract. The price of the up-and-out barrier short forward contract is identical to (A.45) with a negative sign. Note that $\phi(t, x)$ coincides with the price of (11.10) of the standard up-and-out barrier put option in the case $B < K$.

g) Down-and-in barrier short forward contract. The price of the down-and-in barrier short forward contract is identical to (A.46) with a negative sign.

h) Down-and-out barrier short forward contract. The price of the down-and-out barrier short forward contract is identical to (A.47) with a negative sign.

Exercise 11.3 When $B < K$, we find

\[ \text{Vega}_{\text{down-and-out-call}} = S_t \sqrt{\frac{T-t}{2\pi}} e^{-\left(\delta_+^{T-t}(S_t/K)\right)^2/2} \]

\[ - \frac{4r}{\sigma^3} \left( \frac{S_t}{B} \right)^{1-2r/\sigma^2} \left( \frac{B^2}{S_t} \Phi \left( \delta_+^{T-t} \left( \frac{B^2}{KS_t} \right) \right) - K e^{-(T-t)r} \Phi \left( \delta_-^{T-t} \left( \frac{B^2}{KS_t} \right) \right) \right) \log \frac{S_t}{B} \]

\[ - \sqrt{\frac{T-t}{2\pi}} \frac{B^2}{S_t} \left( \frac{S_t}{B} \right)^{1-2r/\sigma^2} e^{-\left(\delta_+^{T-t}(B^2/K/S_t)\right)^2/2}. \]

When $B > K$ we find

\[ \text{Vega}_{\text{down-and-out-call}} = \frac{S_t}{\sqrt{2\pi}} e^{-\left(\delta_+^{T-t}(S_t/K)\right)^2/2} \left( \frac{K/B - 1}{\frac{\delta_-^{T-t}(S_t/B)}{\sigma}} + \sqrt{T-t} \right) + \sqrt{T-t} \]
The corresponding formulas for the down-and-in call option can be obtained from the parity relation (11.4) and the value $S_t \sqrt{\frac{T-t}{2\pi}} e^{-\left(\delta^+_{t}-t\right)\left(S_t/K\right)^2/2}$ of the Black-Scholes Vega, see Table 6.1.

Exercise 11.4 We have

$$
\mathbb{E}^*[C] = \mathbb{E}^* \left[ \mathbbm{1}_{\{S_T \geq K\}} \mathbbm{1}_{\{M_0^T \leq B\}} \right] 
= \mathbb{E}^* \left[ \mathbbm{1}_{\{S_0 e^{\sigma \hat{W}_T} \geq K\}} \mathbbm{1}_{\{S_0 e^{\sigma \hat{X}_0} \leq B\}} \right] 
= \int_{-\infty}^{\infty} \int_{y \geq 0} \mathbbm{1}_{\{S_0 e^{\sigma v} \geq K\}} \mathbbm{1}_{\{S_0 e^{\sigma x} \leq B\}} d\mathcal{P}(\hat{X}_T \leq x, \hat{W}_T \leq y) 
= \int_{-\infty}^{\infty} \int_{y \geq 0} \mathbbm{1}_{\{S_0 e^{\sigma v} \geq K\}} \mathbbm{1}_{\{S_0 e^{\sigma x} \leq B\}} f_{\hat{X}_T, \hat{W}_T}(x, y) dx dy 
= \frac{1}{T} \sqrt{\frac{2}{\pi T}} \int_{-\infty}^{\infty} \sigma^{-1} \log(B/S_0) \int_{y \geq 0} \mathbbm{1}_{\{S_0 e^{\sigma v} \geq K\}} \mathbbm{1}_{\{S_0 e^{\sigma x} \leq B\}} (2x - y) e^{-\mu^2 T/2 + \mu y - (2x-y)^2/(2T)} dx dy 
= \frac{1}{T} \sqrt{\frac{2}{\pi T}} \int_{\sigma^{-1} \log(K/S_0)}^{\infty} \sigma^{-1} \log(B/S_0) \int_{y \geq 0} \mathbbm{1}_{\{S_0 e^{\sigma v} \geq K\}} \mathbbm{1}_{\{S_0 e^{\sigma x} \leq B\}} (2x - y) e^{-\mu^2 T/2 + \mu y - (2x-y)^2/(2T)} dx dy,
$$

if $B \geq S_0$ (otherwise the option price is 0), with $\mu = r/\sigma - \sigma/2$ and $y \geq 0 = \text{Max}(y, 0)$. Next, letting $a = y \geq 0$ and $b = \sigma^{-1} \log(B/S_0)$, we have

$$
\int_{a}^{b} (2x - y) e^{2x(y-x)/T} dx = \frac{T}{2} (1 - e^{2b(y-b)/T}),
$$

hence, letting $c = \sigma^{-1} \log(K/S_0)$, we have

$$
\mathbb{E}^*[C] = e^{-\mu^2 T/2} \frac{1}{\sqrt{2\pi T}} \int_{c}^{b} e^{\mu y - y^2/(2T)} (1 - e^{2b(y-b)/T}) dy 
= e^{-\mu^2 T/2} \frac{1}{\sqrt{2\pi T}} \int_{c}^{b} e^{\mu y - y^2/(2T)} dy 
- e^{-\mu^2 T/2 - 2b^2/T} \frac{1}{\sqrt{2\pi T}} \int_{c}^{b} e^{y(\mu + 2b/T) - y^2/(2T)} dy.
$$

Using the relation

$$
\frac{1}{\sqrt{2\pi T}} \int_{c}^{b} e^{\gamma y - y^2/(2T)} dy = e^{\gamma^2 T/2} \left( \Phi \left( \frac{-c + \gamma T}{\sqrt{T}} \right) - \Phi \left( \frac{-b + \gamma T}{\sqrt{T}} \right) \right),
$$

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we find
\[
\mathbb{E}^*[C] = \mathbb{E}^* \left[ (S_T - K)^+ \mathbbm{1}_{\{M_0^T \leq B\}} \right] \\
= \Phi \left( \frac{-c + \mu T}{\sqrt{T}} \right) - \Phi \left( \frac{-b + \mu T}{\sqrt{T}} \right) \\
- e^{-\mu^2 T/2 - 2b^2 / T + (\mu + 2b/T)^2 T/2} \left( \Phi \left( \frac{-c + (\mu + 2b/T) T}{\sqrt{T}} \right) - \Phi \left( \frac{-b + (\mu + 2b/T) T}{\sqrt{T}} \right) \right) \\
= \Phi \left( \delta^- \left( \frac{S_0}{K} \right) \right) - \Phi \left( \delta^- \left( \frac{S_0}{B} \right) \right) \\
- e^{-\mu^2 T/2 - 2b^2 T + (\mu + 2b/T)^2 T/2} \left( \Phi \left( \delta^- \left( \frac{B^2}{KS_0} \right) \right) - \Phi \left( \delta^- \left( \frac{B}{S_0} \right) \right) \right),
\]

0 \leq x \leq B. Given the relation
\[
- \frac{\mu^2 T}{2} - 2 \frac{b^2}{T} + \frac{T}{2} \left( \mu + \frac{2b}{T} \right)^2 = \left( -1 + \frac{2r}{\sigma^2} \right) \log \frac{B}{S_0},
\]
we get
\[
e^{-r T} \mathbb{E}^*[C] = e^{-r T} \mathbb{E}^* \left[ \mathbb{1}_{\{S_T \geq K\}} \mathbbm{1}_{\{M_0^T \leq B\}} \right] \\
= e^{-r T} \left( \Phi \left( \delta^- \left( \frac{S_0}{K} \right) \right) - \Phi \left( \delta^- \left( \frac{S_0}{B} \right) \right) \right) \\
- \left( \frac{S_0}{B} \right)^{1 - 2r/\sigma^2} \left( \Phi \left( \delta^- \left( \frac{B^2}{KS_0} \right) \right) - \Phi \left( \delta^- \left( \frac{B}{S_0} \right) \right) \right).
\]

Exercise 11.5

a) For \( x = B \) and \( t \in [0, T] \) we check that
\[
g(t, B) = B \left( \Phi \left( \delta^T_{+ t} \left( \frac{B}{K} \right) \right) - \Phi \left( \delta^T_{+ t} (1) \right) \right) \\
- e^{-r(T-t)} K \left( \Phi \left( \delta^-_{t} \left( \frac{B}{K} \right) \right) - \Phi \left( \delta^-_{t} (1) \right) \right) \\
- B \left( \Phi \left( \delta^T_{+ t} \left( \frac{B}{K} \right) \right) - \Phi \left( \delta^T_{+ t} (1) \right) \right) \\
+ e^{-r(T-t)} K \left( \Phi \left( \delta^-_{t} \left( \frac{B}{K} \right) \right) - \Phi \left( \delta^-_{t} (1) \right) \right) \\
= 0,
\]
and the function \( g(t, x) \) is extended to \( x > B \) by letting
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\[ g(t, x) = 0, \quad x > B. \]

b) For \( x = K \) and \( t = T \), we find

\[ \delta^0_\pm (s) = -\infty \times 1_{\{s < 1\}} + \infty \times 1_{\{s > 1\}} = \begin{cases} +\infty & \text{if } s > 1, \\ 0 & \text{if } s = 1, \\ -\infty & \text{if } s < 1, \end{cases} \]

hence when \( x < K < B \) we have

\[ g(T, x) = x \left( \Phi(-\infty) - \Phi(-\infty) \right) - K \left( \Phi(-\infty) - \Phi(-\infty) \right) - B \left( \frac{B}{x} \right)^{2r/\sigma^2} \left( \Phi(+\infty) - \Phi(+\infty) \right) + K \left( \frac{B}{K} \right)^{2r/\sigma^2} \left( \Phi(+\infty) - \Phi(+\infty) \right) = 0, \]

c) when \( K < x < B \), we get

\[ g(T, x) = x \left( \Phi(+\infty) - \Phi(-\infty) \right) - K \left( \Phi(+\infty) - \Phi(-\infty) \right) - B \left( \frac{B}{x} \right)^{2r/\sigma^2} \left( \Phi(+\infty) - \Phi(+\infty) \right) + K \left( \frac{B}{K} \right)^{2r/\sigma^2} \left( \Phi(+\infty) - \Phi(+\infty) \right) = x - K. \]

Finally, for \( x > B \) we obtain

\[ g(T, K) = x \left( \Phi(+\infty) - \Phi(+\infty) \right) - K \left( \Phi(+\infty) - \Phi(+\infty) \right) - B \left( \frac{B}{x} \right)^{2r/\sigma^2} \left( \Phi(-\infty) - \Phi(-\infty) \right) + K \left( \frac{B}{K} \right)^{2r/\sigma^2} \left( \Phi(-\infty) - \Phi(-\infty) \right) = 0. \]
Exercise 11.6

a) The price at time $t \in [0, T]$ of the European knock-out call option is given by

$$E_{KOC}(t) = e^{-(T-t)r} \mathbb{E}^*[\left((S_T - K)^+ \mathbb{1}_{\{S_T \leq B\}} \right] \mid \mathcal{F}_t], \quad 0 \leq t \leq T.$$ 

Fig. S.46: Payoff functions of the European knock-out call option.

Using the relation

$$S_T = S_t e^{(T-t)r + (\tilde{B}_T - \tilde{B}_t)\sigma - (T-t)\sigma^2/2}, \quad 0 \leq t \leq T,$$

we have

$$e^{-(T-t)r} \mathbb{E}^*[\left((S_T - K)^+ \mathbb{1}_{\{S_T \leq B\}} \right] \mid \mathcal{F}_t]$$

$$= e^{-(T-t)r} \mathbb{E}^*[\left((e^{m(x)}+X-K)^+ \mathbb{1}_{\{e^{m(x)}+X \leq B\}} \right] \mid \mathcal{F}_t], \quad 0 \leq t \leq T,$$

where

$$m(x) := (T-t)r - \frac{\sigma^2}{2}(T-t) + \log x$$

and

$$X := (\tilde{B}_T - \tilde{B}_t)\sigma \simeq \mathcal{N}(0, (T-t)\sigma^2)$$

under $\mathbb{P}^*$. Next, as in Lemma 7.8 we note that if $X$ is a centered Gaussian random variable with variance $\nu^2 > 0$ and $B \geq K$, we have

$$\mathbb{E}[(e^{(m+X)} - K)^+ \mathbb{1}_{\{e^{m+X} \leq B\}}]$$

$$= \frac{1}{\sqrt{2\pi \nu^2}} \int_{m+\log B}^{\infty} (e^{(m+X)} - K)^+ \mathbb{1}_{\{e^{m+X} \leq B\}} e^{-x^2/(2\nu^2)} dx$$

$$= \frac{1}{\sqrt{2\pi \nu^2}} \int_{m+\log B}^{\infty} (e^{(m+X)} - K) e^{-x^2/(2\nu^2)} dx$$

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\begin{align*}
&b) \text{Using the call-put parity relation}\ 
\end{align*}

\begin{align*}
&= \frac{e^m}{\sqrt{2\pi v^2}} \int_{-m+\log K}^{-m+\log B} e^{x^2/(2v^2)} \, dx - \frac{K}{\sqrt{2\pi}} \int_{-m+\log K}^{-m+\log B} e^{x^2/(2v^2)} \, dx \\
&= \frac{e^{m+v^2/2}}{\sqrt{2\pi v^2}} \int_{-m+\log K}^{-m+\log B} e^{-(v^2-x^2)/(2v^2)} \, dx - \frac{K}{\sqrt{2\pi}} \int_{-m+\log K}^{-m+\log B} e^{x^2/(2v^2)} \, dx \\
&= \frac{e^{m+v^2/2}}{\sqrt{2\pi v^2}} \int_{-v^2-m+\log K}^{-v^2-m+\log B} e^{-y^2/(2v^2)} \, dy \\
&\quad - K \Phi((m - \log K) / v) - \Phi((m - \log B) / v) \\
&= e^{m+v^2/2} \left( \Phi(v + (m - \log K) / v) - \Phi(v + (m - \log B) / v) \right) \\
&\quad - K \Phi((m - \log K) / v) - \Phi((m - \log B) / v).
\end{align*}

Hence, the price of the European knock-out call option is given, if \( B \geq K \), by

\begin{align*}
\text{EKOC}_t &= e^{-(T-t)r} \mathbb{E}^* \left[ (S_T - K)^+ \mathbb{1}_{\{S_T \leq B\}} \mid \mathcal{F}_t \right] \\
&= e^{-(T-t)r} e^{m(S_t) + \sigma^2(T-t)/2} \left( \Phi \left( v + \frac{m(S_t) - \log K}{v} \right) - \Phi \left( v + \frac{m(S_t) - \log B}{v} \right) \right) \\
&\quad - K e^{-(T-t)r} \left( \Phi \left( \frac{m(S_t) - \log K}{v} \right) - \Phi \left( \frac{m(S_t) - \log B}{v} \right) \right) \\
&= S_t \Phi \left( \frac{(T-t)r + (T-t)\sigma^2/2 + \log(S_t/K)}{\sigma \sqrt{T-t}} \right) \\
&\quad - S_t \Phi \left( \frac{(T-t)r + (T-t)\sigma^2/2 + \log(S_t/B)}{\sigma \sqrt{T-t}} \right) \\
&\quad - K e^{-(T-t)r} \Phi \left( \frac{(T-t)r - (T-t)\sigma^2/2 + \log(S_t/K)}{\sigma \sqrt{T-t}} \right) \\
&\quad + K e^{-(T-t)r} \Phi \left( \frac{(T-t)r - (T-t)\sigma^2/2 + \log(S_t/B)}{\sigma \sqrt{T-t}} \right),
\end{align*}

\( 0 \leq t \leq T \), with \( \text{EKOC}_t = 0 \) if \( B \leq K \).

b) Using the call-put parity relation

\begin{align*}
\text{EKOC}_t - \text{EKOP}_t &= e^{-(T-t)r} \mathbb{E}^* \left[ (S_T - K)^+ \mathbb{1}_{\{S_T \leq B\}} \mid \mathcal{F}_t \right] \\
\end{align*}

we find that, if \( B \leq K \),

\begin{align*}
\text{EKOP}_t &= -e^{-(T-t)r} \mathbb{E}^* \left[ (K - S_T)^+ \mid \mathcal{F}_t \right] \\
&= -S_t \Phi \left( -\frac{(T-t)r + (T-t)\sigma^2/2 + \log(S_t/B)}{\sigma \sqrt{T-t}} \right) \\
&\quad + K e^{-(T-t)r} \Phi \left( -\frac{(T-t)r - (T-t)\sigma^2/2 + \log(S_t/B)}{\sigma \sqrt{T-t}} \right).
\end{align*}

When \( B \geq K \), we have
Fig. S.47: Payoff functions of the European knock-out put option.

\[ \text{EKOP}_t = e^{-(T-t)r} \mathbb{E}^* \left[ (K - S_T)^+ \mathbb{1}_{\{S_T \leq B\}} \mid \mathcal{F}_t \right] \]
\[ = e^{-(T-t)r} \mathbb{E}^* \left[ (K - S_T)^+ \mid \mathcal{F}_t \right] \]
\[ = Ke^{-(T-t)r} \Phi \left( - \frac{(T-t)r - (T-t)\sigma^2/2 + \log(S_t/K)}{\sigma\sqrt{T-t}} \right) \]
\[ - S_t \Phi \left( - \frac{(T-t)r + (T-t)\sigma^2/2 + \log(S_t/K)}{\sigma\sqrt{T-t}} \right), \]
\[ 0 \leq t \leq T. \]

c) Using the in-out parity relation

\[ \text{EKOC}_t + \text{EKIC}_t = e^{-(T-t)r} \mathbb{E}^* \left[ (S_T - K)^+ \mid \mathcal{F}_t \right] \]
\[ = S_t \Phi \left( \frac{(T-t)r + (T-t)\sigma^2/2 + \log(S_t/K)}{\sigma\sqrt{T-t}} \right) \]
\[ -Ke^{-(T-t)r} \Phi \left( \frac{(T-t)r - (T-t)\sigma^2/2 + \log(S_t/K)}{\sigma\sqrt{T-t}} \right), \]

which is the price of a European call put option with strike price \( K \), the price at time \( t \in [0, T] \) of the European knock-in call option is given, if \( B \geq K \), as

\[ \text{EKIC}_t = e^{-(T-t)r} \mathbb{E}^* \left[ (S_T - K)^+ \mathbb{1}_{\{S_T \geq B\}} \mid \mathcal{F}_t \right] \]
\[ = S_t \Phi \left( \frac{(T-t)r + (T-t)\sigma^2/2 + \log(S_t/B)}{\sigma\sqrt{T-t}} \right) \]
\[ -Ke^{-(T-t)r} \Phi \left( \frac{(T-t)r - (T-t)\sigma^2/2 + \log(S_t/B)}{\sigma\sqrt{T-t}} \right), \]
\[ 0 \leq t \leq T. \]

When \( B \leq K \), we have

\[ \text{EKIC}_t = e^{-(T-t)r} \mathbb{E}^* \left[ (S_T - K)^+ \mathbb{1}_{\{S_T \geq B\}} \mid \mathcal{F}_t \right] \]

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Fig. S.48: Payoff functions of the European knock-in call option.

\[ = e^{-(T-t)r} \mathbb{E}^*[\left(S_T - K\right)^+ | \mathcal{F}_t] \]

\[ = S_t \Phi \left( \frac{(T-t)r + (T-t)\sigma^2/2 + \log(S_t/K)}{\sigma \sqrt{T-t}} \right) \]

\[ - K \ e^{-(T-t)r} \Phi \left( \frac{(T-t)r - (T-t)\sigma^2/2 + \log(S_t/K)}{\sigma \sqrt{T-t}} \right). \]

d) Using the in-out parity relation

\[ \mathbb{E}^K_{OP} t + \mathbb{E}^K_{IP} t = e^{-(T-t)r} \mathbb{E}^*[\left(K - S_T\right)^+ | \mathcal{F}_t], \]

which is the price of a European put option with strike price \( K \), we find that the price at time \( t \in [0,T] \) of the European knock-in put option is given, if \( B \leq K \), as

\[ \mathbb{E}^K_{IP} t = e^{-(T-t)r} \mathbb{E}^*[\left(K - S_T\right)^+ | \mathcal{F}_t] - \mathbb{E}^K_{OP} t \]

\[ = K \ e^{-(T-t)r} \Phi \left( \frac{(T-t)r - (T-t)\sigma^2/2 + \log(S_t/K)}{\sigma \sqrt{T-t}} \right) \]

\[ - S_t \Phi \left( \frac{(T-t)r + (T-t)\sigma^2/2 + \log(S_t/K)}{\sigma \sqrt{T-t}} \right) \]

\[ + S_t \Phi \left( \frac{(T-t)r + (T-t)\sigma^2/2 + \log(S_t/B)}{\sigma \sqrt{T-t}} \right) \]

\[ - K \ e^{-(T-t)r} \Phi \left( \frac{(T-t)r - (T-t)\sigma^2/2 + \log(S_t/K)}{\sigma \sqrt{T-t}} \right). \]

When \( B \geq K \) we have

\[ \mathbb{E}^K_{IP} t = e^{-(T-t)r} \mathbb{E}^*[\left(K - S_T\right)^+ 1_{\{S_T \geq B\}} | \mathcal{F}_t] = 0. \]

In addition, by the results of Questions (d) and (e) we can verify the call-put parity relation

\[ \checkmark \]

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Fig. S.49: Payoff functions of the European knock-in put option.

\[ \text{EKIC}_t - \text{EKIP}_t = e^{-(T-t)r} \mathbb{E}^*[\{S_T - K\} \mathbb{1}_{\{S_T > B\}} | \mathcal{F}_t] \]
\[ = S_t \Phi \left( \frac{(T-t)r + (T-t)\sigma^2/2 + \log(S_t/B)}{\sigma \sqrt{T-t}} \right) \]
\[ - Ke^{-(T-t)r} \Phi \left( \frac{(T-t)r - (T-t)\sigma^2/2 + \log(S_t/B)}{\sigma \sqrt{T-t}} \right) . \]

Chapter 12

Exercise 12.1

a) This probability density function is given by
\[ x \mapsto e^{-\sigma x} \frac{2}{\sqrt{2\pi T}} e^{-x^2/(2T)} - \sigma e^{\sigma x} \Phi \left( \frac{-x - \sigma T/2}{\sqrt{T}} \right) , \quad x \in \mathbb{R}_+. \]

b) We have
\[
E \left[ \min_{t \in [0,T]} S_t \right] = S_0 E \left[ \min_{t \in [0,T]} e^{\sigma B_t - \sigma^2 t/2} \right] = S_0 E \left[ \exp \left[ -\sigma \max_{t \in [0,T]}(B_t + \sigma t/2) \right] \right] \]
\[ = S_0 \int_{0}^{\infty} e^{-\sigma x} \left( \frac{2}{\sqrt{2\pi T}} e^{-x^2/(2T)} - \sigma e^{\sigma x} \Phi \left( \frac{-x - \sigma T/2}{\sqrt{T}} \right) \right) dx \]
\[ = \frac{2S_0}{\sqrt{2\pi T}} \int_{0}^{\infty} e^{-(x+\sigma T/2)^2/(2T)} dx - S_0 \sigma \int_{0}^{\infty} \Phi \left( \frac{-x - \sigma T/2}{\sqrt{T}} \right) dx \]
\[ = \frac{2S_0}{\sqrt{2\pi T}} \int_{0}^{\infty} e^{-x^2/(2T)} dx - \frac{\sigma S_0}{\sqrt{2\pi T}} \int_{0}^{\infty} x e^{-(x+\sigma T/2)^2/(2T)} dx \]
\[ = \frac{2S_0}{\sqrt{2\pi T}} \int_{0}^{\infty} e^{-x^2/(2T)} dx - \frac{\sigma S_0}{\sqrt{2\pi T}} \int_{0}^{\infty} (x - \sigma T/2) e^{-x^2/(2T)} dx \]

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\[ = 2S_0(1 + \sigma^2 T/4)\Phi(-\sigma\sqrt{T}/2) - S_0\sigma\sqrt{\frac{T}{2\pi}} e^{-\sigma^2 T/8}. \] (A.48)

c) We have

\[ E\left[ \left( K - \min_{t\in[0,T]} S_t \right)^+ \right] = E\left[ K - \min_{t\in[0,T]} S_t \right] = K - S_0 \left(2(1 + \sigma^2 T/4)\Phi(-\sigma\sqrt{T}/2) - \sigma\sqrt{\frac{T}{2\pi}} e^{-\sigma^2 T/8}\right). \]

Fig. S.50: Expected minimum of geometric Brownian motion over \([0, T]\).

The derivative with respect to time is given by

\[
\frac{\partial}{\partial T} E\left[ \min_{t\in[0,T]} S_t \right] = S_0(\sigma^2/2)\Phi(-\sigma\sqrt{T}/2) - 2S_0 \left(1 + \frac{\sigma^2 T}{4}\right) \frac{\sigma}{4\sqrt{2\pi T}} e^{-\sigma^2 T/8} \\
- \frac{\sigma S_0}{\sqrt{8\pi T}} e^{-\sigma^2 T/8} + \frac{S_0\sigma^3}{8} \sqrt{\frac{T}{2\pi}} e^{-\sigma^2 T/8} \\
= S_0\sigma^2 \frac{2}{2} \Phi(-\sigma\sqrt{T}/2) - \frac{S_0\sigma^3}{\sqrt{2\pi T}} e^{-\sigma^2 T/8} \left(1 + \frac{3\sigma^2 T}{4}\right).
\]
Fig. S.51: Time derivative of the expected minimum of geometric Brownian motion.

On the other hand, when \( r > 0 \) we have

\[
\mathbb{E}^* \left[ m_t^T \mid \mathcal{F}_t \right] = m_t^0 \Phi \left( \frac{S_t - m_t^0}{S_t} \right) - \frac{\sigma^2}{2T} \frac{2r/\sigma^2}{\Phi \left( \frac{S_t - m_t^0}{S_t} \right)} + S_t e^{(T-t)r} \left( 1 + \frac{\sigma^2}{2T} \right) \Phi \left( -\frac{S_t}{m_t^0} \right).
\]

When \( r \) tends to 0, this minimum tends to

\[
m_t^0 \Phi \left( \frac{\log(S_t/m_t^0) - \sigma^2 T/2}{\sigma \sqrt{T}} \right) + S_t \Phi \left( -\frac{\log(S_t/m_t^0) + \sigma^2 T/2}{\sigma \sqrt{T}} \right) + \sigma^2 S_t \lim_{r \to 0} \frac{1}{2r} \left( e^{(T-t)r} \Phi \left( -\frac{S_t}{m_t^0} \right) - \left( \frac{m_t^0}{S_t} \right)^{2r/\sigma^2} \Phi \left( \frac{S_t - m_t^0}{S_t} \right) \right),
\]

where

\[
\lim_{r \to 0} \frac{1}{2r} \left( e^{(T-t)r} \Phi \left( -\frac{S_t}{m_t^0} \right) - \left( \frac{m_t^0}{S_t} \right)^{2r/\sigma^2} \Phi \left( \frac{S_t - m_t^0}{S_t} \right) \right) = \lim_{r \to 0} \frac{1}{2r} \left( (1 + (T-t)r) \Phi \left( -\frac{\log(S_t/m_t^0) + \sigma^2 T/2 + rT}{\sigma \sqrt{T}} \right) \right)
\]

\[
= \frac{1}{2} \left( T - t + \frac{2}{\sigma^2} \log \frac{m_t^0}{S_t} \right) \Phi \left( -\frac{\log(S_t/m_t^0) + \sigma^2 T/2}{\sigma \sqrt{T}} \right)
\]

\[
+ \lim_{r \to 0} \frac{1}{r \sqrt{8 \pi}} \left( \int_{-\infty}^{-(\log(S_t/m_t^0) + \sigma^2 T/2 + rT)/(\sigma \sqrt{T})} e^{-y^2/2} dy \right) - \int_{-\infty}^{-(\log(S_t/m_t^0) - \sigma^2 T/2 + rT)/(\sigma \sqrt{T})} e^{-y^2/2} dy
\]
We have

\[ \frac{1}{2} \left( T - t + \frac{2}{\sigma^2} \log \frac{m^t_0}{S_t} \right) \Phi \left( -\frac{\log(S_t/m^t_0) + \sigma^2 T/2}{\sigma \sqrt{T}} \right) \]

\[ - \lim_{r \to 0} \frac{1}{r \sqrt{8 \pi}} \int_{(-\log(S_t/m^t_0) - \sigma^2 T/2 + rT)/(\sigma \sqrt{T})}^{(-\log(S_t/m^t_0) - \sigma^2 T/2 - rT)/(\sigma \sqrt{T})} e^{-y^2/2} dy \]

\[ = \frac{1}{2} \left( T - t + \frac{2}{\sigma^2} \log \frac{m^t_0}{S_t} \right) \Phi \left( -\frac{\log(S_t/m^t_0) + \sigma^2 T/2}{\sigma \sqrt{T}} \right) \]

\[ - \frac{\sqrt{T}}{\sigma \sqrt{2\pi}} e^{-((\log(S_t/m^t_0) + \sigma^2 T/2)/(\sigma \sqrt{T}))^2/2}, \]

hence

\[ \mathbb{E}^* [m^T_0 \mid \mathcal{F}_t] = m^0_t \Phi \left( \frac{\log(S_t/m^t_0) - \sigma^2 T/2}{\sigma \sqrt{T}} \right) + S_t \Phi \left( -\frac{\log(S_t/m^t_0) + \sigma^2 T/2}{\sigma \sqrt{T}} \right) \]

\[ + \frac{S_t}{2} \left( (T-t)^2 + 2 \log \frac{m^t_0}{S_t} \right) \Phi \left( -\frac{\log(S_t/m^t_0) + \sigma^2 T/2}{\sigma \sqrt{T}} \right) \]

\[ - \sigma S_t \sqrt{\frac{T}{2\pi}} e^{-((\log(S_t/m^t_0) + \sigma^2 T/2)/(\sigma \sqrt{T}))^2/2}. \]

In particular, when \( T \) tends to infinity we find that

\[ \lim_{T \to \infty} \frac{\mathbb{E}^* [m^T_0 \mid \mathcal{F}_t]}{\mathbb{E}^* [S_T \mid \mathcal{F}_t]} = 0, \quad r \geq 0. \]

When \( t = 0 \) we have \( S_0 = m^0_0 \), and we recover

\[ \mathbb{E}^* [m^T_0] = 2 \left( S_0 + \frac{\sigma^2 T}{4} \right) \Phi \left( -\sigma \sqrt{T} / 2 \right) - \sigma S_0 \sqrt{\frac{T}{2\pi}} e^{-\sigma^2 T/8}. \]

Exercise 12.2

a) By (A.48), we have

\[ E \left[ \max_{t \in [0,1]} S_t \right] = E \left[ e^{\sigma \max_{t \in [0,1]} (B_t - \sigma t/2)} \right] \]

\[ = S_0 E \left[ e^{-(-\sigma) \max_{t \in [0,T]} (B_t - (-\sigma)t/2)} \right] \]

\[ = 2S_0 (1 + \sigma^2 T/4) \Phi(\sigma \sqrt{T}/2) + S_0 \sigma \sqrt{\frac{T}{2\pi}} e^{-\sigma^2 T/8}. \]

b) We have

\[ E \left[ S_0 \max_{t \in [0,1]} e^{\sigma B_t - \sigma^2 t/2 - K} \right] = E \left[ S_0 \max_{t \in [0,1]} e^{\sigma B_t - \sigma^2 t/2} \right] - K \]
\[
= 2S_0(1 + \sigma^2T/4)\Phi(\sigma\sqrt{T}/2) + S_0\sigma\sqrt{T/2\pi}e^{-\sigma^2T/8} - K.
\]

Fig. S.52: Expected maximum of geometric Brownian motion over \([0, T]\).

The derivative with respect to time is given by

\[
\frac{\partial}{\partial T}E \left[ \max_{t \in [0, T]} S_t \right] = \frac{S_0\sigma^2}{2} \Phi(\sigma\sqrt{T}/2) + \frac{S_0\sigma}{\sqrt{2\pi T}}e^{-\sigma^2T/8}\left(1 + \frac{3\sigma^2T}{4}\right).
\]

Fig. S.53: Time derivative of the expected maximum of geometric Brownian motion.

Note that when \(r > 0\) we have

\[E^* \left[ M_0^T \mid \mathcal{F}_t \right] = M_0^t\Phi\left(\delta_{-}^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) + S_te^{(T-t)r}\left(1 + \frac{\sigma^2}{2r}\right)\Phi\left(\delta_{+}^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) - S_t\frac{\sigma^2}{2r}\left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(\delta_{-}^{T-t}\left(\frac{M_0^t}{S_t}\right)\right).\]

When \(r\) tends to 0, this maximum tends to
\[ M_0^t \Phi \left( - \frac{\log(S_t/M_0^t) - \sigma^2 T/2}{\sigma \sqrt{T}} \right) + S_t \Phi \left( \frac{\log(S_t/M_0^t) + \sigma^2 T/2}{\sigma \sqrt{T}} \right) \]
\[ + \sigma^2 S_t \lim_{r \to 0} \frac{1}{2r} \left( e^{(T-t)r} \Phi \left( \delta_{T-t} \left( \frac{S_t}{M_0^t} \right) \right) - \left( \frac{M_0^t}{S_t} \right)^{2r/\sigma^2} \Phi \left( -\delta_{T-t} \left( \frac{M_0^t}{S_t} \right) \right) \right), \]

where

\[ \lim_{r \to 0} \frac{1}{2r} \left( e^{(T-t)r} \Phi \left( \delta_{T-t} \left( \frac{S_t}{M_0^t} \right) \right) - \left( \frac{M_0^t}{S_t} \right)^{2r/\sigma^2} \Phi \left( -\delta_{T-t} \left( \frac{M_0^t}{S_t} \right) \right) \right) \]
\[ = \lim_{r \to 0} \frac{1}{2r} \left( (1 + (T-t)r) \Phi \left( \frac{\log(S_t/M_0^t) + \sigma^2 T/2 - rT}{\sigma \sqrt{T}} \right) \right) - \left( 1 + \frac{2r}{\sigma^2} \log \frac{M_0^t}{S_t} \right) \Phi \left( \frac{\log(S_t/M_0^t) + \sigma^2 T/2 - rT}{\sigma \sqrt{T}} \right) \]
\[ = \frac{1}{2} \left( T - t + \frac{2}{\sigma^2} \log \frac{M_0^t}{S_t} \right) \Phi \left( \frac{\log(S_t/M_0^t) + \sigma^2 T/2}{\sigma \sqrt{T}} \right) \]
\[ + \lim_{r \to 0} \frac{1}{r \sqrt{8\pi}} \int_{-\infty}^{(\log(S_t/M_0^t) + \sigma^2 T/2 + rT)/(\sigma \sqrt{T})} e^{-y^2/2} dy \]
\[ - \int_{-\infty}^{(\log(S_t/M_0^t) + \sigma^2 T/2 - rT)/(\sigma \sqrt{T})} e^{-y^2/2} dy \]
\[ = \frac{1}{2} \left( T - t + \frac{2}{\sigma^2} \log \frac{M_0^t}{S_t} \right) \Phi \left( \frac{\log(S_t/M_0^t) + \sigma^2 T/2}{\sigma \sqrt{T}} \right) \]
\[ + \frac{\sqrt{T}}{\sigma \sqrt{2\pi}} e^{-((\log(S_t/M_0^t) + \sigma^2 T/2)/(\sigma \sqrt{T}))^2/2}, \]

hence

\[ \mathbb{E}^* \left[ M_0^T \mid \mathcal{F}_t \right] = M_0^t \Phi \left( - \frac{\log(S_t/M_0^t) - \sigma^2 T/2}{\sigma \sqrt{T}} \right) + S_t \Phi \left( \frac{\log(S_t/M_0^t) + \sigma^2 T/2}{\sigma \sqrt{T}} \right) \]
\[ + \frac{S_t}{2} \left( (T-t)\sigma^2 + 2 \log \frac{M_0^t}{S_t} \right) \Phi \left( \frac{\log(S_t/M_0^t) + \sigma^2 T/2}{\sigma \sqrt{T}} \right) \]
\[ + \sigma S_t \sqrt{\frac{T}{2\pi}} e^{-((\log(S_t/M_0^t) + \sigma^2 T/2)/(\sigma \sqrt{T}))^2/2}. \]

In particular, when \( T \) tends to infinity we find that
\[
\lim_{T \to \infty} \frac{E^* [M_0^T \mid \mathcal{F}_t]}{E^*[S_T \mid \mathcal{F}_t]} = \begin{cases} 
1 + \frac{\sigma^2}{2r} & \text{if } r > 0, \\
+\infty & \text{if } r = 0.
\end{cases}
\]

When \( t = 0 \) we have \( S_0 = M_0^0 \), and we recover

\[
E^* [M_0^T] = 2\left(S_0 + \frac{\sigma^2 T}{4}\right) \Phi(\sigma\sqrt{T}/2) + \sigma S_0 \sqrt{\frac{T}{2\pi}} e^{-\sigma^2 T/8}.
\]

**Exercise 12.3**

a) We have

\[
P \left( \min_{t \in [0,T]} B_t \leq a \right) = 2 \int_{-\infty}^a e^{-x^2/(2T)} \frac{dx}{\sqrt{2\pi T}}, \quad a < 0,
\]

*i.e.* the probability density function \( \varphi \) of \( \sup_{t \in [0,T]} B_t \) is given by

\[
\varphi(a) = \sqrt{\frac{2}{\pi T}} e^{-a^2/(2T)} \mathbb{1}_{(-\infty,0)}(a), \quad a \in \mathbb{R}.
\]

b) We have

\[
E \left[ \min_{t \in [0,T]} S_t \right] = S_0 E \left[ \exp \left( \sigma \min_{t \in [0,T]} B_t \right) \right]
= \frac{2S_0}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{\sigma x - x^2/(2T)} dx = \frac{2S_0}{\sqrt{2\pi \sigma^2 T}} \int_{-\infty}^{0} e^{-(x-\sigma T)^2/(2T) + \sigma^2 T/2} dx
= \frac{2S_0}{\sqrt{2\pi T}} e^{\sigma^2 T/2} \int_{-\infty}^{-\sigma \sqrt{T}} e^{-x^2/(2T)} dx = \frac{2S_0}{\sqrt{2\pi}} e^{\sigma^2 T/2} \int_{-\infty}^{-\sigma \sqrt{T}} e^{-x^2/2} dx
= 2S_0 e^{\sigma^2 T/2} \Phi(-\sigma \sqrt{T}) = 2E[S_T] \left( 1 - \Phi(\sigma \sqrt{T}) \right),
\]

hence

\[
E \left[ S_T - \min_{t \in [0,T]} S_t \right] = E[S_T] - E \left[ \min_{t \in [0,T]} S_t \right] = E[S_T] - 2E[S_T] \left( 1 - \Phi(\sigma \sqrt{T}) \right)
= E[S_T] \left( 2\Phi(\sigma \sqrt{T}) - 1 \right) = 2S_0 e^{\sigma^2 T/2} \left( \Phi(\sigma \sqrt{T}) - \frac{1}{2} \right),
\]

and

\[
e^{-\sigma^2 T/2} E \left[ S_T - \min_{t \in [0,T]} S_t \right] = S_0 \left( 2\Phi(\sigma \sqrt{T}) - 1 \right) = S_0 \left( 1 - 2\Phi(-\sigma \sqrt{T}) \right).
\]

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Remark: We note that as $T$ goes to infinity, the price of the lookback option converges to $S_0$.

Fig. S.54: Lookback call option price as a function of $T$ with $S_0 = 1$.

Exercise 12.4 Lookback options. By (12.4) we find

$$\xi_t = \frac{\partial f}{\partial x}(t, S_t, M_t^0)$$

$$= -1 + \left(1 + \frac{2r}{\sigma^2}\right) \Phi\left(\delta^{T-t} - \left(\frac{S_t}{M_t^0}\right)\right)$$

$$+ e^{-(T-t)r} \left(\frac{M_t^0}{S_t}\right)^{2r/\sigma^2} \left(1 - \frac{\sigma^2}{2r}\right) \Phi\left(-\delta^{T-t} - \left(\frac{M_t^0}{S_t}\right)\right), \quad t \in [0, T],$$

and

$$\eta_t A_t = f(t, S_t, M_t^0) - \xi_t S_t$$

$$= M_t^0 e^{-(T-t)r} \Phi\left(-\delta^{T-t} - \left(\frac{S_t}{M_t^0}\right)\right) - e^{-(T-t)r} \left(M_t^0\right)^{-1+2r/\sigma^2} \Phi\left(-\delta^{T-t} - \left(\frac{M_t^0}{S_t}\right)\right), \quad t \in [0, T].$$

Exercise 12.5 We have

$$\mathbb{E}^*\left[e^{-r\tau} \mathbb{1}_{\{\tau \leq T\}} \mathbb{1}\{M_0^\tau - S_\tau \geq K\}\right]$$

$$= \int_1^T \int_0^\infty \int_{K+x}^\infty e^{-rt} f(\tau, S_\tau, M_\tau)(t, x, y) dy dx dt$$

$$= \int_1^T \int_0^\infty \int_K^{y-K} e^{-rt} f(\tau, S_\tau, M_\tau)(t, x, y) dx dy dt$$

for $T \geq 1$, and $\mathbb{E}^*\left[e^{-r\tau} \mathbb{1}_{\{\tau \leq T\}} \mathbb{1}\{M_0^\tau - S_\tau \geq K\}\right] = 0$ if $T \in [0, 1]$. 

