Chapter 5
Continuous-Time Market Model

In this chapter we review the notions of assets, self-financing portfolios, risk-neutral probability measures, and arbitrage in continuous time. We also derive the Black-Scholes partial differential equation (PDE) for self-financing portfolios, and we solve this equation using the heat kernel method.

5.1 Asset price modeling

The random vector

\[ \mathbf{S}_t = (S_t^{(0)}, S_t^{(1)}, \ldots, S_t^{(d)}) \]

denotes the prices at time \( t \in \mathbb{R}_+ \) of \( d+1 \) assets numbered \( n^0, 1, \ldots, d \), and forms a stochastic process \((\mathbf{S}_t)_{t \in \mathbb{R}_+}\). As in discrete time, the asset \( n^0 \) is a riskless asset (of savings account type) yielding an interest rate \( r \), i.e. we have

\[ S_t^{(0)} = S_0^{(0)} e^{rt}, \quad t \in \mathbb{R}_+. \]

A portfolio strategy is a stochastic process \((\xi_t)_{t \in \mathbb{R}_+} \subset \mathbb{R}^{d+1} \) where \( \xi_t^{(k)} \) denotes the (possibly fractional) quantity of asset \( n^0 \) \( k \) held at time \( t \in \mathbb{R}_+ \). We refer to Figures 2.2 and 2.3 for illustrations of the concept of discounting.

Definition 5.1. Let

\[ \mathbf{X}_t := (\mathbf{S}_t^{(0)}, \mathbf{S}_t^{(1)}, \ldots, \mathbf{S}_t^{(d)}), \quad t \in \mathbb{R}, \]

denote the vector of discounted asset prices, defined as:

\[ \mathbf{X}_t := (\mathbf{S}_t^{(0)}, \mathbf{S}_t^{(1)}, \ldots, \mathbf{S}_t^{(d)}), \quad t \in \mathbb{R}. \]

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The discounted price at time 0 of the portfolio is defined by

$$\tilde{V}_t = e^{-rt} V_t, \quad t \in \mathbb{R}_+.$$ 

For $t \in \mathbb{R}_+$ we have

$$\tilde{V}_t = e^{-rt} \xi_t \cdot \tilde{S}_t$$

$$= e^{-rt} \sum_{k=0}^d \xi_t^{(k)} S_t^{(k)}$$

$$= \sum_{k=0}^d \xi_t^{(k)} \tilde{S}_t^{(k)}$$

$$= \xi_t \cdot \tilde{X}_t,$$

while for $t = 0$ we get

$$\tilde{V}_0 = \xi_1 \cdot \tilde{X}_0 = \xi_1 \cdot \tilde{S}_0.$$ 

The effect of discounting from time $t$ to time 0 is to divide prices by $e^{rt}$, making all prices comparable at time 0.

### 5.2 Arbitrage and Risk-Neutral Measures

In continuous-time, the definition of arbitrage follows the lines of its analogs in the one-step and discrete-time models. In the sequel we will only consider admissible portfolio strategies whose total value $V_t$ remains nonnegative for all times $t \in [0, T]$.

**Definition 5.2.** A portfolio strategy $(\xi_t^{(k)})_{t \in [0, T], k=0,1,..., d}$ with price

$$V_t = \xi_t \cdot \tilde{S}_t = \sum_{k=0}^d \xi_t^{(k)} \tilde{S}_t^{(k)}, \quad t \in \mathbb{R}_+,$$

constitutes an arbitrage opportunity if all three following conditions are satisfied:

i) $V_0 \leq 0$ at time $t = 0$, [start from a zero-cost portfolio or in debt]

ii) $V_T \geq 0$ at time $t = T$, [finish with a nonnegative amount]

iii) $\mathbb{P}(V_T > 0) > 0$ at time $t = T$. [profit made with nonzero probability]

Roughly speaking, (ii) means that the investor wants no loss, (iii) means that he wishes to sometimes make a strictly positive gain, and (i) means that he starts with a zero-cost portfolio or in debt.
that he starts with zero capital or even with a debt.

Next, we turn to the definition of risk-neutral probability measures (or martingale measures) in continuous time, which states that under a risk-neutral probability measure $\mathbb{P}^*$, the return of the risky asset over the time interval $[u, t]$ equals the return of the riskless asset given by

$$S_t^{(0)} = e^{(t-u)r} S_u^{(0)}, \quad 0 \leq u \leq t.$$ 

Recall that the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is generated by Brownian motion $(B_t)_{t \in \mathbb{R}_+}$, i.e.

$$\mathcal{F}_t = \sigma(B_u : 0 \leq u \leq t), \quad t \in \mathbb{R}.$$ 

**Definition 5.3.** A probability measure $\mathbb{P}^*$ on $\Omega$ is called a risk-neutral measure if it satisfies

$$\mathbb{E}^*[S_t^{(k)} | \mathcal{F}_u] = e^{(t-u)r} S_u^{(k)}, \quad 0 \leq u \leq t, \quad k = 1, 2, \ldots, d. \quad (5.1)$$

where $\mathbb{E}^*$ denotes the expectation under $\mathbb{P}^*$.

As in the discrete-time case, $\mathbb{P}^\#$ would be called a risk premium measure if it satisfied

$$\mathbb{E}^\#[S_t^{(k)} | \mathcal{F}_u] > e^{(t-u)r} S_u^{(k)}, \quad 0 \leq u \leq t, \quad k = 1, 2, \ldots, d,$$

meaning that by taking risks in buying $S_t^{(i)}$, one could make an expected return higher than that of the riskless asset

$$S_t^{(0)} = e^{(t-u)r} S_u^{(0)}, \quad 0 \leq u \leq t.$$

Similarly, a negative risk premium measure $\mathbb{P}^\$ satisfies

$$\mathbb{E}^\$[S_t^{(k)} | \mathcal{F}_u] < e^{(t-u)r} S_u^{(k)}, \quad 0 \leq u \leq t, \quad k = 1, 2, \ldots, d.$$ 

From the relation

$$S_t^{(0)} = e^{(t-u)r} S_u^{(0)}, \quad 0 \leq u \leq t,$$

we interpret (5.1) by saying that the expected return of the risky asset $S_t^{(k)}$ under $\mathbb{P}^*$ equals the return of the riskless asset $S_t^{(0)}$, $k = 1, 2, \ldots, d$. Recall that the discounted (in $ at time 0) price $\tilde{S}_t^{(k)}$ of the risky asset n° k is defined by

$$\tilde{S}_t^{(k)} := e^{-rt} S_t^{(k)} = \frac{S_t^{(k)}}{S_t^{(0)} / S_0^{(0)}}, \quad t \in \mathbb{R}_+, \quad k = 0, 1, \ldots, d.$$
i.e. \( S^{(0)}_t / S^{(0)}_0 \) plays the role of a numéraire in the sense of Chapter 15.

**Definition 5.4.** A continuous-time process \((Z_t)_{t \in \mathbb{R}^+}\) of integrable random variables is a martingale under \(\mathbb{P}\) and with respect to the filtration \((\mathcal{F}_t)_{t \in \mathbb{R}^+}\) if

\[
\mathbb{E}[Z_t | \mathcal{F}_s] = Z_s, \quad 0 \leq s \leq t.
\]

Note that when \((Z_t)_{t \in \mathbb{R}^+}\) is a martingale, \(Z_t\) is in particular \(\mathcal{F}_t\)-measurable at all times \(t \in \mathbb{R}^+\).

In continuous-time finance, the martingale property can be used to characterize risk-neutral probability measures, for the derivation of pricing partial differential equations (PDEs), and for the computation of conditional expectations.

As in the discrete-time case, the notion of martingale can be used to characterize risk-neutral probability measures as in the next proposition.

**Proposition 5.5.** The probability measure \(\mathbb{P}^*\) is risk-neutral if and only if the discounted risky asset price process \((\tilde{S}^{(k)}_t)_{t \in \mathbb{R}^+}\) is a martingale under \(\mathbb{P}^*\), \(k = 1, 2, \ldots, d\).

**Proof.** If \(\mathbb{P}^*\) is a risk-neutral probability measure, we have

\[
\mathbb{E}^* \left[ \tilde{S}^{(i)}_t | \mathcal{F}_u \right] = \mathbb{E}^* \left[ e^{-rt} S^{(i)}_t | \mathcal{F}_u \right] = e^{-rt} \mathbb{E}^* \left[ S^{(i)}_t | \mathcal{F}_u \right] = e^{-rt} e^{(t-u)r} S^{(i)}_u = e^{-ru} S^{(i)}_u = \tilde{S}^{(i)}_u, \quad 0 \leq u \leq t,
\]

hence \((\tilde{S}^{(i)}_t)_{t \in \mathbb{R}^+}\) is a martingale under \(\mathbb{P}^*\). Conversely, if \((\tilde{S}^{(i)}_t)_{t \in \mathbb{R}^+}\) is a martingale under \(\mathbb{P}^*\) then

\[
\mathbb{E}^* \left[ S^{(i)}_t | \mathcal{F}_u \right] = \mathbb{E}^* \left[ e^{rt} \tilde{S}^{(i)}_t | \mathcal{F}_u \right] = e^{rt} \mathbb{E}^* \left[ \tilde{S}^{(i)}_t | \mathcal{F}_u \right] = e^{rt} \tilde{S}^{(i)}_u = e^{(t-u)r} S^{(i)}_u, \quad 0 \leq u \leq t, \quad i = 1, 2, \ldots, d,
\]

hence the probability measure \(\mathbb{P}^*\) is risk-neutral according to Definition 5.3. \(\square\)

In the sequel we will only consider probability measures \(\mathbb{P}^*\) that are equivalent to \(\mathbb{P}\) in the sense that they have the share the same events of zero probability.
**Definition 5.6.** A probability measure $P^*$ on $(\Omega, \mathcal{F})$ is said to be equivalent to another probability measure $P$ when

$$P^*(A) = 0 \text{ if and only if } P(A) = 0, \text{ for all } A \in \mathcal{F}. \quad (5.2)$$

Next, we note that the first fundamental theorem of asset pricing also holds in continuous time, and can be used to check for the existence of arbitrage opportunities.

**Theorem 5.7.** A market is without arbitrage opportunity if and only if it admits at least one equivalent risk-neutral probability measure $P^*$.


\[\square\]

### 5.3 Self-Financing Portfolio Strategies

Let $\xi_t^{(i)}$ denote the (possibly fractional) quantity invested at time $t$ over the time period $[t, t + dt)$, in the asset $S_t^{(k)}$, $k = 0, 1, \ldots, d$, and let

$$\xi_t = (\xi_t^{(k)})_{k=0,1,\ldots,d}, \quad S_t = (S_t^{(k)})_{k=0,1,\ldots,d}, \quad t \in \mathbb{R}_+,$$

denote the associated portfolio and asset price processes. The portfolio value $V_t$ at time $t$ is given by

$$V_t = \xi_t \cdot S_t = \sum_{k=0}^d \xi_t^{(k)} S_t^{(k)}, \quad t \in \mathbb{R}_+. \quad (5.3)$$

Our description of portfolio strategies proceeds in four steps which correspond to different interpretations of the self-financing condition.