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https://www.ntu.edu.sg/home/nprivault/indext.html
Exercise 12.6

a)  
1) The boundary condition (12.3a) is explained by the fact that
\begin{align*}
f(t,0,y) &= e^{-(T-t)r} \mathbb{E}^* \left[ M_0^T - S_T \mid S_t = 0, \ M_0^t = y \right] \\
&= e^{-(T-t)r} \mathbb{E}^* \left[ M_0^t - S_T \mid S_t = 0, \ M_0^t = y \right] \\
&= e^{-(T-t)r} \mathbb{E}^* \left[ M_0^t \mid M_0^t = y \right] - e^{-(T-t)r} \mathbb{E}^* [S_T \mid S_t = 0] \\
&= y e^{-(T-t)r},
\end{align*}

since \( \mathbb{E}^*[S_T \mid S_t = 0] = 0 \) as \( S_t = 0 \) implies \( S_T = 0 \) from the relation
\[
S_T = S_t e^{\sigma (B_T - B_t) + (\mu - \sigma^2/2)(T-t)}, \quad 0 \leq t \leq T.
\]

ii) The boundary condition (12.3b), i.e.
\[
\frac{\partial f}{\partial y} (t,x,y)_{y=x} = 0, \quad 0 \leq x \leq y,
\]
is illustrated in the following Figure S.55, cf. also Exercise 12.6

iii) Condition (12.3c) follows from the fact that
\[
f(T,x,y) = \mathbb{E}^* \left[ M_0^T - S_T \mid S_T = x, \ M_0^T = y \right] = y - x.
\]

b)  
1) The boundary condition (12.13a) is explained by the fact that
\begin{align*}
f(t,x,0) &= e^{-(T-t)r} \mathbb{E}^* \left[ S_T - m_0^T \mid S_t = x, \ m_0^t = 0 \right] \\
&= e^{-(T-t)r} \mathbb{E}^* \left[ S_T \mid S_t = x, \ m_0^t = 0 \right] \\
&= e^{-(T-t)r} \mathbb{E}^*[S_T \mid S_t = x] \\
&= e^{-(T-t)r} x, \quad x > 0.
\end{align*}
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ii) Condition (12.13b) follows from the fact that

\[ f(T, x, y) = \mathbb{E}^* \left[ S_T - m_0 T \mid S_T = x, m_0 = y \right] = x - y. \]

We have

\[ f(t, x, x) = x C(T - t), \]

with

\[ C(\tau) = 1 - e^{-r \tau} \Phi(\delta^- (1)) - \left( 1 + \frac{\sigma^2}{2r} \right) \Phi(-\delta^+ (1)) + e^{-r \tau} \frac{\sigma^2}{2r} \Phi(\delta^- (1)), \]

\( \tau > 0, \) hence

\[ \frac{\partial f}{\partial x}(t, x, x) = C(T - t), \quad t \in [0, T], \]

while we also have

\[ \frac{\partial f}{\partial y}(t, x, y) = 0, \quad 0 \leq x \leq y. \]

Chapter 13

Exercise 13.1 We have

\[ \mathbb{E} \left[ \int_T^\tau S_t dt \right] = S_0 \frac{e^{rT} - e^{r\tau}}{r}, \]

and

\[ \mathbb{E} \left[ \left( \int_T^\tau S_t dt \right)^2 \right] = 2S_0^2 \frac{r e^{(\sigma^2 + 2r)T} - (\sigma^2 + 2r) e^{rT} e^{(\sigma^2 + r)\tau} + (\sigma^2 + r) e^{(\sigma^2 + 2r)\tau}}{(\sigma^2 + r)(\sigma^2 + 2r)r}. \]

Exercise 13.2

a) The integral \( \int_0^T r_s ds \) is centered Gaussian with variance

\[ \mathbb{E} \left[ \left( \int_0^T r_s ds \right)^2 \right] = \sigma^2 \mathbb{E} \left[ \int_0^T \int_0^T B_s B_t ds dt \right] = \sigma^2 \int_0^T \int_0^T \mathbb{E}[B_s B_t] ds dt = \sigma^2 \int_0^T \int_0^T \min(s, t) ds dt = 2\sigma^2 \int_0^T \int_0^t s ds dt. \]
b) Since the integral \( \int_0^T r_s ds \) is a random variable with probability density

\[
\varphi(x) = \frac{1}{\sqrt{2\pi T^3/3}} e^{-3x^2/(2\pi T^3)},
\]

we have

\[
e^{-rT} \mathbb{E} \left[ \left( \int_0^T r_u du - \kappa \right)^+ \right] = e^{-rT} \int_{-\infty}^\infty (x - \kappa)^+ \varphi(x) dx
\]

Exercise 13.3 We have

\[
e^{-(T-t)r} \mathbb{E} \left[ \left( \frac{1}{T} \int_0^T S_u du - \kappa \right)^+ \mid \mathcal{F}_t \right] = e^{-(T-t)r} \mathbb{E} \left[ \frac{1}{T} \int_0^T S_u du - \kappa \mid \mathcal{F}_t \right]
\]

\[
e^{-(T-t)r} \mathbb{E} \left[ \int_0^T S_u du - \kappa \mid \mathcal{F}_t \right] - \kappa e^{-(T-t)r}
\]

\[
e^{-(T-t)r} \frac{1}{T} \mathbb{E} \left[ \int_0^T S_u du \mid \mathcal{F}_t \right] + e^{-(T-t)r} \frac{1}{T} \mathbb{E} \left[ \int_t^T S_u du \mid \mathcal{F}_t \right] - \kappa e^{-(T-t)r}
\]

\[
e^{-(T-t)r} \frac{1}{T} \int_0^t S_u du + e^{-(T-t)r} \frac{1}{T} \mathbb{E} \left[ \int_t^T S_u du \mid \mathcal{F}_t \right] - \kappa e^{-(T-t)r}
\]

\[
e^{-(T-t)r} \frac{1}{T} \int_0^t S_u du + e^{-(T-t)r} \frac{1}{T} \int_t^T \mathbb{E} [S_u \mid \mathcal{F}_t] du - \kappa e^{-(T-t)r}
\]

\[
e^{-(T-t)r} \frac{1}{T} \int_0^t S_u du + e^{-(T-t)r} \frac{1}{T} \int_t^T S_t e^{(u-t)r} du - \kappa e^{-(T-t)r}
\]
Exercise 13.4 The geometric mean price $G$ satisfies

$$G = \exp \left( \frac{1}{T} \int_0^T \log S_u du \right) = \exp \left( \frac{1}{T} \int_0^t \log S_u du + \frac{1}{T} \int_t^T \log S_u du \right)$$

$$= \exp \left( \frac{1}{T} \int_0^t \log S_u du + \frac{T-t}{T} \log S_t + \frac{1}{T} \int_t^T \log \frac{S_u}{S_t} du \right)$$

$$= \exp \left( \frac{1}{T} \int_0^t \log S_u du + \frac{T-t}{T} \log S_t + \frac{1}{T} \int_t^T (r(u-t) + (B_u - B_t)\sigma - (u-t)\sigma^2/2) du \right)$$

$$= \exp \left( \frac{1}{T} \int_0^t \log S_u du + \frac{T-t}{T} \log S_t + \frac{1}{T} \int_0^{T-t} (ru - \sigma^2 u^2/2) du + \frac{r}{T} \int_t^T (B_u - B_t) du \right)$$

$$= (S_t)^{(T-t)/T} \exp \left( \frac{1}{T} \int_0^t \log S_u du + \frac{(T-t)^2}{2T} (r - \sigma^2/2) + \frac{1}{T} \int_t^T (B_u - B_t) du \right)$$

where $\int_t^T B_u du$ is centered Gaussian with conditional variance

$$\mathbb{E} \left[ \left( \int_t^T B_u du \right)^2 | \mathcal{F}_t \right] = \mathbb{E} \left[ \left( \int_t^T (B_u - B_t) du \right)^2 | \mathcal{F}_t \right]$$

$$= \mathbb{E} \left[ \left( \int_t^T (B_u - B_t) du \right)^2 \right]$$
Exercise 13.5 Under the above condition we have, taking $m := \frac{1}{T} \int_0^t \log S_u du + \frac{T - t}{T} \log S_t + \frac{(T - t)^2}{2T} (r - \sigma^2/2), \quad X := \frac{\sigma}{T} \int_t^T B_u du,$ and $v^2 = (T - t)\sigma^2/3$, we find
\[
e^{-T} \mathbb{E}^* \left[ \left( \exp \left( \frac{1}{T} \int_0^T \log S_u du \right) - K \right)^+ \mid \mathcal{F}_t \right]
= (S_t)^{(T-t)/T} e^{-(T-t)r} \exp \left( \frac{1}{T} \int_0^t \log S_u du + \frac{(T - t)^2}{4T} (2r - \sigma^2) + \frac{\sigma^2}{6} (T - t) \right)
\times \Phi \left( \frac{(T - t)\sigma^2/3 + \frac{1}{T} \int_0^t \log S_u du + \log \frac{S_0^{(T-t)/T}}{K} + \frac{(T - t)^2}{2T} (r - \sigma^2/2)}{\sigma \sqrt{(T - t)/3}} \right) - K e^{-(T-t)r} \Phi \left( \frac{\frac{1}{T} \int_0^t \log S_u du + \log \frac{S_0^{(T-t)/T}}{K} + \frac{(T - t)^2}{2T} (r - \sigma^2/2)}{\sigma \sqrt{T/3}} \right),
\]
where $0 \leq t \leq T$. In case $t = 0$, we get
\[
e^{-rT} \mathbb{E}^* \left[ \left( \exp \left( \frac{1}{T} \int_0^T \log S_u du \right) - K \right)^+ \right]
= S_0 e^{-T(r+\sigma^2/6)/2} \Phi \left( \frac{\log \frac{S_0}{K} + \frac{T}{2} (r + \sigma^2/6)}{\sigma \sqrt{T/3}} \right) - K e^{-rT} \Phi \left( \frac{\log \frac{S_0}{K} + \frac{T}{2} (r - \sigma^2/2)}{\sigma \sqrt{T/3}} \right) .
\]

Exercise 13.5 Under the above condition we have, taking $t \in [\tau, T]$,
\[
e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T - \tau} \int_\tau^T r_s ds - K \right)^+ \mid \mathcal{F}_t \right]
= e^{-(T-t)r} \mathbb{E}^* \left[ \left( \Lambda_t + \frac{1}{T - \tau} \int_t^T r_s ds - K \right)^+ \mid \mathcal{F}_t \right]
= e^{-(T-t)r} \mathbb{E}^* \left[ \Lambda_t + \frac{1}{T - \tau} \int_t^T r_s ds - K \mid \mathcal{F}_t \right]
= e^{-(T-t)r} (\Lambda_t - K) + e^{-(T-t)r} \mathbb{E}^* \left[ \int_t^T r_s ds \mid \mathcal{F}_t \right]

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Exercise 13.7 Taking Exercise 13.6 This question extends Exercise 7.3 to martingale then for any convex payoff function

\[ \phi \] such as \( \phi \) is convex, hence

\[
\mathbb{E}^*[\phi \left( \frac{S_{T_1} + \cdots + S_{T_n}}{n} \right)] \leq \mathbb{E}^*[\phi(S_{T_1}) + \cdots + \phi(S_{T_n})]
\]

\[
= \frac{\mathbb{E}^*[\phi(S_{T_1})] + \cdots + \mathbb{E}^*[\phi(S_{T_n})]}{n}
\]

\[
= \frac{\mathbb{E}^*[\phi(S_{T_1}) | \mathcal{F}_{T_1}] + \cdots + \mathbb{E}^*[\phi(S_{T_n}) | \mathcal{F}_{T_n}]}{n}
\]

because \( (S_t)_{t \in \mathbb{R}^+} \) is a martingale,

by Jensen’s inequality,

by the tower property,

\[
= \mathbb{E}^*[\phi(S_{T_n})] + \cdots + \mathbb{E}^*[\phi(S_{T_n})]
\]

On the other hand, if \( (S_t)_{t \in \mathbb{R}^+} \) is only a submartingale then the above argument still applies to a convex non-decreasing payoff function \( \phi \) such as \( \phi(x) = (x - K)^+ \).

Exercise 13.7 Taking \( t \in \tau, T \), under the condition

\[
e^{-(T-t)r}(\Lambda_t - K) + \frac{e^{-(T-t)r}}{T - \tau} \int_0^T \mathbb{E}^*[r_s | \mathcal{F}_t] ds, \quad t \in \tau, T,
\]

where

\[
\mathbb{E}^*[r_s | \mathcal{F}_t] = v_t e^{-(s-t)\lambda} + m (1 - e^{-(s-t)\lambda}), \quad 0 \leq s \leq t,
\]

hence

\[
\mathbb{E}^* \left[ \left( \frac{1}{T - \tau} \int_{\tau}^{T} r_s ds - K \right)^+ \bigg| \mathcal{F}_t \right] = \mathbb{E}^* \left[ \left( \Lambda_t + \frac{1}{T - \tau} \int_{t}^{T} r_s ds - K \right)^+ \bigg| \mathcal{F}_t \right]
\]

\[
= \Lambda_t - K + \frac{1}{T - \tau} \int_{t}^{T} \mathbb{E}^*[r_s | \mathcal{F}_t] ds
\]

\[
= \Lambda_t - K + \frac{1}{T - \tau} \int_{t}^{T} \big( r_t e^{-(s-t)\lambda} + m (1 - e^{-(s-t)\lambda}) \big) ds
\]

\[
= \Lambda_t - K + \frac{1}{T - \tau} \big( r_t - m \big) \int_{0}^{T-t} e^{-\lambda s} ds + m(T - t) \frac{e^{-(T-t)r}}{T - \tau}
\]

\[
= \Lambda_t - K + \frac{1 - e^{-(T-t)\lambda}}{(T - \tau)\lambda} (r_t - m) + m \frac{T - t}{T - \tau}.
\]
\[ \Lambda_t := \frac{1}{T - \tau} \int_{\tau}^{t} S_s ds \geq K, \]

we have

\[
\begin{align*}
&\, e^{-r(T-t)} \mathbb{E}^* \left[ \left( \frac{1}{T - \tau} \int_{\tau}^{T} S_s ds - K \right)^+ \right] \\
&\, = e^{-r(T-t)} \mathbb{E}^* \left[ \left( \Lambda_t + \frac{1}{T - \tau} \int_{t}^{T} S_s ds - K \right)^+ \right] \\
&\, = e^{-r(T-t)} \mathbb{E}^* \left[ \Lambda_t + \frac{1}{T - \tau} \int_{t}^{T} S_s ds - K \right] \\
&\, = e^{-r(T-t)} (\Lambda_t - K) + S_t \frac{e^{-r(T-t)}}{T - \tau} \mathbb{E}^* \left[ \int_{t}^{T} S_s ds \right] \\
&\, = e^{-r(T-t)} (\Lambda_t - K) + S_t \frac{e^{-r(T-t)}}{T - \tau} \mathbb{E}^* \left[ \int_{t}^{T} S_s ds \right] \\
&\, = e^{-r(T-t)} (\Lambda_t - K) + S_t \frac{e^{-r(T-t)}}{T - \tau} (e^{r(T-t)} - 1) \\
&\, = e^{-r(T-t)} (\Lambda_t - K) + S_t \frac{1 - e^{-r(T-t)}}{(T - \tau)r}, \quad t \in [\tau, T].
\end{align*}
\]

Exercise 13.8 The Asian option price can be written as

\[
\begin{align*}
&\, e^{-r(T-t)} \mathbb{E}^* \left[ \left( \frac{1}{T} \int_{0}^{T} S_u du - K \right)^+ \right] \\
&\, = S_t \mathbb{E} \left[ (U_T)^+ \right] \\
&\, = S_t h(t, U_t) = S_t g(t, Z_t),
\end{align*}
\]

which shows that

\[ g(t, Z_t) = h(t, U_t), \]

and it remains to use the relation

\[ U_t = \frac{1 - e^{-r(T-t)}}{rT} + e^{-r(T-t)} Z_t, \quad t \in [0, T]. \]

Exercise 13.9

a) When \( \Lambda_t / T \geq K \) we have

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\[ f(t, S_t, \Lambda_t) = e^{-(T-t)r} \left( \frac{\Lambda_t}{T} - K \right) + S_t \frac{1 - e^{-(T-t)r}}{rT}, \]

see Exercise 13.7.

b) When \( \Lambda_t / T \geq K \) we have

\[ \xi_t = \frac{1 - e^{-(T-t)r}}{rT} \quad \text{and} \quad \eta_t A_t = e^{(T-t)r} \left( \frac{\Lambda_t}{T} - K \right), \quad t \in [0, T]. \]

c) At maturity we have \( f(T, S_T, \Lambda_T) = (\Lambda_T / T - K)^+, \) hence \( \xi_T = 0 \) and \( \eta_T A_T = A_T \frac{e^{-rT}}{A_0} \left( \frac{\Lambda_T}{T} - K \right) \mathbb{1}_{\{\Lambda_T > KT\}} = \left( \frac{\Lambda_T}{T} - K \right)^+. \)

d) By Proposition 13.9 we have

\[ \xi_t = \frac{1}{S_t} \left( f(t, S_t, \Lambda_t) - \left( \frac{\Lambda_t}{T} - K \right) \frac{\partial g}{\partial z} \left( t, \frac{1}{S_t} \left( \frac{\Lambda_t}{T} - K \right) \right) \right) \]

where the function \( g(t, z) \) satisfies \( f(t, x, y) = xg(t, (y / T - K) / x)) \) and

\[ g(t, z) = z e^{-(T-t)r} + \frac{1 - e^{-(T-t)r}}{rT}, \quad z > 0, \]

and solves the PDE

\[ \frac{\partial g}{\partial t}(t, z) + \left( \frac{1}{T} - rz \right) \frac{\partial g}{\partial z}(t, z) + \frac{1}{2} \sigma^2 z^2 \frac{\partial^2 g}{\partial z^2}(t, z) = 0, \]

under the terminal condition \( g(T, z) = z^+ \), hence letting

\[ h(t, z) := e^{(T-t)r} \frac{\partial g}{\partial z}(t, z), \]

we have

\[ e^{(T-t)r} \frac{\partial g}{\partial t}(t, z) + e^{(T-t)r} \left( \frac{1}{T} - rz \right) \frac{\partial g}{\partial z}(t, z) + \frac{1}{2} \sigma^2 z^2 \frac{\partial^2 g}{\partial z^2}(t, z) = 0, \]

with \( h(t, z) = 1, \ z > 0, \) hence

\[ e^{(T-t)r} \frac{\partial^2 g}{\partial t \partial z}(t, z) - r e^{(T-t)r} \frac{\partial g}{\partial z}(t, z) + e^{(T-t)r} \left( \frac{1}{T} - rz \right) \frac{\partial^2 g}{\partial z^2}(t, z) \]

\[ + \sigma^2 z e^{(T-t)r} \frac{\partial^2 g}{\partial z^2}(t, z) + \frac{1}{2} e^{(T-t)r} \sigma^2 z^2 \frac{\partial^3 g}{\partial z^3}(t, z) = 0, \]

or
the portfolio self-financing condition reads

\[
\frac{\partial h}{\partial t}(t, z) + \left( \frac{1}{T} + (\sigma^2 - r)z \right) \frac{\partial h}{\partial z}(t, z) + \frac{1}{2} \sigma^2 z^2 \frac{\partial^2 h}{\partial z^2}(t, z) = 0,
\]

with the terminal condition \( h(T, z) = 1_{\{z > 0\}} \). On the other hand, we have

\[
\eta_t = \frac{1}{A_t} (f(t, S_t, \Lambda_t) - \xi_t S_t)
\]

\[
= \frac{1}{A_t} \left( \frac{\Lambda_t}{T} - K \right) \frac{\partial g}{\partial z} \left( t, \frac{1}{S_t} \left( \frac{\Lambda_t}{T} - K \right) \right)
\]

\[
= \frac{e^{-(T-t)r}}{A_t} \left( \frac{\Lambda_t}{T} - K \right) h \left( t, \frac{1}{S_t} \left( \frac{\Lambda_t}{T} - K \right) \right).
\]

Exercise 13.10 Asian options with dividends. When reinvesting dividends, the portfolio self-financing condition reads

\[
dV_t = \eta_t dA_t + \xi_t dS_t + \delta \xi_t S_t dt
\]

trading profit and loss

\[
dV_t = r\eta_t A_t dt + \xi_t ((\mu - \delta) S_t dt + \sigma S_t dB_t) + \delta \xi_t S_t dt
\]

\[
dV_t dt + (\mu - r) \xi_t S_t dt + \sigma \xi_t S_t dB_t, \quad t \in \mathbb{R}_+.
\]

On the other hand, by Itô’s formula we have

\[
dg_{\delta}(t, S_t, \Lambda_t)
\]

\[
= \frac{\partial g_{\delta}}{\partial t}(t, S_t, \Lambda_t) dt + \frac{\partial g_{\delta}}{\partial y}(t, S_t, \Lambda_t) d\Lambda_t + (\mu - \delta) S_t \frac{\partial g_{\delta}}{\partial x}(t, S_t, \Lambda_t) dt
\]

\[
+ \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 g_{\delta}}{\partial x^2}(t, S_t, \Lambda_t) dt + \sigma S_t \frac{\partial g_{\delta}}{\partial x}(t, S_t, \Lambda_t) dB_t
\]

\[
= \frac{\partial g_{\delta}}{\partial t}(t, S_t, \Lambda_t) dt + S_t \frac{\partial g_{\delta}}{\partial y}(t, S_t, \Lambda_t) dt + (\mu - \delta) S_t \frac{\partial g_{\delta}}{\partial x}(t, S_t, \Lambda_t) dt
\]

\[
+ \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 g_{\delta}}{\partial x^2}(t, S_t, \Lambda_t) dt + \sigma S_t \frac{\partial g_{\delta}}{\partial x}(t, S_t, \Lambda_t) dB_t,
\]

hence by identification of the terms in \( dB_t \) and \( dt \) in the expressions of \( dV_t \) and \( dg_{\delta}(t, S_t) \), we get

\[
\xi_t = \frac{\partial g_{\delta}}{\partial x}(t, S_t, \Lambda_t),
\]

and we derive the Black-Scholes PDE with dividend

\[
r g_{\delta}(t, x, y) = \frac{\partial g_{\delta}}{\partial t}(t, x, y) + y \frac{\partial g_{\delta}}{\partial y}(t, x, y)
\]

\[
+ (r - \delta) x \frac{\partial g_{\delta}}{\partial x}(t, x, y) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 g_{\delta}}{\partial x^2}(t, x, y).
\]

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Defining \( f(t, x, y) := e^{(T-t)\delta} g_\delta(t, x, y) \) and substituting
\[
g_\delta(t, x, y) = e^{(T-t)\delta} f(t, x, y)
\]
in (A.49) yields the equation
\[
rf(t, x, y) = \delta f(t, x, y) + y \frac{\partial f}{\partial y}(t, x, y) + \frac{\partial f}{\partial t}(t, x, y)
\]
\[
+ (r - \delta)x \frac{\partial f}{\partial x}(t, x, y) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}(t, x, y),
\]
\[\text{i.e.}\]
\[
(r - \delta)f(t, x, y) = \frac{\partial f}{\partial t}(t, x, y) + y \frac{\partial f}{\partial y}(t, x, y)
\]
\[
+ (r - \delta)x \frac{\partial f}{\partial x}(t, x, y) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}(t, x, y),
\]
whose solution \( f(t, x, y) \) is the Asian option pricing function with modified interest rate \( r - \delta \) and no dividends, under the terminal condition
\[
f(T, x, y) = g_\delta(T, x, y) = \left( \frac{y}{T} - K \right)^{+}.
\]
Therefore the Asian option price \( g_\delta(t, S_t, \Lambda_t) \) with dividend rate \( \delta \) can be recovered from the relation
\[
g_\delta(t, x, y) = e^{(T-t)\delta} f(t, x, y), \quad t \in [0, T], \ x, y > 0.
\]
Note that we can also define
\[
h(t, x, y) := g_\delta(t, x e^{-\delta(T-t)}, y)
\]
and substituting
\[
g_\delta(t, x, y) = h(t, x e^{\delta(T-t)}, y)
\]
in (A.49) yields the equation
\[
rh(t, x, y) = y \frac{\partial h}{\partial y}(t, x, y) + \frac{\partial h}{\partial t}(t, x, y)
\]
\[
+ r x \frac{\partial h}{\partial x}(t, x, y) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 h}{\partial x^2}(t, x, y),
\]
whose solution \( h(t, x, y) \) is the Asian option pricing function with interest rate \( r \) and no dividends, under the terminal condition
\[
h(T, x, y) = g_\delta(T, x, y) = \left( \frac{y}{T} - K \right)^{+}.
\]
Finally, the Asian option price \( g_{\delta}(t, S_t, \Lambda_t) \) with dividend rate \( \delta \) can be also recovered from the relation
\[
g_{\delta}(t, x, y) = h(t, x e^{-(T-t)\delta}, y), \quad t \in [0, T], \ x, y > 0.
\]

Chapter 14

Exercise 14.1

a) This process is a convex function \( x \mapsto -(2-x) \) of the Brownian martingale, hence it is a submartingale.

b) This process can be written as
\[
e^{\sigma B_t - \sigma^2 t/2 + \mu t} = e^{\mu t} e^{\sigma B_t - \sigma^2 t/2}, \quad t \in \mathbb{R}_+,
\]
with \( \sigma = 1 \) and \( \mu = \sigma^2/2 > 0 \), hence it is a submartingale as the geometric Brownian motion \( e^{\sigma B_t - \sigma^2 t/2} \) is a martingale.

c) When \( t > 0 \), the question “is \( \nu > t \)” cannot be answered at time \( t \) without waiting to know the value of \( B_{2t} \) at time \( 2t > t \). Therefore \( \nu \) is not a stopping time.

d) For any \( t \in \mathbb{R}_+ \), the question “is \( \tau > t \)” can be answered based on the observation of the paths of \( (B_s)_{0 \leq s \leq t} \) and of the (deterministic) curve \( (e^{s/2} + \alpha s e^{s/2})_{0 \leq s \leq t} \) up to the time \( t \). Therefore \( \tau \) is a stopping time.

e) Since \( \tau \) is a stopping time and \( (e^{B_{t/2}})_{t \in \mathbb{R}_+} \) is a martingale, the Stopping Time Theorem 14.7 shows that \( (e^{B_{t \wedge \tau} - (t \wedge \tau)/2})_{t \in \mathbb{R}_+} \) is also a martingale and, in particular, its expectation* is constantly equal to 1 for all \( t \geq 0 \). This shows that
\[
E[e^{B_{t \wedge \tau} - (t \wedge \tau)/2}] = E[e^{B_0 - (0 \wedge \tau)/2}] = E[e^{B_0 - 0/2}] = 1
\]

Next, we note that we have \( e^{B_{t \wedge \tau}/2} = 1 + \alpha \tau \), hence
\[
1 + \alpha E[\tau] = E[1 + \alpha \tau] = E[e^{B_{\tau}/2}] = 1,
\]
i.e. \( E[\tau] = (1 - \alpha)/\beta \).

Remark: This argument also recovers \( E[\tau] = 0 \) when \( \alpha = 1 \), however it fails when \( (\alpha > 1 \text{ and } \beta > 0) \) or \( (\alpha < 1 \text{ and } \beta < 0) \) because \( \tau \) is not a.s. finite in those cases.

* We let \( t \wedge \tau = \min(t, \tau) \).

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https://www.ntu.edu.sg/home/nprivault/index.html
Exercise 14.2 Stopping times.

a) When $0 \leq t < 1$ the question “is $\nu > t$?” cannot be answered at time $t$ without waiting to know the value of $B_1$ at time 1. Therefore $\nu$ is not a stopping time.

b) For any $t \in \mathbb{R}_+$, the question “is $\tau > t$?” can be answered based on the observation of the paths of $(B_s)_{0 \leq s \leq t}$ and of the (deterministic) curve $(\alpha e^{-s/2})_{0 \leq s \leq t}$ up to the time $t$. Therefore $\tau$ is a stopping time.

Since $\tau$ is a stopping time and $(B_t)_{t \in \mathbb{R}_+}$ is a martingale, the Stopping Time Theorem 14.7 shows that $(e^{B_{t \wedge \tau} - (t \wedge \tau)/2})_{t \in \mathbb{R}_+}$ is also a martingale and in particular its expectation

$$
\mathbb{E}[e^{B_{t \wedge \tau} - (t \wedge \tau)/2}] = \mathbb{E}[e^{B_{0 \wedge \tau} - (0 \wedge \tau)/2}] = \mathbb{E}[e^{B_0 - 0/2}] = 1
$$

is constantly equal to 1 for all $t$. This shows that

$$
\mathbb{E}[e^{B_{t \wedge \tau} - t/2}] = \mathbb{E}\left[\lim_{t \to \infty} e^{B_{t \wedge \tau} - (t \wedge \tau)/2}\right] = \lim_{t \to \infty} \mathbb{E}[e^{B_{t \wedge \tau} - (t \wedge \tau)/2}] = 1.
$$

Next, we note that we have $e^{B_{t \wedge \tau}} = \alpha e^{-\tau/2}$, hence

$$
\alpha \mathbb{E}[e^{-\tau}] = \mathbb{E}[e^{B_{t \wedge \tau} - t/2}] = 1, \quad i.e. \quad \mathbb{E}[e^{-\tau}] = 1/\alpha \leq 1.
$$

Remark: note that this argument fails when $\alpha < 1$ because in that case $\tau$ is not a.s. finite.

c) For any $t \in \mathbb{R}_+$, the question “is $\tau > t$?” can be answered based on the observation of the paths of $(B_s)_{0 \leq s \leq t}$ and of the (deterministic) curve $(1 + \alpha s)_{0 \leq s \leq t}$ up to the time $t$. Therefore $\tau$ is a stopping time.

Since $\tau$ is a stopping time and $(B_t)_{t \in \mathbb{R}_+}$ is a martingale, the Stopping Time Theorem 14.7 shows that $(B_{t \wedge \tau}^2 - (t \wedge \tau))_{t \in \mathbb{R}_+}$ is also a martingale and in particular its expectation

$$
\mathbb{E}[B_{t \wedge \tau}^2 - (t \wedge \tau)] = \mathbb{E}[B_{0 \wedge \tau}^2 - (0 \wedge \tau)] = \mathbb{E}[B_0^2 - 0] = 0
$$

is constantly equal to 0 for all $t$. This shows that

$$
\mathbb{E}[B_{t \wedge \tau}^2 - \tau] = \mathbb{E}\left[\lim_{t \to \infty} (B_{t \wedge \tau}^2 - (t \wedge \tau))\right] = \lim_{t \to \infty} \mathbb{E}[(B_{t \wedge \tau}^2 - (t \wedge \tau))] = 0.
$$

Next, we note that we have $B_{t \wedge \tau}^2 = 1 + \alpha \tau$, hence

$$
1 + \alpha \mathbb{E}[\tau] = \mathbb{E}[1 + \alpha \tau] = \mathbb{E}[B_{t \wedge \tau}^2] - \mathbb{E}[\tau] = 0,
$$

i.e.

$$
\mathbb{E}[\tau] = 1/(1 - \alpha).
$$

Remark: Note that this argument is valid whenever $\alpha \leq 1$ and yields $\mathbb{E}[\tau] = +\infty$ when $\alpha = 1$, however it fails when $\alpha > 1$ because in that case $\tau$ is not a.s. finite.
Exercise 14.3

a) By the Stopping Time Theorem 14.7, for all \( n \geq 0 \) we have

\[
\begin{align*}
1 & = \mathbb{E} \left[ e^{\sqrt{2r} B_{\tau_L \wedge n} - r(\tau_L \wedge n)} \right] \\
& = \mathbb{E} \left[ e^{\sqrt{2r} B_{\tau_L \wedge n} - r(\tau_L \wedge n)} \mathbb{1}_{\{\tau_L < n\}} \right] + \mathbb{E} \left[ e^{\sqrt{2r} B_{\tau_L \wedge n} - r(\tau_L \wedge n)} \mathbb{1}_{\{\tau_L \geq n\}} \right] \\
& = \mathbb{E} \left[ e^{\sqrt{2r} B_{\tau_L} - r \tau_L} \mathbb{1}_{\{\tau_L < n\}} \right] + \mathbb{E} \left[ e^{\sqrt{2r} B_{\tau_L} - r \tau_L} \mathbb{1}_{\{\tau_L \geq n\}} \right].
\end{align*}
\]

The first term above converges to

\[
e^{L \sqrt{2r}} \mathbb{E} \left[ e^{-r \tau_L} \mathbb{1}_{\{\tau_L < \infty\}} \right] = e^{L \sqrt{2r}} \mathbb{E} \left[ e^{-r \tau_L} \right]
\]

as \( n \) tends to infinity, by dominated or monotone convergence and the fact that \( r > 0 \). The second term can be bounded as

\[
0 \leq \mathbb{E} \left[ e^{\sqrt{2r} B_{\tau_L} - r \tau_L} \mathbb{1}_{\{\tau_L \geq n\}} \right] \leq e^{-rn} \mathbb{E} \left[ e^{L \sqrt{2r}} \mathbb{1}_{\{\tau_L \geq n\}} \right] \leq e^{-rn} e^{L \sqrt{2r}},
\]

which tends to 0 as \( n \) tends to infinity because \( r > 0 \). Therefore we have

\[
1 = \lim_{n \to \infty} \mathbb{E} \left[ e^{\sqrt{2r} B_{\tau_L \wedge n} - r(\tau_L \wedge n)} \right] = e^{L \sqrt{2r}} \mathbb{E} \left[ e^{-r \tau_L} \right],
\]

which yields \( \mathbb{E} [e^{-r \tau_L}] = e^{-L \sqrt{2r}} \) for any \( r \geq 0 \). When \( r < 0 \) we could in fact show that \( \mathbb{E}[e^{-r \tau_L}] = +\infty \).

b) In order to maximize

\[
\mathbb{E} \left[ e^{-r \tau_L} B_{\tau_L} \right] = L \mathbb{E} \left[ e^{-r \tau_L} \right] = L e^{-L \sqrt{2r}}
\]

we differentiate

\[
\frac{\partial}{\partial L} \left( L e^{-L \sqrt{2r}} \right) = e^{-L \sqrt{2r}} - L \sqrt{2r} e^{-L \sqrt{2r}} = 0,
\]

which yields the optimal level \( L^* = 1 / \sqrt{2r} \).

This shows that when the value of \( r \) is “large” the better strategy is to opt for a “small gain” at the level \( L^* = 1 / \sqrt{2r} \) rather than to wait for a longer time.

Exercise 14.4 See e.g. Theorem 6.16 page 161 of Klebaner (2005). By the Itô formula we have

\[
X_t = f(B_t) - \frac{1}{2} \int_0^t f''(B_s) ds
\]
\[ f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2} \int_0^t f''(B_s)ds - \frac{1}{2} \int_0^t f''(B_s)ds \]

\[ = f(B_0) + \int_0^t f'(B_s)dB_s, \]

hence the process \((X_t)_{t \in \mathbb{R}_+}\) is a *martingale*. By the Doob Stopping Time Theorem 14.7 we have

\[ f(x) = \mathbb{E}[X_0 \mid B_0 = x] \]

\[ = \mathbb{E}[X_{\tau \wedge t} \mid B_0 = x] \]

\[ = \mathbb{E}[f(B_{\tau \wedge t}) \mid B_0 = x] - \frac{1}{2} \mathbb{E} \left[ \int_0^{\tau \wedge t} f''(B_s)ds \mid B_0 = x \right] \]

\[ = \mathbb{E}[f(B_{\tau \wedge t}) \mid B_0 = x] + \mathbb{E}[\tau \wedge t \mid B_0 = x], \]

since \(f''(y) = -2\) for all \(y \in \mathbb{R}\). We note that, by dominated convergence,

\[ \mathbb{E} \left[ \tau \mid B_0 = x \right] = \mathbb{E} \left[ \lim_{t \to \infty} (\tau \wedge t) \mid B_0 = x \right] \]

\[ = \lim_{t \to \infty} \mathbb{E}[\tau \wedge t \mid B_0 = x] \]

\[ \leq |f(x)| + \max_{y \in [a,b]} |f(y)| \]

\[ < \infty, \]

hence \(\mathbb{E}[\tau] < \infty\) and therefore \(\mathbb{P}(\tau < \infty) = 1\). Next, we have

\[ f(x) = \lim_{t \to \infty} \mathbb{E}[f(B_{\tau \wedge t}) \mid B_0 = x] + \lim_{t \to \infty} \mathbb{E}[\tau \wedge t \mid B_0 = x] \]

\[ = \mathbb{E} \left[ \lim_{t \to \infty} f(B_{\tau \wedge t}) \mid B_0 = x \right] + \mathbb{E}[\tau \mid B_0 = x] \]

\[ = \mathbb{E}[f(B_{\tau}) \mid B_0 = x] + \mathbb{E}[\tau \mid B_0 = x] \]

\[ = \mathbb{E}[\tau \mid B_0 = x], \]

since \(f''(x) = -2\) and \(f(a) = f(b) = 0\) with \(B_{\tau} \in \{a, b\}\).

**Remarks.**

i) The above exchanges between \(\lim_{t \to \infty}\) and the expectation operator \(\mathbb{E}[\cdot \mid B_0 = x]\) is justified by the *dominated convergence theorem*, since

\[ |f(B_{\tau \wedge t})| \leq \max_{y \in [a,b]} |f(y)|, \quad t \in \mathbb{R}_+. \]

ii) The function \(f(x)\) can be determined by searching for a quadratic solution of the form \(f(x) = \alpha + \beta x + \gamma x^2\), which shows that \(f''(x) = 2\gamma = -2\) hence \(\gamma = -1\), and
\[
\begin{align*}
    f(a) &= \alpha + \beta a - a^2 = 0, \\
    f(b) &= \alpha + \beta b - b^2 = 0,
\end{align*}
\]

hence \( \alpha = -ab \) and \( \beta = a + b \). Therefore, we have

\[
\mathbb{E} [\tau \mid B_0 = x] = f(x) = -ab + (a + b)x - x^2 = (x - a)(b - x).
\]

Exercise 14.5

a) Letting \( A_0 := 0 \),

\[
A_{n+1} := A_n + \mathbb{E}[M_{n+1} - M_n \mid F_n], \quad n \geq 0,
\]

and

\[
N_n := M_n - A_n, \quad n \in \mathbb{N}, \quad (A.50)
\]

we have

(i) for all \( n \in \mathbb{N} \),

\[
\mathbb{E}[N_{n+1} \mid F_n] = \mathbb{E}[M_{n+1} - A_{n+1} \mid F_n]
= \mathbb{E}[M_{n+1} - A_n - \mathbb{E}[M_{n+1} - M_n \mid F_n] \mid F_n]
= \mathbb{E}[M_{n+1} - A_n - \mathbb{E}[M_{n+1} - M_n \mid F_n] \mid F_n]
= \mathbb{E}[M_{n+1} - A_n \mid F_n] - \mathbb{E}[M_{n+1} - M_n \mid F_n]
= - \mathbb{E}[A_n \mid F_n] + \mathbb{E}[M_n \mid F_n]
= M_n - A_n
= N_n,
\]

hence \( (N_n)_{n \in \mathbb{N}} \) is a martingale with respect to \( (F_n)_{n \in \mathbb{N}} \).

(ii) We have

\[
A_{n+1} - A_n = \mathbb{E}[M_{n+1} - M_n \mid F_n]
= \mathbb{E}[M_{n+1} \mid F_n] - \mathbb{E}[M_n \mid F_n]
= \mathbb{E}[M_{n+1} \mid F_n] - M_n \geq 0, \quad n \in \mathbb{N},
\]

since \( (M_n)_{n \in \mathbb{N}} \) is a submartingale.

(iii) By induction we have

\[
A_n = A_{n-1} + \mathbb{E}[M_n - M_{n-1} \mid F_{n-1}], \quad n \geq 1,
\]

which is \( F_{n-1} \)-measurable provided that \( A_n \) is \( F_{n-1} \)-measurable, \( n \geq 1 \).

(iv) This property is obtained by construction in \( (A.50) \).
b) For all bounded stopping times $\sigma$ and $\tau$ such that $\sigma \leq \tau$ a.s., we have

$$\mathbb{E}[M_{\sigma}] = \mathbb{E}[N_{\sigma}] + \mathbb{E}[A_{\sigma}]$$

$$\leq \mathbb{E}[N_{\tau}] + \mathbb{E}[A_{\tau}]$$

$$= \mathbb{E}[N_{\tau}] + \mathbb{E}[A_{\tau}]$$

$$= \mathbb{E}[M_{\tau}],$$

by (14.6), since $(M_n)_{n \in \mathbb{N}}$ is a martingale and $(A_n)_{n \in \mathbb{N}}$ is non-decreasing.

Exercise 14.6 The option payoffs at immediate exercise are given as follows:

<table>
<thead>
<tr>
<th>$K - S_2^+$</th>
<th>$p^*$</th>
<th>$(K - S_2)^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2/3</td>
<td>0</td>
</tr>
<tr>
<td>0.05</td>
<td>2/3</td>
<td>0</td>
</tr>
<tr>
<td>0.3</td>
<td>1/3</td>
<td>0</td>
</tr>
<tr>
<td>0.35</td>
<td>1/3</td>
<td>0.17</td>
</tr>
<tr>
<td>0.44</td>
<td>1/3</td>
<td>0.44</td>
</tr>
</tbody>
</table>

The expected payoffs are given by

$$\mathbb{E}^*[(K - S_2)^+ | S_1 = 1.2] = 0.17/3$$

$$\mathbb{E}^*[(K - S_2)^+ | S_1 = 0.9] = 0.26$$

Consequently, at time $t = 1$ we would exercise immediately if $S_1 = 0.9$, and wait if $S_1 = 1.2$. At time $t = 0$ with $S_0 = 1$ the initial value of the option is $(0.34/3 + 0.35)/3 = 1.39/9 \approx 0.154 < 0.25$ so we would exercise immediately as well.

Exercise 14.7

a) Taking $f(x) := Cx^{-2r/\sigma^2}$, we have

$$rxf'(x) + \frac{1}{2}\sigma^2 x^2 f''(x) = -C\frac{2r^2}{\sigma^2}x^{-2r/\sigma^2} + Cr \left(1 + \frac{2r}{\sigma^2}\right)x^{-2r/\sigma^2}$$
\[= Crx^{-2r/\sigma^2}
\]
\[= rf(x),\]

and the condition \(\lim_{x \to \infty} f(x) = 0\) is satisfied since \(r > 0\).

b) The conditions \(f(L^*) = K - L^*\) and \(f'(L^*) = -1\) read
\[
\begin{align*}
\begin{cases}
C(L^*)^{-2r/\sigma^2} &= K - L^*, \\
-\frac{2r}{\sigma^2} C(L^*)^{-1-2r/\sigma^2} &= -1,
\end{cases}
\end{align*}
\]
i.e.
\[
\begin{align*}
\begin{cases}
C(L^*)^{-2r/\sigma^2} &= K - L^*, \\
\frac{2r}{\sigma^2} (K - L^*) &= L^*,
\end{cases}
\end{align*}
\]
hence
\[
\begin{align*}
\begin{cases}
L^* &= \frac{2rK}{2r + \sigma^2} \\
C &= \frac{K\sigma^2}{2r + \sigma^2} \left( \frac{2rK}{2r + \sigma^2} \right)^{2r/\sigma^2} = \frac{\sigma^2}{2r} \left( \frac{2rK}{2r + \sigma^2} \right)^{1+2r/\sigma^2}.
\end{cases}
\end{align*}
\]

Exercise 14.8

a) Given the value \(-\Phi(-d_+(S^*, T))\) of the Delta of the Black-Scholes put option, the smooth fit condition states that at \(x = S^*\), the left derivative of (14.46) should match the right derivative of (14.47), i.e.
\[
-1 = -\Phi(-d_+(S^*, T)) - \frac{2r\alpha}{\sigma^2} (S^*)^{-1},
\]
which yields
\[
\alpha(S^*) = \frac{\sigma^2 S^*}{2r} (1 - \Phi(-d_+(S^*, T))) = \frac{\sigma^2 S^*}{2r} \Phi(d_+(S^*, T)),
\]
and
\[
f(x, T) \simeq \begin{cases} 
BS_p(x, T) + \frac{\sigma^2(S^*)^{1+2r/\sigma^2}}{2r x^{2r/\sigma^2}} \Phi(d_+(S^*, T)), & \text{if } x > S^*, \\
K - x, & \text{if } x \leq S^*.
\end{cases}
\]

Note that at maturity (\(T = 0\) here) we have \(d_+(S^*, 0) = -\infty\) since \(S^* < K\), hence \(\Phi(d_+(S^*, 0)) = 0\) and \(f(x, 0) = K - x\) as expected.

b) Equating (14.46) to (14.47) at \(x = S^*\) yields the equation
\[
\]
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\[ K - S^* = B_S(x, T) + \alpha(S^*), \]

\textit{i.e.}

\[ 1 = e^{-rT} \Phi(-d_-(S^*, T)) + \frac{S^*}{K} \left( 1 + \frac{\sigma^2}{2r} \right) \Phi(-d_+(S^*, T)), \]

which can be used to determine the value of \( S^* \), and then the corresponding value of \( \alpha \). The proposed strategy is to exercise the put option as soon as the underlying asset price reaches the critical level \( S^* \).