#### Self-financing portfolio update

The portfolio strategy $(\xi_t)_{t \in \mathbb{R}_+}$ is self-financing if the portfolio value remains constant after updating the portfolio from $\xi_t$ to $\xi_{t+dt}$, i.e.

$$\xi_t \cdot S_{t+dt} = \sum_{k=0}^d \xi_t^{(k)} S_{t+dt}^{(k)} = \sum_{k=0}^d \xi_{t+dt}^{(k)} S_{t+dt}^{(k)} = \xi_{t+dt} \cdot S_{t+dt}, \quad (5.4)$$

which is the continuous-time analog of the self-financing condition already encountered in the discrete setting of Chapter 2, see Definition 2.3. A major difference with the discrete-time case of Definition 2.3, however, is that the continuous-time differentials $dS_t$ and $d\xi_t$ do not make pathwise sense.
as continuous-time stochastic integrals are defined by $L^2$ limits, cf. Proposition 4.16, or by convergence in probability.

<table>
<thead>
<tr>
<th>Portfolio value</th>
<th>$\xi_t \cdot S_t$</th>
<th>$\xi_t \cdot S_{t+dt}$ = $\xi_{t+dt} \cdot S_{t+dt}$</th>
<th>$\xi_{t+dt} \cdot S_{t+2dt}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset value</td>
<td>$S_t$</td>
<td>$S_{t+dt}$</td>
<td>$S_{t+2dt}$</td>
</tr>
<tr>
<td>Time scale</td>
<td>$t$</td>
<td>$t + dt$</td>
<td>$t + dt$</td>
</tr>
<tr>
<td>Portfolio allocation</td>
<td>$\xi_t$</td>
<td>$\xi_{t+dt}$</td>
<td>$\xi_{t+2dt}$</td>
</tr>
</tbody>
</table>

Fig. 5.1: Illustration of the self-financing condition (5.4).

Equivalently, Condition (5.4) can be rewritten as

$$\sum_{k=0}^d S_{t+dt}^{(k)} d\xi_{t+dt}^{(k)} = 0,$$

(5.5)

where

$$d\xi_{t+dt}^{(k)} := \xi_{t+dt}^{(k)} - \xi_t^{(k)}, \quad k = 0, 1, \ldots, d,$$

denote the respective changes in portfolio allocations. In other words, (5.5) rewrites as

$$\sum_{k=0}^d S_{t+dt}^{(k)} (\xi_{t+dt}^{(k)} - \xi_t^{(k)}) = 0.$$

(5.6)

Condition (5.6) can be rewritten as

$$S_t^{(k)} (\xi_{t+dt}^{(k)} - \xi_t^{(k)}) + \sum_{k=0}^d S_{t+dt}^{(k)} - S_t^{(k)} (\xi_{t+dt}^{(k)} - \xi_t^{(k)}) = 0,$$

which shows that (5.4) and (5.5) are equivalent to

$$\overline{S}_t d\xi_t + d\overline{S}_t \cdot d\xi_t = \sum_{k=0}^d S_t^{(k)} d\xi_t^{(k)} + \sum_{k=0}^d dS_t^{(k)} \cdot d\xi_t^{(k)} = 0$$

(5.7)

in differential notation.

**Portfolio differential**

In practice, the self-financing portfolio property will be characterized by the following proposition.

**Proposition 5.8.** A portfolio strategy $(\xi_t^{(k)})_{t \in [0,T], k=0,1,\ldots,d}$ with price
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\[ V_t = \xi_t \cdot S_t = \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)}, \quad t \in \mathbb{R}_+ , \]

is self-financing according to (5.4) if and only if the relation

\[ dV_t = \sum_{k=0}^{d} \xi_t^{(k)} dS_t^{(k)} \]

holds.

Proof. By Itô's calculus we have

\[ dV_t = \left( \sum_{k=0}^{d} \xi_t^{(k)} dS_t^{(k)} \right) + \left( \sum_{k=0}^{d} S_t^{(k)} d\xi_t^{(k)} + \sum_{k=0}^{d} dS_t^{(k)} \cdot d\xi_t^{(k)} \right), \]

which shows that (5.7) is equivalent to (5.8). \( \square \)

Market Completeness

**Definition 5.9.** A contingent claim with payoff \( C \) is said to be attainable if there exists a (self-financing) portfolio strategy \( (\xi_t^{(k)})_{t \in [0,T], k=0,1,\ldots,d} \) such that at the maturity time \( T \) the equality

\[ V_T = \tilde{\xi}_T \cdot \tilde{S}_T = \sum_{k=0}^{d} \xi_T^{(k)} S_T^{(k)} = C \]

holds (almost surely) between random variables.

When a claim with payoff \( C \) is attainable, its price at time \( t \) will be given by the value \( V_t \) of a self-financing portfolio hedging \( C \).

**Definition 5.10.** A market model is said to be complete if every contingent claim payoff \( C \) is attainable.

The next result is the continuous-time statement of the second fundamental theorem of asset pricing.

**Theorem 5.11.** A market model without arbitrage opportunities is complete if and only if it admits only one equivalent risk-neutral probability measure \( \mathbb{P}^* \).

Proof. See Harrison and Pliska (1981) and Chapter VII-4a of Shiryaev (1999). \( \square \)

In the Black and Scholes (1973) model, one can show the existence of a unique risk-neutral probability measure, hence the model is without arbitrage and complete.
5.4 Continuous-Time Market Model

From now one we work with $d = 1$, i.e. with a market based on a riskless asset with price $(A_t)_{t \in \mathbb{R}_+}$ and a risky asset with price $(S_t)_{t \in \mathbb{R}_+}$.

The riskless asset price process $(A_t)_{t \in \mathbb{R}_+}$ admits the following equivalent constructions:

$$
\frac{A_t + dt - A_t}{A_t} = rd t, \quad \frac{dA_t}{A_t} = rd t, \quad A_t' = rA_t, \quad t \in \mathbb{R}_+.
$$

with the solution

$$
A_t = A_0 e^{rt}, \quad t \in \mathbb{R}_+, \quad (5.9)
$$

where $r > 0$ is the risk-free interest rate.