![Fig. S.56: Perpetual vs finite expiration American put option price.](https://www.ntu.edu.sg/home/nprivault/indext.html)

The plot in Figure S.56 yields a finite expiration critical price \( S^* = 87.3 \) which is expectedly higher than the perpetual critical price \( L^* = 85.71 \), with \( K = 100 \), \( \sigma = 10\% \), and \( r = 3\% \). The perpetual price, however, appears higher than the finite expiration price.

Exercise 14.9

a) We have

\[ \tau_\epsilon = \begin{cases} 
\epsilon & \text{if } Z = 1, \\
+\infty & \text{if } Z = 0.
\end{cases} \]

b) First, we note that

\[ \mathcal{F}_t = \begin{cases} 
\emptyset, \Omega & \text{if } t = 0, \\
\emptyset, \Omega, \{Z = 0\}, \{Z = 1\} & \text{if } t > 0.
\end{cases} \]

Next, we have

\[ \{\tau_\epsilon > 0\} = \{Z = 0\}, \]

hence

\[ \{\tau_\epsilon > 0\} \notin \mathcal{F}_0 = \{\emptyset, \Omega\}, \]
and therefore $\tau_0$ is not an $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$-stopping time.

c) i) For $t = 0$ we have $\{\tau_\epsilon > 0\} = \{Z = 0\} \cup \{Z = 1\} = \Omega$, hence

\[ \{\tau_\epsilon > 0\} \in \mathcal{F}_0 = \{\emptyset, \Omega\}. \]

ii) For $0 < t < \epsilon$ we have $\{\tau_\epsilon > t\} = \Omega$, hence

\[ \{\tau_\epsilon > t\} \in \mathcal{F}_t = \{\emptyset, \Omega, \{Z = 0\}, \{Z = 1\}\}. \]

iii) For $t > \epsilon$ we have $\{\tau_\epsilon > t\} = \{Z = 0\}$, hence

\[ \{\tau_\epsilon > t\} \in \mathcal{F}_t = \{\emptyset, \Omega, \{Z = 0\}, \{Z = 1\}\}. \]

Therefore $\tau_\epsilon$ is an $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$-stopping time when $\epsilon > 0$.

Note that here the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is not right-continuous, as

\[ \{\emptyset, \Omega\} = \mathcal{F}_0 \neq \mathcal{F}_0^+ := \bigcap_{t > 0} \mathcal{F}_t = \{\emptyset, \Omega, \{Z = 0\}, \{Z = 1\}\}. \]

Exercise 14.10

a) This intrinsic payoff is $\kappa - S_0$.

b) We note that

\[ Z_t = \left( \frac{S_t}{S_0} \right)^\lambda e^{-(r-\delta)\lambda t + \lambda \sigma^2 t/2 - \lambda^2 \sigma^2 t/2} = e^{\lambda \sigma B_t - \lambda^2 \sigma^2 t/2}, \quad t \in \mathbb{R}_+, \]

is a geometric Brownian motion without drift, hence a martingale, under the risk-neutral probability measure $\mathbb{P}^*$.

c) The parameter $\lambda$ should satisfy the equation

\[ r = (r-\delta)\lambda - \frac{\sigma^2}{2} \lambda (1-\lambda), \]

i.e.

\[ \lambda^2 \sigma^2/2 + \lambda (r - \delta - \sigma^2/2) - r = 0. \]

This equation admits two solutions

\[ \lambda_{\pm} = \frac{-(r-\delta-\sigma^2/2) \pm \sqrt{(r-\delta-\sigma^2/2)^2 + 4r\sigma^2/2}}{\sigma^2}, \]

d) Relation (14.25) follows from the fact that $S_t < L$ and $\lambda_- < 0$.

e) By the Stopping Time Theorem 14.7 we have

\[ \mathbb{E}^*[Z_{\tau_L}] = \mathbb{E}^*[Z_0] = 1, \]
which rewrites as
\[
\mathbb{E}^* \left[ \left( \frac{S_{\tau L}}{S_0} \right)^\lambda e^{-(r-\delta)\lambda - \lambda \sigma^2 / 2 + \lambda^2 \sigma^2 / 2} \right] = 1,
\]
or, given the relation \( S_{\tau L} = L \),
\[
\mathbb{E}^* \left[ e^{-(r-\delta)\lambda - \lambda \sigma^2 / 2 + \lambda^2 \sigma^2 / 2} \right] = 1,
\]
i.e.
\[
\mathbb{E}^* \left[ e^{-r\tau L} \right] = \left( \frac{S_0}{L} \right)^\lambda,
\]
provided that we choose \( \lambda \) such that
\[
-((r-\delta)\lambda - \lambda \sigma^2 / 2 + \lambda^2 \sigma^2 / 2) = -r,
\]
i.e.
\[
\lambda = \frac{-(r-\delta - \sigma^2 / 2) \pm \sqrt{(r-\delta - \sigma^2 / 2)^2 + 4r \sigma^2 / 2}}{\sigma^2},
\]
and we choose the negative solution
\[
\lambda = \frac{-(r-\delta - \sigma^2 / 2) - \sqrt{(r-\delta - \sigma^2 / 2)^2 + 4r \sigma^2 / 2}}{\sigma^2}
\]
since \( S_0/L = x/L > 1 \) and the expectation \( \mathbb{E}^*[e^{-r\tau L}] < 1 \) is lower than 1 as \( r \geq 0 \).

f) This follows from (14.25) and the fact that \( r > 0 \). Using the fact that \( S_{\tau L} = L < K \) when \( \tau L < \infty \), we find
\[
\mathbb{E}^* \left[ e^{-r\tau L} (K - S_{\tau L})^+ \big| S_0 = x \right] = \mathbb{E}^* \left[ e^{-r\tau L} (K - S_{\tau L})^+ \mathbb{1}\{\tau L < x\} \big| S_0 = x \right]
= \mathbb{E}^* \left[ e^{-r\tau L} (K - L)^+ \mathbb{1}\{\tau L < x\} \big| S_0 = x \right]
= (K - L) \mathbb{E}^* \left[ e^{-r\tau L} \mathbb{1}\{\tau L < x\} \big| S_0 = x \right]
= (K - L) \mathbb{E}^* \left[ e^{-r\tau L} \big| S_0 = x \right].
\]
Next, noting that \( \tau L = 0 \) if \( S_0 \leq L \), for all \( L \in (0, K) \) we have
\[
\mathbb{E}^* \left[ e^{-r\tau L} (K - S_{\tau L})^+ \big| S_0 = x \right] = \begin{cases}
K - x, & 0 < x \leq L, \\
E \left[ e^{-r\tau L} (K - L)^+ \big| S_0 = x \right], & x \geq L.
\end{cases}
\]

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https://www.ntu.edu.sg/home/nprivault/index.html
\[ g) \text{ In order to compute } L^* \text{ we observe that, geometrically, the slope of } x \mapsto f_L(x) = (K - L)(x/L) = \lambda - (K - L)^+ at x = L^* \text{ is equal to } -1, \text{ i.e.} \]
\[ f_L^*(L^*) = \lambda_-(K - L^*) \frac{(L^*)^{\lambda_- - 1}}{(L^*)^{\lambda_-}} = -1, \]
hence
\[ \lambda_-(K - L^*) = L^*, \text{ or } L^* = \frac{\lambda_-}{\lambda_- - 1} K < K. \quad (A.52) \]
Equivalently we may recover the value of \( L^* \) from the optimality condition
\[ \frac{\partial f_L(x)}{\partial L} = - \left( \frac{x}{L} \right)^{\lambda_-} - \lambda_- x(K - L) \left( \frac{x}{L} \right)^{\lambda_- + 1} = 0, \]
at \( L = L^* \), hence
\[ - \left( \frac{x}{L^*} \right)^{\lambda_-} - \lambda_- (K - L)x^{\lambda_-} L^{\lambda_- - 1} = 0, \]
hence
\[ L^* = \frac{\lambda_-}{1 - \lambda_-} K = \frac{1}{1 - 1/\lambda_-} K, \]
and
\[ \sup_{L \in (0, K)} \mathbb{E}^* \left[ e^{-r\tau_L} (K - S_{\tau_L})^+ \mid S_0 = x \right] = - \frac{1}{\lambda_-} \left( \frac{K}{1 - 1/\lambda_-} \right)^{1 - \lambda_-} x^{\lambda_-}. \]

h) For \( x \geq L \) we have
\[ f_L^*(x) = (K - L^*) \left( \frac{x}{L^*} \right)^{\lambda_-} \]
\[ = \left( K - \frac{\lambda_-}{\lambda_- - 1} K \right) \left( \frac{x}{\lambda_- - 1} K \right)^{\lambda_- - 1} \]
\[ = \left( - \frac{K}{\lambda_- - 1} \right) \left( \frac{x(\lambda_- - 1)}{\lambda_- K} \right)^{\lambda_- - 1} \]
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\begin{align}
&= \left( -\frac{K}{\lambda_- - 1} \right) \left( \frac{x}{-\lambda_-} \right)^{\lambda_-} \left( \frac{\lambda_- - 1}{-K} \right)^{\lambda_-} \\
&= \left( \frac{x}{-\lambda_-} \right)^{\lambda_-} \left( \frac{\lambda_- - 1}{-K} \right)^{\lambda_- - 1} \\
&= \left( \frac{x}{K} \right)^{\lambda_-} \left( \frac{\lambda_- - 1}{\lambda_-} \right)^{\lambda_-} \frac{K}{1 - \lambda_-} . \tag{A.53}
\end{align}

i) Let us check that the relation

\[ f_{L^*}(x) \geq (K - x)^+ \]  \hspace{1cm} (A.54)

holds. For all \( x \leq K \) we have

\begin{align}
&\quad f_{L^*}(x) - (K - x) = \left( \frac{x}{K} \right)^{\lambda_-} \left( \frac{\lambda_- - 1}{\lambda_-} \right)^{\lambda_-} \frac{K}{1 - \lambda_-} + x - K \\
&\quad = K \left( \left( \frac{x}{K} \right)^{\lambda_-} \left( \frac{\lambda_- - 1}{\lambda_-} \right)^{\lambda_-} \frac{1}{1 - \lambda_-} + \frac{x}{K} - 1 \right). 
\end{align}

Hence it suffices to take \( K = 1 \) and to show that for all

\[ L^* = \frac{\lambda_-}{\lambda_- - 1} \leq x \leq 1 \]

we have

\[ f_{L^*}(x) - (1 - x) = \frac{x^{\lambda_-}}{1 - \lambda_-} \left( \frac{\lambda_- - 1}{\lambda_-} \right)^{\lambda_-} + x - 1 \geq 0 . \]

Equality to 0 holds for \( x = \lambda_- / (\lambda_- - 1) \). By differentiation of this relation we get

\begin{align}
&f'_{L^*}(x) - (1 - x)' = \lambda_- x^{\lambda_- - 1} \left( \frac{\lambda_- - 1}{\lambda_-} \right)^{\lambda_-} \frac{1}{1 - \lambda_-} + 1 \\
&\quad = x^{\lambda_- - 1} \left( \frac{\lambda_- - 1}{\lambda_-} \right)^{\lambda_- - 1} + 1 \\
&\quad \geq 0 ,
\end{align}

hence the function \( f_{L^*}(x) - (1 - x) \) is non-decreasing and the inequality holds throughout the interval \([\lambda_- / (\lambda_- - 1), K]\).

On the other hand, using \( (A.51) \) it can be checked by hand that \( f_{L^*} \) given by \( (A.53) \) satisfies the equality

\[ (r - \delta) x f'_{L^*}(x) + \frac{1}{2} \sigma^2 x^2 f''_{L^*}(x) = r f_{L^*}(x) \]  \hspace{1cm} (A.55)
for \( x \geq L^* = \frac{\lambda_-}{\lambda_- - 1} K \). In case

\[
0 \leq x \leq L^* = \frac{\lambda_-}{\lambda_- - 1} K < K,
\]

we have

\[
f_{L^*}(x) = K - x = (K - x)^+,
\]

hence the relation

\[
\left( rf_{L^*}(x) - (r - \delta) x f'_{L^*}(x) - \frac{1}{2} \sigma^2 x^2 f''_{L^*}(x) \right) (f_{L^*}(x) - (K - x)^+) = 0
\]

always holds. On the other hand, in that case we also have

\[
(r - \delta) x f'_{L^*}(x) + \frac{1}{2} \sigma^2 x^2 f''_{L^*}(x) = -(r - \delta) x,
\]

and to conclude we need to show that

\[
(r - \delta) x f'_{L^*}(x) + \frac{1}{2} \sigma^2 x^2 f''_{L^*}(x) \leq rf_{L^*}(x) = r(K - x), \tag{A.56}
\]

which is true if

\[
\delta x \leq rK.
\]

Indeed by (A.51) we have

\[
(r - \delta) \lambda_- = r + \lambda_- (\lambda_- - 1) \sigma^2 / 2
\]

\[
\geq r,
\]

hence

\[
\delta \frac{\lambda_-}{\lambda_- - 1} \leq r,
\]

since \( \lambda_- < 0 \), which yields

\[
\delta x \leq \delta L^* \leq \delta \frac{\lambda_-}{\lambda_- - 1} K \leq rK.
\]

j) By Itô’s formula and the relation

\[
dS_t = (r - \delta) S_t dt + \sigma S_t d\tilde{B}_t
\]

we have

\[
d(f_{L^*}(S_t)) = -r e^{-rt} f_{L^*}(S_t) dt + e^{-rt} df_{L^*}(S_t)
\]

\[
= -r e^{-rt} f_{L^*}(S_t) dt + e^{-rt} f'_{L^*}(S_t) dS_t + \frac{1}{2} e^{-rt} \sigma^2 S_t^2 f''_{L^*}(S_t)
\]
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\[
e^{-rt} \left( -rf_{L^*}(S_t) + (r - \delta)S_t f'_{L^*}(S_t) + \frac{1}{2} \sigma^2 S_t^2 f''_{L^*}(S_t) \right) dt + e^{-rt} \sigma S_t f'_{L^*}(S_t) d\bar{B}_t,
\]

and from Equations (A.55) and (A.56) we have

\[
(r - \delta) x f'_{L^*}(x) + \frac{1}{2} \sigma^2 x^2 f''_{L^*}(x) \leq r f_{L^*}(x),
\]

hence

\[
t \mapsto e^{-rt} f_{L^*}(S_t)
\]

is a supermartingale.

k) By the supermartingale property of

\[
t \mapsto e^{-rt} f_{L^*}(S_t),
\]

for all stopping times \( \tau \) we have

\[
f_{L^*}(S_0) \geq \mathbb{E}^* \left[ e^{-r\tau} f_{L^*}(S_\tau) \ \big| \ S_0 \right] \geq \mathbb{E}^* \left[ e^{-r\tau}(K - S_\tau)^+ \ \big| \ S_0 \right],
\]

by (A.54), hence

\[
f_{L^*}(S_0) \geq \sup_{\tau \text{ stopping time}} \mathbb{E}^* \left[ e^{-r\tau}(K - S_\tau)^+ \ \big| \ S_0 \right]. \tag{A.57}
\]

l) The stopped process

\[
t \mapsto e^{-r(t \wedge \tau_{L^*})} f_{L^*}(S_{t \wedge \tau_{L^*}})
\]

is a martingale since it has vanishing drift up to time \( \tau_{L^*} \) by (A.55), and it is constant after time \( \tau_{L^*} \), hence by the Stopping Time Theorem 14.7 we find

\[
f_{L^*}(S_0) = \mathbb{E}^* \left[ e^{-r\tau_{L^*}} f_{L^*}(S_{\tau_{L^*}}) \ \big| \ S_0 \right] = \mathbb{E}^* \left[ e^{-r\tau_{L^*}} (L^*) \ \big| \ S_0 \right] = \mathbb{E}^* \left[ e^{-r\tau_{L^*}} (K - S_{\tau_{L^*}})^+ \ \big| \ S_0 \right] \leq \sup_{\tau \text{ stopping time}} \mathbb{E}^* \left[ e^{-r\tau}(K - S_\tau)^+ \ \big| \ S_0 \right].
\]

m) By combining the above results and conditioning at time \( t \) instead of time \( 0 \) we deduce that

\[
f_{L^*}(S_t) = \mathbb{E}^* \left[ e^{-r(\tau_{L^*} - t)}(K - S_{\tau_{L^*}})^+ \ \big| \ S_t \right]
\]
for all $t \in \mathbb{R}_+$, where

$$
\tau_{L^*} = \inf\{u \geq t : S_u \leq L\}.
$$

We note that the perpetual put option price does not depend on the value of $t \geq 0$.

Exercise 14.11

a) We have

$$
Z_t^{(\lambda)} = (S_t)^{\lambda} e^{-t((r-\delta)\lambda-\lambda(1-\lambda)\sigma^2/2)} = (S_0)^{\lambda} e^{\lambda \sigma \tilde{B}_t - \lambda^2 \sigma^2 t/2},
$$

which is a driftless geometric Brownian motion, and therefore a martingale under $\mathbb{P}^*$.

b) The condition is $r = (r-\delta)\lambda - \lambda(1-\lambda)\sigma^2/2$, with solutions

$$
\lambda_- = \frac{\delta - r + \sigma^2/2 - \sqrt{(\delta - r + \sigma^2/2)^2 + 2r\sigma^2}}{\sigma^2} \leq 0,
$$

$$
\lambda_+ = \frac{\delta - r + \sigma^2/2 + \sqrt{(\delta - r + \sigma^2/2)^2 + 2r\sigma^2}}{\sigma^2} \geq 1.
$$

c) The inequality

$$
0 \leq Z_t^{(\lambda_+)} = (S_t)^{\lambda_+} e^{-r t} \leq L^{\lambda_+}
$$

holds because $\lambda_+ > 0$ and $S_t \leq L$, $0 \leq t < \tau_L$. Hence, since $\lim_{t \to \infty} Z_t^{(\lambda_+)} = 0$, we have

$$
L^{\lambda_+} \mathbb{E}^* \left[ e^{-r \tau_L} \right] = \mathbb{E}^* \left[ (S_{\tau_L})^{\lambda_+} e^{-r \tau_L} \mathbb{1}_{\{\tau_L < \infty\}} \right] = \mathbb{E}^* \left[ Z_{\tau_L}^{(\lambda_+)} \mathbb{1}_{\{\tau_L < \infty\}} \right] = \mathbb{E}^* \left[ Z_{\tau_L}^{(\lambda_+)} \mathbb{1}_{\{\tau_L < \infty\}} \right] = \mathbb{E}^* \left[ Z_0^{(\lambda_+)} \right] = (S_0)^{\lambda_+},
$$

hence

$$
\mathbb{E}^* \left[ e^{-r \tau_L} (S_{\tau_L} - K)^+ \mid S_0 = x \right] = \mathbb{E}^* \left[ (S_{\tau_L} - K) e^{-r \tau_L} \mathbb{1}_{\{\tau_L < \infty\}} \mid S_0 = x \right] = (L - K) \mathbb{E}^* \left[ e^{-r \tau_L} \mathbb{1}_{\{\tau_L < \infty\}} \mid S_0 = x \right] = (L - K) \mathbb{E}^* \left[ e^{-r \tau_L} \mid S_0 = x \right]
$$
when \( S_0 = x > L \). In order to maximize

\[
\sup_{L \in (0,K)} \mathbb{E}^* \left[ e^{-r\tau L} (K - S_{\tau_L})^+ \mid S_0 = x \right],
\]

we differentiate \( L \mapsto (L - K) \left( \frac{x}{L} \right)^{\lambda^+} \) with respect to \( L \), to find

\[
\left( \frac{x}{L} \right)^{\lambda^+} - \lambda_+ (L - K) x^{\lambda^+} L^{-\lambda^+ - 1} = 0,
\]

hence

\[
L^*_\delta = \frac{\lambda_+}{\lambda_+ - 1} K = \frac{K}{1 - 1/\lambda_+},
\]

and

\[
\sup_{L \in (0,K)} \mathbb{E}^* \left[ e^{-r\tau L} (K - S_{\tau_L})^+ \mid S_0 = x \right] = \frac{1}{\lambda_+} \left( \frac{K}{1 - 1/\lambda_+} \right)^{1-\lambda_+} x^{\lambda^+}.
\]

We note that as \( \delta \downarrow 0 \) we have \( \lambda_+ \downarrow 1 \) and \( L^*_\delta \uparrow \infty \), and since

\[
\left( \frac{K}{1 - 1/\lambda_+} \right)^{1-\lambda_+} = \exp \left( (\lambda_+ - 1) \log \frac{\lambda_+ - 1}{\lambda_+ K} \right) \to 1,
\]

we find that the perpetual American call option price without dividend \( (\delta = 0) \) is \( S_0 = x \).

Exercise 14.12

a) By the definition (14.51) of \( S_1(t) \) and \( S_2(t) \) we have

\[
Z_t = e^{-rt} S_2(t) \left( \frac{S_1(t)}{S_2(t)} \right)^\alpha
\]

\[
= e^{-rt} S_1(t)^\alpha S_2(t)^{1-\alpha}
\]

\[
= S_1(0)^\alpha S_2(0)^{1-\alpha} e^{(\alpha \sigma_1 + (1-\alpha)\sigma_2) W_t - \sigma_2^2 t/2},
\]

which is a martingale when

\[
\sigma_2^2 = (\alpha \sigma_1 + (1-\alpha)\sigma_2)^2,
\]

i.e.

\[
\alpha \sigma_1 + (1-\alpha)\sigma_2 = \pm \sigma_2,
\]

which yields either \( \alpha = 0 \) or
In order to maximize (A.60) as a function of b) We have

\[ \mathbb{E}[e^{-r\tau_L}(S_1(\tau_L) - S_2(\tau_L))^{+}] = \mathbb{E}[e^{-r\tau_L}(LS_2(\tau_L) - S_2(\tau_L))^{+}] = (L - 1)^+ \mathbb{E}[e^{-r\tau_L}S_2(\tau_L)]. \]  

(A.58)

c) Since \( \tau_L \wedge t \) is a bounded stopping time we can write

\[
S_2(0) \left( \frac{S_1(0)}{S_2(0)} \right)^\alpha = \mathbb{E} \left[ e^{-r(\tau_L \wedge t)} S_2(\tau_L \wedge t) \left( \frac{S_1(\tau_L \wedge t)}{S_2(\tau_L \wedge t)} \right)^\alpha \right] \]

(A.59)

We have

\[
e^{-rt} S_2(t) \left( \frac{S_1(t)}{S_2(t)} \right)^\alpha \mathbbm{1}_{\{\tau_L > t\}} \leq e^{-rt} S_2(t)L^\alpha \mathbbm{1}_{\{\tau_L > t\}} \leq e^{-rt} S_2(t)L^\alpha,
\]

hence by a uniform integrability argument,

\[
\lim_{t \to \infty} \mathbb{E} \left[ e^{-rt} S_2(t) \left( \frac{S_1(t)}{S_2(t)} \right)^\alpha \mathbbm{1}_{\{\tau_L > t\}} \right] = 0,
\]

and letting \( t \) go to infinity in (A.59) shows that

\[
S_2(0) \left( \frac{S_1(0)}{S_2(0)} \right)^\alpha = \mathbb{E} \left[ e^{-r\tau_L} S_2(\tau_L) \left( \frac{S_1(\tau_L)}{S_2(\tau_L)} \right)^\alpha \right] = L^\alpha \mathbb{E} \left[ e^{-r\tau_L} S_2(\tau_L) \right] ,
\]

since \( S_1(\tau_L)/S_2(\tau_L) = L/L = 1 \). The conclusion

\[
\mathbb{E}[e^{-r\tau_L}(S_1(\tau_L) - S_2(\tau_L))^{+}] = (L - 1)^+ L^{-\alpha} S_2(0) \left( \frac{S_1(0)}{S_2(0)} \right)^\alpha \]  

(A.60)

then follows by an application of (A.58).
d) In order to maximize (A.60) as a function of \( L \) we consider the derivative

\[
\frac{\partial}{\partial L} \frac{L - 1}{L^\alpha} = \frac{1}{L^\alpha} - \alpha(L - 1)L^{-\alpha - 1} = 0,
\]

which vanishes for

\[
L^* = \frac{\alpha}{\alpha - 1},
\]

and we substitute \( L \) in (A.60) with the value of \( L^* \).
e) In addition to \( r = \sigma^2/2 \) it is sufficient to let \( S_1(0) = \kappa \) and \( \sigma_1 = 0 \) which yields \( \alpha = 2, L^* = 2 \), and we find

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\[
\sup_{\tau \text{ stopping time}} \mathbb{E} [e^{-r\tau} (\kappa - S_2(\tau))^+] = \frac{1}{S_2(0)} \left( \frac{\kappa}{2} \right)^2,
\]

which coincides with the result of Proposition 14.10.

Exercise 14.13

a) It suffices to check the sign of the quantity

\[
(\lambda - 1)(\lambda + 2r/\sigma^2),
\]  

in (14.53), which is positive when \( \lambda \in (-\infty, -2r/\sigma^2] \cup [1, \infty) \), and negative when \(-2r/\sigma^2 \leq \lambda \leq 1\).

b) The sign of (A.61) is positive when \( \lambda \in (-\infty, 1] \cup [-2r/\sigma^2, \infty) \), and negative when \( 1 \leq \lambda \leq -2r/\sigma^2 \).

c) By the Stopping Time Theorem 14.7, for any \( n \geq 0 \) we have

\[
x^\lambda = \mathbb{E}^* \left[ e^{-r(\tau_L \wedge n)} Z_{\tau_L \wedge n}^{(\lambda)} \mid S_0 = x \right]
\]

\[
= \mathbb{E}^* \left[ Z_{\tau_L}^{(\lambda)} \mathbb{1}_{\{\tau_L < n\}} \mid S_0 = x \right] + e^{-rn} \mathbb{E}^* \left[ Z_n^{(\lambda)} \mathbb{1}_{\{\tau_L > n\}} \mid S_0 = x \right]
\]

\[
\geq \mathbb{E}^* \left[ e^{-r\tau_L} (S_{\tau_L})^{\lambda} \mathbb{1}_{\{\tau_L < n\}} \mid S_0 = x \right] = L^\lambda \mathbb{E}^* \left[ e^{-r\tau_L} \mathbb{1}_{\{\tau_L < n\}} \mid S_0 = x \right].
\]

By the results of Questions (a)-(b), the process \( (Z_t^{(\lambda)})_{t \in \mathbb{R}_+} \) is a martingale when \( \lambda \in \{1, -2r\sigma^2/2\} \). Next, letting \( n \) to infinity, by monotone convergence we find

\[
\mathbb{E}^* \left[ e^{-r\tau_L} \mathbb{1}_{\{\tau_L < \infty\}} \mid S_0 = x \right] \leq \left( \frac{x}{L} \right)^{\lambda} \leq \begin{cases} 
\left( \frac{x}{L} \right)^{\max\{1,-2r/\sigma^2\}}, & x \geq L, \\
\left( \frac{x}{L} \right)^{\min\{1,-2r/\sigma^2\}}, & 0 < x \leq L.
\end{cases}
\]

d) We note that \( \mathbb{P}^*(\tau_L < \infty) = 1 \) by (14.14), hence if \(-\sigma^2/2 \leq r < 0\) we have

\[
\mathbb{E}^* \left[ e^{-r\tau_L} (K - S_{\tau_L})^+ \mathbb{1}_{\{\tau_L < \infty\}} \mid S_0 = x \right] = (K - L) \mathbb{E}^* \left[ e^{-r\tau_L} \mid S_0 = x \right] \leq \begin{cases} 
(K - L) \left( \frac{x}{L} \right)^{-2r/\sigma^2}, & x \geq L, \\
(K - L) \frac{x}{L}, & 0 < x \leq L.
\end{cases}
\]

Similarly, if \( r \leq -\sigma^2/2 \) we have
\[ \mathbb{E}^* \left[ e^{-r\tau L} (K - S_{\tau L})^+ \mathbb{1}_{\{\tau_L < \infty\}} \mid S_0 = x \right] = (K - L) \mathbb{E}^* \left[ e^{-r\tau L} \mid S_0 = x \right] \]

\[ \leq \begin{cases} 
(K - L) \frac{x}{L}, & x \geq L, \\
(K - L) \left( \frac{x}{L} \right)^{-2r/\sigma^2}, & 0 < x \leq L.
\end{cases} \]

e) This follows by noting that \((K - L)(x/L) = (K/L - 1)x\) increases to \(\infty\) 
when \(L\) tends to zero.
f) If \(-\sigma^2/2 \leq r < 0\) we have

\[ \mathbb{E}^* \left[ e^{-r\tau L} (S_{\tau L} - K)^+ \mathbb{1}_{\{\tau_L < \infty\}} \mid S_0 = x \right] = (L - K) \mathbb{E}^* \left[ e^{-r\tau L} \mathbb{1}_{\{\tau_L < \infty\}} \mid S_0 = x \right] \]

\[ \leq \begin{cases} 
(L - K) \left( \frac{x}{L} \right)^{-2r/\sigma^2}, & x \geq L, \\
(L - K) \frac{x}{L}, & 0 < x \leq L.
\end{cases} \]

If \(r \leq -\sigma^2/2\) we have

\[ \mathbb{E}^* \left[ e^{-r\tau L} (S_{\tau L} - K)^+ \mathbb{1}_{\{\tau_L < \infty\}} \mid S_0 = x \right] = (L - K) \mathbb{E}^* \left[ e^{-r\tau L} \mathbb{1}_{\{\tau_L < \infty\}} \mid S_0 = x \right] \]

\[ \leq \begin{cases} 
(L - K) \left( \frac{x}{L} \right), & x \geq L, \\
(L - K) \left( \frac{x}{L} \right)^{-2r/\sigma^2}, & 0 < x \leq L.
\end{cases} \]

g) This follows by noting that for fixed \(x > 0\), the quantity \((L - K)x/L = (1 - K/L)x\) increases to \(x\) when \(L\) tends to infinity.


a) Similarly, for \(x \geq K\), immediate exercise is the optimal strategy and we have \(C^{Am}_{b}(t, x) = 1\). When \(x < K\) the optimal exercise level of the perpetual American binary call option is \(L^* = K\) with the optimal exercise time \(\tau_K\), and by e.g. (4.4.22) page 135 we have

\[ C^{Am}_{b}(t, x) = \sup_{\tau \geq t} \mathbb{E}^* \left[ e^{-(\tau - t)r} \mathbb{1}_{\{S_{\tau} \geq K\}} \mid S_t = x \right] \]

\[ = \mathbb{E}^* \left[ e^{-(\tau K - t)r} \mathbb{1}_{\{S_{\tau} \geq K\}} \mid S_t = x \right] \]

\[ = \frac{x}{K}, \quad x < K. \]
b) For $x \leq K$, immediate exercise is the optimal strategy and we have $P^\text{Am}_b(t, x) = 1$. When $x > K$ the optimal exercise level of the perpetual American binary put option is $L^* = K$ with the optimal exercise time $\tau_K$, and by e.g. (4.4.11) page 125 we have

\[
P^\text{Am}_b(t, x) = \sup_{\tau \geq t} \mathbb{E}^* \left[ e^{-(\tau-t)r} 1_{\{S_\tau \leq K\}} \mid S_t = x \right]
\]

\[
= \mathbb{E}^* \left[ e^{-(\tau-K-t)r} \mid S_t = x \right]
\]

\[
= \left( \frac{x}{K} \right)^{-2r/\sigma^2}, \quad x > K.
\]

Exercise 14.15 Finite expiration American binary options.

a) The optimal strategy is as follows:

(i) if $S_t \geq K$, then exercise immediately.

(ii) if $S_t < K$, then wait.

b) The optimal strategy is as follows:
(i) if $S_t > K$, then wait.
(ii) if $S_t \leq K$, exercise immediately.

c) Based on the answers to Question (a) we set
$$C_d^{Am}(t, T, K) = 1, \quad 0 \leq t < T,$$
and
$$C_d^{Am}(T, T, x) = 0, \quad 0 \leq x < K.$$

d) Based on the answers to Question (b), we set
$$P_d^{Am}(t, T, K) = 1, \quad 0 \leq t < T,$$
and
$$P_d^{Am}(T, T, x) = 0, \quad x > K.$$

e) Starting from $S_t \leq K$, the maximum possible payoff is clearly reached as soon as $S_t$ hits the level $K$ before the expiration date $T$, hence the discounted optimal payoff of the option is $e^{-r(\tau_K - t)}\mathbb{1}_{\{\tau_K < T\}}$.

f) From Relation (10.9) we find
$$\mathbb{P}(\tau_a \leq u) = \Phi \left( \frac{a - \mu u}{\sqrt{u}} - e^{2\mu a} \Phi \left( \frac{a - \mu u}{\sqrt{u}} \right) \right), \quad u > 0,$$
and by differentiation with respect to $u$ this yields the probability density function
$$f_{\tau_a}(u) = \frac{\partial}{\partial u} \mathbb{P}(\tau_a \leq u) = \frac{a}{\sqrt{2\pi u^3}} e^{-(a - \mu u)^2/(2u)} \mathbb{1}_{[0, \infty)}(u)$$
of the first hitting time of level $a$ by Brownian motion with drift $\mu$. Given the relation
$$S_u = S_t e^{(B_u - B_t)\sigma - (u - t)\sigma^2/2 + (u - t)\mu}, \quad u \geq t,$$
we find that the probability density function of the first hitting time of level $K$ after time $t$ by $(S_u)_{u \in [t, \infty)}$ is given by
$$s \rightarrow \frac{a}{\sqrt{2\pi(s - t)^3}} e^{-(a - (s - t)\mu)^2/(2(s - t))}, \quad s > t,$$
with
$$\mu := \sigma^{-1} \left( r - \sigma^2/2 \right) \quad \text{and} \quad a := \frac{1}{\sigma} \log \frac{K}{x},$$
given that $S_t = x$. Hence for $x \in (0, K)$ we have
$$C_d^{Am}(t, T, x) = \mathbb{E}[e^{-r(\tau_K - t)}\mathbb{1}_{\{\tau_K < T\}} | S_t = x]$$
\[
= \int_t^T e^{-r(s-t)} \frac{a}{\sqrt{2\pi (s-t)^3}} e^{-(a-(s-t)\mu)^2/(2(s-t))} ds \\
= \int_0^{T-t} e^{-rs} \frac{a}{\sqrt{2\pi s^3}} e^{-(a-\mu s)^2/(2s)} ds \\
= \int_0^{T-t} \log(K/x) \exp\left(-rs - \frac{1}{2\sigma^2 s} \left(- \left(r - \frac{\sigma^2}{2}\right)^2 s + \log K_x^2\right)\right) ds \\
= \left(\frac{K}{x}\right)^{(r/\sigma^2-1/2)\pm(r/\sigma^2+1/2)} \\
\times \int_0^{T-t} \frac{\log(K/x)}{\sigma\sqrt{2\pi s^3}} \exp\left(- \frac{1}{2\sigma^2 s} \left(\pm \left(r + \frac{\sigma^2}{2}\right)^2 s + \log K_x^2\right)\right) ds \\
= \frac{1}{\sqrt{2\pi}} \frac{x}{K} \int_{y^-}^{\infty} e^{-y^2/2} dy + \frac{1}{\sqrt{2\pi}} \frac{K}{x} \int_{y^+}^{\infty} e^{-y^2/2} dy \\
= \frac{x}{K} \Phi\left(\frac{(r+\sigma^2/2)(T-t)+\log(x/K)}{\sigma\sqrt{T-t}}\right) - 2r/\sigma^2 \Phi\left(\frac{(r-\sigma^2/2)(T-t)+\log(x/K)}{\sigma\sqrt{T-t}}\right), \quad 0 < x < K,
\]
\[
\text{where} \quad y^\pm = \frac{1}{\sigma\sqrt{T-t}} \left(\pm \left(r + \frac{\sigma^2}{2}\right)(T-t) + \log K_x\right),
\]
and we used the decomposition
\[
\log K_x = \frac{1}{2} \left(\left(r + \frac{\sigma^2}{2}\right)s + \log K_x\right) + \frac{1}{2} \left(- \left(r + \frac{\sigma^2}{2}\right)s + \log K_x\right).
\]
We check that
\[
C_d^{Am}(t,T,K) = \Phi(\infty) + \Phi(-\infty) = 1,
\]
and
\[
C_d^{Am}(T,T,x) = \frac{x}{K} \Phi(-\infty) - 2r/\sigma^2 \Phi(-\infty) = 0, \quad x < K,
\]
since \( t = T \), which is consistent with the answers to Question (c).

In addition, as \( T \) tends to infinity we have
\[
\lim_{T \to \infty} C_d^{Am}(t,T,x) = \frac{x}{K} \lim_{T \to \infty} \Phi\left(\frac{(r+\sigma^2/2)(T-t)+\log(x/K)}{\sigma\sqrt{T-t}}\right) \\
+ \left(\frac{x}{K}\right)^{-2r/\sigma^2} \lim_{T \to \infty} \Phi\left(\frac{-(r+\sigma^2/2)(T-t)+\log(x/K)}{\sigma\sqrt{T-t}}\right).
\]
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\[ = \frac{x}{K}, \quad 0 < x < K, \]

which is consistent with the answer to Question (a) of Exercise 14.14.

![American Binary Call Price Map](image-url)

Fig. S.59: Finite expiration American binary call price map with \( K = 100 \).

g) Starting from \( S_t \geq K \), the maximum possible payoff is clearly reached as soon as \( S_t \) hits the level \( K \) before the expiration date \( T \), hence the discounted optimal payoff of the option is \( e^{-r(\tau_K - t)} \mathbb{1}_{\{\tau_K < T\}} \).

h) Using the notation and answer to Question (f), for \( x > K \) we find

\[
P_{d}^{Am}(t, T, x) = \mathbb{E}[e^{-r(\tau_K - t)} \mathbb{1}_{\{\tau_K < T\}} | S_t = x]
\]

\[ = \int_{0}^{T-t} e^{-rs} \frac{a}{\sqrt{2\pi s}} e^{-(a-\mu s)^2/2s} ds
\]

\[ = \int_{0}^{T-t} \frac{\log(x/K)}{\sigma \sqrt{2\pi s^3}} \exp \left( -rs - \frac{1}{2\sigma^2 s} \left( \frac{r - \sigma^2}{2} s + \log \frac{x}{K} \right)^2 \right) ds
\]

\[ = \left( \frac{K}{x} \right)^{\left( \frac{1}{\sigma^2} - \frac{1}{2} \right)} (\frac{1}{\sigma^2} + \frac{1}{2})
\]

\[ \times \int_{0}^{T-t} \frac{\log(x/K)}{\sigma \sqrt{2\pi s^3}} \exp \left( -\frac{1}{2\sigma^2 s} \left( \mp \frac{r + \sigma^2}{2} s + \log \frac{x}{K} \right)^2 \right) ds
\]

\[ = \frac{1}{\sqrt{2\pi}} \frac{x}{K} \int_{y^-}^{\infty} e^{-y^2/2} dy + \frac{1}{\sqrt{2\pi}} \left( \frac{x}{K} \right)^{2r/\sigma^2} \int_{y^+}^{\infty} e^{-y^2/2} dy
\]

\[ = \frac{x}{K} \Phi \left( -\frac{(r + \sigma^2/2)(T-t) - \log(x/K)}{\sigma \sqrt{T-t}} \right)
\]

\[ + \left( \frac{x}{K} \right)^{-2r/\sigma^2} \Phi \left( \frac{(r + \sigma^2/2)(T-t) - \log(x/K)}{\sigma \sqrt{T-t}} \right), \quad x > K,
\]

with

\[ y^\pm = \frac{1}{\sigma \sqrt{T-t}} \left( \mp \frac{r + \sigma^2}{2} (T-t) + \log \frac{x}{K} \right), \]

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We check that

\[ P_{d}^{Am}(t, T, K) = \Phi(-\infty) + \Phi(\infty) = 1, \]

and

\[ P_{d}^{Am}(T, T, x) = \frac{x}{K} \Phi(-\infty) + \left(\frac{x}{K}\right)^{-2r/\sigma^2} \Phi(-\infty) = 0, \quad 0 < x < K, \]

since \( t = T \), which is consistent with the answers to Question (c).

In addition, as \( T \) tends to infinity we have

\[
\lim_{T \to \infty} P_{d}^{Am}(t, T, x) = \frac{x}{K} \lim_{T \to \infty} \Phi \left( \frac{-(r + \sigma^2/2)(T - t) - \log(x/K)}{\sigma \sqrt{T - t}} \right) \\
+ \left(\frac{x}{K}\right)^{-2r/\sigma^2} \lim_{T \to \infty} \Phi \left( \frac{(r + \sigma^2/2)(T - t) - \log(x/K)}{\sigma \sqrt{T - t}} \right) \\
= \left(\frac{x}{K}\right)^{-2r/\sigma^2}, \quad x > K,
\]

which is consistent with the answer to Question (b) of Exercise 14.14.