Self-Financing Portfolio Strategies

Let $\xi_t$ and $\eta_t$ denote the (possibly fractional) quantities invested at time $t$ over the time period $[t, t + dt)$, respectively in the assets $S_t$ and $A_t$, and let

$$
\tilde{\xi}_t = (\eta_t, \xi_t), \quad \tilde{S}_t = (A_t, S_t), \quad t \in \mathbb{R}_+,
$$

denote the associated portfolio and asset price processes. The portfolio value $V_t$ at time $t$ is given by

$$
V_t = \tilde{\xi}_t \cdot \tilde{S}_t = \eta_t A_t + \xi_t S_t, \quad t \in \mathbb{R}_+.
$$

Our description of portfolio strategies proceeds in four steps which correspond to different interpretations of the self-financing condition.

Self-financing portfolio update

The portfolio strategy $(\eta_t, \xi_t)_{t \in \mathbb{R}_+}$ is self-financing if the portfolio value remains constant after updating the portfolio from $(\eta_t, \xi_t)$ to $(\eta_{t+dt}, \xi_{t+dt})$, i.e.

$$
\tilde{\xi}_t \cdot \tilde{S}_{t+dt} = \eta_t A_{t+dt} + \xi_t S_{t+dt} = \eta_{t+dt} A_{t+dt} + \xi_{t+dt} S_{t+dt} = \tilde{\xi}_{t+dt} \cdot \tilde{S}_{t+dt}.
$$

(5.10)

* "Anyone who believes exponential growth can go on forever in a finite world is either a madman or an economist", Kenneth E. Boulding, Boulding (1973), page 248.
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Fig. 5.2: Illustration of the self-financing condition (5.10).

Equivalently, Condition (5.10) can be rewritten as

\[ A_{t+dt} d\eta_t + S_{t+dt} d\xi_t = 0, \]  

(5.11)

where

\[ d\eta_t := \eta_t - \eta_{t+dt} \quad \text{and} \quad d\xi_t := \xi_{t+dt} - \xi_t \]

denote the respective changes in portfolio allocations. In other words, we have

\[ A_{t+dt} (\eta_t - \eta_{t+dt}) = S_{t+dt} (\xi_{t+dt} - \xi_t). \]  

(5.12)

In other words, when one sells a (possibly fractional) quantity \( \eta_t - \eta_{t+dt} > 0 \) of the riskless asset valued \( A_{t+dt} \) at the end of the time period \([t, t+dt]\) for the total amount \( A_{t+dt} (\eta_t - \eta_{t+dt})\), one should entirely spend this income to buy the corresponding quantity \( \xi_{t+dt} - \xi_t > 0 \) of the risky asset for the same amount \( S_{t+dt} (\xi_{t+dt} - \xi_t) > 0 \).

Similarly, if one sells a quantity \(-d\xi_t > 0\) of the risky asset \( S_{t+dt} \) between the time periods \([t, t+dt]\) and \([t+dt, t+2dt]\) for a total amount \( -S_{t+dt} d\xi_t\), one should entirely use this income to buy a quantity \( d\eta_t > 0\) of the riskless asset for an amount \( A_{t+dt} d\eta_t > 0\), i.e.

\[ A_{t+dt} d\eta_t = -S_{t+dt} d\xi_t. \]

Condition (5.12) can be rewritten as

\[ S_t (\xi_{t+dt} - \xi_t) + A_t (\eta_t - \eta_{t+dt}) + (S_{t+dt} - S_t)(\xi_{t+dt} - \xi_t) \]
\[ + (A_{t+dt} - A_t) \cdot (\eta_{t+dt} - \eta_t) = 0, \]

which shows that (5.10) and (5.11) are equivalent to

\[ S_t d\xi_t + A_t d\eta_t + dS_t \cdot d\xi_t + dA_t \cdot d\eta_t = 0 \]  

(5.13)

in differential notation, with

\[ dA_t \cdot d\eta_t \simeq (A_{t+dt} - A_t) \cdot (\eta_{t+dt} - \eta_t) = r A_t (dt \cdot d\eta_t) = 0 \]
in the sense of the Itô calculus by the Itô Table 4.1. The following proposition is consequence of Proposition 5.8.

**Proposition 5.12.** A portfolio allocation \((\xi_t, \eta_t)_{t \in \mathbb{R}_+}\) with price

\[ V_t = \eta_t A_t + \xi_t S_t, \quad t \in \mathbb{R}_+, \]

is self-financing according to (5.10) if and only if the relation

\[ dV_t = \eta_t dA_t + \xi_t dS_t \]  \hspace{1cm} (5.14)

holds.

**Proof.** By Itô’s calculus we have

\[ dV_t = [\eta_t dA_t + \xi_t dS_t] + [S_t d\xi_t + A_t d\eta_t + dS_t \cdot d\xi_t + dA_t \cdot d\eta_t], \]

which shows that (5.13) is equivalent to (5.14). \(\square\)

Let

\[ \tilde{V}_t := e^{-rt} V_t \quad \text{and} \quad \tilde{S}_t := e^{-rt} S_t, \quad t \in \mathbb{R}_+, \]

respectively denote the discounted portfolio value and discounted risky asset price at time \(t \geq 0\).

**Geometric Brownian motion**

The risky asset price process \((S_t)_{t \in \mathbb{R}_+}\) will be modeled using a geometric Brownian motion defined from the equation

\[ \frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \quad t \in \mathbb{R}_+, \]  \hspace{1cm} (5.15)

see Section 5.5.

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1 N=2000; t <- 1:N; dt <- 1.0/N; mu=0.5; sigma=0.2; nsim <- 10
2 X <- matrix(rnorm(nsim*N,mean=0,sd=sqrt(dt)), nsim, N-1)
3 X <- cbind(rep(0, nsim), t(apply(X, 1, cumsum)))
4 for (i in 1:nsim){X[i,] <- exp(mu*t*dt+sigma*X[i,]-sigma*sigma*t*dt/2)}
5 plot(t*dt, rep(0, N), xlab = "time", ylab = "Geometric Brownian motion", lwd=2, ylim = c(min(X),max(X)), type = "l", col = 0)
6 for (i in 1:nsim){lines(t*dt, X[i, ], lwd=2, type = "l", col = i)}

By Proposition 5.15 we have

\[ S_t = S_0 \exp \left( \sigma B_t + \left( \mu - \frac{1}{2} \sigma^2 \right) t \right), \quad t \in \mathbb{R}_+. \]

The next Figure 5.3 presents sample paths of geometric Brownian motion.

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https://www.ntu.edu.sg/home/nprivault/index.html
Lemma 5.13. Discounting lemma. Consider an asset price process \((S_t)_{t \in \mathbb{R}_+}\) be as in (5.15), i.e.
\[
dS_t = \mu_S t dt + \sigma_S dB_t, \quad t \in \mathbb{R}_+.
\]
Then the discounted asset prices process \((\tilde{S}_t)_{t \in \mathbb{R}_+}\) satisfies the discounted equation
\[
d\tilde{S}_t = (\mu - r)\tilde{S}_t dt + \sigma\tilde{S}_t dB_t.
\]

Proof. We have
\[
d\tilde{S}_t = d(e^{-rt} S_t) \\
= S_t d(e^{-rt}) + e^{-rt} dS_t + (d e^{-rt}) \cdot dS_t \\
= -r e^{-rt} S_t dt + e^{-rt} dS_t + (-r e^{-rt} S_t dt) \cdot dS_t \\
= -r e^{-rt} S_t dt + \mu e^{-rt} S_t dt + \sigma e^{-rt} S_t dB_t \\
= (\mu - r)\tilde{S}_t dt + \sigma\tilde{S}_t dB_t.
\]

In the next Lemma 5.14, which is the continuous-time analog of Lemma 3.2, we show that when a portfolio is self-financing, its discounted value is a gain process given by the sum over time of discounted trading profits and losses (number of risky assets \(\xi_t\) times discounted price variation \(d\tilde{S}_t\)). Note in Equation (5.16) below, no profit and loss arises from trading the discounted riskless asset \(\tilde{A}_t := e^{-rt} A_t = A_0\) because its value is constant over time.

Lemma 5.14. Let \((\eta_t, \xi_t)_{t \in \mathbb{R}_+}\) be a portfolio strategy with value
\[
V_t = \eta_t A_t + \xi_t S_t, \quad t \in \mathbb{R}_+.
\]
The following statements are equivalent:

(i) the portfolio strategy \((\eta_t, \xi_t)_{t \in \mathbb{R}^+}\) is self-financing,

(ii) the discounted portfolio value \(\widetilde{V}_t\) can be written as the stochastic integral sum

\[
\widetilde{V}_t = \widetilde{V}_0 + \int_0^t \xi_u \widetilde{d}S_u, \quad t \in \mathbb{R}^+,
\]  

of discounted profits and losses.

Proof. Assuming that (i) holds, the self-financing condition and (5.9)-(5.15) show that

\[
dV_t = \eta_t dA_t + \xi_t dS_t
d = r\eta_t A_t dt + \mu\xi_t S_t dt + \sigma\xi_t S_t dB_t
= rV_t dt + (\mu - r)\xi_t S_t dt + \sigma\xi_t S_t dB_t, \quad t \in \mathbb{R}^+,
\]

hence

\[
e^{-rt}dV_t = r e^{-rt}V_t dt + (\mu - r) e^{-rt} \xi_t S_t dt + \sigma e^{-rt} \xi_t S_t dB_t, \quad t \in \mathbb{R}^+,
\]

and

\[
d\widetilde{V}_t = d(e^{-rt}V_t)
d = -r e^{-rt}V_t dt + e^{-rt} dV_t
= (\mu - r)\xi_t e^{-rt} S_t dt + \sigma\xi_t e^{-rt} S_t dB_t
= (\mu - r)\xi_t \widetilde{S}_t dt + \sigma\xi_t \widetilde{S}_t dB_t
= \xi_t d\widetilde{S}_t, \quad t \in \mathbb{R}^+,
\]

i.e. (5.16) holds by integrating on both sides as

\[
\widetilde{V}_t - \widetilde{V}_0 = \int_0^t d\widetilde{V}_u = \int_0^t \xi_u d\widetilde{S}_u, \quad t \in \mathbb{R}^+.
\]

(ii) Conversely, if (5.16) is satisfied we have

\[
dV_t = d(e^{rt}\widetilde{V}_t)
d = r e^{rt}\widetilde{V}_t dt + e^{rt} d\widetilde{V}_t
= r e^{rt}\widetilde{V}_t dt + e^{rt} \xi_t d\widetilde{S}_t
= rV_t dt + e^{rt} \xi_t d\widetilde{S}_t
= rV_t dt + \xi_t \widetilde{S}_t ((\mu - r) dt + \sigma dB_t)
= rV_t dt + \xi_t S_t ((\mu - r) dt + \sigma dB_t)
= r\eta_t A_t dt + \mu\xi_t S_t dt + \sigma\xi_t S_t dB_t
\]
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\[ \eta_t dA_t + \xi_t dS_t, \]

hence the portfolio is self-financing according to Definition 5.8. □

As a consequence of Relation (5.16), the problem of hedging a claim payoff \( C \) with maturity \( T \) also reduces to that of finding the process \((\xi_t)_{t \in [0,T]}\) appearing in the decomposition of the discounted claim payoff \( \tilde{C} = e^{-rT}C \) as a stochastic integral:

\[ \tilde{C} = \tilde{V}_T = \tilde{V}_0 + \int_0^T \xi_t d\tilde{S}_t, \]

see Section 7.5 on hedging by the martingale method. Simple examples of stochastic integral decompositions include the relations

\[ B^2_T = T + 2 \int_0^T B_t dB_t, \]

and

\[ B^3_T = 3 \int_0^T (T - t + B^2_t) dB_t, \]

cf. Exercise 4.5. Note that according to (5.16), the (non-discounted) self-financing portfolio price \( V_t \) can be written as

\[ V_t = e^{rt}V_0 + (\mu - r) \int_0^t e^{(t-u)r} \xi_u S_u du + \sigma \int_0^t e^{(t-u)r} \xi_u S_u dB_u, \quad t \in \mathbb{R}_+. \]  

(5.17)

5.5 Geometric Brownian Motion

In this section we solve the stochastic differential equation

\[ dS_t = \mu S_t dt + \sigma S_t dB_t \]

which is used to model the \( S_t \) the risky asset price at time \( t \), where \( \mu \in \mathbb{R} \) and \( \sigma > 0 \). This equation is rewritten in integral form as

\[ S_t = S_0 + \mu \int_0^t S_u du + \sigma \int_0^t S_u dB_u, \quad t \in \mathbb{R}_+. \]  

(5.18)

It can be solved by applying Itô’s formula to the Itô process \((S_t)_{t \in \mathbb{R}_+}\) as in (4.21) with \( v_t = \mu S_t \) and \( u_t = \sigma S_t \), and by taking \( f(S_t) = \log S_t \) with \( f(x) = \log x \), from which we derive the log-return dynamics

\[ d \log S_t = \mu S_t f'(S_t) dt + \sigma S_t f'(S_t) dB_t + \frac{1}{2} \sigma^2 S_t^2 f''(S_t) dt \]

\[ = \mu dt + \sigma dB_t - \frac{1}{2} \sigma^2 dt, \]
hence

\[
\log S_t - \log S_0 = \int_0^t d\log S_r
\]

\[
= \left(\mu - \frac{1}{2}\sigma^2\right) \int_0^t ds + \sigma \int_0^t dB_s
\]

\[
= \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t, \quad t \in \mathbb{R}_,
\]

and

\[
S_t = S_0 \exp \left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t\right), \quad t \in \mathbb{R}_+.
\]

The above calculation provides a proof for the next proposition.

**Proposition 5.15.** The solution of the stochastic differential equation

\[
dS_t = \mu S_t dt + \sigma S_t dB_t
\]

is given by

\[
S_t = S_0 \exp \left(\sigma B_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right), \quad t \in \mathbb{R}_+.
\]

**Proof.** Let us provide an alternative proof by searching for a solution of the form

\[
S_t = f(t, B_t)
\]

where \(f(t, x)\) is a function to be determined. By Itô’s formula (4.24) we have

\[
dS_t = df(t, B_t) = \frac{\partial f}{\partial t}(t, B_t) dt + \frac{\partial f}{\partial x}(t, B_t) dB_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, B_t) dt.
\]

Comparing this expression to (5.19) and identifying the terms in \(dB_t\) we get

\[
\begin{cases}
\frac{\partial f}{\partial x}(t, B_t) = \sigma S_t, \\
\frac{\partial f}{\partial t}(t, B_t) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, B_t) = \mu S_t.
\end{cases}
\]

Using the relation \(S_t = f(t, B_t)\), these two equations rewrite as

\[
\begin{cases}
\frac{\partial f}{\partial x}(t, B_t) = \sigma f(t, B_t), \\
\frac{\partial f}{\partial t}(t, B_t) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, B_t) = \mu f(t, B_t).
\end{cases}
\]
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Since $B_t$ is a Gaussian random variable taking all possible values in $\mathbb{R}$, the equations should hold for all $x \in \mathbb{R}$, as follows:

\[
\begin{align*}
\frac{\partial f}{\partial x}(t, x) &= \sigma f(t, x), \\
\frac{\partial f}{\partial t}(t, x) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, x) &= \mu f(t, x).
\end{align*}
\tag{5.22a}
\tag{5.22b}
\]

To find the solution $f(t, x) = f(t, 0) e^{\sigma x}$ of (5.22a) we let $g(t, x) = \log f(t, x)$ and rewrite (5.22a) as

\[
\frac{\partial g}{\partial x}(t, x) = \frac{\partial \log f}{\partial x}(t, x) = \frac{1}{f(t, x)} \frac{\partial f}{\partial x}(t, x) = \sigma,
\]

i.e.

\[
\frac{\partial g}{\partial x}(t, x) = \sigma,
\]

which is solved as

\[
g(t, x) = g(t, 0) + \sigma x,
\]

hence

\[
f(t, x) = e^{g(t, 0)} e^{\sigma x} = f(t, 0) e^{\sigma x}.
\]

Plugging back this expression into the second equation (5.22b) yields

\[
e^{\sigma x} \frac{\partial f}{\partial t}(t, 0) + \frac{1}{2} \sigma^2 e^{\sigma x} f(t, 0) = \mu f(t, 0) e^{\sigma x},
\]

i.e.

\[
\frac{\partial f}{\partial t}(t, 0) = \left( \mu - \frac{\sigma^2}{2} \right) f(t, 0).
\]

In other words, we have $\frac{\partial g}{\partial t}(t, 0) = \mu - \sigma^2 / 2$, which yields

\[
g(t, 0) = g(0, 0) + \left( \mu - \frac{\sigma^2}{2} \right) t,
\]

i.e.

\[
f(t, x) = e^{g(t,x)} = e^{g(t,0)+\sigma x} = e^{g(0,0)+\sigma x+(\mu-\sigma^2/2)t} = f(0, 0) e^{\sigma x+(\mu-\sigma^2/2)t}, \quad t \in \mathbb{R}_+.
\]

We conclude that

\[
S_t = f(t, B_t) = f(0, 0) e^{\sigma B_t+(\mu-\sigma^2/2)t},
\]

\( \copyright \)
and the solution to (5.19) is given by

$$S_t = S_0 e^{\sigma B_t + (\mu - \sigma^2/2)t}, \quad t \in \mathbb{R}_+.$$  

□

The next Figure 5.4 presents an illustration of the geometric Brownian process of Proposition 5.15.