![Figure S.60: Finite expiration American binary put price map with \( K = 100 \).](https://www.ntu.edu.sg/home/nprivault/indext.html)

i) The call-put parity does not hold for American binary options since for \( x \in (0, K) \) we have

\[
C_{d}^{Am}(t, T, x) + P_{d}^{Am}(t, T, x) = 1 + \frac{x}{K} \Phi \left( \frac{(r + \sigma^2/2)(T - t) + \log(x/K)}{\sigma \sqrt{T - t}} \right) \\
+ \left(\frac{x}{K}\right)^{-2r/\sigma^2} \Phi \left( \frac{-(r + \sigma^2/2)(T - t) + \log(x/K)}{\sigma \sqrt{T - t}} \right),
\]

while for \( x > K \) we find
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\[
C^\text{Am}_d(t, T, x) + P^\text{Am}_d(t, T, x) = 1 + \frac{x}{K} \Phi \left( \frac{-(r + \sigma^2/2)(T - t) - \log(x/K)}{\sigma \sqrt{T - t}} \right) + \left( \frac{x}{K} \right)^{-2r/\sigma^2} \Phi \left( \frac{(r + \sigma^2/2)(T - t) - \log(x/K)}{\sigma \sqrt{T - t}} \right).
\]

Exercise 14.16 American forward Contracts.

a) For all stopping times \( \tau \) such that \( t \leq \tau \leq T \) we have

\[
\mathbb{E}^* \left[ e^{-r(\tau-t)}(K - S_\tau) \mid S_t \right] = K \mathbb{E}^* \left[ e^{-r(\tau-t)} \mid S_t \right] - \mathbb{E}^* \left[ e^{-r(\tau-t)}S_\tau \mid S_t \right] = e^{-r(\tau-t)}K - S_t,
\]

since \( \tau \in [t, T] \) is bounded and \( (e^{-rt}S_t)_{t \in \mathbb{R}_+} \) is a martingale, and the above quantity is clearly maximized by taking \( \tau = t \). Hence we have

\[
f(t, S_t) = \sup_{t \leq \tau \leq T} \mathbb{E}^* \left[ e^{-r(\tau-t)}(K - S_\tau) \mid S_t \right] = K - S_t,
\]

and the optimal strategy is to exercise immediately (or avoiding to buy the option) at time \( t \).

b) Similarly we have

\[
\mathbb{E}^* \left[ e^{-r(\tau-t)}(S_\tau - K) \mid S_t \right] = \mathbb{E}^* \left[ e^{-r(\tau-t)}S_\tau \mid S_t \right] - K \mathbb{E}^* \left[ e^{-r(\tau-t)} \mid S_t \right] = S_t - K \mathbb{E}^* \left[ e^{-r(\tau-t)} \mid S_t \right],
\]

since \( \tau \in [t, T] \) is bounded and \( (e^{-rt}S_t)_{t \in \mathbb{R}_+} \) is a martingale, and the above quantity is clearly maximized by taking \( \tau = T \). Hence we have

\[
f(t, S_t) = \sup_{t \leq \tau \leq T} \mathbb{E}^* \left[ e^{-r(\tau-t)}(S_\tau - K) \mid S_t \right] = S_t - e^{-r(T-t)}K,
\]

and the optimal strategy is to wait until the maturity time \( T \) in order to exercise.

c) Concerning the perpetual American long forward contract, since \( u \mapsto e^{-r(u-t)}S_u \) is a martingale, for all stopping times \( \tau \) we have

\[
\mathbb{E}^* \left[ e^{-r(\tau-t)}(S_\tau - K) \mid S_t \right] = \mathbb{E}^* \left[ e^{-r(\tau-t)}S_\tau \mid S_t \right] - K \mathbb{E}^* \left[ e^{-r(\tau-t)} \mid S_t \right] = S_t - K \mathbb{E}^* \left[ e^{-r(\tau-t)} \mid S_t \right] \leq S_t, \quad t \geq 0.
\]

On the other hand, for all fixed \( T > 0 \) we have

\[
\text{by Fatou's Lemma 22.3.}
\]

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\[ \mathbb{E}^* \left[ e^{-r(T-t)}(S_T - K) \mid S_t \right] = e^{-r(T-t)} \mathbb{E}^* \left[ S_T \mid S_t \right] - e^{-r(T-t)} \mathbb{E}^* \left[ K \mid S_t \right] \\
= S_t - e^{-r(T-t)}K, \quad t \in [0, T], \]

hence

\[ \sup_{t \leq \tau \leq T} \mathbb{E}^* \left[ e^{-r(\tau-t)}(S_\tau - K) \mid S_t \right] \geq (S_t - e^{-r(T-t)}K), \quad T \in [t, \infty), \]

and letting \( T \to \infty \) we get

\[ \sup_{\tau \geq t} \mathbb{E}^* \left[ e^{-r(\tau-t)}(S_\tau - K) \mid S_t \right] \geq \lim_{T \to \infty} (S_t - e^{-r(T-t)}K) \]

\[ = S_t, \]

hence we have

\[ f(t, S_t) = \sup_{t \leq \tau \leq T} \mathbb{E}^* \left[ e^{-r(\tau-t)}(S_\tau - K) \mid S_t \right] = S_t, \]

and the optimal strategy \( \tau^* = +\infty \) is to wait indefinitely.

Concerning the perpetual American short forward contract we have

\[ f(t, S_t) = \sup_{t \leq \tau \leq T} \mathbb{E}^* \left[ e^{-r(\tau-t)}(K - S_\tau) \mid S_t \right] \]

\[ \leq \sup_{t \leq \tau \leq T} \mathbb{E}^* \left[ e^{-r(\tau-t)}(K - S_\tau)^+ \mid S_t \right] \]

\[ = f_{L^*}(S_t). \]

On the other hand, for \( \tau = \tau_{L^*} \) we have

\[ (K - S_{\tau_{L^*}}) = (K - L^*) = (K - L^*)_+ \]

since \( 0 < L^* = 2Kr/(2r + \sigma^2) < K \), hence

\[ f_{L^*}(S_t) = \mathbb{E}^* \left[ e^{-r(\tau_{L^*}-t)}(K - S_{\tau_{L^*}})^+ \mid S_t \right] \]

\[ = \mathbb{E}^* \left[ e^{-r(\tau_{L^*}-t)}(K - S_{\tau_{L^*}}) \mid S_t \right] \]

\[ \leq \sup_{t \leq \tau \leq T} \mathbb{E}^* \left[ e^{-r(\tau-t)}(K - S_\tau) \mid S_t \right] \]

\[ = f(t, S_t), \]

which shows that

\[ f(t, S_t) = f_{L^*}(S_t), \]
i.e. the perpetual American short forward contract has same price and exercise strategy as the perpetual American put option.

Exercise 14.17

a) We have

\[ Y_t = e^{-rt} S_0 e^{rt + \sigma B_t - \sigma^2 t/2} e^{-2r/\sigma^2} \]
\[ = S_0 e^{-2r/\sigma^2} e^{-rt - 2\sigma^2 t/2 + 2r B_t / \sigma + rt} \]
\[ = S_0 e^{-2r/\sigma^2} e^{2r B_t / \sigma - (2\sigma/r)^2 t/2} \]

and

\[ Z_t = e^{-rt} S_t = S_0 e^{\sigma B_t - \sigma^2 t/2}, \]

which are both martingales under \( \mathbb{P}^* \) because they are standard geometric Brownian motions with respective volatilities \( \sigma \) and \( 2\sigma/r \).

b) Since \( Y_t \) and \( Z_t \) are both martingales and \( \tau_L \) is a stopping time we have

\[ S_0 e^{-2r/\sigma^2} = \mathbb{E}^*[Y_0] \]
\[ = \mathbb{E}^*[Y_{\tau_L}] \]
\[ = \mathbb{E}^*[e^{-r\tau_L} S_{\tau_L}^{2\sigma/\sigma^2}] \]
\[ = \mathbb{E}^*[e^{-r\tau_L} L^{-2\sigma/\sigma^2}] \]
\[ = L^{-2\sigma/\sigma^2} \mathbb{E}^*[e^{-r\tau_L}], \]

hence

\[ \mathbb{E}^*[e^{-r\tau_L}] = \left( \frac{x}{L} \right)^{-2\sigma/\sigma^2} \]

if \( S_0 = x \geq L \) (note that in this case \( \tau_{\tau_L \wedge t} \) remains bounded by \( L^{-2\sigma/\sigma^2} \)), and

\[ S_0 = \mathbb{E}^*[Z_0] = \mathbb{E}^*[Z_{\tau_L}] = \mathbb{E}^*[e^{-r\tau_L} S_{\tau_L}] = \mathbb{E}^*[e^{-r\tau_L} L] = L \mathbb{E}^*[e^{-r\tau_L}], \]

hence

\[ \mathbb{E}^*[e^{-r\tau_L}] = \frac{x}{L} \]

if \( S_0 = x \leq L \). Note that in this case \( Z_{\tau_L \wedge t} \) remains bounded by \( L \).

c) We find

\[ \mathbb{E} \left[ e^{-r\tau_L} (K - S_{\tau_L}) \mid S_0 = x \right] = (K - L) \mathbb{E}^* \left[ e^{-r\tau_L} \mid S_0 = x \right] \]
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\[
\begin{cases}
    x \frac{K - L}{L}, & 0 < x \leq L, \\
    (K - L) \left( \frac{x}{L} \right)^{-2r/\sigma^2}, & x > L.
\end{cases}
\]

(A.62)

d) By differentiating

\[
\frac{\partial}{\partial L} \mathbb{E} \left[ e^{-r \tau_L} (K - S_{\tau_L}) \mid S_0 = x \right] = \begin{cases} 
    \left( \frac{x}{L} \right)^{-2r/\sigma^2} \left( \frac{2r}{\sigma^2} \left( \frac{K}{L} - 1 \right) - 1 \right), & 0 < L < x, \\
    - \frac{Kx}{L^2}, & L > x,
\end{cases}
\]

and check that the minimum occurs for \( L^* = x \).

e) The value \( L^* = x \) shows that the optimal strategy for the American finite expiration short forward contract is to exercise immediately starting from \( S_0 = x \), which is consistent with the result of Exercise 14.16-(a), since given any stopping time \( \tau \) upper bounded by \( T \) we have

\[
\mathbb{E} \left[ e^{-r \tau} (K - S_{\tau}) \right] = K \mathbb{E} \left[ e^{-r \tau} \right] - \mathbb{E} \left[ e^{-r \tau} S_{\tau} \right] = K \mathbb{E} \left[ e^{-r \tau} \right] - S_0 \leq K - S_0.
\]

Exercise 14.18

a) The option payoff equals \((\kappa - S_t)^p\) if \( S_t \leq L \).

b) We have

\[
f_L(S_t) = \mathbb{E}^* \left[ e^{-r(\tau_L-t)}(\kappa - S_{\tau_L})^+ \mid S_t \right] = \mathbb{E}^* \left[ e^{-r(\tau_L-t)}(\kappa - L)^+ \mid S_t \right] = (\kappa - L)^p \mathbb{E}^* \left[ e^{-r(\tau_L-t)} \mid S_t \right].
\]

c) We have

\[
f_L(x) = \mathbb{E}^* \left[ e^{-r(\tau_L-t)}(\kappa - S_{\tau_L})^+ \mid S_t = x \right] = \begin{cases} (\kappa - x)^p, & 0 < x \leq L, \\
\frac{(k - L)p \left( \frac{L}{x} \right)^{2r/\sigma^2}}{x}, & x > L.
\end{cases}
\]

(A.63)

d) By the differentiation \( \frac{d}{dx} (\kappa - x)^p = -p(\kappa - x)^{p-1} \) we find

\( \Box \)
\[
\frac{\partial f_L(x)}{\partial L} = \frac{2r}{\sigma^2 L} (\kappa - L)^p \left( \frac{L}{x} \right)^{2r/\sigma^2} - p(\kappa - L)^{p-1} \left( \frac{L}{x} \right)^{2r/\sigma^2},
\]

hence the condition \( \frac{\partial f_L^*(x)}{\partial L} \big|_{x=L^*} = 0 \) reads

\[
\frac{2r}{\sigma^2 L^*} (\kappa - L^*) - p = 0, \quad \text{or} \quad L^* = \frac{2r}{2r + p\sigma^2} \kappa < \kappa.
\]

e) By (A.63) the price can be computed as

\[
f(t, S_t) = f_{L^*}(S_t) = \begin{cases} 
(\kappa - S_t)^p, & 0 < S_t \leq L^*, \\
\left( \frac{p\sigma^2 \kappa}{2r + p\sigma^2} \right)^p \left( \frac{2r + p\sigma^2 S_t}{2r \kappa} \right)^{-2r/\sigma^2}, & S_t \geq L^*, 
\end{cases}
\]

using (14.12) as in the proof of Proposition 14.10, since the process

\[
u \mapsto e^{-ru} f_{L^*}(S_u), \quad u \geq t,
\]

is a nonnegative supermartingale.

Exercise 14.19

a) The option payoff is \( \kappa - (S_t)^p \).
b) We have

\[
f_L(S_t) = \mathbb{E}^* \left[ e^{-r(\tau_L-t)} (\kappa - (S_{\tau_L})^p) \right| S_t] \\
= \mathbb{E}^* \left[ e^{-r(\tau_L-t)} (\kappa - L^p) \right| S_t] \\
= (\kappa - L^p) \mathbb{E}^* \left[ e^{-r(\tau_L-t)} \right| S_t].
\]

c) We have

\[
f_L(x) = \mathbb{E}^* \left[ e^{-r(\tau_L-t)} (\kappa - (S_{\tau_L})^p) \right| S_t = x] \\
= \begin{cases} 
\kappa - x^p, & 0 < x \leq L, \\
(\kappa - L^p) \left( \frac{x}{L} \right)^{-2r/\sigma^2}, & x \geq L.
\end{cases}
\]

d) We have

\[
f_{L^*}'(L^*) = -\frac{2r}{\sigma^2} (\kappa - (L^*)^p) \left( \frac{L^*}{(L^*)}-2r/\sigma^2-1 \right) = -p(L^*)^{p-1},
\]

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\[ \frac{2r}{\sigma^2} (\kappa - (L^*)^p) = p(L^*)^p, \]

or

\[ L^* = \left( \frac{2r \kappa}{2r + p\sigma^2} \right)^{1/p} < (\kappa)^{1/p}. \]  (A.64)

Remark: We may also compute \( L^* \) by maximizing \( L \mapsto f_L(x) \) for all fixed \( x \). The derivative \( \partial f_L(x)/\partial L \) can be computed as

\[ \frac{\partial f_L(x)}{\partial L} = \frac{\partial}{\partial L} \left( (\kappa - L^p) \left( \frac{L}{x} \right)^{2r/\sigma^2} \right) = -pL^{p-1} \left( \frac{L}{x} \right)^{2r/\sigma^2} + \frac{2r}{\sigma^2} L^{-1} (\kappa - L^p) \left( \frac{L}{x} \right)^{2r/\sigma^2}, \]

and equating \( \partial f_L(x)/\partial L \) to 0 at \( L = L^* \) yields

\[ -p(L^*)^{p-1} + \frac{2r}{\sigma^2} (L^*)^{-1}(\kappa - (L^*)^p) = 0, \]

which recovers (A.64).

e) We have

\[ f_{L^*}(S_t) = \begin{cases} 
\kappa - (S_t)^p, & 0 < S_t \leq L^*, \\
(\kappa - (L^*)^p) \left( \frac{S_t}{(L^*)^{-2r/\sigma^2}} \right)^{2r/\sigma^2}, & S_t > L^*
\end{cases} \]

\[ = \begin{cases} 
\kappa - (S_t)^p, & 0 < S_t \leq L^*, \\
\frac{\sigma^2}{2r} p(S_t)^{-2r/\sigma^2} (L^*)^{p+2r/\sigma^2}, & S_t > L^*
\end{cases} \]

\[ = \begin{cases} 
\kappa - (S_t)^p, & 0 < S_t \leq L^*, \\
\frac{p\sigma^2 \kappa}{2r + p\sigma^2} \left( \frac{2r + p\sigma^2 S_t^p}{2r \kappa} \right)^{-2r/(p\sigma^2)} < \kappa, & S_t > L^*
\end{cases} \]

however we cannot conclude as in Exercise 14.18-(e) since the process

\[ u \mapsto e^{-ru} f_{L^*}(S_u), \quad u \geq t, \]

does not remain nonnegative when \( p > 1 \), so that (14.12) cannot be applied as in the proof of Proposition 14.10.
Chapter 15

Exercise 15.1

a) We have

\[
d\tilde{X}_t = d \left( \frac{X_t}{N_t} \right)
\]

\[
= \frac{X_0}{N_0} \left( e^{(\sigma - \eta)B_t - (\sigma^2 - \eta^2)t/2} \right)
\]

\[
= \frac{X_0}{N_0} (\sigma - \eta) e^{(\sigma - \eta)B_t - (\sigma^2 - \eta^2)t/2} dB_t
\]

\[
+ \frac{X_0}{2N_0} (\sigma - \eta)^2 e^{(\sigma - \eta)B_t - (\sigma^2 - \eta^2)t/2} dt
\]

\[
= - \frac{X_t}{2N_t} (\sigma^2 - \eta^2) dt + \frac{X_t}{N_t} (\sigma - \eta) dB_t + \frac{X_t}{2N_t} (\sigma - \eta)^2 dt
\]

\[
= - \frac{X_t}{N_t} \eta (\sigma - \eta) dt + \frac{X_t}{N_t} (\sigma - \eta) dB_t
\]

\[
= \frac{X_t}{N_t} (\sigma - \eta) (dB_t - \eta dt)
\]

\[
= (\sigma - \eta) \frac{X_t}{N_t} d\tilde{B}_t = (\sigma - \eta) \tilde{X}_t d\tilde{B}_t,
\]

where \(d\tilde{B}_t = dB_t - \eta dt\) is a standard Brownian motion under \(\tilde{\mathbb{P}}\).

b) By change of numéraire, we have

\[
\mathbb{E}\left[(X_T - \lambda N_T)^+\right] = \tilde{\mathbb{E}}\left[\frac{N_0}{N_T} (X_T - \lambda N_T)^+\right] = N_0 \tilde{\mathbb{E}}[(\tilde{X}_T - \lambda)^+].
\]

Next, by the result of Question (a), \(\tilde{X}_t\) is a driftless geometric Brownian motion with volatility \(\sigma - \eta\) under \(\tilde{\mathbb{P}}\), hence we have

\[
\tilde{\mathbb{E}}[(\tilde{X}_T - \lambda)^+] = \tilde{X}_0 \Phi \left( \frac{\log(\tilde{X}_0/\lambda)}{\tilde{\sigma} \sqrt{T}} + \frac{\tilde{\sigma} \sqrt{T}}{2} \right) - \lambda \Phi \left( \frac{\log(\tilde{X}_0/\lambda)}{\tilde{\sigma} \sqrt{T}} - \frac{\tilde{\sigma} \sqrt{T}}{2} \right),
\]

by the Black-Scholes formula with zero interest rate and volatility parameter \(\tilde{\sigma} = \sigma - \eta\). By multiplication by \(N_0\) and the relation \(X_0 = N_0 \tilde{X}_0\) we conclude to (15.34), i.e.

\[
\mathbb{E} \left[(X_T - \lambda N_T)^+\right] = N_0 \tilde{\mathbb{E}}[(\tilde{X}_T - \lambda)^+] = N_0 \tilde{X}_0 \Phi(d_+) - \lambda N_0 \Phi(d_-)
\]
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\[ = X_0 \Phi(d_+) - \lambda N_0 \Phi(d^-). \]

c) We have \( \hat{\sigma} = \sigma - \eta. \)

Exercise 15.2

a) By the Girsanov Theorem, the processes

\[
d\hat{B}^{(1)}_t = dB^{(1)}_t - \frac{1}{N_t} dN_t \cdot dB^{(1)}_t = dB^{(1)}_t - \frac{1}{S^{(2)}_t} dS^{(2)}_t \cdot dB^{(1)}_t = dB^{(1)}_t - \eta \rho dt,
\]

and

\[
d\hat{B}^{(2)}_t = dB^{(2)}_t - \frac{1}{N_t} dN_t \cdot dB^{(2)}_t = dB^{(2)}_t - \frac{1}{S^{(2)}_t} dS^{(2)}_t \cdot dB^{(2)}_t = dB^{(2)}_t - \eta dt
\]

are standard Brownian motions (and martingales) under \( \hat{P}_2. \)

b) We have

\[
d\hat{S}^{(1)}_t = \left( \begin{array}{c} S^{(1)}_t \\ S^{(2)}_t \end{array} \right)
\]

\[
= \frac{S^{(1)}_0}{S^{(2)}_0} d \left( e^{\sigma B^{(1)}_t - \eta B^{(2)}_t - (\sigma^2 - \eta^2) t / 2} \right)
\]

\[
= \frac{S^{(1)}_0}{S^{(2)}_0} e^{\sigma B^{(1)}_t - \eta B^{(2)}_t - (\sigma^2 - \eta^2) t / 2}
\]

\[
\times \left( \sigma dB^{(1)}_t + \frac{\sigma^2}{2} dt - \eta dB^{(2)}_t + \frac{\eta^2}{2} dt - \frac{\sigma^2 - \eta^2}{2} dt - \sigma \eta \rho dt \right)
\]

\[
= \hat{S}^{(1)}_t (\sigma dB^{(1)}_t - \eta dB^{(2)}_t).
\]

c) We apply the change of numéraire formula

\[
e^{-rT} E^* \left[ (S^{(1)}_T - \lambda S^{(2)}_T)^+ \right] = S^{(2)}_0 \hat{E} \left[ (\hat{S}^{(1)}_T - \lambda)^+ \right]
\]

and the Black-Scholes formula to the driftless geometric Brownian motion \((\hat{S}^{(1)}_t)_{t \in \mathbb{R}}\) under \( \hat{P}_2. \)

d) We have

\[
\hat{\sigma}^2 dt = \left( \sigma dB^{(1)}_t - \eta dB^{(2)}_t \right) \cdot \left( \sigma dB^{(1)}_t - \eta dB^{(2)}_t \right) = (\sigma^2 + \eta^2 - 2 \sigma \eta \rho) dt,
\]

hence \( \hat{\sigma}^2 = \sigma^2 + \eta^2 - 2 \sigma \eta \rho. \)
Exercise 15.3 We have \( N_t = P(t, T) \) and from (16.22) and the relations \( P(t, T) = F(t, r_t) \) and \( P(t, S) = G(t, r_t) \) we find

\[
\begin{cases}
\frac{dP(t, S)}{P(t, S)} = r_t dt + \sigma(t, r_t) \frac{\partial \log G}{\partial x}(t, r_t) dW_t, \\
\frac{dN_t}{N_t} = \frac{dP(t, T)}{P(t, T)} = r_t dt + \sigma(t, r_t) \frac{\partial \log F}{\partial x}(t, r_t) dW_t.
\end{cases}
\]

By the Girsanov Theorem (15.10) we also have

\[
d\tilde{W}_t = dW_t - \frac{dN_t}{N_t} \cdot dW_t = dW_t - \sigma(t, r_t) \frac{\partial \log F}{\partial x}(t, r_t) dt,
\]

hence

\[
\frac{dP(t, S)}{P(t, S)} = r_t dt + \sigma^2(t, r_t) \frac{\partial \log F}{\partial x}(t, r_t) \frac{\partial \log G}{\partial x}(t, r_t) dt + \sigma(t, r_t) \frac{\partial \log G}{\partial x}(t, r_t) d\tilde{W}_t.
\]

Using the relation \( P(t, S) = G(t, r_t) \) we can also write

\[
dP(t, S) = r_t P(t, S) dt + \sigma^2(t, r_t) \frac{\partial \log F}{\partial x}(t, r_t) \frac{\partial G}{\partial x}(t, r_t) dt + \sigma(t, r_t) \frac{\partial G}{\partial x}(t, r_t) d\tilde{W}_t.
\]

Exercise 15.4 Forward contract. Taking \( N_t := P(t, T), t \in [0, T] \), we have

\[
\mathbb{E}^* \left[ \exp \left( - \int_t^T r_s ds \right) (P(T, S) - K) \mid F_t \right] = N_t \hat{\mathbb{E}} \left[ \frac{(P(T, S) - K)}{N_T} \mid F_t \right]
\]

\[
= P(t, T) \hat{\mathbb{E}} \left[ \frac{(P(T, S) - K)}{P(T, T)} \mid F_t \right]
\]

\[
= P(t, T) \hat{\mathbb{E}}[P(T, S) - K \mid F_t]
\]

\[
= P(t, T) \hat{\mathbb{E}}[P(T, S) \mid F_t] - KP(t, T)
\]

\[
= P(t, T) \frac{P(t, S)}{P(t, T)} - KP(t, T)
\]

\[
= P(t, S) - KP(t, T),
\]

since

\[
t \mapsto \frac{P(t, T)}{N_t} = \frac{P(t, S)}{P(t, T)}
\]

is a martingale under the forward measure \( \hat{\mathbb{P}} \). The corresponding (static) hedging strategy is given by buying one bond with maturity \( S \) and by short
selling \( K \) units of the bond with maturity \( T \).

Remark: The above result can also be obtained by a direct argument using the tower property of conditional expectations:

\[
\begin{align*}
\mathbb{E}^{*} \left[ \exp \left( - \int_{t}^{T} r_{s} ds \right) \left( P(T, S) - K \right) \mid \mathcal{F}_{t} \right] &= \mathbb{E}^{*} \left[ \exp \left( - \int_{t}^{T} r_{s} ds \right) \left( \mathbb{E}^{*} \left[ \exp \left( - \int_{T}^{S} r_{s} ds \right) \mid \mathcal{F}_{T} \right] - K \right) \mid \mathcal{F}_{t} \right] \\
&= \mathbb{E}^{*} \left[ \exp \left( - \int_{t}^{T} r_{s} ds \right) \mathbb{E}^{*} \left[ \exp \left( - \int_{T}^{S} r_{s} ds \right) - K \mid \mathcal{F}_{T} \right] \mid \mathcal{F}_{t} \right] \\
&= \mathbb{E}^{*} \left[ \mathbb{E}^{*} \left[ \exp \left( - \int_{t}^{S} r_{s} ds \right) - K \exp \left( - \int_{t}^{T} r_{s} ds \right) \mid \mathcal{F}_{T} \right] \mid \mathcal{F}_{t} \right] \\
&= \mathbb{E}^{*} \left[ \exp \left( - \int_{t}^{S} r_{s} ds \right) - K \exp \left( - \int_{t}^{T} r_{s} ds \right) \mid \mathcal{F}_{t} \right] \\
&= P(t, S) - K P(t, T), \quad t \in [0, T].
\end{align*}
\]

Exercise 15.5

i) We choose \( N_{t} := S_{t} \) as numéraire because this allows us to write the option payoff as \((S_{T}(S_{T} - K))^{+} = N_{T}(S_{T} - K)^{+}, \) therefore the forward measure \( \hat{\mathbb{P}} \) satisfies

\[
\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = e^{-rT} \frac{N_{T}}{N_{0}} = e^{-rT} \frac{S_{T}}{S_{0}},
\]

or

\[
\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} \mid \mathcal{F}_{t} = e^{-(T-t)r} \frac{N_{T}}{N_{t}} = e^{-(T-t)r} \frac{S_{T}}{S_{t}}, \quad 0 \leq t \leq T.
\]

ii) By the change of numéraire formula the option price becomes

\[
e^{-(T-t)r} \mathbb{E}^{*} \left[ (S_{T}(S_{T} - K))^{+} \mid \mathcal{F}_{t} \right] = \mathbb{E}^{*} \left[ e^{-(T-t)r} N_{T}(S_{T} - K)^{+} \mid \mathcal{F}_{t} \right] = N_{t} \hat{\mathbb{E}} \left[ (S_{T} - K)^{+} \mid \mathcal{F}_{t} \right] = S_{t} \hat{\mathbb{E}} \left[ (S_{T} - K)^{+} \mid \mathcal{F}_{t} \right]. \quad (A.65)
\]

iii) In order to compute \((A.65)\) it remains to determine the dynamics of \((S_{t})_{t \in \mathbb{R}^{+}}\) under \( \hat{\mathbb{P}}. \) We have

\[
\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = e^{\sigma B_{T} - \sigma^{2}T/2},
\]

hence by the Girsanov Theorem \( \hat{B}_{t} := B_{t} - \sigma t \) is a standard Brownian motion under \( \hat{\mathbb{P}}, \) with
\[ S_T = S_0 e^{rT+\sigma B_T + \sigma^2 T/2} = S_0 e^{(r+\sigma^2)T+\sigma \widehat{B}_T - \sigma^2 T/2} = S_t e^{(r+\sigma^2)(T-t)+(\widehat{B}_T-\widehat{B}_t)\sigma-(T-t)\sigma^2/2}, \quad 0 \leq t \leq T. \]

iv) According to the above, (A.65) becomes
\[
e^{-(T-t)r} \mathbb{E}^* \left[ \left( S_T (S_T - K) \right)^+ \mid F_t \right] = S_t \hat{\mathbb{E}} \left[ \left( S_T - K \right)^+ \mid F_t \right] = S_t \hat{\mathbb{E}} \left[ \left( S_0 e^{rT+\sigma^2 T+\sigma \widehat{B}_T - \sigma^2 T/2 - K} \right)^+ \mid F_t \right] = S_t \hat{\mathbb{E}} \left[ \left( S_t e^{(r+\sigma^2)(T-t)+(\widehat{B}_T-\widehat{B}_t)\sigma-(T-t)\sigma^2/2 - K} \right)^+ \mid F_t \right] = S_t e^{(T-t)(r+\sigma^2)} \text{Bl}(S_t, r + \sigma^2, \sigma, K, T - t), \quad 0 \leq t \leq T.
\]

Remarks:

i) The option price can be rewritten using other Black-Scholes parametrizations, such as for example
\[ S_t \text{Bl}(S_t e^{(T-t)(r+\sigma^2)}, 0, \sigma, K, T - t), \]

or
\[ S_t e^{(T-t)(r+\sigma^2)} \text{Bl}(S_t, 0, \sigma, K e^{-(T-t)(r+\sigma^2)}, T - t), \]

however we prefer to choose the simplest possibility.

ii) Deflated (or forward) processes such as \( S_t / N_1 = 1 \) or
\[
e^{-(T-t)r} \mathbb{E}^* \left[ (S_T (S_T - K))^+ \mid F_t \right] = \hat{\mathbb{E}} \left[ (S_T - K)^+ \mid F_t \right], \quad 0 \leq t \leq T,
\]

are martingales under the forward measure \( \hat{\mathbb{P}} \).

iii) This option can also be priced via an integral calculation instead of using change of numéraire, as follows:
\[
e^{-(T-t)r} \mathbb{E}^* [S_T (S_T - K)^+ \mid F_t] = e^{-(T-t)r} \mathbb{E}^* \left[ S_t e^{(T-t)r+(B_T-B_t)\sigma-(T-t)\sigma^2/2} \times (S_t e^{(T-t)r+(B_T-B_t)\sigma-(T-t)\sigma^2/2 - K})^+ \mid F_t \right] = S_t e^{-(T-t)\sigma^2/2} \mathbb{E}^* \left[ (S_t e^{(T-t)r+2(B_T-B_t)\sigma-(T-t)\sigma^2/2 - K} e^{(B_T-B_t)\sigma})^+ \mid F_t \right] = S_t e^{-(T-t)\sigma^2/2} \mathbb{E}^* \left[ (S_t e^{(T-t)r+2(B_T-B_t)\sigma-(T-t)\sigma^2/2} - K e^{(B_T-B_t)\sigma})^+ \right]_{x=S_t} = S_t e^{-(T-t)\sigma^2/2} \mathbb{E}^* \left[ (e^{m(x)+2X} - K e^X)^+ \right]_{x=S_t}, \quad 0 \leq t \leq T,
\]

where \( X \sim \mathcal{N}(0, v^2) \) with \( v^2 = (T-t)\sigma^2 \) and \( m(x) = (T-t)r - (T-t)\sigma^2/2 + \log x \). Next, we note that
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\[ \mathbb{E} \left( (e^{m+2X} - K e^X)^+ \right) = \frac{1}{\sqrt{2\pi v^2}} \int_{-\infty}^{\infty} (e^{m+2x} - K e^x)^+ e^{-x^2/(2v^2)} \, dx \]

\[ = \frac{1}{\sqrt{2\pi v^2}} \int_{-\infty}^{\infty} (e^{m+2x} - K e^x) e^{-x^2/(2v^2)} \, dx \]

\[ = \frac{e^m}{\sqrt{2\pi v^2}} \int_{-\infty}^{\infty} e^{2x-x^2/(2v^2)} \, dx - \frac{K}{\sqrt{2\pi v^2}} \int_{-\infty}^{\infty} e^{-x^2/(2v^2)} \, dx \]

\[ = \frac{e^m+2v^2}{\sqrt{2\pi v^2}} \int_{-\infty}^{\infty} e^{-(v^2-x^2)/(2v^2)} \, dx \]

\[ = \frac{e^m+2v^2}{\sqrt{2\pi v^2}} \int_{-\infty}^{\infty} e^{-v^2/(2v^2)} \, dx - \frac{K e^{v^2/2}}{\sqrt{2\pi v^2}} \int_{-\infty}^{\infty} e^{-x^2/(2v^2)} \, dx \]

\[ = \frac{e^m+2v^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/(2v^2)} \, dx - \frac{K e^{v^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/(2v^2)} \, dx \]

\[ = \frac{e^m+2v^2}{\sqrt{2\pi v^2}} \int_{-\infty}^{\infty} e^{-x^2/2} \, dx - \frac{K e^{v^2/2}}{\sqrt{2\pi v^2}} \int_{-\infty}^{\infty} e^{-x^2/2} \, dx \]

\[ = \frac{e^m+2v^2}{\sqrt{2\pi v^2}} \Phi \left( 2v + \frac{m - \log K}{v} \right) - K e^{v^2/2} \Phi \left( v + \frac{m - \log K}{v} \right), \]

hence

\[ e^{-(T-t)} \mathbb{E}^* \left[ S_T (S_T - K)^+ \mid \mathcal{F}_t \right] = S_t e^{-(T-t)\sigma^2/2} \mathbb{E}^* \left[ (e^{m(X)} + 2X - K e^X)^+ \right]_{X=S_t} \]

\[ = S_t^2 e^{(T-t)(r+\sigma^2)} \Phi \left( \frac{(T-t)(r+\sigma^2) + (T-t)\sigma^2/2 + \log(S_t/K)}{\sigma \sqrt{T-t}} \right) - K S_t \Phi \left( \frac{(T-t)(r+\sigma^2) - (T-t)\sigma^2/2 + \log(S_t/K)}{\sigma \sqrt{T-t}} \right) \]

\[ = S_t e^{(T-t)(r+\sigma^2)} B_l(S_t, r + \sigma^2, \sigma, K, T-t), \quad 0 \leq t \leq T. \]

Exercise 15.6 Bond options.

a) Itô’s formula yields

\[ d \left( \frac{P(t,S)}{P(t,T)} \right) = \frac{P(t,S)}{P(t,T)} \left( \zeta^S(t) - \zeta^T(t) \right) \, (dW_t - \zeta^T(t) \, dt) \]

\[ = \frac{P(t,S)}{P(t,T)} \left( \zeta^S(t) - \zeta^T(t) \right) \widehat{dW}_t, \quad (A.66) \]

where \((\widehat{W}_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion under \(\hat{P}\) by the Girsanov Theorem.

b) From (A.66) or (18.7) we have

\[ \frac{P(t,S)}{P(t,T)} = \frac{P(0,S)}{P(0,T)} \exp \left( \int_0^t (\zeta^S(s) - \zeta^T(s)) \, d\widehat{W}_s - \frac{1}{2} \int_0^t |\zeta^S(s) - \zeta^T(s)|^2 \, ds \right), \]

\[ \diamond \]

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hence

\[ \frac{P(u,S)}{P(u,T)} = \frac{P(t,S)}{P(t,T)} \exp \left( \int_t^u (\zeta^S(s) - \zeta^T(s))d\widehat{W}_s - \frac{1}{2} \int_t^u |\zeta^S(s) - \zeta^T(s)|^2 ds \right), \]

\( t \in [0,u] \), and for \( u = T \) this yields

\[ P(T,S) = \frac{P(t,S)}{P(t,T)} \exp \left( \int_t^T (\zeta^S(s) - \zeta^T(s))d\widehat{W}_s - \frac{1}{2} \int_t^T |\zeta^S(s) - \zeta^T(s)|^2 ds \right), \]

since \( P(T,T) = 1 \). Let \( \hat{P} \) denote the forward measure associated to the numéraire

\[ N_t := P(t,T), \quad 0 \leq t \leq T. \]

c) For all \( S \geq T > 0 \) we have

\[ \mathbb{E} \left[ e^{-\int_t^T r_s ds} (P(T,S) - K)^+ \middle| \mathcal{F}_t \right] \]

\[ = P(t,T) \hat{\mathbb{E}} \left[ \left( \frac{P(t,S)}{P(t,T)} \exp \left( X - \frac{1}{2} \int_t^T |\zeta^S(s) - \zeta^T(s)|^2 ds \right) \right)^+ \middle| \mathcal{F}_t \right] \]

\[ = P(t,T) \hat{\mathbb{E}} \left[ \left( e^{X + m(t,T,S)} - K \right)^+ \middle| \mathcal{F}_t \right], \]

where \( X \) is a centered Gaussian random variable with variance

\[ v^2(t,T,S) = \int_t^T |\zeta^S(s) - \zeta^T(s)|^2 ds \]

given \( \mathcal{F}_t \), and

\[ m(t,T,S) = -\frac{1}{2}v^2(t,T,S) + \log \frac{P(t,S)}{P(t,T)}. \]

Recall that when \( X \) is a centered Gaussian random variable with variance \( v^2 \), the expectation of \( (e^{m+X} - K)^+ \) is given, as in the standard Black-Scholes formula, by

\[ \mathbb{E}[(e^{m+X} - K)^+] = e^{m+v^2/2} \Phi(v + (m - \log K)/v) - K \Phi((m - \log K)/v), \]

where

\[ \Phi(z) = \int_{-\infty}^z e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}, \quad z \in \mathbb{R}, \]

denotes the Gaussian cumulative distribution function and for simplicity of notation we dropped the indices \( t,T,S \) in \( m(t,T,S) \) and \( v^2(t,T,S) \).

Consequently we have

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\[
\mathbb{E} \left[ e^{-\int_t^T r_s \, ds} (P(T, S) - K)^+ \bigg| \mathcal{F}_t \right] \\
= P(t, S) \Phi \left( \frac{\nu}{2} + \frac{1}{\nu} \log \frac{P(t, S)}{KP(t, T)} \right) - KP(t, T) \Phi \left( -\frac{\nu}{2} + \frac{1}{\nu} \log \frac{P(t, S)}{KP(t, T)} \right).
\]

d) The self-financing hedging strategy that hedges the bond option is obtained by holding a (possibly fractional) quantity

\[
\Phi \left( \frac{\nu}{2} + \frac{1}{\nu} \log \frac{P(t, S)}{KP(t, T)} \right)
\]

of the bond with maturity \( S \), and by shorting a quantity

\[
K \Phi \left( -\frac{\nu}{2} + \frac{1}{\nu} \log \frac{P(t, S)}{KP(t, T)} \right)
\]

of the bond with maturity \( T \).

Exercise 15.7

a) The process

\[
e^{-rt} S_2(t) = S_2(0) e^{\sigma_2 W_t + (\mu - r)t}
\]

is a martingale if

\[r - \mu = \frac{1}{2} \sigma_2^2.\]

b) We note that

\[
e^{-rt} X_t = e^{-rt} e^{(r-\mu)t - \sigma_1^2 t/2} S_1(t)
\]

is a martingale, where

\[
X_t = e^{(r-\mu)t - \sigma_1^2 t/2} S_1(t) = e^{(\sigma_2^2 - \sigma_1^2) t/2} S_1(t).
\]

c) By (15.36) we have

\[
\hat{X}(t) = \frac{X_t}{N_t}
\]

\[
= e^{(\sigma_2^2 - \sigma_1^2) t/2} \frac{S_1(t)}{S_2(t)}
\]

\[\circ \]

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given that

\[
\frac{S_1(0)}{S_2(0)} e^{(\sigma_2^2-\sigma_1^2)t/2 + (\sigma_1-\sigma_2)W_t}
= S_1(0) e^{(\sigma_2^2-\sigma_1^2)t/2 + (\sigma_1-\sigma_2)\widehat{W}_t + \sigma_2(\sigma_1-\sigma_2)t}
= S_1(0) e^{(\sigma_1-\sigma_2)\widehat{W}_t + \sigma_2\sigma_1t - (\sigma_2^2+\sigma_1^2)t/2}
= S_1(0) e^{(\sigma_1-\sigma_2)\widehat{W}_t - (\sigma_1-\sigma_2)^2t/2},
\]

where

\[
\widehat{W}_t := W_t - \sigma_2t
\]
is a standard Brownian motion under the forward measure \(\widehat{\mathbb{P}}\) defined by

\[
\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} = e^{-\int_0^T r_s ds N_T}
= e^{-rT} \frac{S_2(T)}{S_2(0)}
= e^{-rT} e^{\sigma_2 W_T + \mu T}
= e^{\sigma_2 W_T + (\mu - r)T}
= e^{\sigma_2 W_T - \sigma_2^2 t/2}.
\]

d) Given that \(X_t = e^{(\sigma_2^2-\sigma_1^2)t/2} S_1(t)\) and \(\widehat{X}(t) = X_t / N_t = X_t / S_2(t)\), we have

\[
e^{-rT} \mathbb{E}[(S_1(T) - \kappa S_2(T))^+] = e^{-rT} \mathbb{E}[(e^{-(\sigma_2^2-\sigma_1^2)T/2} X_T - \kappa S_2(T))^+]
= e^{-rT} e^{-(\sigma_2^2-\sigma_1^2)T/2} \mathbb{E}[(X_T - \kappa e^{(\sigma_2^2-\sigma_1^2)T/2} S_2(T))^+]
= S_2(0) e^{-(\sigma_2^2-\sigma_1^2)T/2} \mathbb{E}[(\widehat{X}_0 e^{(\sigma_1-\sigma_2)\widehat{W}_T - (\sigma_1-\sigma_2)^2T/2 - \kappa \sigma_2(\sigma_2^2-\sigma_1^2)T/2)^+]
= S_2(0) e^{-(\sigma_2^2-\sigma_1^2)T/2} (\widehat{X}_0 \Phi^0_+(T, \widehat{X}_0) - \kappa e^{(\sigma_2^2-\sigma_1^2)T/2} \Phi^0_0(T, \widehat{X}_0))
= S_2(0) e^{-(\sigma_2^2-\sigma_1^2)T/2} \widehat{X}_0 \Phi^0_+(T, \widehat{X}_0)
- \kappa S_2(0) e^{-(\sigma_2^2-\sigma_1^2)T/2} e^{(\sigma_2^2-\sigma_1^2)T/2} \Phi^0_0(T, \widehat{X}_0)
= X_0 e^{-(\sigma_2^2-\sigma_1^2)T/2} \Phi^0_+(T, \widehat{X}_0) - \kappa S_2(0) \Phi^0_0(T, \widehat{X}_0)
= S_1(0) e^{-(\sigma_2^2-\sigma_1^2)T/2} \Phi^0_+(T, \widehat{X}_0) - \kappa S_2(0) \Phi^0_0(T, \widehat{X}_0),
\]

where

\[
\Phi^0_+(T, x) = \Phi \left( \frac{\log(x/\kappa)}{\sigma_1 - \sigma_2 \sqrt{T}} + \frac{(\sigma_1 - \sigma_2)^2 - (\sigma_2^2 - \sigma_1^2)}{2|\sigma_1 - \sigma_2| \sqrt{T}} \right).
\]
Exercise 15.8 We have

\[
\Phi \left( \frac{\log(x/\kappa)}{|\sigma_1 - \sigma_2| \sqrt{T}} + \sigma_1 \sqrt{T} \right), \quad \sigma_1 > \sigma_2,
\]

\[
\Phi \left( \frac{\log(x/\kappa)}{|\sigma_1 - \sigma_2| \sqrt{T}} - \sigma_1 \sqrt{T} \right), \quad \sigma_1 < \sigma_2,
\]

and

\[
\Phi_0(T, x) = \Phi \left( \frac{\log(x/\kappa)}{|\sigma_1 - \sigma_2| \sqrt{T}} - \frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2^2 - \sigma_1^2)}{2|\sigma_1 - \sigma_2| \sqrt{T}} \right)
\]

\[
\Phi \left( \frac{\log(x/\kappa)}{|\sigma_1 - \sigma_2| \sqrt{T}} + \sigma_2 \sqrt{T} \right), \quad \sigma_1 > \sigma_2,
\]

\[
\Phi \left( \frac{\log(x/\kappa)}{|\sigma_1 - \sigma_2| \sqrt{T}} - \sigma_2 \sqrt{T} \right), \quad \sigma_1 < \sigma_2,
\]

if \( \sigma_1 \neq \sigma_2 \). In case \( \sigma_1 = \sigma_2 \), we find

\[
e^{-rT} \mathbb{E}[(S_1(T) - \kappa S_2(T))^+] = e^{-rT} \mathbb{E}[S_1(T)(1 - \kappa S_2(0)/S_1(0))^+]
\]

\[
= (1 - \kappa S_2(0)/S_1(0))^+ e^{-rT} \mathbb{E}[S_1(T)]
\]

\[
= (S_1(0) - \kappa S_2(0)) \mathbb{1}_{\{S_1(0) > \kappa S_2(0)\}}.
\]

Exercise 15.8 We have

\[
e^{-(T-t)r} \mathbb{E}^* \left[ \mathbb{1}_{\{R_T \geq \kappa\}} \mid R_t \right] = e^{-(T-t)r} \mathbb{P}^* \left( R_T \geq \kappa \mid R_t \right)
\]

\[
= e^{-(T-t)r} \mathbb{P}^* \left( R_t e^{(W_T - W_t) \sigma + (r-r^f)(T-t) - (T-t)\sigma^2/2} \geq \kappa \mid R_t \right)
\]

\[
= e^{-(T-t)r} \mathbb{P}^* \left( x e^{(W_T - W_t) \sigma + (r-r^f)(T-t) - (T-t)\sigma^2/2 \geq \kappa} \right)_{x=R_t}
\]

\[
= e^{-(T-t)r} \Phi \left( \frac{(r-r^f)(T-t) - (T-t)\sigma^2/2 - \log(\kappa/R_t)}{\sigma \sqrt{T-t}} \right),
\]

after applying the hint provided, with

\[
\eta^2 := (T-t)\sigma^2 \quad \text{and} \quad \mu := (r-r^f)(T-t) - (T-t)\sigma^2/2.
\]

**Remark:** Binary options are often proposed at the money, i.e. \( \kappa = R_t \), with a short time to maturity, for example the small value

\[
T - t \simeq 30 \text{ seconds} = 0.000000951 = 9.51 \times 10^{-7} \text{ year}^{-1},
\]

in which case we have

\[
e^{-(T-t)r} \mathbb{E}^* \left[ \mathbb{1}_{\{R_T \geq \kappa\}} \mid R_t \right] = e^{-(T-t)r} \Phi \left( \left( \frac{r-r^f}{\sigma} - \frac{\sigma}{2} \right) \sqrt{T-t} \right)
\]
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\[
\simeq \Phi(0) = \frac{1}{2}.
\]
Taking for example \( r - r^f = 0.02 = 2\% \) and \( \sigma = 0.3 = 30\% \), we have
\[
\left( \frac{r - r^f}{\sigma} - \frac{\sigma}{2} \right) \sqrt{T - t} = \left( \frac{0.02}{0.3} - \frac{0.3}{2} \right) \sqrt{9.51 \times 10^{-7}} = -0.000081279
\]
and
\[
e^{-(T-t)r} \mathbb{E}^{*} \left[ \mathbb{1}_{\{R_T \geq \kappa\}} \mid R_t \right] = e^{-(T-t)r} \Phi(-0.000081279) = e^{-r \times 0.00000951} \times 0.499968 = 0.49996801 
\quad \simeq \frac{1}{2},
\]
with \( r = 0.02 = 2\% \).