---

Fig. 5.4: Geometric Brownian motion started at $S_0 = 1$, with $r = 1$ and $\sigma^2 = 0.5$.∗

Conversely, taking $S_t = f(t, B_t)$ with $f(t, x) = S_0 e^{\sigma x - \sigma^2 t/2 + \mu t}$ we may apply Itô’s formula to check that

$$dS_t = df(t, B_t)$$

$$= \frac{\partial f}{\partial t}(t, B_t)dt + \frac{\partial f}{\partial x}(t, B_t)dB_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, B_t)dt$$

$$= (\mu - \sigma^2/2) S_0 e^{\sigma B_t + (\mu - \sigma^2/2)t}dt + \sigma S_0 e^{\sigma B_t + (\mu - \sigma^2/2)t}dB_t$$

$$+ \frac{1}{2} \sigma^2 S_0 e^{\sigma B_t + (\mu - \sigma^2/2)t}dt$$

$$= \mu S_t dt + \sigma S_t dB_t.$$

* The animation works in Acrobat Reader on the entire pdf file.
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Exercises

Exercise 5.1 Show that at any time $T > 0$, the random variable $S_T := S_0 e^{\sigma B_T + (\mu - \sigma^2/2)T}$ has the lognormal distribution with probability density function

$$x \mapsto f(x) = \frac{1}{x \sigma \sqrt{2\pi T}} e^{\left(-\left(\frac{\mu - \sigma^2 + \log(x/S_0)}{2}\right)^2 / (2\sigma^2 T)\right)}$$

and log-variance $\sigma^2$.

Exercise 5.2 Consider the price process $(S_t)_{t \in \mathbb{R}_+}$ given by the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dB_t.$$

Find the stochastic integral decomposition of the random variable $S_T$, *i.e.*, find the constant $C(S_0, r, T) \in \mathbb{R}$ and the process $(\zeta_t, T)_{t \in [0, T]}$ such that

$$S_T = C(S_0, r, T) + \int_0^T \zeta_t dB_t.$$  (5.23)

Exercise 5.3 Consider $(B_t)_{t \in \mathbb{R}_+}$ a standard Brownian motion generating the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ and the process $(S_t)_{t \in \mathbb{R}_+}$ defined by

$$S_t = S_0 \exp \left( \int_0^t \sigma_s dB_s + \int_0^t u_s ds \right), \quad t \in \mathbb{R}_+,$$

where $(\sigma_t)_{t \in \mathbb{R}_+}$ and $(u_t)_{t \in \mathbb{R}_+}$ are $(\mathcal{F}_t)_{t \in [0, T]}$-adapted processes.

a) Compute $dS_t$ using Itô calculus.
b) Show that $S_t$ satisfies a stochastic differential equation to be determined.

c) Show that $S_t = e^{\sigma B_t - \sigma^2 t/2}$ for $t \in \mathbb{R}_+$.

Exercise 5.4 Consider $(B_t)_{t \in \mathbb{R}_+}$ a standard Brownian motion generating the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$, and let $\sigma > 0$.

a) Compute the mean and variance of the random variable $S_t$ defined as

$$S_t := 1 + \sigma \int_0^t e^{\sigma B_s - \sigma^2 s/2} dB_s, \quad t \in \mathbb{R}_+.$$

b) Express $d \log(S_t)$ using the Itô formula.
c) Show that $S_t = e^{\sigma B_t - \sigma^2 t/2}$ for $t \in \mathbb{R}_+$.

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https://www.ntu.edu.sg/home/nprivault/index.html
Exercise 5.5

a) Solve the ordinary differential equation \( df(t) = cf(t)dt \) and the stochastic differential equation \( dS_t = rS_t dt + \sigma S_t dB_t, \quad t \in \mathbb{R}_+ \), where \( r, \sigma \in \mathbb{R} \) are constants and \( (B_t)_{t \in \mathbb{R}_+} \) is a standard Brownian motion.

b) Show that
\[
\mathbb{E}[S_t] = S_0 e^{rt} \quad \text{and} \quad \text{Var}[S_t] = S_0^2 e^{2rt}(e^{\sigma^2 t} - 1), \quad t \in \mathbb{R}_+.
\]

c) Compute \( d\log S_t \) using the Itô formula.

d) Assume that \( (W_t)_{t \in \mathbb{R}_+} \) is another standard Brownian motion, correlated to \( (B_t)_{t \in \mathbb{R}_+} \) according to the Itô rule \( dW_t \cdot dB_t = \rho dt \), for \( \rho \in [-1, 2] \), and consider the solution \( (Y_t)_{t \in \mathbb{R}_+} \) of the stochastic differential equation
\[
dY_t = \mu Y_t dt + \eta Y_t dW_t, \quad t \in \mathbb{R}_+, \quad \mu, \eta \in \mathbb{R} \text{ are constants. Compute } f(S_t, Y_t), \text{ for } f \text{ a } C^2 \text{ function of } \mathbb{R}^2.
\]

Exercise 5.6

We consider a leveraged fund with factor \( \beta : 1 \) on an index \( (S_t)_{t \in \mathbb{R}_+} \) modeled as the geometric Brownian motion
\[
dS_t = rS_t dt + \sigma S_t dB_t, \quad t \in \mathbb{R}_+, \quad \text{under the risk-neutral probability measure } \mathbb{P}^*.
\]

a) Find the portfolio allocation \((\xi_t, \eta_t)\) of the leveraged fund value \( F_t = \xi_t S_t + \eta_t A_t \), \( t \in \mathbb{R}_+ \), where \( A_t := A_0 e^{rt} \) is the risk-free money market account.

b) Find the stochastic differential equation satisfied by \((F_t)_{t \in \mathbb{R}_+}\) under the self-financing condition \( dF_t = \xi_t dS_t + \eta_t dA_t \).

c) Find the relation between the fund value \( F_t \) and the index \( S_t \) by solving the stochastic differential equation obtained for \( F_t \) in Question (b). For simplicity we take \( F_0 := S_0^\beta \).

Exercise 5.7

Solve the stochastic differential equation
\[
dX_t = h(t)X_t dt + \sigma X_t dB_t,
\]
where \( \sigma > 0 \) and \( h(t) \) is a deterministic, integrable function of \( t \in \mathbb{R}_+ \).

Hint: Look for a solution of the form \( X_t = f(t) e^{\sigma B_t - \sigma^2 t/2} \), where \( f(t) \) is a function to be determined, \( t \in \mathbb{R}_+ \).

Exercise 5.8

Let \((B_t)_{t \in \mathbb{R}_+}\) denote a standard Brownian motion generating the filtration \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\).
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a) Consider the Itô formula

\[ f(X_t) = f(X_0) + \int_0^t u_s \frac{\partial f}{\partial x}(X_s) dB_s + \int_0^t v_s \frac{\partial f}{\partial x}(X_s) ds + \frac{1}{2} \int_0^t u_s^2 \frac{\partial^2 f}{\partial x^2}(X_s) ds, \]

where \( X_t = X_0 + \int_0^t u_s dB_s + \int_0^t v_s ds. \)

Compute \( S_t := e^{X_t} \) by the Itô formula (5.24) applied to \( f(x) = e^x \) and \( X_t = \sigma B_t + \nu t, \sigma > 0, \nu \in \mathbb{R}. \)

b) Let \( r > 0 \). For which value of \( \nu \) does \((S_t)_{t \in \mathbb{R}_+}\) satisfy the stochastic differential equation

\[ dS_t = rS_t dt + \sigma S_t dB_t \]

\( ? \)

c) Given \( \sigma > 0 \), let \( X_t := (B_T - B_t)\sigma \), and compute \( \text{Var}[X_t], t \in [0, T]. \)

d) Let the process \((S_t)_{t \in \mathbb{R}_+}\) be defined by \( S_t = S_0 e^{\sigma B_t + \nu t}, t \in \mathbb{R}_+. \) Using the result of Exercise A.2, show that the conditional probability that \( S_T > K \) given \( S_t = x \) can be computed as

\[ \mathbb{P}(S_T > K \mid S_t = x) = \Phi \left( \frac{\log(x/K) + (T-t)\nu}{\sigma \sqrt{T-t}} \right), \quad t \in [0, T). \]

\( \text{Hint: Use the time splitting decomposition} \)

\[ S_T = S_t \frac{S_T}{S_t} = S_t e^{(B_T - B_t)\sigma + (T-t)\nu}, \quad t \in [0, T]. \]

Problem 5.9 \( \text{Stop-loss start-gain strategy (Lipton (2001) § 8.3.3., Exercise 4.14 continued). Let} \ (B_t)_{t \in \mathbb{R}_+} \text{ be a standard Brownian motion started at} \ B_0 \in \mathbb{R}. \)

a) We consider a simplified foreign exchange model in which the AUD is a risky asset and the AUD/SGD exchange rate at time \( t \) is modeled by \( B_t \), \( i.e. \) AU$1 equals SG$1 at time \( t \). A foreign exchange (FX) European call option gives to its holder the right (but not the obligation) to receive AU$1 in exchange for \( K = \text{SG$1} \) at maturity \( T \). Give the option payoff at maturity, quoted in SGD.

In the sequel, for simplicity we assume no time value of money (\( r = 0 \)), \( i.e. \) the (riskless) SGD account is priced \( A_t = A_0 = 1, t \in [0, T]. \)

b) Consider the following hedging strategy for the European call option of Question (a):

i) If \( B_0 > 1 \), charge the premium \( B_0 - 1 \) at time 0, borrow SG$1 and purchase AU$1.

ii) If \( B_0 < 1 \), issue the option for free.
iii) From time 0 to time $T$, purchase∗ AU$1 every time $B_t$ crosses $K = 1$ from below, and sell† AU$1 each time $B_t$ crosses $K = 1$ from above.

Show that this strategy effectively hedges the foreign exchange European call option at maturity $T$.

Hint. Note that it suffices to consider four scenarios based on $B_0 < 1$ vs $B_0 < 1$ and $B_T > 1$ vs $B_T < 1$.

c) Determine the quantities $\eta_t$ of SGD cash and $\xi_t$ of (risky) AUDs to be held in the portfolio and express the portfolio value

$$V_t = \eta_t + \xi_t B_t$$

at all times $t \in [0, T]$.

d) Compute the integral summation

$$\int_0^t \eta_s dA_s + \int_0^t \xi_s dB_s$$

of portfolio profits and losses at any time $t \in [0, T]$.

Hint. Apply the result of Question (e).

e) Is the portfolio strategy $(\eta_t, \xi_t)_{t \in [0, T]}$ self-financing? How to interpret the answer in practice?

∗ We need to borrow SG$1 if this is the first AUD purchase.
† We use the SG$1 product of the sale to refund the loan.