**Exercise 15.9**

a) It suffices to check that the definition of \((W^N_t)_{t \in \mathbb{R}_+}\) implies the correlation identity \(dW^S_t \cdot dW^N_t = \rho dt\) by Itô’s calculus.

b) We let
\[
\tilde{\sigma}_t = \sqrt{(\sigma^S_t)^2 - 2\rho \sigma^R_t \sigma^S_t + (\sigma^R_t)^2}
\]
and
\[
dW^X_t = \frac{\sigma^S_t}{\tilde{\sigma}_t} dW^S_t - \sqrt{1 - \rho^2} \frac{\sigma^N_t}{\tilde{\sigma}_t} dW_t, \quad t \in \mathbb{R}_+,
\]
which defines a standard Brownian motion under \(\mathbb{P}^*\) due to the definition of \(\tilde{\sigma}_t\).

**Exercise 15.10**

a) We have \(\tilde{\sigma} = \sqrt{(\sigma^S)^2 - 2\rho \sigma^R \sigma^S + (\sigma^R)^2}\).

b) Letting \(\tilde{X}_t = e^{-rt} X_t = e^{(a-r)t} S_t / R_t, t \in \mathbb{R}_+\), we have
\[
\mathbb{E}^{*} \left[ \left( \frac{S_T}{R_T} - \kappa \right)^+ \mid \mathcal{F}_t \right] = e^{-aT} \mathbb{E}^{*} \left[ \left( X_T - e^{aT} \kappa \right)^+ \mid \mathcal{F}_t \right]
\]
\[
= e^{-(a-r)T} \mathbb{E}^{*} \left[ \tilde{X}_T \Phi \left( \frac{r - a + \tilde{\sigma}^2/2}{\tilde{\sigma} \sqrt{T-t}} + \frac{1}{\tilde{\sigma} \sqrt{T-t}} \log \frac{S_t}{\kappa R_t} \right) \mid \mathcal{F}_t \right]
\]

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Exercise 16.1  We have

\[
-\kappa e^{(a-r)T} \Phi \left( \left( \frac{r - a - \sigma^2/2}{\sigma_\tilde{T}} \right) \left( T - t \right) + \frac{1}{\sigma_\tilde{T}} \log \frac{S_t}{\kappa R_t} \right) \\
= \frac{S_t}{R_t} e^{(r-a)(T-t)} \Phi \left( \left( \frac{r - a + \sigma^2/2}{\sigma_\tilde{T}} \right) \left( T - t \right) + \frac{1}{\sigma_\tilde{T}} \log \frac{S_t}{\kappa R_t} \right) \\
- \kappa \Phi \left( \left( \frac{r - a - \sigma^2/2}{\sigma_\tilde{T}} \right) \left( T - t \right) + \frac{1}{\sigma_\tilde{T}} \log \frac{S_t}{\kappa R_t} \right),
\]

hence the price of the quanto option is

\[
e^{-r(T-t)} \mathbb{E}^* \left[ \left( \frac{S_T}{R_T} - \kappa \right)^+ \mid \mathcal{F}_t \right] = \frac{S_t}{R_t} e^{-a(T-t)} \Phi \left( \left( \frac{r - a + \sigma^2/2}{\sigma_\tilde{T}} \right) \left( T - t \right) + \frac{1}{\sigma_\tilde{T}} \log \frac{S_t}{\kappa R_t} \right) - \kappa e^{-r(T-t)} \Phi \left( \left( \frac{r - a - \sigma^2/2}{\sigma_\tilde{T}} \right) \left( T - t \right) + \frac{1}{\sigma_\tilde{T}} \log \frac{S_t}{\kappa R_t} \right).
\]

Chapter 16

Exercise 16.1  We have

\[
dr_t = r_0 e^{-bt} + \frac{a}{b} d(1 - e^{-bt}) + \sigma d \left( e^{-bt} \int_0^t e^{bs} dB_s \right)
= -br_0 e^{-bt} dt + \sigma e^{-bt} \int_0^t e^{bs} dB_s + \sigma \int_0^t e^{bs} dB_s e^{-bt}
= -br_0 e^{-bt} dt + \sigma e^{-bt} dB_t - \sigma b \int_0^t e^{bs} dB_s e^{-bt} dt
= -br_0 e^{-bt} dt + \sigma e^{-bt} dt + \sigma dB_t - \sigma b \int_0^t e^{bs} dB_s e^{-bt} dt
= -br_0 e^{-bt} dt + \sigma e^{-bt} dt + \sigma dB_t - b \left( r_t - r_0 e^{-bt} - \frac{a}{b} (1 - e^{-bt}) \right) dt
= (a - br_t) dt + \sigma dB_t,
\]

which shows that \( r_t \) solves (16.41).

Exercise 16.2  An estimator of \( \sigma \) can be obtained from the orthogonality relation

\[
\sum_{l=0}^{n-1} \left( (\tilde{r}_{t_{l+1}} - a \Delta t - (1 - b \Delta t) \tilde{r}_{t_l})^2 - \sigma^2 \Delta t (\tilde{r}_{t_l})^2 \right) = \sigma^2 \sum_{l=0}^{n-1} (\tilde{r}_{t_l})^2 \left( (Z_l)^2 - \Delta t \right) \approx 0,
\]

which is due to the independence of \( t_{t_l} \) and \( Z_l, l = 0, \ldots, n - 1 \), and yields

\( \Diamond \)
\[
\hat{\sigma}^2 = \frac{\sum_{l=0}^{n-1} (\tilde{r}_{t+l} - a\Delta t - (1 - b\Delta t)\tilde{r}_t)^2}{\Delta t \sum_{l=0}^{n-1} (\tilde{r}_t)^{2\gamma}}.
\]

Regarding the estimation of \(\gamma\), we can combine the above relation with the second orthogonality relation

\[
\sum_{l=0}^{n-1} \left( (\tilde{r}_{t+l} - a\Delta t - (1 - b\Delta t)\tilde{r}_t)^2 - \sigma^2 \Delta t (\tilde{r}_t)^{2\gamma} \right) \tilde{r}_t
= \sigma^2 \sum_{l=0}^{n-1} (\tilde{r}_t)^{2\gamma+1} ((Z_t)^2 - \Delta t)
\sim 0,
\]

cf. § 2.2 of Faff and Gray (2006). One may also attempt to minimize the residual

\[
\sum_{l=0}^{n-1} \left( (\tilde{r}_{t+l} - a\Delta t - (1 - b\Delta t)\tilde{r}_t)^2 - \sigma^2 \Delta t (\tilde{r}_t)^{2\gamma} \right)^2
\]

by equating the following derivatives to zero, as

\[
\frac{\partial}{\partial \sigma} \sum_{l=0}^{n-1} \left( (\tilde{r}_{t+l} - a\Delta t - (1 - b\Delta t)\tilde{r}_t)^2 - \sigma^2 \Delta t (\tilde{r}_t)^{2\gamma} \right)^2
= -4\sigma \sum_{l=0}^{n-1} (\tilde{r}_t)^{2\gamma} \left( (\tilde{r}_{t+l} - a\Delta t - (1 - b\Delta t)\tilde{r}_t)^2 - \sigma^2 \Delta t (\tilde{r}_t)^{2\gamma} \right)
= 0,
\]

hence

\[
\sum_{l=0}^{n-1} (\tilde{r}_t)^{2\gamma} \left( (\tilde{r}_{t+l} - a\Delta t - (1 - b\Delta t)\tilde{r}_t)^2 - \sigma^2 \Delta t \sum_{l=0}^{n-1} (\tilde{r}_t)^{4\gamma} \right) = 0,
\]

which yields

\[
\hat{\sigma}^2 = \frac{\sum_{l=0}^{n-1} (\tilde{r}_t)^{2\gamma} (\tilde{r}_{t+l} - a\Delta t - (1 - b\Delta t)\tilde{r}_t)^2}{\Delta t \sum_{l=0}^{n-1} (\tilde{r}_t)^{4\gamma}}.
\]

We also have

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\[
\frac{\partial}{\partial \gamma} \sum_{t=0}^{n-1} \left( (\tilde{r}_{t+1} - a \Delta t - (1 - b \Delta t) \tilde{r}_t)^2 - \sigma^2 \Delta t (\tilde{r}_t)^{2\gamma} \right)^2
\]

\[
= -4 \sigma^2 \Delta t \sum_{t=0}^{n-1} (\tilde{r}_t)^{2\gamma} ( (\tilde{r}_{t+1} - a \Delta t - (1 - b \Delta t) \tilde{r}_t)^2 - \sigma^2 \Delta t (\tilde{r}_t)^{2\gamma} ) \log \tilde{r}_t
\]

\[
= 0,
\]

which yields

\[
\hat{\sigma}^2 = \frac{\sum_{t=0}^{n-1} (\tilde{r}_t)^{2\gamma} ( (\tilde{r}_{t+1} - a \Delta t - (1 - b \Delta t) \tilde{r}_t)^2 \log \tilde{r}_t}{\Delta t \sum_{t=0}^{n-1} (\tilde{r}_t)^{4\gamma} \log \tilde{r}_t}
\]

and shows that \( \gamma \) can be estimated by matching the relation

\[
\frac{\sum_{t=0}^{n-1} (\tilde{r}_t)^{2\gamma} ( (\tilde{r}_{t+1} - a \Delta t - (1 - b \Delta t) \tilde{r}_t)^2 \log \tilde{r}_t}{\sum_{t=0}^{n-1} (\tilde{r}_t)^{4\gamma}}
\]

\[
= \frac{\sum_{t=0}^{n-1} (\tilde{r}_t)^{2\gamma} ( (\tilde{r}_{t+1} - a \Delta t - (1 - b \Delta t) \tilde{r}_t)^2 \log \tilde{r}_t}{\sum_{t=0}^{n-1} (\tilde{r}_t)^{4\gamma} \log \tilde{r}_t
\]

\text{Remarks.}

i) Estimators of \( a \) and \( b \) can be obtained by minimizing the residual

\[
\sum_{t=0}^{n-1} (\tilde{r}_{t+1} - a \Delta t - (1 - b \Delta t) \tilde{r}_t)^2
\]

as in the Vasicek model, i.e. from the equations

\[
\sum_{t=0}^{n-1} (\tilde{r}_{t+1} - a \Delta t - (1 - b \Delta t) \tilde{r}_t) = 0
\]

and

\[
\sum_{t=0}^{n-1} (\tilde{r}_{t+1} - a \Delta t - (1 - b \Delta t) \tilde{r}_t) \tilde{r}_t = 0.
\]
ii) Instead of the (generalised) method of moments, parameter estimation for stochastic differential equations can be achieved by maximum likelihood estimation, see e.g. Lindström (2007) and references therein.

Exercise 16.3
a) We have \( r_t = r_0 + at + \sigma B_t \), and

\[
F(t, r_t) = F(t, r_0 + at + \sigma B_t),
\]

hence by Proposition 16.2 the PDE satisfied by \( F(t, x) \) is

\[
-xF(t, x) + \frac{\partial F}{\partial t}(t, x) + a \frac{\partial F}{\partial x}(t, x) + \frac{1}{2} \sigma^2 \frac{\partial^2 F}{\partial x^2}(t, x) = 0,
\]

(A.67)

with terminal condition \( F(T, x) = 1 \).

b) We have \( r_t = r_0 + at + \sigma B_t \) and

\[
F(t, r_t) = \mathbb{E}^* \left[ \exp \left( - \int_t^T r_s ds \right) \mid \mathcal{F}_t \right] \\
= \mathbb{E}^* \left[ \exp \left( -r_0(T-t) - a \int_t^T sds - \sigma \int_t^T B_s ds \right) \mid \mathcal{F}_t \right] \\
= \mathbb{E}^* \left[ e^{-r_0(T-t)-a(T^2-t^2)/2} \exp \left( -(T-t)B_t - \sigma \int_t^T (T-s)dB_s \right) \mid \mathcal{F}_t \right] \\
= e^{-r_0(T-t)-a(T^2-t^2)/2-(T-t)\sigma B_t} \mathbb{E}^* \left[ \exp \left( -\sigma \int_t^T (T-s)dB_s \right) \mid \mathcal{F}_t \right] \\
= e^{-r_0(T-t)-a(T-t)(T+t)/2-(T-t)\sigma B_t} \mathbb{E}^* \left[ \exp \left( -\sigma \int_t^T (T-s)dB_s \right) \right] \\
= \exp \left( -(T-t)r_t - a(T-t)^2/2 + \sigma^2/2 \int_t^T (T-s)^2ds \right) \\
= \exp \left( -(T-t)r_t - a(T-t)^2/2 + (T-t)^3\sigma^2/6 \right),
\]

hence \( F(t, x) = \exp \left( -(T-t)x - a(T-t)^2/2 + (T-t)^3\sigma^2/6 \right) \).

Note that the PDE (A.67) can also be solved by looking for a solution of the form \( F(t, x) = e^{A(T-t)+xC(T-t)} \), in which case one would find \( A(s) = -as^2/2 + \sigma^2s^3/6 \) and \( C(s) = -s \).

c) We check that the function \( F(t, x) \) of Question (b) satisfies the PDE (A.67) of Question (a), since \( F(T, x) = 1 \) and

\[
-xF(t, x) + \left( x + a(T-t) - \frac{\sigma^2}{2}(T-t)^2 \right) F(t, x) - a(T-t)F(t, x) \\
+ \frac{\sigma^2}{2}(T-t)^2F(t, x) = 0.
\]
Exercise 16.4 We check from (16.45) that

\[
\begin{align*}
    dr_t &= \alpha \beta d \left( S_t \int_0^t \frac{1}{S_u} du \right) + r_0 dt S_t \\
    &= \alpha \beta S_t d \int_0^t \frac{1}{S_u} du + \alpha \beta S_t \int_0^t \frac{1}{S_u} dudS_t + r_0 dt S_t \\
    &= \alpha \beta S_t dt + \alpha \beta \int_0^t \frac{S_t}{S_u} du \frac{dS_t}{S_t} + r_0 dt S_t \\
    &= \alpha \beta dt + (r_t - r_0) \frac{dS_t}{S_t} + r_0 dt S_t \\
    &= \alpha \beta dt + r_t (-\beta dt + \sigma dB_t) \\
    &= \beta (\alpha - r_t) dt + \sigma dB_t, \quad t \in \mathbb{R}_+.
\end{align*}
\]

Exercise 16.5 By Itô’s formula we have

\[
\begin{align*}
    d \left( e^{-\int_0^t r_s ds} P(t, T) \right) &= -r_t e^{-\int_0^t r_s ds} P(t, T) dt + e^{-\int_0^t r_s ds} dP(t, T) \\
    &= -r_t e^{-\int_0^t r_s ds} F(t, r_t) dt + e^{-\int_0^t r_s ds} dF(t, r_t) \\
    &= -r_t e^{-\int_0^t r_s ds} F(t, r_t) dt + e^{-\int_0^t r_s ds} \frac{\partial F}{\partial x} (t, r_t) ((\beta r_t^{-1-\gamma} + \alpha r_t) dt + \sigma r_t^{\gamma/2} dB_t) \\
    &\quad + e^{-\int_0^t r_s ds} \left( \frac{1}{2} \sigma^2 r_t^{\gamma} \frac{\partial^2 F}{\partial x^2} (t, r_t) + \frac{\partial F}{\partial t} (t, r_t) \right) dt \\
    &= e^{-\int_0^t r_s ds} \sigma r_t^{\gamma/2} \frac{\partial F}{\partial x} (t, r_t) dB_t \\
    &\quad + e^{-\int_0^t r_s ds} \left( -r_t F(t, r_t) + (\beta r_t^{-1-\gamma} + \alpha r_t) \frac{\partial F}{\partial x} (t, r_t) \\
    &\quad + \frac{1}{2} \sigma^2 r_t^{\gamma} \frac{\partial^2 F}{\partial x^2} (t, r_t) + \frac{\partial F}{\partial t} (t, r_t) \right) dt.
\end{align*}
\]

Given that \( t \mapsto e^{-\int_0^t r_s ds} P(t, T) \) is a martingale, the above expression (A.68) should only contain terms in \( dB_t \) and all terms in \( dt \) should vanish inside (A.68). This leads to the identities

\[
\begin{align*}
    r_t F(t, r_t) &= (\beta r_t^{-1-\gamma} + \alpha r_t) \frac{\partial F}{\partial x} (t, r_t) + \frac{1}{2} \sigma^2 r_t^{\gamma} \frac{\partial^2 F}{\partial x^2} (t, r_t) + \frac{\partial F}{\partial t} (t, r_t) \\
    d \left( e^{-\int_0^t r_s ds} P(t, T) \right) &= \sigma e^{-\int_0^t r_s ds} r_t^{\gamma/2} \frac{\partial F}{\partial x} (t, r_t) dB_t,
\end{align*}
\]

and to the PDE
\[ xF(t, x) = \frac{\partial F}{\partial t}(t, x) + (\beta x^{-(1-\gamma)} + \alpha x) \frac{\partial F}{\partial x}(t, x) + \frac{\sigma^2}{2} x^\gamma \frac{\partial^2 F}{\partial x^2}(t, x). \]

Exercise 16.6

a) The process \( e^{-\int_0^t r_s ds} F(t, r_t) \) is a martingale and we have

\[
d\left( e^{-\int_0^t r_s ds} F(t, r_t) \right) = e^{-\int_0^t r_s ds} \left( -r_t F(t, r_t)dt + \frac{\partial F}{\partial t}(t, r_t)dt + \frac{\partial F}{\partial x}(t, r_t)dr_t + \frac{\sigma^2}{2} \frac{\partial^2 F}{\partial x^2}(t, r_t)(dr_t)^2 \right)
\]

\[
= e^{-\int_0^t r_s ds} \left( -r_t F(t, r_t)dt + \frac{\partial F}{\partial t}(t, r_t)dt + \frac{\partial F}{\partial x}(t, r_t)(-ar_t dt + \sigma \sqrt{r_t} dB_t) \right)
\]

\[+r_t e^{-\int_0^t r_s ds} \frac{\sigma^2}{2} \frac{\partial^2 F}{\partial x^2}(t, r_t)dt, \]

hence

\[-xF(t, x) + \frac{\partial F}{\partial t}(t, x) - ax \frac{\partial F}{\partial x}(t, x) + x \frac{\sigma^2}{2} \frac{\partial^2 F}{\partial x^2}(t, x) = 0. \] (A.70)

b) Plugging \( F(t, x) = e^{A(T-t)+xC(T-t)} \) into the PDE (A.70) shows that

\[ e^{A(T-t)+xC(T-t)} \left( -x - A'(T-t) - xC'(T-t) - axC(T-t) + \frac{\sigma^2 x}{2} C^2(T-t) \right) \]

\[= 0, \]

hence

\[
\begin{align*}
-1 - C'(T-t) - aC(T-t) + \frac{\sigma^2}{2} C^2(T-t) &= 0, \\
A'(T-t) &= 0.
\end{align*}
\]

Remark: The initial condition \( A(0) = 0 \) shows that \( A(s) = 1, \) and it can be shown from the condition \( C(0) = 0 \) that

\[ C(T-t) = \frac{2(1 - e^\gamma(T-t))}{2\gamma + (a + \gamma)(e^\gamma(T-t) - 1)}, \quad t \in [0, T], \]

with \( \gamma = \sqrt{a^2 + 2\sigma^2}, \) see e.g. Eq. (3.25) page 66 of Brigo and Mercurio (2006).

Exercise 16.7

a) The payoff of the convertible bond is given by \( \text{Max}(\alpha S_T, P(T, T)) \).

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b) We have
\begin{align*}
\text{Max}(\alpha S_\tau, P(\tau, T)) &= P(\tau, T)\mathbb{1}_{\{\alpha S_\tau \leq P(\tau, T)\}} + \alpha S_\tau \mathbb{1}_{\{\alpha S_\tau > P(\tau, T)\}} \\
&= P(\tau, T) + (\alpha S_\tau - P(\tau, T))\mathbb{1}_{\{\alpha S_\tau > P(\tau, T)\}} \\
&= P(\tau, T) + (\alpha S_\tau - P(\tau, T))^+ \\
&= P(\tau, T) + \alpha(S_\tau - P(\tau, T)/\alpha)^+,
\end{align*}
where the latter European call option payoff has the strike price \( K := P(\tau, T)/\alpha \).

c) From the Markov property applied at time \( t \in [0, \tau] \), we will write the corporate bond price as a function \( C(t, S_t, r_t) \) of the underlying asset price and interest rate, hence we have
\[ C(t, S_t, r_t) = \mathbb{E}^*[e^{-\int_t^\tau r_s ds \text{Max}(\alpha S_\tau, P(\tau, T))} \mid \mathcal{F}_t]. \]
The martingale property follows from the equalities
\begin{align*}
&\quad e^{-\int_0^t r_s ds}C(t, S_t, r_t) = e^{-\int_0^t r_s ds} \mathbb{E}^*[e^{-\int_t^\tau r_s ds \text{Max}(\alpha S_\tau, P(\tau, T))} \mid \mathcal{F}_t] \\
&\quad = \mathbb{E}^*[e^{-\int_0^\tau r_s ds \text{Max}(\alpha S_\tau, P(\tau, T))} \mid \mathcal{F}_t].
\end{align*}
d) We have
\begin{align*}
d \left( e^{-\int_0^t r_s ds}C(t, S_t, r_t) \right) &= -r_t e^{-\int_0^t r_s ds}C(t, S_t, r_t)dt + e^{-\int_0^t r_s ds} \frac{\partial C}{\partial t}(t, S_t, r_t)dt \\
&\quad + e^{-\int_0^t r_s ds} \frac{\partial C}{\partial x}(t, S_t, r_t)(rS_tdt + \sigma S_tdB_t^{(1)}) \\
&\quad + e^{-\int_0^t r_s ds} \frac{\partial C}{\partial y}(t, S_t, r_t)(\gamma(t, r_t)dt + \eta(t, S_t)dB_t^{(2)}) \\
&\quad + e^{-\int_0^t r_s ds} \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 C}{\partial x^2}(t, S_t, r_t)dt + e^{-\int_0^t r_s ds} \eta^2(t, r_t)\frac{1}{2} \frac{\partial^2 C}{\partial y^2}(t, S_t, r_t)dt \\
&\quad + \rho \sigma S_t \eta(t, r_t)e^{-\int_0^t r_s ds} \frac{\partial^2 C}{\partial x \partial y}(t, S_t, r_t)dt,
\end{align*}

hence by the martingale property of \( \left( e^{-\int_0^t r_s ds}C(t, S_t, r_t) \right)_{t \in \mathbb{R}_+} \), the associated PDE reads
\begin{align*}
0 &= -yC(t, x, y) + \frac{\partial C}{\partial t}(t, x, y)dt + ry \frac{\partial C}{\partial x}(t, x, y) + \gamma(t, y) \frac{\partial C}{\partial y}(t, x, y) \\
&\quad + \frac{\sigma^2}{2} x^2 \frac{\partial^2 C}{\partial x^2}(t, x, y) + \eta^2(t, y)\frac{1}{2} \frac{\partial^2 C}{\partial y^2}(t, x, y) + \rho \sigma x \eta(t, y) \frac{\partial^2 C}{\partial x \partial y}(t, x, y),
\end{align*}
with the terminal condition

\[ C(\tau, x, y) = \text{Max}(\alpha x, F(\tau, y)), \quad \text{where} \quad F(\tau, r_T) = P(\tau, T) \]

is the bond pricing function.

e) The convertible bond is priced as

\[
\mathbb{E}^* \left[ e^{-\int_t^\tau r_s ds} \text{Max}(\alpha S_\tau, P(\tau, T)) \right] | \mathcal{F}_t \\
= \mathbb{E}^* \left[ e^{-\int_t^\tau r_s ds} P(\tau, T) \right] | \mathcal{F}_t + \alpha \mathbb{E}^* \left[ e^{-\int_t^\tau r_s ds} (S_\tau - P(\tau, T)/\alpha)^+ \right] | \mathcal{F}_t \\
= P(t, T) + \alpha P(t, T) \mathbb{E}_T \left[ (S_\tau / P(\tau, T) - 1/\alpha)^+ \right] | \mathcal{F}_t \\
\tag{A.71}
\]

f) By Proposition 15.6 we find

\[
dZ_t = (\sigma - \sigma_B(t))Z_t d\hat{W}_t,
\]

where \((\hat{W}_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion under the forward measure \(\hat{P}_T\).

g) By modeling \((Z_t)_{t \in \mathbb{R}_+}\) as

\[
dZ_t = \sigma(t)Z_t d\hat{W}_t,
\]

(A.71) shows that the convertible bond is priced as

\[
P(t, T) + S_t \Phi(d_) - P(t, T) \Phi(d_-),
\]

where

\[
d_+ = \frac{1}{v(t, T)} \left( \log \frac{S_t}{P(t, T)} + \frac{v^2(t, \tau)}{2} \right),
\]

\[
d_- = \frac{1}{v(t, T)} \left( \log \frac{S_t}{P(t, T)} - \frac{v^2(t, \tau)}{2} \right),
\]

and \(v^2(t, T) = \int_t^T \sigma^2(s, T) ds, 0 < t < T\).

Exercise 16.8 We have

\[
\frac{\partial}{\partial r} P_c(0, n) = \frac{\partial}{\partial r} \left( \frac{1}{(1+r)^n} + \frac{c}{r} \left( 1 - \frac{1}{(1+r)^n} \right) \right) \\
= -\frac{n}{(1+r)^{n+1}} - \frac{c}{r^2} \left( 1 - \frac{1}{(1+r)^n} \right) + \frac{nc}{r(1+r)^{n+1}},
\]

hence

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\[ D_c(0,n) = -\frac{1+r}{P_c(0,n)} \partial_r P_c(0,n) \]
\[ = -\frac{n}{(1+r)^n} - \frac{(1+r)c}{r^2} \left( 1 - \frac{1}{(1+r)^n} \right) + \frac{nc}{r(1+r)^n} \]
\[ = \frac{1}{(1+r)^n} + \frac{c}{r} \left( 1 - \frac{1}{(1+r)^n} \right) - \frac{nr - \frac{1+r}{r} (c((1+r)^n - 1)) + nc}{r + c((1+r)^n - 1)} \]
\[ = 1 + r - nr - \frac{1+r}{r} (r + c((1+r)^n - 1)) + nc \]
\[ = 1 + r - \frac{1+r + n(c-r)}{r + c((1+r)^n - 1)} \]
\[ = \frac{(1-c/r)n}{1+c((1+r)^n - 1)/r} + \frac{1+r}{r} \left( \frac{c((1+r)^n - 1)}{r + c((1+r)^n - 1)} \right), \]

with \( D_0(0,n) = n \). We note that
\[
\lim_{n \to \infty} D_c(0,n) = \lim_{n \to \infty} \left( \frac{1+r}{r} - \frac{1+r + n(c-r)}{r + c((1+r)^n - 1)} \right) = 1 + \frac{1}{r}.
\]

When \( n \) becomes large, the duration (or relative sensitivity) of the bond price converges to \( 1 + 1/r \) whenever the (positive) coupon \( c \) is nonzero, otherwise the bond duration of \( P_c(0,n) \) is \( n \). In particular, the presence of a nonzero coupon makes the duration (or relative sensitivity) of the bond price bounded as \( n \) increases, whereas the duration \( n \) of \( P_0(0,n) \) goes to infinity as \( n \) increases.

As a consequence, the presence of the coupon tends to put an upper limit the risk and sensitivity of bond prices with respect to the market interest rate \( r \) as \( n \) becomes large, which can be used for bond immunization. Note that the duration \( D_c(0,n) \) can also be written as the relative average
\[
D_c(0,n) = \frac{1}{P_c(0,n)} \left( \frac{n}{(1+r)^n} + c \sum_{k=1}^{n} \frac{k}{(1+r)^k} \right)
\]
of zero coupon bond durations weighted by their respective zero-coupon prices.

Exercise 16.9

a) We have

\[ \]
\[ P(t, T) = P(s, T) \exp \left( \int_s^t r_u du + \int_s^t \sigma_u T dB_u - \frac{1}{2} \int_s^t |\sigma_u|^2 du \right), \]

\[ 0 \leq s \leq t \leq T. \]

b) We have
\[ d \left( e^{-\int_0^t r_s ds} P(t, T) \right) = e^{-\int_0^t r_s ds} \sigma_t^T P(t, T) dB_t, \]
which gives a martingale after integration, from the properties of the Itô integral.

c) By the martingale property of the previous question we have
\[ \mathbb{E} \left[ e^{-\int_0^T r_s ds} \mid \mathcal{F}_t \right] = \mathbb{E} \left[ P(T, T) e^{-\int_0^T r_s ds} \mid \mathcal{F}_t \right] = P(t, T) e^{-\int_0^t r_s ds}, \quad 0 \leq t \leq T. \]

d) By the previous question we have
\[ P(t, T) = e^{\int_0^t r_s ds} \mathbb{E} \left[ e^{-\int_0^T r_s ds} \mid \mathcal{F}_t \right] = \mathbb{E} \left[ e^{\int_0^t r_s ds} e^{-\int_0^T r_s ds} \mid \mathcal{F}_t \right] = \mathbb{E} \left[ e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T, \]
since \( e^{-\int_0^t r_s ds} \) is an \( \mathcal{F}_t \)-measurable random variable.

e) We have
\[ \frac{P(t, S)}{P(t, T)} = \frac{P(s, S)}{P(s, T)} \exp \left( \int_s^t (\sigma_u^S - \sigma_u^T) dB_u - \frac{1}{2} \int_s^t (|\sigma_u^S|^2 - |\sigma_u^T|^2) du \right) \]
\[ = \frac{P(s, S)}{P(s, T)} \exp \left( \int_s^t (\sigma_u^S - \sigma_u^T) dB_u^T - \frac{1}{2} \int_s^t (\sigma_u^S - \sigma_u^T)^2 du \right), \]
\[ 0 \leq t \leq T, \] hence letting \( s = t \) and \( t = T \) in the above expression we have
\[ P(T, S) = \frac{P(t, S)}{P(t, T)} \exp \left( \int_t^T (\sigma_u^S - \sigma_u^T) dB_u^T - \frac{1}{2} \int_t^T (\sigma_u^S - \sigma_u^T)^2 du \right). \]

f) We have
\[ P(t, T) \mathbb{E}_T \left[ (P(T, S) - \kappa)^+ \right] \]
\[ = P(t, T) \mathbb{E}_T \left[ \left( \frac{P(t, S)}{P(t, T)} e^{\int_t^T (\sigma_u^S - \sigma_u^T) dB_u^T - \frac{1}{2} \int_t^T (\sigma_u^S - \sigma_u^T)^2 du / 2 - \kappa} \right)^+ \right] \]
\[ = P(t, T) \mathbb{E}_T^+ \left[ (e^X - \kappa)^+ \mid \mathcal{F}_t \right] \]
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\[ P(t, T) e^{m_t + v_t^2/2} \Phi \left( \frac{v_t}{2} + \frac{1}{v_t} (m_t + v_t^2/2 - \log \kappa) \right) \]

\[ -\kappa P(t, T) \Phi \left( -\frac{v_t}{2} + \frac{1}{v_t} (m_t + v_t^2/2 - \log \kappa) \right) , \]

with

\[ m_t = \log(P(t, S)/P(t, T)) - \frac{1}{2} \int_t^T (\sigma_s^2 - \sigma_s^T)^2 ds \]

and

\[ v_t^2 = \int_t^T (\sigma_s^2 - \sigma_s^T)^2 ds, \]

i.e.

\[ P(t, T) \mathbb{E}_T \left[ (P(T, S) - \kappa)^+ \right] = P(t, S) \Phi \left( \frac{v_t}{2} + \frac{1}{v_t} \log \frac{P(t, S)}{\kappa P(t, T)} \right) - \kappa P(t, T) \Phi \left( -\frac{v_t}{2} + \frac{1}{v_t} \log \frac{P(t, S)}{\kappa P(t, T)} \right) . \]

Exercise 16.10 (Exercise 4.13 continued). From Proposition 16.2, the bond pricing PDE is

\[ \begin{cases} \frac{\partial F}{\partial t}(t, x) = xF(t, x) - (\alpha - \beta x) \frac{\partial F}{\partial x}(t, x) - \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 F}{\partial x^2}(t, x) \\ F(T, x) = 1. \end{cases} \]

We search for a solution of the form

\[ F(t, x) = e^{A(T-t)-xB(T-t)}, \]

with \( A(0) = B(0) = 0 \), which implies

\[ \begin{cases} A'(s) = 0 \\ B'(s) + \beta B(s) + \frac{1}{2} \sigma^2 B^2(s) = 1, \end{cases} \]

hence in particular \( A(s) = 0, s \in \mathbb{R} \), and \( B(s) \) solves a Riccati equation, whose solution can be checked to be

\[ B(s) = \frac{2(e^{s\gamma} - 1)}{2\gamma + (\beta + \gamma)(e^{s\gamma} - 1)}, \]

with \( \gamma = \sqrt{\beta^2 + 2\sigma^2}. \)

Exercise 16.11

a) We have
\[
\begin{aligned}
    y_{0,1} &= -\frac{1}{T_1} \log P(0, T_1) = 9.53\%, \\
    y_{0,2} &= -\frac{1}{T_2} \log P(0, T_2) = 9.1\%, \\
    y_{1,2} &= -\frac{1}{T_2 - T_1} \log \frac{P(0, T_2)}{P(T_1, T_2)} = 8.6\%,
\end{aligned}
\]
with \(T_1 = 1\) and \(T_2 = 2\).

b) We have
\[
P_c(1, 2) = (\$1 + 0.1) \times P_0(1, 2) = (\$1 + 0.1) \times e^{-(T_2-T_2)y_{1,2}} = \$1.00914,
\]
and
\[
P_c(0, 2) = (\$1 + 0.1) \times P_0(0, 2) + 0.1 \times P_0(0, 1) \\
= (\$1 + 0.1) \times e^{-(T_2-T_2)y_{0,2}} + 0.1 \times e^{-(T_2-T_2)y_{0,1}} \\
= \$1.00831.
\]

Exercise 16.12

a) The discretization \(r_{t_{k+1}} := r_{t_k} + (a - br_{t_k})\Delta t \pm \sigma \sqrt{\Delta t}\), does not lead to a binomial tree as \(r_{t_2}\) could be obtained in four different ways from \(r_{t_0}\) as
\[
\begin{aligned}
    r_{t_2} &= r_{t_1} (1 - b\Delta t) + a\Delta t \pm \sigma \sqrt{\Delta t} \\
    &= \left\{ \begin{array}{l}
        (r_{t_0} (1 - b\Delta t) + a\Delta t + \sigma \sqrt{\Delta t}) (1 - b\Delta t) + a\Delta t + \sigma \sqrt{\Delta t} \\
        (r_{t_0} (1 - b\Delta t) + a\Delta t + \sigma \sqrt{\Delta t}) (1 - b\Delta t) + a\Delta t - \sigma \sqrt{\Delta t} \\
        (r_{t_0} (1 - b\Delta t) + a\Delta t - \sigma \sqrt{\Delta t}) (1 - b\Delta t) + a\Delta t + \sigma \sqrt{\Delta t} \\
        (r_{t_0} (1 - b\Delta t) + a\Delta t - \sigma \sqrt{\Delta t}) (1 - b\Delta t) + a\Delta t - \sigma \sqrt{\Delta t}.
        \end{array} \right.
    \end{aligned}
\]

b) By the Girsanov Theorem, the process \( (r_t / \sigma)_{t \in [0,T]} \) with
\[
\frac{dr_t}{\sigma} = \frac{a - br_t}{\sigma} dt + dB_t
\]
is a standard Brownian motion under the probability measure \(Q\) with density
\[
\frac{dQ}{dP} = \exp \left( -\frac{1}{\sigma} \int_0^T (a - br_t) dB_t - \frac{1}{2\sigma^2} \int_0^T (a - br_t)^2 dt \right) \\
\simeq \exp \left( -\frac{1}{\sigma^2} \int_0^T (a - br_t) (dr_t - (a - br_t) dt) - \frac{1}{2\sigma^2} \int_0^T (a - br_t)^2 dt \right)
\]

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= \exp\left(-\frac{1}{\sigma^2} \int_0^T (a - br_t) \, dr_t + \frac{1}{2\sigma^2} \int_0^T (a - br_t)^2 \, dt\right).

In other words, if we generate \( (r_t/\sigma)_{t\in[0,T]} \) and the increments \( \sigma^{-1} \, dr_t \simeq \pm \sqrt{\Delta t} \) as a standard Brownian motion under \( Q \), then, under the probability measure \( P \) such that

\[
\frac{dP}{dQ} = \exp\left(\frac{1}{\sigma^2} \int_0^T (a - br_t) \, dr_t - \frac{1}{2\sigma^2} \int_0^T (a - br_t)^2 \, dt\right)
\]

\[
\simeq 2^{T/\Delta T} \prod_{0 < t < T} \left(\frac{1}{2} \pm \frac{a - br_t}{2\sigma} \sqrt{\Delta t}\right),
\]

the process

\[
 dB_t = \frac{dr_t}{\sigma} - \frac{a - br_t}{\sigma} \, dt
\]

will be a standard Brownian motion under \( P \), and the samples

\[
dr_t = (a - br_t) \, dt + \sigma dB_t
\]

of \( (r_t)_{t\in[0,T]} \) will be distributed as a Vasicek process under \( P \).

c) We check that

\[
\mathbb{E}[\Delta r_{t_1}] = (a - br_{t_0}) \Delta t
\]

\[
= p(r_{t_0}) \sigma \sqrt{\Delta t} - (1 - p(r_{t_0})) \sigma \sqrt{\Delta t}
\]

\[
= \sigma p(r_{t_0}) \sqrt{\Delta t} - \sigma q(r_{t_0}) \sqrt{\Delta t},
\]

with

\[
p(r_{t_0}) = \frac{1}{2} + \frac{a - br_{t_0}}{2\sigma} \sqrt{\Delta t} \quad \text{and} \quad q(r_{t_0}) = \frac{1}{2} - \frac{a - br_{t_0}}{2\sigma} \sqrt{\Delta t}.
\]

Similarly, we have

\[
\mathbb{E}[\Delta r_{t_2}] = (a - br_{t_1}) \Delta t
\]

\[
= p(r_{t_1}) \sigma \sqrt{\Delta t} - (1 - p(r_{t_1})) \sigma \sqrt{\Delta t}
\]

\[
= \sigma p(r_{t_1}) \sqrt{\Delta t} - \sigma q(r_{t_1}) \sqrt{\Delta t},
\]

with

\[
p(r_{t_1}) = \frac{1}{2} + \frac{a - br_{t_1}}{2\sigma} \sqrt{\Delta t}, \quad q(r_{t_1}) = \frac{1}{2} - \frac{a - br_{t_1}}{2\sigma} \sqrt{\Delta t},
\]

and

\[
p(r_{t_1}) = \frac{1}{2} + \frac{a - br_{t_1}}{2\sigma} \sqrt{\Delta t}, \quad q(r_{t_1}) = \frac{1}{2} - \frac{a - br_{t_1}}{2\sigma} \sqrt{\Delta t}.
\]
Exercise 16.13

a) We have
\[ P(1, 2) = \mathbb{E}^* \left[ \frac{100}{1 + r_1} \right] = \frac{100}{2(1 + r_1^u)} + \frac{100}{2(1 + r_1^d)}. \]

b) We have
\[ P(0, 2) = \frac{100}{2(1 + r_0)(1 + r_1^u)} + \frac{100}{2(1 + r_0)(1 + r_1^d)}. \]

c) We have \( P(0, 1) = 91.74 = 100/(1 + r_0), \) hence
\[ r_0 = \frac{100 - P(0, 1)}{P(0, 1)} = 100/91.74 - 1 = 0.090037061 \approx 9\%. \]

d) We have
\[ 83.40 = P(0, 2) = \frac{P(0, 1)}{2(1 + r_1^d)} \]
and \( r_1^u / r_1^d = e^{2\sigma \sqrt{\Delta t}}, \) hence
\[ 83.40 = P(0, 2) = \frac{P(0, 1)}{2(1 + r_1^d e^{2\sigma \sqrt{\Delta t}})} + \frac{P(0, 1)}{2(1 + r_1^d)} \]
or
\[ e^{2\sigma \sqrt{\Delta t}} (r_1^d)^2 + 2r_1^d e^{\sigma \sqrt{\Delta t}} \cosh (\sigma \sqrt{\Delta t}) \left( 1 - \frac{P(0, 1)}{2P(0, 2)} \right) + 1 - \frac{P(0, 1)}{P(0, 2)} = 0, \]
and
\[ r_1^d = e^{-\sigma \sqrt{\Delta t}} \left( \cosh (\sigma \sqrt{\Delta t}) \left( \frac{P(0, 1)}{2P(0, 2)} - 1 \right) \right) \]
\[ \pm \sqrt{ \left( \frac{P(0, 1)}{2P(0, 2)} - 1 \right)^2 \cosh^2 (\sigma \sqrt{\Delta t}) + \frac{P(0, 1)}{P(0, 2)} - 1 } \]
\[ = 0.078684844 \approx 7.87\%, \]
and
\[ r_1^u = r_1^d e^{2\sigma \sqrt{\Delta t}} \]
\[ = e^{\sigma \sqrt{\Delta t}} \left( \cosh (\sigma \sqrt{\Delta t}) \left( \frac{P(0, 1)}{2P(0, 2)} - 1 \right) \right) \]
\[ \pm \sqrt{\left( \frac{P(0, 1)}{2P(0, 2)} - 1 \right)^2 \cosh^2 (\sigma \sqrt{\Delta t}) + \frac{P(0, 1)}{P(0, 2)} - 1} \]

\[ = 0.122174525 \approx 12.22\% . \]

We also find

\[ \mu = \frac{1}{\Delta t} \left( \sigma \sqrt{\Delta t} + \log \frac{r^d}{r_0} \right) = \frac{1}{\Delta t} \left( -\sigma \sqrt{\Delta t} + \log \frac{r^u}{r_0} \right) = 0.085229181 \approx 8.52\% . \]

**Exercise 16.14**

a) When \( n = 1 \) the relation (16.48) shows that \( \tilde{f}(t, t, T_1) = f(t, t, T_1) \) with \( F(t, x) = c_1 e^{-(T_1-1)x} \) and \( P(t, T_1) = c_1 e^{f(t, t, T_1)} \), hence

\[ D(t, T_1) := -\frac{1}{P(t, T_1)} \frac{\partial F}{\partial x}(t, f(t, t, T_1)) = T_1 - t, \quad 0 \leq t \leq T_1. \]

b) In general, we have

\[ D(t, T_n) = -\frac{1}{P(t, T_n)} \frac{\partial F}{\partial x}(t, \tilde{f}(t, t, T_n)) = \frac{1}{P(t, T_n)} \sum_{k=1}^{n} (T_k - t)c_k e^{-(T_k-t)\tilde{f}(t, t, T_n)} = \sum_{k=1}^{n} w_k(T_k - t), \]

where

\[ w_k := \frac{c_k}{P(t, T_n)} e^{-(T_k-t)\tilde{f}(t, t, T_n)} = \frac{c_k e^{-(T_k-t)f(t, t, T_n)}}{\sum_{l=1}^{n} c_l e^{-(T_l-t)f(t, t, T_l)}} = \frac{c_k e^{-(T_k-t)f(t, t, T_n)}}{\sum_{l=1}^{n} c_l e^{-(T_l-t)f(t, t, T_n)}}, \quad k = 1, 2, \ldots, n, \]

and the weights \( w_1, w_2, \ldots, w_n \) satisfy

\[ \sum_{k=1}^{n} w_k = 1. \]
c) We have
\[ C(t, T_n) = \frac{1}{P(t, T_n)} \frac{\partial^2 F}{\partial x^2}(t, \tilde{f}(t, t, T_n)) \]
\[ = \sum_{k=1}^{n} w_k (T_k - t)^2 \]
\[ = \sum_{k=1}^{n} w_k (T_k - t - D(t, T_n))^2 + 2D(t, T_n) \sum_{k=1}^{n} w_k (T_k - t) - (D(t, T_n))^2 \]
\[ = (D(t, T_n))^2 + \sum_{k=1}^{n} w_k (T_k - t - D(t, T_n))^2 \]
\[ = (S(t, T_n))^2 + (D(t, T_n))^2, \]
with
\[ (S(t, T_n))^2 := \sum_{k=1}^{n} w_k (T_k - t - D(t, T_n))^2. \]

d) We have
\[ D(t, T_n) = \frac{1}{P(t, T_n)} \sum_{k=1}^{n} c_k B(T_k - t) e^{A(T_k-t)+B(T_k-t)\alpha(t,t,T_n)} \]
\[ = \frac{1}{e^{A(T_k-t)+B(T_n-t)\alpha(t,t,T_n)}} \sum_{k=1}^{n} c_k B(T_k - t) e^{A(T_k-t)+B(T_k-t)\alpha(t,t,T_n)} \]
\[ = \sum_{k=1}^{n} c_k B(T_k - t) e^{B(T_k-t)-B(T_n-t)\tilde{f}_\alpha(t,t,T_n)}. \]

e) We have
\[ D(t, T_n) = \frac{1}{b} \sum_{k=1}^{n} c_k (1 - e^{-b(T_k-t)}) e^{(e^{-b(T_n-t)} - e^{-b(T_k-t)})\tilde{f}_\alpha(t,t,T_n)/b} \]
\[ = \frac{1}{b} \sum_{k=1}^{n} c_k (1 - e^{-b(T_k-t)}) (P(t, t + \alpha(T_n - t)))^\frac{e^{-b(T_n-t)} - e^{-b(T_k-t)}}{\alpha b(T_n-t)}. \]

Chapter 17

Exercise 17.1 We have
\[ P(0, T_2) = \exp \left( - \int_0^{T_2} f(t, s) ds \right) = e^{-r_1T_1-r_2(T_2-T_1)}, \quad t \in [0, T_2], \]
and
\[ P(T_1, T_2) = \exp \left( - \int_{T_1}^{T_2} f(t, s) \, ds \right) = e^{-r_2(T_2-T_1)}, \quad t \in [0, T_2], \]
from which we deduce
\[ r_2 = -\frac{1}{T_2-T_1} \log P(T_1, T_2), \]
and
\[ r_1 = -r_2 \frac{T_2-T_1}{T_1} - \frac{1}{T_1} \log P(0, T_2) \]
\[ = \frac{1}{T_1} \log P(T_1, T_2) - \frac{1}{T_1} \log P(0, T_2) \]
\[ = -\frac{1}{T_1} \log \frac{P(0, T_2)}{P(T_1, T_2)}. \]

Exercise 17.2 (Exercise 4.10 continued).

a) We check that \( P(T, T) = e^{X_T^T} = 1. \)

b) We have
\[ f(t, T, S) = -\frac{1}{S-T} \left( X_t^S - X_t^T - \mu(S-T) \right) \]
\[ = \mu - \sigma \frac{1}{S-T} \left( (S-t) \int_0^t \frac{1}{S-s} dB_s - (T-t) \int_0^t \frac{1}{T-s} dB_s \right) \]
\[ = \mu - \sigma \frac{1}{S-T} \int_0^t \frac{(S-t) - (T-t)}{S-s} dB_s \]
\[ = \mu - \sigma \frac{1}{S-T} \int_0^t \frac{(T-s)(S-t) - (T-t)(S-s)}{(S-s)(T-s)} dB_s \]
\[ = \mu + \frac{\sigma}{S-T} \int_0^t \frac{(s-t)(S-T)}{(S-s)(T-s)} dB_s. \]

c) We have
\[ f(t, T) = \mu - \sigma \int_0^t \frac{t-s}{(T-s)^2} dB_s. \]
d) We note that
\[ \lim_{T \to t} f(t, T) = \mu - \sigma \int_0^t \frac{1}{t-s} dB_s \]
does not exist in \( L^2(\Omega). \)
e) By Itô’s calculus we have
\[
\frac{dP(t, T)}{P(t, T)} = \sigma dB_t + \frac{1}{2} \sigma^2 dt + \mu dt - \frac{X_T^t}{T-t} dt
\]
\[
= \sigma dB_t + \frac{1}{2} \sigma^2 dt - \frac{\log P(t, T)}{T-t} dt, \quad t \in [0, T].
\]

f) Letting
\[
r_t^T := \mu + \frac{1}{2} \sigma^2 - \frac{X_T^t}{T-t}
\]
\[
= \mu + \frac{1}{2} \sigma^2 - \sigma \int_0^t \frac{dB_s}{T-s},
\]
by Question (e) we find that
\[
\frac{dP(t, T)}{P(t, T)} = \sigma dB_t + r_t^T dt, \quad 0 \leq t \leq T.
\]

g) The equation of Question (f) can be solved as
\[
P(t, T) = P(0, T) \exp \left( \sigma B_t - \frac{\sigma^2 t}{2} + \int_0^t r_s^T ds \right), \quad 0 \leq t \leq T,
\]
hence the process
\[
P(t, T) \exp \left( - \int_0^t r_s^T ds \right) = P(0, T) \exp \left( \sigma B_t - \frac{\sigma^2 t}{2} \right), \quad 0 \leq t \leq T,
\]
is a martingale under \( P^* \), with the relation
\[
P(t, T) \exp \left( - \int_0^t r_s^T ds \right) = \mathbb{E}^* \left[ P(T, T) \exp \left( - \int_0^T r_s^T ds \right) \mid \mathcal{F}_t \right]
\]
\[
= \mathbb{E}^* \left[ \exp \left( - \int_0^T r_s^T ds \right) \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T,
\]
showing that
\[
P(t, T) = \exp \left( \int_0^t r_s^T ds \right) \mathbb{E}^* \left[ \exp \left( - \int_0^T r_s^T ds \right) \mid \mathcal{F}_t \right]
\]
\[
= \mathbb{E}^* \left[ \exp \left( \int_0^t r_s^T ds \right) \exp \left( - \int_0^T r_s^T ds \right) \mid \mathcal{F}_t \right]
\]
\[
= \mathbb{E}^* \left[ \exp \left( - \int_t^T r_s^T ds \right) \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T.
\]
h) By Question (g) we have
\[
\mathbb{E} \left[ \frac{dP_T}{dP} \middle| \mathcal{F}_t \right] = \frac{P(t, T)}{P(0, T)} e^{\int_0^t r_s^T ds} = e^{\sigma B_t - \sigma^2 t/2}, \quad 0 \leq t \leq T.
\]

i) By the Girsanov Theorem, the process \( \tilde{B}_t := B_t - \sigma t \) is a standard Brownian motion under \( P_T \).

j) We have
\[
\log P(T, S) = -\mu(S-T) + \sigma \int_0^T \frac{S-T}{S-s} dB_s
\]
\[
= -\mu(S-T) + \sigma \int_0^t \frac{S-T}{S-s} dB_s + \sigma \int_t^T \frac{S-T}{S-s} dB_s
\]
\[
= \frac{S-T}{S-t} \log P(t, S) + \sigma \int_t^T \frac{S-T}{S-s} dB_s
\]
\[
= \frac{S-T}{S-t} \log P(t, S) + \sigma \int_t^T \frac{S-T}{S-s} d\tilde{B}_s + \sigma^2 \int_t^T \frac{S-T}{S-s} ds
\]
\[
= \frac{S-T}{S-t} \log P(t, S) + \sigma \int_t^T \frac{S-T}{S-s} d\tilde{B}_s + (S-T)\sigma^2 \log \frac{S-t}{S-T},
\]
\( 0 < T < S \).

k) We have
\[
P(t, T) \mathbb{E}_T \left[ (P(T, S) - K)^+ \middle| \mathcal{F}_t \right]
= P(t, T) \mathbb{E} \left[ (e^X - K)^+ \middle| \mathcal{F}_t \right]
= P(t, T) e^{m_t + v_t^2/2} \Phi \left( \frac{v_t}{2} + \frac{1}{v_t} (m_t + v_t^2/2 - \log \kappa) \right)
\]
\[
- \kappa P(t, T) \Phi \left( -\frac{v_t}{2} + \frac{1}{v_t} (m_t + v_t^2/2 - \log \kappa) \right)
= P(t, T) e^{m_t + v_t^2/2} \Phi \left( v_t + \frac{1}{v_t} (m_t - \log \kappa) \right) - \kappa P(t, T) \Phi \left( \frac{1}{v_t} (m_t - \log \kappa) \right),
\]
with
\[
m_t = \frac{S-T}{S-t} \log P(t, S) + (S-T)\sigma^2 \log \frac{S-t}{S-T}
\]
and
\[
v_t^2 = \sigma^2 \int_t^T \frac{(S-T)^2}{(S-s)^2} ds
= (S-T)^2 \sigma^2 \left( \frac{1}{S-T} - \frac{1}{S-t} \right)
= (S-T)\sigma^2 \frac{(T-t)}{(S-t)}.
\]
hence

\[ P(t,T) \mathbb{E}_T \left[(P(T,S) - K)^+ | \mathcal{F}_t\right] \]

\[ = P(t,T) (P(t,S))^{(S-T)(S-t)} \left( \frac{S-t}{S-T} \right)^{(S-T)\sigma^2} e^{\frac{v_t^2}{2}} \]

\[ \times \Phi \left(v_t + \frac{1}{v_t} \log \left( \frac{(P(t,S))^{(S-T)(S-t)}}{\kappa} \left( \frac{S-t}{S-T} \right)^{(S-T)\sigma^2} \right) \right) \]

\[ - \kappa P(t,T) \Phi \left( \frac{1}{v_t} \log \left( \frac{(P(t,S))^{(S-T)(S-t)}}{\kappa} \left( \frac{S-t}{S-T} \right)^{(S-T)\sigma^2} \right) \right). \]

Exercise 17.3

a) In the Vasicek model we have

\[ P(t,T) = \mathbb{E} \left[ \exp \left( - \int_t^T r_s ds \right) \right] \]

\[ = \mathbb{E} \left[ \exp \left( - \int_t^T h(s) ds - \int_t^T X_s ds \right) \right] \]

\[ = \exp \left( - \int_t^T h(s) ds \right) \mathbb{E} \left[ \exp \left( - \int_t^T X_s ds \right) \right] \]

\[ = \exp \left( - \int_t^T h(s) ds + A(T-t) + X_tC(T-t) \right), \quad 0 \leq t \leq T, \]

hence, since \( X_0 = 0 \) we find

\[ P(0,T) = \exp \left( - \int_0^T h(s) ds + A(T) \right). \]

b) By the identification

\[ P(t,T) = \exp \left( - \int_t^T h(s) ds + A(T-t) + X_tC(T-t) \right) \]

\[ = \exp \left( - \int_t^T f(t,s) ds \right), \]

we find

\[ \int_t^T h(s) ds = \int_t^T f(t,s) ds + A(T-t) + X_tC(T-t), \]

and by differentiation with respect to \( T \) this yields

\[ h(T) = f(t,T) + A'(T-t) + X_tC'(T-t), \quad t \in [t,T], \]

where
\[ A(T-t) = \frac{4ab - 3\sigma^2}{4b^3} + \frac{\sigma^2 - 2ab}{2b^2} (T-t) + \frac{\sigma^2 - ab}{b^3} e^{-b(T-t)} - \frac{\sigma^2}{4b^3} e^{-2b(T-t)}. \]

Given an initial market data curve \( f^M(0, T) \), the matching \( f^M(0, T) = f(0, T) \) can be achieved at time \( t = 0 \) by letting

\[ h(T) := f^M(0, T) + A'(T) = f^M(0, T) + \frac{\sigma^2 - 2ab}{2b^2} - \frac{\sigma^2 - ab}{b^3} e^{-bT} + \frac{\sigma^2}{2b^2} e^{-2bT}, \]

\( T > 0 \). Note however that in general, at time \( t \in (0, T] \) we will have

\[ h(T) = f(t, T) + A'(T - t) + X_tC'(T - t) = f^M(0, T) + A'(T), \]

and the relation

\[ f(t, T) = f^M(0, T) + A'(T) - A'(T - t) - X_tC'(T - t), \quad t \in [0, T], \]

will allow us to match market data at time \( t = 0 \) only, i.e. for the initial curve. In any case, model calibration is to be done at time \( t = 0 \).

**Exercise 17.4** From the definition

\[ L(t, t, T) = \frac{1}{T-t} \left( \frac{1}{P(t,T)} - 1 \right), \]

we have

\[ P(t, T) = \frac{1}{1 + (T-t)L(t, t, T)}, \]

and similarly

\[ P(t, S) = \frac{1}{1 + (S-t)L(t, t, S)}. \]

Hence we get

\[ L(t, T, S) = \frac{1}{S-T} \left( \frac{P(t, T)}{P(t, S)} - 1 \right) = \frac{1}{S-T} \left( \frac{1 + (S-t)L(t, t, S)}{1 + (T-t)L(t, t, T)} - 1 \right) = \frac{1}{S-T} \left( \frac{(S-t)L(t, t, S) - (T-t)L(t, t, T)}{1 + (T-t)L(t, t, T)} \right). \]

When \( T = \) one year and \( L(0, 0, T) = 2\% \), \( L(0, 0, 2T) = 2.5\% \) we find

\[ L(t, T, S) = \frac{1}{T} \left( \frac{2TL(0, 0, 2T) - TL(0, 0, T)}{1 + TL(0, 0, T)} \right) = \frac{2 \times 0.025 - 0.02}{1 + 0.02} = 2.94\%, \]

\[ \bullet \]
so we would not prefer a spot rate at \( L(T, T, 2T) = 2\% \) over a forward contract with rate \( L(0, T, 2T) = 2.94\% \).

Exercise 17.5 (Exercise 16.3 continued).

a) We have
\[
\begin{align*}
f(t, T, S) &= \frac{1}{S - T} (\log P(t, T) - \log P(t, S)) \\
&= \frac{1}{S - T} \left( \left( -(T-t)r_t + \frac{\sigma^2}{6}(T-t)^3 \right) - \left( -(S-t)r_t + \frac{\sigma^2}{6}(S-t)^3 \right) \right) \\
&= r_t + \frac{1}{S - T} \frac{\sigma^2}{6} ((T-t)^3 - (S-t)^3).
\end{align*}
\]

b) We have
\[
f(t, T) = -\frac{\partial}{\partial T} \log P(t, T) = r_t - \frac{\sigma^2}{2} (T-t)^2.
\]

c) We have
\[
d_t f(t, T) = (T-t)\sigma^2 dt + adt + \sigma dB_t.
\]

d) The HJM condition (17.23) is satisfied since the drift of \( d_t f(t, T) \) equals \( \sigma \int_t^T \sigma ds \).

Exercise 17.6

a) We have
\[
f(t, x) = f(0, x) + \alpha \int_0^t x^2 ds + \sigma \int_0^t d_s B(s, x) = r + \alpha tx^2 + \sigma B(t, x).
\]

b) We have
\[
r_t = f(t, 0) = r + B(t, 0) = r.
\]

c) We have
\[
P(t, T) = \exp \left( -\int_t^T f(t, s) ds \right)
\]
\[
= \exp \left( -r(T-t) - \alpha t \int_0^{T-t} s^2 ds - \sigma \int_0^{T-t} B(t, x) dx \right)
\]
\[
= \exp \left( -r(T-t) - \frac{\alpha}{3} t(T-t)^3 - \sigma \int_0^{T-t} B(t, x) dx \right), \quad t \in [0, T].
\]

d) We have
\[ \mathbb{E} \left[ \left( \int_0^{T-t} B(t, x) dx \right)^2 \right] = \mathbb{E} \left[ \int_0^{T-t} B(t, x) dx \int_0^{T-t} B(t, y) dy \right] \\
= \mathbb{E} \left[ \int_0^{T-t} \int_0^{T-t} B(t, y) B(t, x) dx dy \right] \\
= \int_0^{T-t} \int_0^{T-t} \mathbb{E} [B(t, y) B(t, x)] dx dy \\
= \int_0^{T-t} \int_0^{T-t} \mathbb{E} [B(t, x) B(t, y)] dx dy \\
= t \int_0^{T-t} \int_0^{T-t} \min(x, y) dx dy \\
= 2t \int_0^{T-t} \int_y x dx dy = \frac{1}{3} t(T-t)^3. \]

e) We have

\[ \mathbb{E}[P(t, T)] = \mathbb{E} \left[ \exp \left( -r(T-t) - \frac{\alpha}{3} t(T-t)^3 - \sigma \int_0^{T-t} B(t, x) dx \right) \right] \\
= \exp \left( -r(T-t) - \frac{\alpha}{3} t(T-t)^3 + \frac{\sigma^2}{6} t(T-t)^3 \right), \quad t \in [0, T]. \]

f) We need to take \( \alpha = \sigma^2/2. \)

Remark: In order to derive an analog of the HJM absence of arbitrage condition in this stochastic string model, one would have to check whether the discounted bond price \( e^{-rt} P(t, T) \) can be a martingale by doing stochastic calculus with respect to the Brownian sheet \( B(t, x) \).

g) We have

\[ \mathbb{E} \left[ \exp \left( - \int_0^T r_s ds \right) (P(T, S) - K)^+ \right] \\
= e^{-rT} \mathbb{E} \left[ \left( \exp \left( -r(S-T) - \frac{\alpha}{3} (S-T)^3 + \sigma \int_0^{S-T} B(T, x) dx \right) - K \right)^+ \right] \\
= e^{-rT} \mathbb{E} \left[ (x e^{m+X} - K)^+ \right], \]

where \( x = e^{-r(S-T)} \), \( m = -\alpha T(S-T)^3/3 \), and

\[ X = \sigma \int_0^{S-T} B(T, x) dx \approx \mathcal{N}(0, \sigma^2 t(T-t)^3/3). \]

Given the relation \( \alpha = \sigma^2/2 \) this yields

\[ \mathbb{E} \left[ \exp \left( - \int_0^T r_s ds \right) (P(T, S) - K)^+ \right] \]
\[
= e^{-rS} \Phi \left( \sigma \sqrt{T(S-T)^3/12} + \frac{\log(e^{-r(S-T)}/K)}{\sigma \sqrt{T(S-T)^3/3}} \right)
\]

\[
-K e^{-rT} \Phi \left( -\sigma \sqrt{T(S-T)^3/12} + \frac{\log(e^{-r(S-T)}/K)}{\sigma \sqrt{T(S-T)^3/3}} \right)
\]

\[
= P(0, S) \Phi \left( \sigma \sqrt{T(S-T)^3/12} + \frac{\log(e^{-r(S-T)}/K)}{\sigma \sqrt{T(S-T)^3/3}} \right)
\]

\[
-K P(0, T) \Phi \left( -\sigma \sqrt{T(S-T)^3/12} + \frac{\log(e^{-r(S-T)}/K)}{\sigma \sqrt{T(S-T)^3/3}} \right).
\]

**Chapter 18**

Exercise 18.1

a) We take \( t = 0, T_1 = 9 \) months, \( T_2 = 1 \) year, \( \kappa = 4.5\% \), and the LIBOR rate \( (L(t, T_1, T_2))_{t \in [0, T_1]} \) is modeled as a driftless geometric Brownian motion with volatility coefficient \( \hat{\sigma} = \sigma_{1,2}(t) = 0.1 \) under the forward measure \( \mathcal{P}_2 \). The discount factors are given by

\[
P(0, T_1) = (1 + r)^{-9/12} \approx 0.964413249
\]

and

\[
P(0, T_2) = \frac{1}{1+r} \approx 0.952834683,
\]

with \( r = 4.95\% \).

b) By (18.17) the price of the floorlet is

\[
\mathbb{E}^* \left[ e^{-\int_0^{T_2} r_s ds} (\kappa - L(T_1, T_1, T_2))^+ \right]
\]

\[
= P(0, T_2) \left( \kappa \Phi \left( -d_-(T_1) \right) - L(0, T_1, T_2) \Phi \left( -d_+(T_1) \right) \right), \quad (A.72)
\]

where

\[
d_+(T_1) = \frac{\log(L(0, T_1, T_2)/\kappa) + \sigma^2 T_1/2}{\sigma \sqrt{T_1}},
\]

and

\[
d_-(T_1) = \frac{\log(L(0, T_1, T_2)/\kappa) - \sigma^2 T_1/2}{\sigma \sqrt{T_1 - t}},
\]

are given in Proposition 18.4, and the LIBOR rate \( L(0, T_1, T_2) \) is given by

\[
L(0, T_1, T_2) = \frac{P(0, T_1) - P(0, T_2)}{(T_2 - T_1)P(0, T_2)}
\]
\[
= \frac{(1 + r)^{-3/4} - (1 + r)^{-1}}{0.3(1 + r)^{-1}}
= \frac{(1 + r)^{1/4} - 1}{0.3}
\approx 4.05%.
\]

Hence, we have
\[
d_+(T_1) = \frac{\log(0.0405/0.045) + (0.1)^2 \times 0.75/2}{0.1 \times \sqrt{0.75}} \approx -1.259899712,
\]
and
\[
d_-(T_1) = \frac{\log(0.0405/0.045) - (0.1)^2 \times 0.75/2}{0.1 \times \sqrt{0.75}} \approx -1.173297171,
\]

hence
\[
\mathbb{E}^* \left[ e^{-\int_0^{T_2} r_s ds} (\kappa - L(T_1, T_1, T_2))^+ \right] = 0.952834683 \times (0.045 \times 0.879662 - 0.0405 \times 0.896147) \approx 0.3135623%.
\]

Finally, we need to multiply (A.72) by the notional principal amount of \$1 million, i.e. to multiply by 10,000 when (A.72) is quoted in percentage points, which yields \$3,135.62

Exercise 18.2

a) We take \(t = 0\), \(T_1 = 4\) years, \(T_2 = 5\) years, \(T_3 = 6\) years, \(T_4 = 7\) years, \(\kappa = 5\%\), and the swap rate \((S(t, T_1, T_4))_{t \in [0, T_1]}\) is modeled as a driftless geometric Brownian motion with volatility coefficient \(\tilde{\sigma} = \sigma_{1,4}(t) = 0.2\) under the forward swap measure \(P_{1.4}\). The discount factors are given by 
\[P(0, T_1) = (1 + r)^{-4}, \quad P(0, T_2) = (1 + r)^{-5}, \quad P(0, T_3) = (1 + r)^{-6}, \quad P(0, T_4) = (1 + r)^{-7},\]
where \(r = 5\%\).

b) By Proposition 18.10 the price of the swaption is
\[P(0, T_1) - P(0, T_4) \Phi_+(d_+(T_1 - t)) - \kappa \Phi_-(d_-(T_1)) (P(0, T_2) + P(0, T_3) + P(0, T_4)), (A.73)\]

where \(d_+(T_1)\) and \(d_-(T_1)\) are given in Proposition 18.10, and the LIBOR swap rate \(S(0, T_1, T_4)\) is given by
\[S(0, T_1, T_4) = \frac{P(0, T_1) - P(0, T_4)}{P(0, T_1, T_4)}\]
\[ \frac{P(0, T_1) - P(0, T_4)}{P(0, T_2) + P(0, T_3) + P(0, T_4)} = \frac{(1 + r)^{-4} - (1 + r)^{-7}}{(1 + r)^{-5} + (1 + r)^{-6} + (1 + r)^{-7}} = \frac{(1 + r)^3 - 1}{(1 + r)^2 + (1 + r) + 1}. \]

Finally, we need to multiply (A.73) by the notional principal amount of $10 million, \text{i.e.} to multiply by 100,000 when (A.73) is quoted in percentage points.

**Exercise 18.3**

a) We have

\[
\begin{align*}
&\frac{d}{dt}\left( \frac{P(t, T_2)}{P(t, T_1)} \right) = \frac{dP(t, T_2)}{P(t, T_1)} - \frac{P(t, T_2)}{P(t, T_1)} \frac{dP(t, T_1)}{(P(t, T_1))^2} \\
&+ \frac{2}{P(t, T_2)} \frac{(dP(t, T_1))^2}{(P(t, T_1))^3} - \frac{dP(t, T_1) \cdot dP(t, T_2)}{(P(t, T_1))^2} \\
&= r_1 P(t, T_2) dt + \zeta_2(t) P(t, T_2) dW_t \\
&- P(t, T_2) r_1 P(t, T_1) dt + \zeta_1(t) P(t, T_1) dW_t \\
&+ P(t, T_2) (r_1 P(t, T_1) dt + \zeta_1(t) P(t, T_1) dW_t)^2 \\
&- (r_1 P(t, T_1) dt + \zeta_1(t) P(t, T_1) dW_t) \cdot (r_1 P(t, T_2) dt + \zeta_2(t) P(t, T_2) dW_t) \\
&= \zeta_2(t) \frac{P(t, T_2)}{P(t, T_1)} dW_t - \zeta_1(t) \frac{P(t, T_2)}{P(t, T_1)} dW_t \\
&+ (\zeta_1(t))^2 \frac{P(t, T_2)}{P(t, T_1)} dt - \zeta_1(t) \zeta_2(t) \frac{P(t, T_2)}{P(t, T_1)} dt \\
&= - \frac{P(t, T_2)}{P(t, T_1)} \zeta_1(t) (\zeta_2(t) - \zeta_1(t)) dt + \frac{P(t, T_2)}{P(t, T_1)} (\zeta_2(t) - \zeta_1(t)) dW_t \\
&= \frac{P(t, T_2)}{P(t, T_1)} (\zeta_2(t) - \zeta_1(t)) (dW_t - \zeta_1(t) dt) \\
&= (\zeta_2(t) - \zeta_1(t)) \frac{P(t, T_2)}{P(t, T_1)} d\hat{W}_t = (\zeta_2(t) - \zeta_1(t)) \frac{P(t, T_2)}{P(t, T_1)} d\hat{W}_t,
\end{align*}
\]

where \( d\hat{W}_t = dW_t - \zeta_1(t) dt \) is a standard Brownian motion under the \( T_1 \)-forward measure \( \hat{P} \).

b) From Question (a) or (18.7) we have

\[ 990 \]
\begin{align*}
P(T_1, T_2) &= \frac{P(T_1, T_2)}{P(T_1, T_1)} \\
&= \frac{P(t, T_2)}{P(t, T_1)} \exp \left( \int_t^{T_1} (\zeta_2(s) - \zeta_1(s)) \, d\widehat{W}_s - \frac{1}{2} \int_t^{T_1} (\zeta_2(s) - \zeta_1(s))^2 \, ds \right) \\
&= \frac{P(t, T_2)}{P(t, T_1)} \exp (X - v^2/2),
\end{align*}

where $X$ is a centered Gaussian random variable with variance $v^2 = \int_t^{T_1} (\zeta_2(s) - \zeta_1(s))^2 \, ds$, independent of $\mathcal{F}_t$ under $\hat{\mathbb{P}}$. Hence by the hint below we find

\begin{align*}
\mathbb{E}^* \left[ e^{\int_0^{T_1} r_s \, ds} (K - P(T_1, T_2))^+ \mid \mathcal{F}_t \right] &= P(t, T_1) \widehat{\mathbb{E}} \left[ (K - P(T_1, T_2))^+ \mid \mathcal{F}_t \right] \\
&= P(t, T_1) (\kappa \Phi(v/2 + (\log(\kappa/x))/v) - \frac{P(t, T_2)}{P(t, T_1)} \Phi(-v/2 + (\log(\kappa/x))/v)) \\
&= P(t, T_1) \kappa \Phi(v/2 + (\log(\kappa/x))/v) - P(t, T_2) \Phi(-v/2 + (\log(\kappa/x))/v),
\end{align*}

with $x = P(t, T_2)/P(t, T_1)$.

Exercise 18.4

a) The forward measure $\hat{\mathbb{P}}_S$ is defined from the numéraire $N_t := P(t, S)$ and this gives

\[ F_t = P(t, S) \widehat{\mathbb{E}}[(\kappa - L(T, T, S))^+ \mid \mathcal{F}_t]. \]

b) The LIBOR rate $L(t, T, S)$ is a driftless geometric Brownian motion with volatility $\sigma$ under the forward measure $\hat{\mathbb{P}}_S$. Indeed, the LIBOR rate $L(t, T, S)$ can be written as the forward price $L(t, T, S) = \hat{X}_t = X_t / N_t$ where $X_t = (P(t, T) - P(t, S)) / (S - T)$ and $N_t = P(t, S)$. Since both discounted bond prices $e^{-\int_0^t r_s \, ds} P(t, T)$ and $e^{-\int_0^t r_s \, ds} P(t, S)$ are martingales under $\mathbb{P}^*$, the same is true of $X_t$. Hence $L(t, T, S) = X_t / N_t$ becomes a martingale under the forward measure $\hat{\mathbb{P}}_S$ by Proposition 2.1, and computing its dynamics under $\hat{\mathbb{P}}_S$ amounts to removing any “$dt$” term in (18.31), i.e.

\[ dL(t, T, S) = \sigma L(t, T, S) d\widehat{W}_t, \quad 0 \leq t \leq T, \]

hence $L(t, T, S) = L(0, T, S) e^{\sigma \widehat{W}_t - \sigma^2 t/2}$, where $(\widehat{W}_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion under $\hat{\mathbb{P}}_S$.

c) We find

\begin{align*}
F_t &= P(t, S) \widehat{\mathbb{E}}[(\kappa - L(T, T, S))^+ \mid \mathcal{F}_t] \\
&= P(t, S) \widehat{\mathbb{E}}[(\kappa - L(t, T, S) e^{-(T-t)\sigma^2/2 + (\widehat{W}_T - \widehat{W}_t)\sigma})^+ \mid \mathcal{F}_t] \\
&= P(t, S) (\kappa \Phi(-d_-(T-t)) - \hat{X}_t \Phi(-d_+(T-t))).
\end{align*}
Exercise 18.5 The swaption can be priced as

\[ e^m = L(t, T, S) e^{-(T-t)\sigma^2/2}, \quad v^2 = (T-t)\sigma^2, \]

where \( e^m = L(t, T, S) e^{-(T-t)\sigma^2/2}, \) \( v^2 = (T-t)\sigma^2, \) and

\[
d_+(T-t) = \frac{\log(L(t, T, S)/\kappa)}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2},
\]

and

\[
d_-(T-t) = \frac{\log(L(t, T, S)/\kappa)}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2},
\]

because \( L(t, T, S) \) is a driftless geometric Brownian motion with volatility \( \sigma \) under the forward measure \( \hat{P}_S. \)

Exercise 18.5 The swaption can be priced as

\[
\mathbb{E}^* \left[ e^{-\int_t^T r_s ds} (P(T_i, T_j) - P(T_i, T_j) - \kappa P(T_i, T_j)) | \mathcal{F}_t \right]
\]

\[
= \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} \left( 1 - \kappa \sum_{k=i}^{j-1} c_{k+1} P(T_i, T_{k+1}) \right) | \mathcal{F}_t \right]
\]

\[
= \kappa \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} \left( \sum_{k=i}^{j-1} c_{k+1} (F_{k+1}(T_i, \gamma_k) - F_{k+1}(T_i, r_{T_i})) | \mathcal{F}_t \right]\right]
\]

\[
= \kappa \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} \left( \sum_{k=i}^{j-1} c_{k+1} \mathbbm{1}_{\{r_{T_i} \leq \gamma_k\}} (F_{k+1}(T_i, \gamma_k) - F_{k+1}(T_i, r_{T_i})) | \mathcal{F}_t \right]\right]
\]

\[
= \kappa \sum_{k=i}^{j-1} c_{k+1} \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} \left( F_{k+1}(T_i, \gamma_k) - P(T_i, T_{k+1}) \right) | \mathcal{F}_t \right]
\]

\[
= \kappa \sum_{k=i}^{j-1} c_{k+1} P(t, T_i) \mathbb{E} [ (F_{k+1}(T_i, \gamma_k) - P(T_i, T_{k+1})) | \mathcal{F}_t ],
\]

which is a weighted sum of bond put option prices with strike prices \( F_{k+1}(T_i, \gamma_k), \) \( k = i, i+1, \ldots, j-1. \)
Exercise 18.6

a) We have

\[ dX_t = \sum_{i=2}^{n} c_i d\hat{P}(t, T_i) = \sum_{i=2}^{n} c_i \sigma(t) \hat{P}(t, T_i) d\hat{W}_t = \sigma_t X_t d\hat{W}_t, \]

from which we obtain

\[ \sigma_t = \frac{1}{X_t} \sum_{i=2}^{n} c_i \sigma_i(t) \hat{P}(t, T_i) = \frac{\sum_{i=2}^{n} c_i \sigma_i(t) \hat{P}(t, T_i)}{\sum_{i=2}^{n} c_i \hat{P}(t, T_i)} \cdot \sum_{i=2}^{n} c_i \sigma_i(t) P(t, T_i) \]

b) Approximating the random process \( \sigma_t \) by the deterministic function of time

\[ \hat{\sigma}(t) := \frac{1}{X_0} \sum_{i=2}^{n} c_i \sigma_i(t) \hat{P}(0, T_i) = \frac{\sum_{i=2}^{n} c_i \sigma_i(t) \hat{P}(0, T_i)}{\sum_{i=2}^{n} c_i \hat{P}(0, T_i)} = \frac{\sum_{i=2}^{n} c_i \sigma_i(t) P(0, T_i)}{\sum_{i=2}^{n} c_i P(0, T_i)} \]

we find

\[ P(0, T_1) \mathbb{E}[ (X_{T_0} - \kappa)^+] \]

\[ \simeq P(0, T_1) \mathbb{E} \left[ \left( X_0 e^{\int_0^{T_0} |\hat{\sigma}(t)|^2 dt - \frac{1}{2} \int_0^{T_0} |\hat{\sigma}(t)|^2 dt} - \kappa \right)^+ \right] \]

\[ = P(0, T_1) \text{Blput} \left( \kappa, X_0, \sqrt{\frac{1}{T_0} \int_0^{T_0} |\hat{\sigma}(t)|^2 dt}, 0, T \right) \]

\[ = P(0, T_1) \left( X_0 \Phi(v/2 + (\log(\kappa/X_0))/v) - \kappa \Phi(-v/2 + (\log(\kappa/X_0))/v) \right) \]

\[ = \sum_{i=2}^{n} c_i P(0, T_i) \Phi \left( \frac{v}{2} - \frac{1}{v} \log \left( \sum_{i=2}^{n} \frac{c_i P(0, T_i)}{\kappa P(0, T_1)} \right) \right) \]

\[ - \kappa P(0, T_1) \Phi \left( -\frac{v}{2} - \frac{1}{v} \log \left( \sum_{i=2}^{n} \frac{c_i P(0, T_i)}{\kappa P(0, T_1)} \right) \right), \]

with \( v := \sqrt{\int_{0}^{T_0} |\hat{\sigma}(t)|^2 dt} \).

Exercise 18.7

a) We have

\[ \text{Blput} \left( \kappa, X_0, \sqrt{\frac{1}{T_0} \int_0^{T_0} |\hat{\sigma}(t)|^2 dt}, 0, T \right) \]

\[ = P(0, T_1) \left( X_0 \Phi(v/2 + (\log(\kappa/X_0))/v) - \kappa \Phi(-v/2 + (\log(\kappa/X_0))/v) \right) \]

\[ = \sum_{i=2}^{n} c_i P(0, T_i) \Phi \left( \frac{v}{2} - \frac{1}{v} \log \left( \sum_{i=2}^{n} \frac{c_i P(0, T_i)}{\kappa P(0, T_1)} \right) \right) \]

\[ - \kappa P(0, T_1) \Phi \left( -\frac{v}{2} - \frac{1}{v} \log \left( \sum_{i=2}^{n} \frac{c_i P(0, T_i)}{\kappa P(0, T_1)} \right) \right), \]

with \( v := \sqrt{\int_{0}^{T_0} |\hat{\sigma}(t)|^2 dt} \).
\[
\frac{dP(t, T_i)}{P(t, T_i)} = r_t dt + \zeta^i(t) dB_t, \quad i = 1, 2,
\]
and
\[
P(T, T_i) = P(t, T_i) \exp \left( \int_t^T r_s ds + \int_t^T \zeta^i(s) dB_s - \frac{1}{2} \int_t^T |\zeta^i(s)|^2 ds \right),
\]
\(0 \leq t \leq T \leq T_i, i = 1, 2,\) hence
\[
\log P(T, T_i) = \log P(t, T_i) + \int_t^T r_s ds + \int_t^T \zeta^i(s) dB_s - \frac{1}{2} \int_t^T |\zeta^i(s)|^2 ds,
\]
\(0 \leq t \leq T \leq T_i, i = 1, 2,\) and
\[
d \log P(t, T_i) = r_t dt + \zeta^i(t) dB_t - \frac{1}{2} |\zeta^i(t)|^2 dt, \quad i = 1, 2.
\]
In the present model, we have
\[
d r_t = \sigma dB_t,
\]
where \((B_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion under \(\mathbb{P}\), by the solution of Exercise 16.3 and (16.22) we have
\[
\zeta^i(t) = -(T_i - t) \sigma, \quad 0 \leq t \leq T_i, \quad i = 1, 2.
\]
Letting
\[
d B^{(i)}_t = dB_t - \zeta^i(t) dt,
\]
defines a standard Brownian motion under \(\mathbb{P}_i, i = 1, 2\), and we have
\[
\frac{P(T, T_1)}{P(T, T_2)} = \frac{P(t, T_1)}{P(t, T_2)} \exp \left( \int_t^T (\zeta^1(s) - \zeta^2(s)) dB_s - \frac{1}{2} \int_t^T (|\zeta^1(s)|^2 - |\zeta^2(s)|^2) ds \right)
\]
\[
= \frac{P(t, T_1)}{P(t, T_2)} \exp \left( \int_t^T (\zeta^1(s) - \zeta^2(s)) dB_s^{(2)} - \frac{1}{2} \int_t^T (\zeta^1(s) - \zeta^2(s))^2 ds \right),
\]
which is an \(\mathcal{F}_t\)-martingale under \(\mathbb{P}_2\) and under \(\mathbb{P}_{1,2}\), and
\[
\frac{P(T, T_2)}{P(T, T_1)} = \frac{P(t, T_2)}{P(t, T_1)} \exp \left( - \int_t^T (\zeta^1(s) - \zeta^2(s)) dB_s^{(1)} - \frac{1}{2} \int_t^T (\zeta^1(s) - \zeta^2(s))^2 ds \right),
\]
which is an \(\mathcal{F}_t\)-martingale under \(\mathbb{P}_1\).
\[\text{b) We have}\]
\[
f(t, T_1, T_2) = - \frac{1}{T_2 - T_1} \left( \log P(t, T_2) - \log P(t, T_1) \right)
\]
\[
= r_t + \frac{1}{T_2 - T_1} \frac{\sigma^2}{6} ((T_1 - t)^3 - (T_2 - t)^3).
\]
c) We have

\[
df(t, T_1, T_2) = -\frac{1}{T_2 - T_1} d \log \frac{P(t, T_2)}{P(t, T_1)}
\]

\[
= -\frac{1}{T_2 - T_1} \left( (\zeta^2(t) - \zeta^1(t)) dB_t - \frac{1}{2} (|\zeta^2(t)|^2 - |\zeta^1(t)|^2) dt \right)
\]

\[
= -\frac{1}{T_2 - T_1} \left( (\zeta^2(t) - \zeta^1(t)) dB_t^{(2)} + \zeta^2(t) dt - \frac{1}{2} (|\zeta^2(t)|^2 - |\zeta^1(t)|^2) dt \right)
\]

\[
= -\frac{1}{T_2 - T_1} \left( (\zeta^2(t) - \zeta^1(t)) dB_t^{(2)} - \frac{1}{2} (\zeta^2(t) - \zeta^1(t))^2 dt \right).
\]

d) We have

\[
f(T, T_1, T_2) = -\frac{1}{T_2 - T_1} \log \frac{P(T, T_2)}{P(T, T_1)}
\]

\[
= f(t, T_1, T_2) - \frac{1}{T_2 - T_1} \left( \int_t^T (\zeta^2(s) - \zeta^1(s)) dB_s - \frac{1}{2} (|\zeta^2(s)|^2 - |\zeta^1(s)|^2) ds \right)
\]

\[
= f(t, T_1, T_2) - \frac{1}{T_2 - T_1} \left( \int_t^T (\zeta^2(s) - \zeta^1(s)) dB_s^{(2)} - \frac{1}{2} \int_t^T (\zeta^2(s) - \zeta^1(s))^2 ds \right)
\]

\[
= f(t, T_1, T_2) - \frac{1}{T_2 - T_1} \left( \int_t^T (\zeta^2(s) - \zeta^1(s)) dB_s^{(1)} + \frac{1}{2} \int_t^T (\zeta^2(s) - \zeta^1(s))^2 ds \right).
\]

Hence \( f(T, T_1, T_2) \) has a Gaussian distribution given \( \mathcal{F}_t \) with conditional mean

\[
m_1 := f(t, T_1, T_2) - \frac{1}{2} \int_t^T (\zeta^2(s) - \zeta^1(s))^2 ds
\]

under \( \mathbb{P}_1 \), resp.

\[
m_2 := f(t, T_1, T_2) + \frac{1}{2} \int_t^T (\zeta^2(s) - \zeta^1(s))^2 ds
\]

under \( \mathbb{P}_2 \), and variance

\[
v^2 = \frac{1}{(T_2 - T_1)^2} \int_t^T (\zeta^2(s) - \zeta^1(s))^2 ds.
\]

Hence

\[
(T_2 - T_1) \mathbb{E}^* \left[ e^{-\int_t^{T_2} r_s ds} (f(T_1, T_1, T_2) - \kappa)^+ | \mathcal{F}_t \right]
\]

\[
= (T_2 - T_1) P(t, T_2) \mathbb{E}_2 \left[ (f(T_1, T_1, T_2) - \kappa)^+ | \mathcal{F}_t \right]
\]

\[
= (T_2 - T_1) P(t, T_2) \mathbb{E}_2 \left[ (m_2 + X - \kappa)^+ | \mathcal{F}_t \right]
\]

\[
= (T_2 - T_1) P(t, T_2) \left( \frac{v}{\sqrt{2\pi}} e^{-(\kappa - m_2)^2/(2v^2)} + (m_2 - \kappa) \Phi((m_2 - \kappa)/v) \right),
\]
see Exercise A.4.

e) We have

\[ L(T, T_1, T_2) = S(T, T_1, T_2) \]

\[ = \frac{1}{T_2 - T_1} \left( \frac{P(T, T_1)}{P(T, T_2)} - 1 \right) \]

\[ = \frac{1}{T_2 - T_1} \left( \frac{P(t, T_1)}{P(t, T_2)} \exp \left( \int_t^T (\zeta^1(s) - \zeta^2(s)) dB_s - \frac{1}{2} \int_t^T (|\zeta^1(s)|^2 - |\zeta^2(s)|^2) ds \right) - 1 \right) \]

\[ = \frac{1}{T_2 - T_1} \left( \frac{P(t, T_1)}{P(t, T_2)} \exp \left( \int_t^T (\zeta^1(s) - \zeta^2(s)) dB_s^{(2)} - \frac{1}{2} \int_t^T (\zeta^1(s) - \zeta^2(s))^2 ds \right) - 1 \right) \]

\[ = \frac{1}{T_2 - T_1} \left( \frac{P(t, T_1)}{P(t, T_2)} \exp \left( \int_t^T (\zeta^1(s) - \zeta^2(s)) dB_s^{(1)} + \frac{1}{2} \int_t^T (\zeta^1(s) - \zeta^2(s))^2 ds \right) - 1 \right), \]

and, by Itô calculus,

\[ dS(t, T_1, T_2) = \frac{1}{T_2 - T_1} d \left( \frac{P(t, T_1)}{P(t, T_2)} \right) \]

\[ = \frac{1}{T_2 - T_1} \frac{P(t, T_1)}{P(t, T_2)} \left( (\zeta^1(t) - \zeta^2(t)) dB_t + \frac{1}{2} (\zeta^1(t) - \zeta^2(t))^2 dt \right. \]

\[ \left. - \frac{1}{2} (|\zeta^1(t)|^2 - |\zeta^2(t)|^2) dt \right) \]

\[ = \left( \frac{1}{T_2 - T_1} + S(t, T_1, T_2) \right) \left( (\zeta^1(t) - \zeta^2(t)) dB_t + \zeta^2(t)(\zeta^2(t) - \zeta^1(t)) dt \right) \]

\[ = \left( \frac{1}{T_2 - T_1} + S(t, T_1, T_2) \right) \left( (\zeta^1(t) - \zeta^2(t)) dB_t^{(1)} + (|\zeta^2(t)|^2 - |\zeta^1(t)|^2) dt \right) \]

\[ = \left( \frac{1}{T_2 - T_1} + S(t, T_1, T_2) \right) (\zeta^1(t) - \zeta^2(t)) dB_t^{(2)}, \quad t \in [0, T_1], \]

hence \( t \mapsto \frac{1}{T_2 - T_1} + S(t, T_1, T_2) \) is a geometric Brownian motion, with

\[ \frac{1}{T_2 - T_1} + S(T, T_1, T_2) \]

\[ = \left( \frac{1}{T_2 - T_1} + S(t, T_1, T_2) \right) \exp \left( \int_t^T (\zeta^1(s) - \zeta^2(s)) dB_s^{(2)} - \frac{1}{2} \int_t^T (\zeta^1(s) - \zeta^2(s))^2 ds \right), \]

\[ 0 \leq t \leq T \leq T_1. \]

f) We have

\[ (T_2 - T_1) \mathbb{E}^* \left[ e^{-\int_{t}^{T_2} r_s ds} (L(T_1, T_1, T_2) - \kappa)^+ | \mathcal{F}_t \right] \]

\[ = (T_2 - T_1) \mathbb{E}^* \left[ e^{-\int_{t}^{T_1} r_s ds} P(T_1, T_2) (L(T_1, T_1, T_2) - \kappa)^+ | \mathcal{F}_t \right] \]
We use change of numéraire under the forward measure. 

Exercise 18.8

(a) We have

\[ L(t, T_1, T_2) = S(t, T_1, T_2) e^{\int_t^T r_s ds}, \quad 0 \leq t \leq T_2, \]

and the forward swap measure is defined by

\[ \mathbb{E}^* \left[ \frac{d\mathbb{P}_{1,2}}{d\mathbb{P}} \bigg| \mathcal{F}_t \right] = \frac{P(t, T_2)}{P(0, T_2)} e^{\int_0^t r_s ds}, \quad 0 \leq t \leq T_1, \]

hence \( \mathbb{P}_2 \) and \( \mathbb{P}_{1,2} \) coincide up to time \( T_1 \) and \( (B_t^{(2)})_{t \in [0, T_1]} \) is a standard Brownian motion until time \( T_1 \) under \( \mathbb{P}_2 \) and under \( \mathbb{P}_{1,2} \), consequently under \( \mathbb{P}_{1,2} \) we have

\[ L(T, T_1, T_2) = S(T, T_1, T_2) \]

\[ = -\frac{1}{T_2 - T_1} + \left( \frac{1}{T_2 - T_1} + S(t, T_1, T_2) \right) e^{\int_t^T (\zeta^1(s) - \zeta^2(s)) dB^{(2)}_s - \frac{1}{2} \int_t^T (\zeta^1(s) - \zeta^2(s))^2 ds}, \]

has same distribution as

\[ \frac{1}{T_2 - T_1} \left( \frac{P(t, T_1)}{P(t, T_2)} e^{X - \frac{1}{2} \text{Var}[X]} - 1 \right), \]

where \( X \) is a centered Gaussian random variable with variance

\[ \int_t^{T_1} (\zeta^1(s) - \zeta^2(s))^2 ds \]

given \( \mathcal{F}_t \). Hence

\[ (T_2 - T_1) \mathbb{E}^* \left[ e^{-\int_t^{T_2} r_s ds} (L(T_1, T_1, T_2) - \kappa)^+ \bigg| \mathcal{F}_t \right] \]

\[ = P(t, T_1, T_2) \times \text{Bl} \left( \frac{1}{T_2 - T_1} + S(t, T_1, T_2), \frac{\int_t^{T_1} (\zeta^1(s) - \zeta^2(s))^2 ds}{T_1 - t}, \kappa + \frac{1}{T_2 - T_1}, T_1 - t \right). \]

Exercise 18.8

(a) We have

\[ L(t, T_1, T_2) = L(0, T_1, T_2) e^{\int_0^t \gamma_1(s) dW^{(2)}_s - \frac{1}{2} \int_0^t |\gamma_1(s)|^2 ds}, \quad 0 \leq t \leq T_1, \]

and \( L(t, T_2, T_3) = b \). Note that we have \( P(t, T_2) / P(t, T_3) = 1 + \delta b \) hence \( \mathbb{P}_2 = \mathbb{P}_3 = \mathbb{P}_{1,2} \) up to time \( T_1 \).

(b) We use change of numéraire under the forward measure \( \mathbb{P}_2 \).
c) We have

\[
\mathbb{E}^* \left[ e^{-\int_{t}^{T_2} r_s ds} (L(T_1, T_1, T_2) - \kappa)^+ \bigg| \mathcal{F}_t \right] = P(t, T_2) \mathbb{E}_2 \left[ (L(T_1, T_1, T_2) - \kappa)^+ \bigg| \mathcal{F}_t \right] = P(t, T_2) \mathbb{E}_2 \left[ (L(t, T_1, T_2) e^{\int_{t}^{T_1} \gamma_1(s) dW_s^{(2)} - \frac{1}{2} \int_{t}^{T_1} |\gamma_1(s)|^2 ds} - \kappa)^+ \bigg| \mathcal{F}_t \right]
\]

\[
= P(t, T_2) \mathbb{B}_1(\kappa, L(t, T_1, T_2), \sigma_1(t), 0, T_1 - t),
\]

where

\[
\sigma_1^2(t) = \frac{1}{T_1 - t} \int_{t}^{T_1} |\gamma_1(s)|^2 ds.
\]

d) We have

\[
\frac{P(t, T_1)}{P(t, T_1, T_3)} = \frac{P(t, T_1)}{P(t, T_2) + \delta P(t, T_3)} = \frac{P(t, T_1)}{\delta P(t, T_2, 1 + P(t, T_3) / P(t, T_2)) = \frac{1 + \delta b}{\delta (2 + \delta b)} (1 + \delta L(t, T_1, T_2)), \quad 0 \leq t \leq T_1,
\]

and

\[
\frac{P(t, T_3)}{P(t, T_1, T_3)} = \frac{P(t, T_3)}{P(t, T_2) + P(t, T_3)} = \frac{1}{1 + P(t, T_2) / P(t, T_3)} = \frac{1}{\delta} \frac{1}{2 + \delta b}, \quad 0 \leq t \leq T_2.
\]

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e) We have

\[
S(t, T_1, T_3) = \frac{P(t, T_1)}{P(t, T_1, T_3)} - \frac{P(t, T_3)}{P(t, T_1, T_3)} = \frac{1 + \delta b}{\delta (2 + \delta b)} (1 + \delta L(t, T_1, T_2)) - \frac{1}{\delta (2 + \delta b)}
\]

\[
= \frac{1}{2 + \delta b} (b + (1 + \delta b) L(t, T_1, T_2)), \quad 0 \leq t \leq T_2.
\]

We have

\[
dS(t, T_1, T_3) = \frac{1 + \delta b}{2 + \delta b} L(t, T_1, T_2) \gamma_1(t) dW_t^{(2)} = \left( S(t, T_1, T_3) - \frac{b}{2 + \delta b} \right) \gamma_1(t) dW_t^{(2)}
\]

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\[ = S(t, T_1, T_3)\sigma_{1,3}(t)dW_t^{(2)}, \quad 0 \leq t \leq T_2, \]

with

\[
\sigma_{1,3}(t) = \left( 1 - \frac{b}{S(t, T_1, T_3)(2 + \delta b)} \right) \gamma_1(t)
\]

\[
= \left( 1 - \frac{b}{b + (1 + \delta b)L(t, T_1, T_2)} \right) \gamma_1(t)
\]

\[
= \frac{(1 + \delta b)L(t, T_1, T_2)}{b + (1 + \delta b)L(t, T_1, T_2)} \gamma_1(t)
\]

\[
= \frac{(1 + \delta b)L(t, T_1, T_2)}{(2 + \delta b)S(t, T_1, T_3)} \gamma_1(t).
\]

f) The process \((W^{(2)})_{t \in \mathbb{R}^+}\) is a standard Brownian motion under \(\hat{P}_2 = \hat{P}_{1,3}\)
and

\[
P(t, T_1, T_3) \hat{E}_{1,3} \left[ (S(T_1, T_1, T_3) - \kappa)^+ | \mathcal{F}_t \right] = P(t, T_2)B(I, \kappa, S(t, T_1, T_2), \tilde{\sigma}_{1,3}(t), 0, T_1 - t),
\]

where \(\tilde{\sigma}_{1,3}(t)^2\) is the approximation of the volatility

\[
\frac{1}{T_1 - t} \int_t^{T_1} |\sigma_{1,3}(s)|^2 ds = \frac{1}{T_1 - t} \int_t^{T_1} \left( \frac{(1 + \delta b)L(s, T_1, T_2)}{(2 + \delta b)S(s, T_1, T_3)} \right)^2 \gamma_1(s) ds
\]

obtained by freezing the random component of \(\sigma_{1,3}(s)\) at time \(t\), i.e.

\[
\tilde{\sigma}_{1,3}^2(t) = \frac{1}{T_1 - t} \left( \frac{(1 + \delta b)L(t, T_1, T_2)}{(2 + \delta b)S(t, T_1, T_3)} \right)^2 \int_t^{T_1} |\gamma_1(s)|^2 ds.
\]

**Exercise 18.9** Bond option hedging.

a) We have

\[
\mathbb{E}^* \left[ e^{-\int_t^T r_s ds} (P(T, S) - \kappa)^+ | \mathcal{F}_t \right] = V_T = V_0 + \int_0^T dV_t
\]

\[
= P(0, T) \mathbb{E}_T \left[ (P(T, S) - \kappa)^+ \right] + \int_0^T \xi^T_s dP(s, T) + \int_0^T \xi_s S dP(s, S).
\]

b) We have

\[
\tilde{V}_t = d \left( e^{\int_0^t r_s ds} V_t \right)
\]

\[
= -r_t e^{\int_0^t r_s ds} V_t dt + e^{\int_0^t r_s ds} dV_t
\]

\[
= -r_t e^{\int_0^t r_s ds} (\xi^T_t P(t, T))
\]
\[ + \xi_t^S P(t, S) dt + e^{-\int_0^t \sigma_S^T d\xi_t^T} dP(t, T) + e^{-\int_0^t \sigma_S^T \xi_t^T dP(t, S)} = \xi_t^T d\tilde{P}(t, T) + \xi_t^S d\tilde{P}(t, S). \]

c) By Itô’s formula we have
\[
\mathbb{E}_T [(P(T, S) - \kappa)^+ | \mathcal{F}_t] = C(X_t, v(t, T)) \\
= C(X_0, v(0, T)) + \int_0^t \frac{\partial C}{\partial x}(X_s, v(s, T)) dX_s \\
= \mathbb{E}_T [(P(T, S) - \kappa)^+] + \int_0^t \frac{\partial C}{\partial x}(X_s, v(s, T)) dX_s,
\]
since the process
\[
t \mapsto \mathbb{E}_T [(P(T, S) - \kappa)^+ | \mathcal{F}_t]
\]
is a martingale under \( \tilde{\mathbb{P}} \).

d) We have
\[
d\tilde{V}_t = d\left( \frac{V_t}{P(t, T)} \right) \\
= d\mathbb{E}T [(P(T, S) - \kappa)^+ | \mathcal{F}_t] \\
= \frac{\partial C}{\partial x}(X_t, v(t, T)) dX_t \\
= \frac{P(t, S)}{P(t, T)} \frac{\partial C}{\partial x}(X_t, v(t, T))(\sigma_t^S - \sigma_t^T) dB_t^T.
\]
e) We have
\[
dV_t = d\left( \frac{P(t, T)}{V_t} \right) \\
= P(t, T) d\tilde{V}_t + \tilde{V}_t dP(t, T) + d\tilde{V}_t \cdot dP(t, T) \\
= P(t, S) \frac{\partial C}{\partial x}(X_t, v(t, T))(\sigma_t^S - \sigma_t^T) dB_t^T + \tilde{V}_t dP(t, T) \\
+ P(t, S) \frac{\partial C}{\partial x}(X_t, v(t, T))(\sigma_t^S - \sigma_t^T) \sigma_t^T dt \\
= P(t, S) \frac{\partial C}{\partial x}(X_t, v(t, T))(\sigma_t^S - \sigma_t^T) dB_t + \tilde{V}_t dP(t, T).
\]
f) We have
\[
d\tilde{V}_t = d(e^{-\int_0^t \sigma_S^T \xi_t^T dP(t, S)}) \\
= -r_t e^{-\int_0^t \sigma_S^T \xi_t^T dP(t, S)} dt + e^{-\int_0^t \sigma_S^T \xi_t^T dP(t, S)} d\tilde{V}_t \\
= \tilde{P}(t, S) \frac{\partial C}{\partial x}(X_t, v(t, T))(\sigma_t^S - \sigma_t^T) dB_t + \tilde{V}_t d\tilde{P}(t, T).
\]

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g) We have

\[ d\tilde{V}_t = \tilde{P}(t,S) \frac{\partial C}{\partial x}(X_t,v(t,T))(\sigma^S_t - \sigma^T_t)dB_t + \tilde{V}_t d\tilde{P}(t,T) \]

\[ = \frac{\partial C}{\partial x}(X_t,v(t,T))d\tilde{P}(t,S) \]

\[ - P(t,S) \frac{\partial C}{P(t,T)} \frac{\partial C}{\partial x}(X_t,v(t,T))d\tilde{P}(t,T) + \tilde{V}_t d\tilde{P}(t,T) \]

\[ = \left( \tilde{V}_t - \frac{P(t,S)}{P(t,T)} \frac{\partial C}{\partial x}(X_t,v(t,T)) \right) d\tilde{P}(t,T) \]

\[ + \frac{\partial C}{\partial x}(X_t,v(t,T))d\tilde{P}(t,S), \]

hence the hedging strategy \((\xi^T_t, \xi^S_t)_{t \in [0,T]}\) of the bond option is given by

\[ \xi^T_t = \tilde{V}_t - \frac{P(t,S)}{P(t,T)} \frac{\partial C}{\partial x}(X_t,v(t,T)) \]

\[ = C(X_t,v(t,T)) - \frac{P(t,S)}{P(t,T)} \frac{\partial C}{\partial x}(X_t,v(t,T)), \]

and

\[ \xi^S_t = \frac{\partial C}{\partial x}(X_t,v(t,T)); \quad 0 \leq t \leq T. \]

h) We have

\[ \frac{\partial C}{\partial x}(x,v) = \frac{\partial}{\partial x} \left( x\Phi \left( \frac{v}{2} + \frac{1}{v} \log \frac{x}{\kappa} \right) - \kappa \Phi \left( -\frac{v}{2} + \frac{1}{v} \log \frac{x}{\kappa} \right) \right) \]

\[ = x \frac{\partial}{\partial x} \Phi \left( \frac{v}{2} + \frac{1}{v} \log \frac{x}{\kappa} \right) - \kappa \frac{\partial}{\partial x} \Phi \left( -\frac{v}{2} + \frac{1}{v} \log \frac{x}{\kappa} \right) + \Phi \left( \frac{v}{2} + \frac{1}{v} \log \frac{x}{\kappa} \right) \]

\[ = x \frac{e^{-\frac{1}{2} \left( \frac{v}{2} + \frac{1}{v} \log \frac{x}{\kappa} \right)^2}}{\sqrt{2\pi v x}} + \kappa \frac{e^{-\frac{1}{2} \left( -\frac{v}{2} + \frac{1}{v} \log \frac{x}{\kappa} \right)^2}}{\sqrt{2\pi v x}} + \Phi \left( \frac{v}{2} + \frac{1}{v} \log \frac{x}{\kappa} \right) \]

\[ = \Phi \left( \log \left( \frac{x}{\kappa} \right) + \frac{v^2}{2} / v \right). \]

As a consequence, we get

\[ \xi^T_t = C(X_t,v(t,T)) - \frac{P(t,S)}{P(t,T)} \frac{\partial C}{\partial x}(X_t,v(t,T)) \]

\[ = \frac{P(t,S)}{P(t,T)} \Phi \left( \frac{v^2(t,T) / 2 + \log X_t}{v(t,T)} \right) \]

\[ - \kappa \Phi \left( -\frac{v(t,T)}{2} + \frac{1}{v(t,T)} \log \frac{P(t,S)}{\kappa P(t,T)} \right). \]
Exercise 18.10

a) The LIBOR rate $L(t, T, S)$ is a driftless geometric Brownian motion with deterministic volatility function $\sigma(t)$ under the forward measure $\hat{\mathbb{P}}_S$.

Explanation: The LIBOR rate $L(t, T, S)$ can be written as the forward price $L(t, T, S) = \hat{X}_t = X_t/N_t$ where $X_t = (P(t, T) - P(t, S))/(S - T)$ and $N_t = P(t, S)$. Since both discounted bond prices $e^{-\int_0^t r_s ds} P(t, T)$ and $e^{-\int_0^t r_s ds} P(t, S)$ are martingales under $\mathbb{P}^*$, the same is true of $X_t$. Hence $L(t, T, S) = X_t/N_t$ becomes a martingale under the forward measure $\hat{\mathbb{P}}_S$ by Proposition 2.1, and computing its dynamics under $\hat{\mathbb{P}}_S$ amounts to removing any “$dt$” term in the original SDE defining $L(t, T, S)$, i.e. we find

$$dL(t, T, S) = \sigma(t)L(t, T, S)d\hat{W}_t,$$

$0 \leq t \leq T$. 

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b) Choosing the annuity numéraire

\[ L(t, T, S) = L(0, T, S) \exp \left( \int_0^t \sigma(s) d\hat{W}_s - \int_0^t \sigma^2(s) ds / 2 \right), \]

where \((\hat{W}_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion under \(\hat{P}_S\).

b) Choosing the annuity numéraire \(N_t = P(t, S)\), we have

\[
\mathbb{E}^* \left[ e^{-\int_t^T r_s ds} \phi(L(T, T, S)) \big| \mathcal{F}_t \right] = \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} N_S \phi(L(T, T, S)) \big| \mathcal{F}_t \right] \\
= N_t \mathbb{E} \left[ \phi(L(T, T, S)) \big| \mathcal{F}_t \right] \\
= P(t, S) \mathbb{E} \left[ \phi(L(T, T, S)) \big| \mathcal{F}_t \right].
\]

c) Given the solution

\[
L(T, T, S) = L(0, T, S) \exp \left( \int_0^T \sigma(s) d\hat{W}_s - \int_0^T \sigma^2(s) ds / 2 \right) \\
= L(t, T, S) \exp \left( \int_t^T \sigma(s) d\hat{W}_s - \int_t^T \sigma^2(s) ds / 2 \right),
\]

we find

\[
P(t, S) \mathbb{E} \left[ \phi(L(T, T, S)) \big| \mathcal{F}_t \right] \\
= P(t, S) \mathbb{E} \left[ \phi \left( L(t, T, S) e^{\int_t^T \sigma(s) d\hat{W}_s - \int_t^T \sigma^2(s) ds / 2} \right) \big| \mathcal{F}_t \right] \\
= P(t, S) \int_{-\infty}^{\infty} \phi \left( L(t, T, S) e^{x - \eta^2 / 2} \right) e^{-x^2 / (2\eta^2)} \frac{dx}{\sqrt{2\pi\eta^2}},
\]

because \(\int_t^T \sigma(s) d\hat{W}_s\) is a centered Gaussian variable with variance \(\eta^2 := \int_t^T \sigma^2(s) ds\), independent of \(\mathcal{F}_t\) under the forward measure \(\hat{P}\).

Exercise 18.11

a) Choosing the annuity numéraire \(N_t = P(t, T_i, T_j)\), we have

\[
\mathbb{E}^* \left[ e^{-\int_t^{T_1} \mathbb{R}_+ r_s ds} P(T_i, T_i, T_j) \phi(S(T_i, T_i, T_j)) \big| \mathcal{F}_t \right] \\
= N_t \mathbb{E}_{i,j} \left[ \frac{P(T_i, T_i, T_j)}{N_{T_i}} \phi(S(T_i, T_i, T_j)) \big| \mathcal{F}_t \right] \\
= P(t, T_i, T_j) \mathbb{E}_{i,j} [\phi(S(T_i, T_i, T_j)) \big| \mathcal{F}_t].
\]

b) Since \(S(t, T_i, T_j)\) is a forward price for the numéraire \(P(t, T_i, T_j)\), it is a martingale under the forward swap measure \(\hat{P}_{i,j}\) and we have
\[ S(t, T_i, T_j) = \sigma S(t, T_i, T_j) d\tilde{W}^{i,j}_t, \quad 0 \leq t \leq T_i, \]

where \((\tilde{W}^{i,j}_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion under the forward swap measure \(\tilde{\mathbb{P}}_{i,j}\).

c) Given the solution
\[ S(T_i, T_i, T_j) = S(0, T_i, T_j) e^{\tilde{W}_{T_i} - \sigma^2 T_i/2} = S(t, T_i, T_j) e^{(\tilde{W}_{T_i} - \tilde{W}_t)\sigma - (T_i - t)\sigma^2/2} \]
of (18.35), we find
\[
P(t, T_i, T_j) \tilde{E}_{i,j} [\phi(S(T_i, T_i, T_j)) | \mathcal{F}_t] \\
= P(t, T_i, T_j) \tilde{E}_{i,j} \left[ \phi \left( S(t, T_i, T_j) e^{(\tilde{W}_{T_i} - \tilde{W}_t)\sigma - (T_i - t)\sigma^2/2} \right) | \mathcal{F}_t \right] \\
= P(t, T_i, T_j) \int_{-\infty}^{\infty} \phi \left( S(t, T_i, T_j) e^{\sigma x - (T_i - t)\sigma^2/2} \right) e^{-x^2/(2(T_i - t))} \frac{dx}{\sqrt{2\pi(T_i - t)}},
\]
because \(\tilde{W}_{T_i} - \tilde{W}_t\) is a centered Gaussian variable with variance \(T_i - t\), independent of \(\mathcal{F}_t\) under the forward measure \(\tilde{\mathbb{P}}_{i,j}\).

d) We find
\[
P(t, T_i, T_j) \tilde{E}_{i,j} [(\kappa - S(T_i, T_i, T_j))^+ | \mathcal{F}_t] \\
= P(t, T_i, T_j) \tilde{E}_{i,j} \left[ (\kappa - S(t, T_i, T_j) e^{-(T_i - t)\sigma^2/2}(\tilde{W}_{T_i} - \tilde{W}_t)\sigma)^+ | \mathcal{F}_t \right] \\
= P(t, T_i, T_j) (\kappa \Phi(-d_-(T_i - t)) - \tilde{X}_t \Phi(-d_+(T_i - t))) \\
= P(t, T_i, T_j) \kappa \Phi(-d_-(T_i - t)) - P(t, T_i, T_j) S(t, T_i, T_j) \Phi(-d_+(T_i - t)) \\
= P(t, T_i, T_j) \kappa \Phi(-d_-(T_i - t)) - (P(t, T_i) - P(t, T_j)) \Phi(-d_+(T_i - t)),
\]
where \(e^m = S(t, T_i, T_j) e^{-(T-t)\sigma^2/2}, v^2 = (T - t)\sigma^2\), and
\[
d_+(T_i - t) = \frac{\log(S(t, T_i, T_j) / \kappa)}{\sigma \sqrt{T_i - t}} + \frac{\sigma \sqrt{T_i - t}}{2},
\]
and
\[
d_-(T_i - t) = \frac{\log(S(t, T_i, T_j) / \kappa)}{\sigma \sqrt{T_i - t}} - \frac{\sigma \sqrt{T_i - t}}{2},
\]
because \(S(t, T_i, T_j)\) is a driftless geometric Brownian motion with volatility \(\sigma\) under the forward measure \(\tilde{\mathbb{P}}_{i,j}\).

Exercise 18.12

a) Apply the Itô formula to \(d(P(t, T_1) / P(t, T_2))\).

b) We have
\[ L_{T_1} = L_t \exp \left( \int_t^{T_1} \sigma(s) d\tilde{B}_s - \frac{1}{2} \int_t^{T_1} |\sigma(s)|^2 ds \right). \]

c) We have
\[
P(t, T_2) \tilde{E} \left[ (L_{T_1} - \kappa)^+ | \mathcal{F}_t \right]
= P(t, T_2) \tilde{E} \left[ \left( L_t e^{\int_t^{T_1} \sigma(s) d\tilde{B}_s - \frac{1}{2} \int_t^{T_1} |\sigma(s)|^2 ds} - \kappa \right)^+ | \mathcal{F}_t \right]
= P(t, T_2) \text{Bl} \left( \kappa, v(t, T_1) / \sqrt{T_1 - t}, 0, T_1 - t \right)
= P(t, T_2) \left( L_t \Phi \left( \frac{\log(x/K)}{v(t, T_1)} + \frac{v(t, T_1)}{2} \right) - \kappa \Phi \left( \frac{\log(x/K)}{v(t, T_1)} - \frac{v(t, T_1)}{2} \right) \right).
\]

d) Integrate the self-financing condition (18.40) between 0 and \( t \).

e) We have
\[
d\tilde{V}_t = d \left( e^{-\int_0^t r_s ds} V_t \right)
= -r_t e^{-\int_0^t r_s ds} V_t dt + e^{-\int_0^t r_s ds} dV_t
= -r_t e^{-\int_0^t r_s ds} \xi(t) P(t, T_1) - r_t e^{-\int_0^t r_s ds} \xi(t) P(t, T_2) dt
+ e^{-\int_0^t r_s ds} \xi(t) P(t, T_1) + e^{-\int_0^t r_s ds} \xi(t) P(t, T_2)
= \xi(t) d\tilde{P}(t, T_1) + \xi(t) d\tilde{P}(t, T_2), \quad 0 \leq t \leq T_1.
\]

since
\[
\frac{d\tilde{P}(t, T_1)}{P(t, T_1)} = \xi_1(t) dt, \quad \frac{d\tilde{P}(t, T_2)}{P(t, T_2)} = \xi_2(t) dt.
\]

f) We apply the Itô formula and the fact that
\[ t \mapsto \tilde{E} \left[ (L_{T_1} - \kappa)^+ | \mathcal{F}_t \right]\]
and \((L_t)_{t \in \mathbb{R}_+}\) are both martingales under \( \tilde{P} \). Next we use the fact that
\[ \tilde{V}_t = \tilde{E} \left[ (L_{T_1} - \kappa)^+ | \mathcal{F}_t \right], \]
and apply the result of Question (f).

g) Apply the Itô rule to \( V_t = P(t, T_2) \tilde{V}_t \) using Relation (18.38) and the result of Question (f).

h) We have
\[
dV_t = \frac{\partial C}{\partial x} (L_t, v(t, T_1)) P(t, T_1) (\xi_1(t) - \xi_2(t)) dB_t + \tilde{V}_t d\tilde{P}(t, T_2)
= \frac{\partial C}{\partial x} (L_t, v(t, T_1)) P(t, T_1) (\xi_1(t) - \xi_2(t)) dB_t + \tilde{V}_t \xi_2(t) P(t, T_2) dB_t
\]
\begin{align*}
&= (1 + L_t \frac{\partial C}{\partial x} (L_t, v(t, T_1))) P(t, T_2)(\zeta_1(t) - \zeta_2(t))dB_t + \hat{V}_t \zeta_2(t)P(t, T_2)dB_t \\
&= L_t \zeta_1(t) \frac{\partial C}{\partial x} (L_t, v(t, T_1)) P(t, T_2)dB_t - L_t \frac{\partial C}{\partial x} (L_t, v(t, T_1)) P(t, T_2)\zeta_2(t)dB_t \\
&\quad + \frac{\partial C}{\partial x} (L_t, v(t, T_1)) P(t, T_2)(\zeta_1(t) - \zeta_2(t))dB_t + \hat{V}_t \zeta_2(t)P(t, T_2)dB_t \\
&= L_t \frac{\partial C}{\partial x} (L_t, v(t, T_1)) P(t, T_2)\zeta_1(t)dB_t + \left( \hat{V}_t - L_t \frac{\partial C}{\partial x} (L_t, v(t, T_1)) \right) P(t, T_2)\zeta_2(t)dB_t \\
&\quad + \frac{\partial C}{\partial x} (L_t, v(t, T_1)) P(t, T_2)(\zeta_1(t) - \zeta_2(t))dB_t,
\end{align*}

hence

\begin{align*}
\text{d}\hat{V}_t &= L_t \zeta_1(t) \frac{\partial C}{\partial x} (L_t, v(t, T_1)) \tilde{P}(t, T_2)dB_t \\
&\quad + \left( \hat{V}_t - L_t \frac{\partial C}{\partial x} (L_t, v(t, T_1)) \right) \tilde{P}(t, T_2)\zeta_2(t)dB_t \\
&\quad + \frac{\partial C}{\partial x} (L_t, v(t, T_1)) \tilde{P}(t, T_2)(\zeta_1(t) - \zeta_2(t))dB_t \\
&= (\zeta_1(t) \tilde{P}(t, T_1) - \zeta_2(t) \tilde{P}(t, T_2)) \frac{\partial C}{\partial x} (L_t, v(t, T_1))dB_t \\
&\quad + \left( \hat{V}_t - L_t \frac{\partial C}{\partial x} (L_t, v(t, T_1)) \right) d\tilde{P}(t, T_2) \\
&\quad + \frac{\partial C}{\partial x} (L_t, v(t, T_1)) d(\tilde{P}(t, T_1) - \tilde{P}(t, T_2)) + \left( \hat{V}_t - L_t \frac{\partial C}{\partial x} (L_t, v(t, T_1)) \right) d\tilde{P}(t, T_2).
\end{align*}

Exercise 18.13

a) We have

\[ S(T_i, T_i, T_j) = S(t, T_i, T_j) \exp \left( \int_t^{T_i} \sigma_{i,j}(s) dB^{i,j}_s - \frac{1}{2} \int_t^{T_i} |\sigma_{i,j}(s)|^2(s) ds \right). \]

b) We have

\[ P(t, T_i, T_j) \mathbb{E}_{i,j} [(S(T_i, T_i, T_j) - \kappa)^+ | \mathcal{F}_t] \]

\[ = P(t, T_i, T_j) \mathbb{E}_{i,j} \left[ (S(t, T_i, T_j) e^{\int_t^{T_i} \sigma_{i,j}(s) dB^{i,j}_s - \frac{1}{2} \int_t^{T_i} |\sigma_{i,j}(s)|^2(s) ds - \kappa)^+ | \mathcal{F}_t \right] \]

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\[ P(t, T_i, T_j) = P(t, T_i, T_j) \mathcal{B} \left( \kappa, v(t, T_i) / \sqrt{T_i - t}, 0, T_i - t \right) \]

\[ \times \left( S(t, T_i, T_j) \mathcal{F} \left( \frac{\log(x/K)}{v(t, T_i)} + \frac{v(t, T_i)}{2} \right) - \kappa \mathcal{F} \left( \frac{\log(x/K)}{v(t, T_i)} - \frac{v(t, T_i)}{2} \right) \right), \]

where

\[ v^2(t, T_i) = \int_t^{T_i} |\sigma_{i,j}|^2(s) ds. \]

c) Integrate the self-financing condition (18.45) between 0 and \( t \).

d) We have

\[
\frac{d\tilde{V}_t}{\tilde{V}_t} = d \left( e^{-\int_0^t r_s ds} V_t \right)
\]

\[ = -r_t e^{-\int_0^t r_s ds} V_t dt + e^{-\int_0^t r_s ds} dV_t \]

\[ = -r_t e^{-\int_0^t r_s ds} \sum_{k=i}^j \xi^{(k)}_t P(t, T_k) dt + e^{-\int_0^t r_s ds} \sum_{k=i}^j \xi^{(k)}_t dP(t, T_k) \]

\[ = \sum_{k=i}^j \xi^{(k)}_t d\tilde{P}(t, T_k), \quad 0 \leq t \leq T_i. \]

since

\[ \frac{d\tilde{P}(t, T_k)}{\tilde{P}(t, T_k)} = \zeta_k(t) dt, \quad k = i, \ldots, j. \]

e) We apply the Itô formula and the fact that

\[ t \mapsto \mathbb{E}_{i,j} \left[ (S(T_i, T_i, T_j) - \kappa)^+ \mid \mathcal{F}_t \right] \]

and \( (S_t)_{t \in \mathbb{R}^+} \) are both martingales under \( \mathbb{P}_{i,j} \).

f) Use the fact that

\[ \tilde{V}_t = \mathbb{E}_{i,j} \left[ (S(T_i, T_i, T_j) - \kappa)^+ \mid \mathcal{F}_t \right], \]

and apply the result of Question (e).

g) Apply the Itô rule to \( V_t = P(t, T_i, T_j) \tilde{V}_t \) using Relation (18.42) and the result of Question (f).

h) From the expression (18.44) of the swap rate volatilities we have

\[ dV_t = S_t \frac{\partial C}{\partial x} (S_t, v(t, T_i)) \]

\[ \times \left( \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) (\zeta_i(t) - \zeta_{k+1}(t)) + P(t, T_j) (\zeta_i(t) - \zeta_j(t)) \right) dB_t \]

\[ + \tilde{V}_t dP(t, T_i, T_j) \]

\[ \diamond \]

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\[
= S_t \frac{\partial C}{\partial x}(S_t, v(t, T_i)) \\
\times \left( \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1})(\zeta_i(t) - \zeta_{k+1}(t)) + P(t, T_j)(\zeta_i(t) - \zeta_j(t)) \right) dB_t \\
+ \tilde{V}_i \sum_{k=i}^{j-1} (T_{k+1} - T_k) \zeta_{k+1}(t) P(t, T_{k+1}) dB_t \\
= S_t \frac{\partial C}{\partial x}(S_t, v(t, T_i)) \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) \zeta_{k+1}(t) dB_t \\
- S_t \frac{\partial C}{\partial x}(S_t, v(t, T_i)) \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) \zeta_{k+1}(t) dB_t \\
+ \frac{\partial C}{\partial x}(S_t, v(t, T_i)) P(t, T_j)(\zeta_i(t) - \zeta_j(t)) dB_t \\
+ \tilde{V}_i \sum_{k=i}^{j-1} (T_{k+1} - T_k) \zeta_{k+1}(t) P(t, T_{k+1}) dB_t \\
= S_t \zeta_i(t) \frac{\partial C}{\partial x}(S_t, v(t, T_i)) \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) dB_t \\
+ \left( \tilde{V}_i - S_t \frac{\partial C}{\partial x}(S_t, v(t, T_i)) \right) \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) \zeta_{k+1}(t) dB_t \\
+ \frac{\partial C}{\partial x}(S_t, v(t, T_i)) P(t, T_j)(\zeta_i(t) - \zeta_j(t)) dB_t. 
\]

i) We have

\[
\tilde{d} \tilde{V}_t = S_t \zeta_i(t) \frac{\partial C}{\partial x}(S_t, v(t, T_i)) \sum_{k=i}^{j-1} (T_{k+1} - T_k) \tilde{P}(t, T_{k+1}) dB_t \\
+ \left( \tilde{V}_i - S_t \frac{\partial C}{\partial x}(S_t, v(t, T_i)) \right) \sum_{k=i}^{j-1} (T_{k+1} - T_k) \tilde{P}(t, T_{k+1}) \zeta_{k+1}(t) dB_t
\]
\[ + \frac{\partial C}{\partial x} (S_t, v(t, T_i)) \tilde{P}(t, T_j) (\zeta_i(t) - \zeta_j(t)) dB_t \]
\[ = (\tilde{P}(t, T_i) - \tilde{P}(t, T_j)) \zeta_i(t) \frac{\partial C}{\partial x} (S_t, v(t, T_i)) dB_t \]
\[ + \left( \tilde{V}_i - S_t \frac{\partial C}{\partial x} (S_t, v(t, T_i)) \right) d\tilde{P}(t, T_i, T_j) \]
\[ + \frac{\partial C}{\partial x} (S_t, v(t, T_i)) (\tilde{P}(t, T_j) (\zeta_i(t) - \zeta_j(t)) dB_t \]
\[ = (\zeta_i(t) \tilde{P}(t, T_i) - \zeta_j(t) \tilde{P}(t, T_j)) \frac{\partial C}{\partial x} (S_t, v(t, T_i)) dB_t \]
\[ + \left( \tilde{V}_i - S_t \frac{\partial C}{\partial x} (S_t, v(t, T_i)) \right) d\tilde{P}(t, T_i, T_j) \]
\[ = \frac{\partial C}{\partial x} (S_t, v(t, T_i)) d(\tilde{P}(t, T_i) - \tilde{P}(t, T_j)) \]
\[ + \left( \tilde{V}_i - S_t \frac{\partial C}{\partial x} (S_t, v(t, T_i)) \right) d\tilde{P}(t, T_i, T_j). \]

j) We have

\[
\frac{\partial C}{\partial x} (x, \tau, v) = \frac{\partial}{\partial x} \left( x \Phi \left( \frac{v}{2} + \frac{1}{v} \log \frac{x}{\kappa} \right) - \kappa \Phi \left( \frac{-v}{2} + \frac{1}{v} \log \frac{x}{\kappa} \right) \right) \\
= x \frac{\partial}{\partial x} \Phi \left( \frac{v}{2} + \frac{1}{v} \log \frac{x}{\kappa} \right) - \kappa \frac{\partial}{\partial x} \Phi \left( \frac{-v}{2} + \frac{1}{v} \log \frac{x}{\kappa} \right) + \Phi \left( \frac{v}{2} + \frac{1}{v} \log \frac{x}{\kappa} \right) \\
= \frac{1}{v \sqrt{2\pi}} e^{(-v/2+v^{-1} \log(x/\kappa))^2/2} - \frac{\kappa}{v \sqrt{2\pi}} e^{(-v/2+v^{-1} \log(x/\kappa))^2/2} \\
+ \Phi \left( \frac{\log(x/\kappa)}{v} + \frac{v}{2} \right) = \Phi \left( \frac{\log(x/\kappa)}{v} + \frac{v}{2} \right). 
\]

k) We have

\[
d\tilde{V}_i = \frac{\partial C}{\partial x} (S_t, v(t, T_i)) d(\tilde{P}(t, T_i) - \tilde{P}(t, T_j)) \\
+ \left( \tilde{V}_i - S_t \frac{\partial C}{\partial x} (S_t, v(t, T_i)) \right) d\tilde{P}(t, T_i, T_j) \\
= \Phi \left( \frac{\log(S_t/K)}{v(t, T_i)} + \frac{v(t, T_i)}{2} \right) d(\tilde{P}(t, T_i) - \tilde{P}(t, T_j)) \\
- \kappa \Phi \left( \frac{\log(S_t/K)}{v(t, T_i)} - \frac{v(t, T_i)}{2} \right) d\tilde{P}(t, T_i, T_j). 
\]

l) We compare the results of Questions (d) and (k).
Chapter 19

Exercise 19.1

a) When $t \in [0, T_1)$ the equation reads

$$dS_t = -\eta \lambda S_t \, dt = -\eta \lambda S_t \, dt,$$

which is solved as $S_t = S_0 e^{-\eta \lambda t}$, $0 \leq t < T_1$. Next, at the first jump time $t = T_1$ we have

$$\Delta S_t := S_t - S_{t^-} = \eta S_t - dN_t = \eta S_t^-,$$

which yields $S_t = (1 + \eta) S_t^-$, hence $S_{T_1} = (1 + \eta) S_{T_1^-} = S_0 (1 + \eta) e^{-\eta \lambda T_1}$. Repeating this procedure over the $N_t$ jump times contained in the interval $[0, t]$ we get

$$S_t = S_0 (1 + \eta)^{N_t} e^{-\lambda \eta t}, \quad t \in \mathbb{R}_+.$$

b) When $t \in [0, T_1)$ the equation reads

$$dS_t = -\eta \lambda S_t \, dt = -\eta \lambda S_t \, dt,$$

which is solved as $S_t = S_0 e^{-\eta \lambda t}$, $0 \leq t < T_1$. Next, at the first jump time $t = T_1$ we have

$$dS_t = S_t - S_{t^-} = dN_t = 1,$$

which yields $S_t = 1 + S_{t^-}$, hence $S_{T_1} = 1 + S_{T_1^-} = 1 + S_0 e^{-\eta \lambda T_1}$, and for $t \in [T_1, T_2)$ we will find

$$S_t = (1 + S_0 e^{-\eta \lambda T_1}) e^{-(t-T_1)\eta \lambda}, \quad t \in [T_1, T_2).$$

More generally, the equation can be solved by letting $Y_t := e^{\eta \lambda t} S_t$ and noting that $(Y_t)_{t \in \mathbb{R}_+}$ satisfies $dY_t = e^{\lambda \eta t} dN_t$, which has the solution

$$Y_t = Y_0 + \int_0^t e^{\eta \lambda s} dN_s, \quad t \in \mathbb{R}_+,$$

hence in general we have

$$S_t = e^{-\eta \lambda t} S_0 + \int_0^t e^{-(t-s)\eta \lambda} dN_s, \quad t \in \mathbb{R}_+.$$
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\[ X_t = \begin{cases} 
X_0 e^{\alpha t}, & 0 \leq t < T_1, \\
(X_0 e^{\alpha T_1} + \sigma) e^{(t-T_1)\alpha} = X_0 e^{\alpha t} + \sigma e^{(t-T_1)\alpha}, & T_1 \leq t < T_2, \\
\left(\left(X_0 e^{\alpha T_1} + \sigma\right) e^{(T_2-T_1)\alpha} + \sigma\right) e^{(t-T_2)\alpha} = X_0 e^{\alpha t} + \sigma e^{(t-T_1)\alpha} + \sigma e^{(t-T_2)\alpha}, & T_2 \leq t < T_3, 
\end{cases} \]

and more generally the solution \((X_t)_{t \in \mathbb{R}_+}\) can be written as

\[ X_t = X_0 e^{\alpha t} + \sigma \sum_{k=1}^{N_t} e^{(t-T_k)\alpha} = X_0 e^{\alpha t} + \sigma \int_0^t e^{(t-s)\alpha} dN_s, \quad t \in \mathbb{R}_+. \tag{A.75} \]

b) Letting \(f(t) := \mathbb{E}[X_t]\) and taking expectation on both sides of the stochastic differential equation \(dX_t = \alpha X_t dt + \sigma dN_t\) we find

\[ df(t) = \alpha f(t) dt + \sigma \lambda dt, \]

or

\[ f'(t) = \alpha f(t) + \sigma \lambda. \]

Letting \(g(t) = f(t) e^{-\alpha t}\) we check that

\[ g'(t) = \sigma \lambda e^{-\alpha t}, \]

hence

\[ g(t) = g(0) + \int_0^t g'(s) ds = g(0) + \sigma \lambda \int_0^t e^{-\alpha s} ds = f(0) + \sigma \frac{\lambda}{\alpha}(1 - e^{-\alpha t}), \]

and

\[ f(t) = \mathbb{E}[X_t] = g(t) e^{\alpha t} = f(0) e^{\alpha t} + \sigma \frac{\lambda}{\alpha}(e^{\alpha t} - 1) = X_0 e^{\alpha t} + \sigma \frac{\lambda}{\alpha}(e^{\alpha t} - 1), \quad t \in \mathbb{R}_+. \]

We could also take the expectation on both sides of (A.75) and directly find

\[ f(t) = \mathbb{E}[X_t] = X_0 e^{\alpha t} + \sigma \lambda \int_0^t e^{(t-s)\alpha} ds = X_0 e^{\alpha t} + \sigma \frac{\lambda}{\alpha}(e^{\alpha t} - 1), \quad t \in \mathbb{R}_+. \]

Exercise 19.3
a) We have \( X_t = X_0 \prod_{k=1}^{N_t} (1 + \sigma) = X_0 (1 + \sigma)^{N_t} = (1 + \sigma)^{N_t} , \ t \in \mathbb{R}_+ \).

b) By stochastic calculus and using the relation \( dX_t = \sigma X_t \, dN_t \), we have

\[
dS_t = d \left( S_0 X_t + r X_t \int_0^t X_s^{-1} \, ds \right) = S_0 dX_t + r d \left( X_t \int_0^t X_s^{-1} \, ds \right)
\]

\[
= S_0 dX_t + r X_t \left( \int_0^t X_s^{-1} \, ds \right) + r \left( \int_0^t X_s^{-1} \, ds \right) dX_t + r dX_t \cdot d \left( \int_0^t X_s^{-1} \, ds \right)
\]

\[
= S_0 dX_t + r X_t \int_0^t X_s^{-1} \, dt + r \left( \int_0^t X_s^{-1} \, ds \right) dX_t + r dX_t \cdot (X_t^{-1} dt)
\]

\[
= S_0 dX_t + r dt + r \left( \int_0^t X_s^{-1} \, ds \right) dX_t = r dt + \left( S_0 + r \int_0^t X_s^{-1} \, ds \right) dX_t
\]

\[
= r dt + \sigma \left( S_0 X_t - + r X_t - \int_0^t X_s^{-1} \, ds \right) dN_t = r dt + \sigma S_t - dN_t.
\]

c) We have

\[
\mathbb{E}[X_t/X_s] = \mathbb{E}[(1 + \sigma)^{N_t-N_s}]
\]

\[
= \sum_{k \geq 0} (1 + \sigma)^k \mathbb{P}(N_t - N_s = k)
\]

\[
= e^{-(t-s)} \lambda \sum_{k \geq 0} (1 + \sigma)^k \frac{(t-s)\lambda)^k}{k!} = e^{-(t-s)} \lambda \sum_{k \geq 0} \frac{(t-s)\lambda(1 + \sigma)^k}{k!}
\]

\[
= e^{-(t-s)} \lambda e^{(t-s)\lambda(1+\sigma)} = e^{(t-s)\lambda}, \quad 0 \leq s \leq t.
\]

**Remarks:** We could also let \( f(t) = \mathbb{E}[X_t] \) and take expectation in the equation \( dX_t = \sigma X_t \, dN_t \) to get \( f'(t) = \sigma \lambda f(t) \, dt \) and \( f(t) = \mathbb{E}[X_t] = \int_0^t e^{\lambda \sigma t} = e^{\lambda \sigma t} \). Note that the relation \( \mathbb{E}[X_t/X_s] = \mathbb{E}[X_t]/\mathbb{E}[X_s] \), which happens to be true here, is **wrong** in general.

d) We have

\[
\mathbb{E}[S_t] = \mathbb{E} \left[ S_0 X_t + r X_t \int_0^t X_s^{-1} \, ds \right] = S_0 \mathbb{E}[X_t] + r \int_0^t \mathbb{E}[X_t/X_s] \, ds
\]

\[
= S_0 e^{\lambda \sigma t} + r \int_0^t e^{(t-s)\lambda \sigma} \, ds = S_0 e^{\lambda \sigma t} + r \int_0^t e^{\lambda \sigma s} \, ds
\]

\[
= S_0 e^{\lambda \sigma t} + \frac{(e^{\lambda \sigma t} - 1) r}{\lambda \sigma}, \quad t \in \mathbb{R}_+.
\]

**Exercise 19.4**

a) Since \( \mathbb{E}[N_t] = \lambda t \), the expectation \( \mathbb{E}[N_t - 2\lambda t] = -\lambda t \) is a decreasing function of \( t \in \mathbb{R}_+ \), and \( (N_t - 2\lambda t)_{t \in \mathbb{R}_+} \) is a supermartingale.

b) We have

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\[ S_t = S_0 e^{r-t-\lambda t} (1 + \sigma)^N_t, \quad t \in \mathbb{R}_+. \]

c) The stochastic differential equation

\[ dS_t = rS_t dt + \sigma S_t (dN_t - \lambda dt) \]

contains a martingale component \((dN_t - \lambda dt)\) and a positive drift \(rS_t dt\), therefore \((S_t)_{t \in \mathbb{R}_+}\) is a submartingale.

d) Given that \(\sigma > 0\) we have

\[ ((1 + \sigma)^k - 1)^+ = (1 + \sigma)^k - 1, \]

hence

\[
e^{-rT} \mathbb{E}^*[(S_T - K)^+] = e^{-rT} \mathbb{E}^*[(S_0 e^{(r-\sigma\lambda)T} (1 + \sigma)^N_T - K)^+]
\]
\[
= e^{-rT} \mathbb{E}^*[(S_0 e^{(r-\sigma\lambda)T} (1 + \sigma)^N_T - S_0 e^{(r-\lambda\sigma)T})^+]
\]
\[
= S_0 e^{-\sigma\lambda T} \mathbb{E}^*[((1 + \sigma)^N_T - 1)^+]
\]
\[
= S_0 e^{-\sigma\lambda T} \sum_{k \geq 0} ((1 + \sigma)^k - 1)^+ \mathbb{P}(N_T = k)
\]
\[
= S_0 e^{-\sigma\lambda T} \sum_{k \geq 0} (1 + \sigma)^k \mathbb{P}(N_T = k) - S_0 e^{-\sigma\lambda T} \sum_{k \geq 0} \mathbb{P}(N_T = k)
\]
\[
= S_0 e^{-\sigma\lambda T - \lambda T} \sum_{k \geq 0} (T\lambda(1 + \sigma))^{k-1} \mathbb{P}(N_T = k) - S_0 e^{-\sigma\lambda T}
\]
\[
= S_0 (1 - e^{-\sigma\lambda T}),
\]

where we applied the exponential identity

\[ e^x = \sum_{k \geq 0} \frac{x^k}{k!} \]

to \(x := T\lambda(1 + \sigma)\).

Exercise 19.5

a) For all \(k = 1, 2, \ldots, N_t\) we have

\[ X_{T_k} - X_{T_{k-1}} = a + \sigma X_{T_{k-1}}, \]

hence

\[ X_{T_k} = a + (1 + \sigma) X_{T_{k-1}}, \]

and continuing by induction, we obtain

\[ X_{T_k} = a + a(1 + \sigma) + \cdots + a(1 + \sigma)^{k-1} + X_0(1 + \sigma)^k. \]
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\[
= a \frac{(1 + \sigma)^k - 1}{\sigma} + X_0(1 + \sigma)^k, \]

which shows that

\[
X_t = X_{T_{N_t}} = X_0(1 + \sigma)^{N_t} + a \frac{(1 + \sigma)^{N_t} - 1}{\sigma}, \quad t \in \mathbb{R}_+.\]

This result can also be obtained by noting that

\[
X_{T_k} + \frac{a}{\sigma} = (1 + \sigma) \left( X_{T_k} - \frac{a}{\sigma} \right), \quad k = 1, 2, \ldots, N_t. \]

b) We have

\[
\mathbb{E}[(1 + \sigma)^{N_t}] = e^{-\lambda t} \sum_{n \geq 0} (1 + \sigma)^k \frac{(\lambda t)^k}{k!} = e^{\sigma \lambda t}, \quad t \in \mathbb{R}_+, \]

hence

\[
\mathbb{E}[X_t] = X_0 e^{\lambda \sigma t} + a \frac{e^{\lambda \sigma t} - 1}{\sigma} = e^{\lambda \sigma t} \left( X_0 + \frac{a}{\sigma} \right) - \frac{a}{\sigma}, \quad t \in \mathbb{R}_+. \]

Exercise 19.6 We have

\[
S_t = S_0 e^{rt} \prod_{k=1}^{N_t} (1 + \eta Z_k), \quad t \in \mathbb{R}_+. \]

Exercise 19.7 We have

\[
\text{Var} [Y_T] = \mathbb{E} \left[ \left( \sum_{k=1}^{N_T} Z_k - \mathbb{E}[Y_T] \right)^2 \right] = e^{-\lambda t} \sum_{n \geq 0} \frac{\lambda^n t^n}{n!} \mathbb{E} \left[ \left( \sum_{k=1}^{n} Z_k - \lambda t \mathbb{E}[Z_1] \right)^2 \right] = e^{-\lambda t} \sum_{n \geq 0} \frac{\lambda^n t^n}{n!} \mathbb{E} \left[ \left( \sum_{k=1}^{n} Z_k \right)^2 - 2\lambda t \mathbb{E}[Z_1] \sum_{k=1}^{n} Z_k + \lambda^2 t^2 (\mathbb{E}[Z_1])^2 \right] = e^{-\lambda t} \sum_{n \geq 0} \frac{\lambda^n t^n}{n!} \]

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Exercise 19.8

a) Applying the Itô formula (19.20) to the function $f(x) = e^x$ and to the process $X_t = \mu t + \sigma W_t + Y_t$, we find

$$
\begin{align*}
\frac{dS_t}{S_t} &= \left(\mu + \frac{1}{2} \sigma^2\right) dt + \sigma dW_t + (S_t - S_{t-}) dN_t \\
&= \left(\mu + \frac{1}{2} \sigma^2\right) dt + \sigma S_t dW_t + (S_t - S_{t-}) dN_t \\
&= \left(\mu + \frac{1}{2} \sigma^2\right) S_t dt + \sigma S_t dW_t + (S_0 e^{\mu + \sigma W_t + Y_t} - S_0 e^{\mu + \sigma W_t + Y_{t-}}) dN_t \\
&= \left(\mu + \frac{1}{2} \sigma^2\right) S_t dt + \sigma S_t dW_t + (S_0 e^{\mu + \sigma W_t + Y_{t-}} + Z_{N_t} - e^{\mu + \sigma W_t + Y_{t-}}) dN_t \\
&= \left(\mu + \frac{1}{2} \sigma^2\right) S_t dt + \sigma S_t dW_t + S_{t-}(e^{Z_{N_t}} - 1) dN_t,
\end{align*}
$$

hence the jumps of $S_t$ are given by the sequence $(e^{Z_k} - 1)_{k \geq 1}$.

b) The discounted process $e^{-rt} S_t$ satisfies

$$
\frac{d(e^{-rt} S_t)}{e^{-rt} S_t} = e^{-rt} \left(\mu - r + \frac{1}{2} \sigma^2\right) dt + \sigma e^{-rt} S_t dW_t + e^{-rt} S_{t-}(e^{Z_{N_t}} - 1) dN_t.
$$

Hence by the Girsanov Theorem 19.18, choosing $u$, $\tilde{\lambda}$, $\tilde{\nu}$ such that
\[
\mu - r + \frac{1}{2} \sigma^2 = \sigma u - \tilde{\lambda} \mathbb{E}_\tilde{\nu} [e^{Z_1} - 1],
\]

shows that
\[
d(e^{-rt} S_t) = \sigma e^{-rt} S_t (dW_t + u dt) + e^{-rt} S_t - ((e^{Z_N} - 1) dN_t - \tilde{\lambda} \mathbb{E}_\tilde{\nu} [e^{Z_1} - 1] dt)
\]
is a martingale under \((\mathbb{P}_u, \tilde{\lambda}, \tilde{\nu})\).

**Exercise 19.9**

a) We have
\[
S_t = S_0 e^{\mu t} \prod_{k=1}^{N_t} (1 + Z_k) = S_0 \exp \left( \mu t + \sum_{k=1}^{N_t} X_k \right), \quad t \in \mathbb{R}_+.
\]

b) We have
\[
e^{-rt} S_t = S_0 \exp \left( (\mu - r) t + \sum_{k=1}^{N_t} X_k \right), \quad t \in \mathbb{R}_+,
\]
which is a martingale if
\[
0 = \mu - r + \lambda \mathbb{E}[Z_k] = \mu - r + \lambda \mathbb{E}[e^{X_k} - 1] = \mu - r + (e^{\sigma^2/2} - 1) \lambda.
\]

c) We have
\[
e^{-(T-t)r} \mathbb{E}[(S_T - \kappa)^+ | S_t]
\]
\[
= e^{-(T-t)r} \mathbb{E} \left[ \left( S_0 \exp \left( \mu T + \sum_{k=1}^{N_T} X_k \right) - \kappa \right)^+ | S_t \right]
\]
\[
= e^{-(T-t)r} \mathbb{E} \left[ \left( S_t \exp \left( \mu (T-t) + \sum_{k=N_t+1}^{N_T} X_k \right) - \kappa \right)^+ | S_t \right]
\]
\[
= e^{-(T-t)r} \mathbb{E} \left[ \left( x \exp \left( \mu (T-t) + \sum_{k=N_t+1}^{N_T} X_k \right) - \kappa \right)^+ \right]_{x=S_t}
\]
\[
= e^{-(T-t)r} \sum_{n \geq 0} \mathbb{E} \left[ \left( x e^{\mu (T-t) + \sum_{k=1}^{n} X_k - \kappa} \right)^+ \right]_{x=S_t} \mathbb{P}(N_T - N_t = n)
\]

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\[ e^{-(r+\lambda)(T-t)} \sum_{n \geq 0} \mathbb{E} \left[ \left( x e^{\mu(T-t) + \sum_{k=1}^{n} X_k - \kappa} \right)^+ \right] \frac{((T-t)\lambda)^n}{n!} \]
\[ = e^{-(T-t)t} \sum_{n \geq 0} \frac{((T-t)\lambda)^n}{n!} \frac{\text{Bl}(S_t e^{(\mu-r)(T-t)} + n\sigma^2/2, n, n\sigma^2/(T-t), \kappa, T-t)}{n!} \]
\[ = e^{-(T-t)t} \sum_{n \geq 0} \left( S_t e^{(\mu-r)(T-t)} + n\sigma^2/2 \Phi(d_+) - \kappa e^{-(T-t)t} \Phi(d_-) \right) \frac{((T-t)\lambda)^n}{n!}, \]

with

\[ d_+ = \frac{\log(S_t e^{(\mu-r)(T-t)} + n\sigma^2/2/\kappa) + (T-t)r + n\sigma^2/2}{\sigma\sqrt{n}} \]
\[ = \frac{\log(S_t/\kappa) + \mu(T-t) + n\sigma^2}{\sigma\sqrt{n}}, \]

\[ d_- = \frac{\log(S_t e^{(\mu-r)(T-t)} + n\sigma^2/2/\kappa) + (T-t)r - n\sigma^2/2}{\sigma\sqrt{n}} \]
\[ = \frac{\log(S_t/\kappa) + \mu(T-t)}{\sigma\sqrt{n}}, \]

and \( \mu = r + (1 - e^{\sigma^2/2})\lambda. \)

**Exercise 19.10**

a) We have

\[ d(e^{\alpha t} S_t) = \sigma e^{\alpha t} (dN_t - \beta dt), \]

hence

\[ e^{\alpha t} S_t = S_0 + \sigma \int_0^t e^{\alpha s} (dN_s - \beta ds), \]

and

\[ S_t = S_0 e^{-\alpha t} + \sigma \int_0^t e^{-(t-s)\alpha} (dN_s - \beta ds), \quad t \in \mathbb{R}_+. \quad (A.76) \]

b) We have

\[ f(t) = \mathbb{E}[S_t] \]
\[ = S_0 e^{-\alpha t} + \sigma \mathbb{E} \left[ \int_0^t e^{-(t-s)\alpha} dN_s \right] - \beta \sigma \int_0^t e^{-(t-s)\alpha} ds \]
\[ = S_0 e^{-\alpha t} + \lambda \sigma \int_0^t e^{-(t-s)\alpha} ds - \beta \sigma \int_0^t e^{-(t-s)\alpha} ds \]
\[ = S_0 e^{-\alpha t} + (\lambda - \beta) \sigma \frac{1 - e^{-\alpha t}}{\alpha} \]
\[ = \sigma \frac{\lambda - \beta}{\alpha} + \left( S_0 + \sigma \frac{\beta - \lambda}{\alpha} \right) e^{-\alpha t}, \quad t \in \mathbb{R}_+. \]

c) By rewriting (A.76) as

\[ S_t = S_0 - \alpha S_0 \int_0^t e^{-(t-s)\alpha} ds + \sigma \int_0^t e^{-(t-s)\alpha} (dN_s - \beta ds) \]

\[ = S_0 + \sigma \int_0^t e^{-(t-s)\alpha} (dN_s - (\beta + \alpha S_0/\sigma) ds) \]

\[ = S_0 + \sigma \int_0^t e^{-(t-s)\alpha} (\lambda - \beta - \alpha S_0/\sigma) ds + \sigma \int_0^t e^{-(t-s)\alpha} (dN_s - \lambda ds), \]

\[ t \in \mathbb{R}_+, \] we check that the process \((S_t)_{t \in \mathbb{R}_+}\) is a submartingale, provided that \(\lambda - \beta - \alpha S_0/\sigma \geq 0\), i.e. \(S_0 + (\beta - \lambda)\sigma/\alpha \leq 0\). We also check that this condition makes the expectation \(f(t) = \mathbb{E}[S_t]\) decreasing in Question (b).

d) Since, given that \(N_T = n\) the jump times \((T_1, T_2, \ldots, T_n)\) of the Poisson process \((N_t)_{t \in \mathbb{R}_+}\) are independent uniformly distributed random variables over \([0, T]\), hence we can write

\[ \mathbb{E}[\phi(S_T)] = \mathbb{E} \left[ \phi \left( S_0 e^{-\alpha T} + \sigma \int_0^T e^{-(T-s)\alpha} (dN_s - \beta ds) \right) \right] \]

\[ = \sum_{n \geq 0} \mathbb{P}(N_T = n) \times \mathbb{E} \left[ \phi \left( S_0 e^{-\alpha T} + \sigma \int_0^T e^{-(T-s)\alpha} (dN_s - \beta ds) \right) \mid N_T = n \right] \]

\[ = e^{-\lambda T} \sum_{n \geq 0} \frac{(\lambda T)^n}{n!} \times \mathbb{E} \left[ \phi \left( S_0 e^{-\alpha T} + \sigma \sum_{k=1}^n e^{-(T-T_k)\alpha} - \sigma \beta \int_0^T e^{-(T-s)\alpha} ds \right) \mid N_T = n \right] \]

\[ = e^{-\lambda T} \sum_{n \geq 0} \frac{\lambda^n}{n!} \times \int_0^T \cdots \int_0^T \phi \left( S_0 e^{-\alpha T} + \sigma \sum_{k=1}^n e^{-(T-s_k)\alpha} - \sigma \beta \frac{1 - e^{-\alpha T}}{\alpha} \right) ds_1 \cdots ds_k, \]

\( T \geq 0. \)

Exercise 19.11

a) From the decomposition \(Y_t - \lambda t(t + \mathbb{E}[Z_1]) = Y_t - \lambda \mathbb{E}[Z_1]t - \lambda t^2\) as the sum of a martingale and a decreasing function, we conclude that \(t \mapsto Y_t - \lambda t(t + \mathbb{E}[Z_1])\) is a supermartingale.

b) Writing

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\[
dS_t = \mu S_t \, dt + \sigma S_t \, dY_t \\
= r S_t \, dt + \sigma S_t \left( dY_t - \frac{r - \mu}{\sigma} \, dt \right) \\
= r S_t \, dt + \sigma S_t \left( dY_t - \tilde{\lambda} \mathbb{E}[Z_1] \, dt \right), \quad 0 \leq t \leq T,
\]
we conclude that \((S_t)_{t \in [0,T]}\) is a martingale under \(\mathbb{P}_{\tilde{\lambda}}\) provided that
\[
\frac{\mu - r}{\sigma} = -\tilde{\lambda} \mathbb{E}[Z_1] \, dt,
\]
i.e.
\[
\tilde{\lambda} = \frac{r - \mu}{\sigma \mathbb{E}[Z_1]}.
\]
We note that \(\tilde{\lambda} < 0\) if \(\mu < r\), hence in this case there is no risk-neutral probability measure and the market admits arbitrage opportunities as the risky asset always overperforms the risk-free interest rate \(r\).

c) We have
\[
e^{-(T-t)r} \mathbb{E}_\tilde{\lambda}[S_T - \kappa | \mathcal{F}_t] = e^{rt} \mathbb{E}_\tilde{\lambda}[e^{-rT} S_T | \mathcal{F}_t] - K e^{-(T-t)r} \\
= S_t - K e^{-(T-t)r},
\]
since \((S_t)_{t \in [0,T]}\) is a martingale under \(\mathbb{P}_{\tilde{\lambda}}\).

Exercise 19.12

a) We have
\[
S_t = S_0 e^{\mu t} \prod_{k=1}^{N_t} (1 + Z_k), \quad t \in \mathbb{R}_+.
\]

b) Letting \(X_k = \log(1 + Z_k)\), \(k \geq 1\), we find that
\[
e^{-rt} S_t = S_0 \exp \left( (\mu - r)t + \sum_{k=1}^{N_t} X_k \right), \quad t \in \mathbb{R}_+,
\]
and (19.37) can be rewritten for the discounted price process
\[
\tilde{S}_t := e^{-rt} S_t, \quad t \in \mathbb{R}_+,
\]
as
\[
d\tilde{S}_t = (\mu - r + \lambda \mathbb{E}[Z_1]) \tilde{S}_t \, dt + \tilde{S}_t \left( dY_t + \lambda \mathbb{E}[Z_1] \right),
\]
which becomes a martingale if
\[
0 = \mu - r + \lambda \mathbb{E}[Z_1] = \mu - r + \lambda \int_{-\infty}^{\infty} z \nu(\,dz).
\]
c) We have
\[ e^{-(T-t)r} \mathbb{E}[(S_T - \kappa)^+ | S_t] \]
\[ = e^{-(T-t)r} \mathbb{E} \left[ \left( S_0 e^{\mu T} \prod_{k=1}^{N_T} Z_k - \kappa \right)^+ | S_t \right] \]
\[ = e^{-(T-t)r} \sum_{n \geq 0} \mathbb{E} \left[ \left( S_t e^{\mu(T-t)} \prod_{k=N_t+1}^{N_T} Z_k - \kappa \right)^+ | S_t \right] \mathbb{P}(N_T - N_t = n) \]
\[ = e^{-(r+\lambda)(T-t)} \sum_{n \geq 0} \frac{((T-t)\lambda)^n}{n!} \times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( S_t e^{\mu(T-t)} \prod_{k=1}^{n} z_k - \kappa \right)^+ \nu(dz_1) \cdots \nu(dz_n). \]

Exercise 19.13

a) The discounted price process \( (e^{-rt}S_t)_{t \in [0,T]} \) is a martingale, hence it is both a submartingale and a supermartingale.
b) The discounted price process \( (e^{-rt}S_t)_{t \in [0,T]} \) is a supermartingale.
c) The discounted price process \( (e^{-rt}S_t)_{t \in [0,T]} \) is a submartingale.
d) Under the probability measure \( \tilde{\mathbb{P}}_{\lambda} \), the discounted price process \( (e^{-rt}S_t)_{t \in [0,T]} \) is a martingale, hence it is both a submartingale and a supermartingale.

Chapter 20

Exercise 20.1

a) We have \( \mathbb{E}[N_t - \alpha t] = \mathbb{E}[N_t] - \alpha t = \lambda t - \alpha t \), hence \( N_t - \alpha t \) is a martingale if and only if \( \alpha = \lambda \). Given that
\[ d(e^{-rt}S_t) = \eta e^{-rt}S_t - (dN_t - \alpha dt), \]
we conclude that the discounted price process \( e^{-rt}S_t \) is a martingale if and only if \( \alpha = \lambda \).
b) Since we are pricing under the risk-neutral probability measure we take \( \alpha = \lambda \). Next, we note that
\[ S_T = S_0 e^{(r-\eta\lambda)T} (1+\eta)^{N_T} = S_t e^{(r-\eta\lambda)(T-t)} (1+\eta)^{N_T - N_t}, \quad 0 \leq t \leq T, \]
hence the price at time $t$ of the option is
\[ e^{-r(T-t)} \mathbb{E}[|S_T|^2 \mid \mathcal{F}_t] \]
\[ = e^{-r(T-t)} \mathbb{E}[|S_t|^2 e^{2(r-\eta \lambda)(T-t)}(1+\eta)^2(N_T-N_t) \mid \mathcal{F}_t] \]
\[ = |S_t|^2 e^{(r-2\eta \lambda)(T-t)} \mathbb{E}[(1+\eta)^2(N_T-N_t) \mid \mathcal{F}_t] \]
\[ = |S_t|^2 e^{(r-2\eta \lambda)(T-t)} \mathbb{E}[(1+\eta)^2(N_T-N_t)] \]
\[ = |S_t|^2 e^{(r-2\eta \lambda)(T-t)} \sum_{n \geq 0} (1+\eta)^2 n \mathbb{P}(N_T-N_t = n) \]
\[ = |S_t|^2 e^{(r-2\eta \lambda-\lambda)(T-t)} \sum_{n \geq 0} (1+\eta)^2 \frac{(\lambda(T-t))^n}{n!} \]
\[ = |S_t|^2 e^{(r-2\eta \lambda-\lambda)(T-t)+(1+\eta)^2 \lambda(T-t)} \]
\[ = |S_t|^2 e^{(r+\eta^2 \lambda)(T-t)}, \quad 0 \leq t \leq T. \]

Exercise 20.2

a) Regardless of the choice of a particular risk-neutral probability measure $\mathbb{P}_{u, \tilde{\lambda}, \tilde{\nu}}$, we have
\[ e^{-r(T-t)} \mathbb{E}_{u, \tilde{\lambda}, \tilde{\nu}}[S_T - K \mid \mathcal{F}_t] = e^{rt} \mathbb{E}_{u, \tilde{\lambda}, \tilde{\nu}}[e^{-rT}S_T \mid \mathcal{F}_t] - K e^{-r(T-t)} \]
\[ = e^{rt} e^{-rt} S_t - K e^{-r(T-t)} \]
\[ = S_t - K e^{-r(T-t)} \]
\[ = f(t, S_t), \]
for
\[ f(t, x) = x - K e^{-r(T-t)}, \quad t, x > 0. \]

b) Clearly, holding one unit of the risky asset and shorting a (possibly fractional) quantity $K e^{-rT}$ of the riskless asset will hedge the payoff $S_T - K$, and this (static) hedging strategy is self-financing because it is constant in time.

c) Since $\frac{\partial f}{\partial x}(t, x) = 1$ we have
\[ \xi_t = \frac{\sigma^2 \frac{\partial f}{\partial x}(t, S_t^-) + \frac{a \tilde{\lambda}}{S_t^-} (f(t, S_t^- (1+a)) - f(t, S_t^-))}{\sigma^2 + a^2 \tilde{\lambda}} \]
\[ = \frac{\sigma^2 + \frac{a \tilde{\lambda}}{S_t^-} (S_t^- (1+a) - S_t^-)}{\sigma^2 + a^2 \tilde{\lambda}} \]
\[ = 1, \quad 0 \leq t \leq T, \]
which coincides with the result of Question (b).
Exercise 20.3

a) We have
\[ S_t = S_0 \exp \left( \mu t + \sigma B_t - \frac{1}{2} \sigma^2 t \right) (1 + \eta)^{N_t}. \]

b) We have
\[ \tilde{S}_t = S_0 \exp \left( (\mu - r) t + \sigma B_t - \frac{1}{2} \sigma^2 t \right) (1 + \eta)^{N_t}, \]
and
\[ d\tilde{S}_t = (\mu - r + \lambda \eta) \tilde{S}_t dt + \eta \tilde{S}_t (dN_t - \lambda dt) + \sigma \tilde{S}_t dW_t, \]
hence we need to take
\[ \mu - r + \lambda \eta = 0, \]
since the compensated Poisson process \((N_t - \lambda t)_{t \in \mathbb{R}_+}\) is a martingale.

c) We have
\[
e^{-r(T-t)} \mathbb{E}^*[(S_T - \kappa)^+ | S_t] \\
= e^{-r(T-t)} \mathbb{E}^* \left[ \left( S_0 \exp \left( \mu T + \sigma B_T - \frac{1}{2} \sigma^2 T \right) (1 + \eta)^{N_T - \kappa} \right)^+ | S_t \right] \\
= e^{-r(T-t)} \mathbb{E}^* \left[ \left( S_t e^{\mu(T-t)+(B_T-B_t)\sigma-(T-t)\sigma^2/2} (1 + \eta)^{N_T-N_t - \kappa} \right)^+ | S_t \right] \\
= e^{-r(T-t)} \sum_{n \geq 0} \mathbb{P}(N_T - N_t = n) \times \mathbb{E}^* \left[ \left( S_t e^{(r-\lambda\eta)(T-t)+(B_T-B_t)\sigma-(T-t)\sigma^2/2} (1 + \eta)^{n - \kappa} \right)^+ | S_t \right] \\
= e^{-(r+\lambda)(T-t)} \sum_{n \geq 0} \frac{(\lambda(T-t))^n}{n!} \times \mathbb{E}^* \left[ \left( S_t e^{-(r-\lambda\eta)(T-t)+(B_T-B_t)\sigma-(T-t)\sigma^2/2} (1 + \eta)^{n - \kappa} \right)^+ | S_t \right] \\
= e^{-\lambda(T-t)} \sum_{n \geq 0} \text{Bl}(S_t e^{-\lambda\eta(T-t)} (1 + \eta)^n, r, \sigma^2, T-t, \kappa) \frac{(\lambda(T-t))^n}{n!} \\
= e^{-\lambda(T-t)} \sum_{n \geq 0} \left( S_t e^{-\lambda\eta(T-t)} (1 + \eta)^n \Phi(d_+) - \kappa e^{-r(T-t)} \Phi(d_-) \right) \frac{(\lambda(T-t))^n}{n!},
\]
with
\[ d_+ = \frac{\log(S_t e^{-\lambda\eta(T-t)} (1 + \eta)^n / \kappa) + (r + \sigma^2 / 2)(T-t)}{\sigma \sqrt{T-t}}. \]
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\[
\log(S_t(1 + \eta)^n / \kappa) + (r - \lambda \eta + \sigma^2 / 2)(T - t)
\]

\[
\sigma \sqrt{T - t}
\]

and

\[
d_- = \log(S_t e^{-\lambda (T-t)}(1 + \eta)^n / \kappa) + (r - \sigma^2 / 2)(T - t)
\]

\[
\sigma \sqrt{T - t}
\]

Exercise 20.4

a) The discounted process \( \tilde{S}_t = e^{-rt} S_t \) satisfies the equation

\[d\tilde{S}_t = Y_{N_t} \tilde{S}_t - dN_t,\]

and it is a martingale since the compound Poisson process \( Y_{N_t} dN_t \) is centered with independent increments as \( \mathbb{E}[Y_1] = 0 \).

b) We have

\[S_T = S_0 e^{rT} \prod_{k=1}^{N_T} (1 + Y_k),\]

hence

\[e^{-rT} \mathbb{E}[(S_T - \kappa)^+] = e^{-rT} \mathbb{E}\left[\left(S_0 e^{rT} \prod_{k=1}^{N_T} (1 + Y_k) - \kappa\right)^+\right]\]

\[= e^{-rT} \sum_{n \geq 0} \mathbb{E}\left[\left(S_0 e^{rT} \prod_{k=1}^{n} (1 + Y_k) - \kappa\right)^+ \mid N_T = n\right] \mathbb{P}(N_T = n)\]

\[= e^{-rT - \lambda T} \sum_{n \geq 0} \mathbb{E}\left[\left(S_0 e^{rT} \prod_{k=1}^{n} (1 + Y_k) - \kappa\right)^+ \mid (\lambda T)^n / n!\right] \]

\[= e^{-rT - \lambda T} \sum_{n \geq 0} \frac{(\lambda T)^n}{2^n n!} \int_{-1}^{1} \cdots \int_{-1}^{1} \left(S_0 e^{rT} \prod_{k=1}^{n} (1 + y_k) - \kappa\right)^+ dy_1 \cdots dy_n.\]

Exercise 20.5

a) We find \( \alpha = \lambda \) where \( \lambda \) is the intensity of the Poisson process \( (N_t)_{t \in \mathbb{R}_+}. \)

b) We have

\[e^{-r(T-t)} \mathbb{E}[S_T - \kappa \mid \mathcal{F}_t] = e^{rt} \mathbb{E}[e^{-rT} S_T \mid \mathcal{F}_t] - e^{-r(T-t)} \mathbb{E}[\kappa \mid \mathcal{F}_t]\]

\[= e^{rt} \mathbb{E}[e^{-rT} S_t \mid \mathcal{F}_t] - e^{-r(T-t)} \kappa\]

\[= S_t - e^{-r(T-t)} \kappa,\]
since the process \((e^{-rt}S_t)_{t \in \mathbb{R}^+}\) is a martingale.

**Exercise 20.6**

a) We have
\[
S_t = S_0 e^{(r-\lambda)t} (1 + \alpha)^{N_t}, \quad t \in \mathbb{R}^+.
\]

b) We have
\[
e^{-r(T-t)} \mathbb{E}^*\left[\phi(S_T) \mid \mathcal{F}_t\right] = e^{-r(T-t)} \mathbb{E}^*\left[\phi(xS_T / S_t)\right]_{x=S_t}
= e^{-r(T-t)} \mathbb{E}^*\left[\phi(x e^{(r-\lambda)(T-t)}(1 + \alpha)^{N_T-N_t})\right]_{x=S_t}
= e^{-(r+\lambda)(T-t)} \sum_{k=0}^{\infty} \frac{(T-t)^k}{k!} \phi\left(S_t e^{(r-\lambda)(T-t)}(1 + \alpha)^k\right), \quad 0 \leq t \leq T.
\]

c) We have
\[
dV_t = r\eta_te^{rt}dt + \xi_t dS_t
= r\eta_te^{rt}dt + \xi_t(rS_tdt + \alpha S_t -(dN_t - \lambda dt))
= rV_tdt + \alpha \xi_t S_t - (dN_t - \lambda dt)
= rf(t, S_t)dt + \alpha \xi_t S_t - (dN_t - \lambda dt).
\]

(A.77)

d) We apply the Itô formula with jumps and make use of the martingale property of \(t \mapsto e^{-rt}f(t, S_t)\) to get the expression
\[
\begin{aligned}
df(t, S_t) &= rf(t, S_t)dt \\
&+ (f(t, S_t(1 + \alpha)) - f(t, S_t))dN_t - \lambda(f(t, S_t(1 + \alpha)) - f(t, S_t))dt.
\end{aligned}
\]

Finally, we identify the terms in the above formula with those appearing in (A.77).

**Exercise 20.7**

a) We have
\[
\mathbb{E}[N_t \mid \mathcal{F}_s] = e^{\theta Y_s - sm(\theta)} \mathbb{E}\left[e^{\theta(Y_t-Y_s)-(t-s)m(\theta)} \mid \mathcal{F}_s\right]
= e^{\theta Y_s - sm(\theta)} \mathbb{E}\left[e^{\theta(Y_t-Y_s)-(t-s)m(\theta)}\right] = N_s, \quad 0 \leq s \leq t.
\]

b) We have
\[
\mathbb{E}^\theta\left[e^{-rt}S_t \mid \mathcal{F}_s\right] = \mathbb{E}\left[e^{Y_t \frac{N_T}{N_s}} \mid \mathcal{F}_s\right]
= \mathbb{E}\left[e^{Y_t \frac{N_t}{N_s}} \mid \mathcal{F}_s\right]
\]

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\[ e^Y_s \mathbb{E} \left[ e^{Y_t - Y_s} e^{\theta (Y_t - Y_s) - (t-s)m(\theta)} \mid \mathcal{F}_s \right] = e^{Y_s} e^{-(t-s)m(\theta)} \mathbb{E} \left[ e^{(1+\theta)(Y_t - Y_s)} \right] = e^{Y_s} e^{-(t-s)m(\theta)} e^{(t-s)m(\theta + 1)}, \]

hence we should have \( m(\theta) = m(\theta + 1) \). For example, when \( (Y_t)_{t \in \mathbb{R}_+} = (N_t - t)_{t \in \mathbb{R}_+} \) is a compensated Poisson process we have \( m(\theta) = e^{\theta - \theta - 1} \), i.e. \( \theta = -\log(e - 1) \).

c) We have

\[ e^{-(T-t)r} \mathbb{E}^\theta \left[ (S_T - K)^+ \mid \mathcal{F}_t \right] = e^{-(T-t)r} \mathbb{E} \left[ \left( S_T - K \right)^+ \frac{N_T}{N_t} \mid \mathcal{F}_t \right] = e^{-(T-t)((1+r)\theta + m(\theta))} \mathbb{E} \left[ (S_T - K)^+ \left( \frac{S_T}{S_t} \right) ^\theta \mid \mathcal{F}_t \right]. \]

Background on Probability Theory

Exercise A.1 We have

\[ \mathbb{E}[X] = \sum_{k \geq 0} k \mathbb{P}(X = k) = e^{-\lambda} \sum_{k \geq 0} k \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k \geq 1} \frac{\lambda^k}{(k-1)!} = \lambda. \]

Exercise A.2 We have

\[ \mathbb{P}(e^X > c) = \mathbb{P}(X > \log c) = \int_{\log c}^{\infty} e^{-y^2/(2\eta^2)} \frac{dy}{\sqrt{2\pi}\eta^2} = \int_{(\log c) / \eta}^{\infty} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}} = 1 - \Phi \left( (\log c) / \eta \right) = \Phi \left( -(\log c) / \eta \right). \]

Exercise A.3

a) If \( \mu = 0 \) we have

\[ \mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) dx = \frac{1}{\sqrt{2\pi}\sigma^2} \int_{-\infty}^{\infty} x e^{-x^2/(2\sigma^2)} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{-y^2/2} dy = \frac{1}{\sqrt{2\pi}} \lim_{A \to +\infty} \int_{-A}^{A} y e^{-y^2/2} dy = 0, \]

by symmetry of the function \( y \mapsto y e^{-y^2/2} \). Note that we have
\[
\int_{-\infty}^{\infty} |y| e^{-y^2/2} dy = \lim_{A \to +\infty} \int_{-A}^{A} |y| e^{-y^2/2} dy = 2 \lim_{A \to +\infty} \int_{0}^{A} y e^{-y^2/2} dy
\]

\[
= -2 \lim_{A \to +\infty} \left[ e^{-y^2/2} \right]_{0}^{A} = 2 \lim_{A \to +\infty} (1 - e^{-A^2/2}) = 2 < \infty,
\]

hence the function \( y \mapsto ye^{-y^2/2} \) is integrable on \( \mathbb{R} \) and the above computation of \( \mathbb{E}[X] \) is valid. Next, for all \( \mu \in \mathbb{R} \) we have

\[
\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x e^{-(x-\mu)^2/(2\sigma^2)} dx
\]

\[
= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (y + \mu) e^{-y^2/(2\sigma^2)} dy
\]

\[
= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} y e^{-y^2/(2\sigma^2)} dy + \frac{\mu}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-y^2/(2\sigma^2)} dy
\]

\[
= \frac{\mu}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-y^2/(2\sigma^2)} dy = \mu \int_{-\infty}^{\infty} f(y) dy = \mu \mathbb{P}(X \in \mathbb{R}) = \mu.
\]