Chapter 5
Continuous-Time Market Model

In this chapter we review the notions of assets, self-financing portfolios, risk-neutral probability measures, and arbitrage in continuous time. We also derive the Black-Scholes partial differential equation (PDE) for self-financing portfolios, and we solve this equation using the heat kernel method.

5.1 Asset price modeling

The prices at time $t \in \mathbb{R}_+$ of $d + 1$ assets numbered $n^0, 0, 1, \ldots, d$ is denoted by the random vector

$$\vec{S}_t = (S_t^{(0)}, S_t^{(1)}, \ldots, S_t^{(d)})$$

which forms a stochastic process $(\vec{S}_t)_{t \in \mathbb{R}_+}$. As in discrete time, the asset $n^0 0$ is a riskless asset (of savings account type) yielding an interest rate $r$, i.e. we have

$$S_t^{(0)} = S_0^{(0)} e^{rt}, \quad t \in \mathbb{R}_+.$$

**Definition 5.1.** Discounting. Let

$$\vec{X}_t := (\vec{S}_t^{(0)}, \vec{S}_t^{(1)}, \ldots, \vec{S}_t^{(d)}), \quad t \in \mathbb{R},$$

denote the vector of discounted asset prices, defined as:
\[ \tilde{S}_t^{(k)} = e^{-rt} S_t^{(k)}, \quad t \in \mathbb{R}, \quad k = 0, 1, \ldots, d. \]

We can also write
\[ \mathcal{X}_t := e^{-rt} \tilde{S}_t, \quad t \in \mathbb{R}_+. \]

We refer to Figures 2.2 and 2.3 for illustrations of the concept of discounting.

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My portfolio \( S_t \) grew by \( b = 5\% \) this year.

Q: Did I achieve a positive return?

A:

(a) Scenario A.

(b) Scenario B.

---

My portfolio \( S_t \) grew by \( b = 5\% \) this year.

The risk-free or inflation rate is \( r = 10\% \).

Q: Did I achieve a positive return?

A:

(a) Without inflation adjustment.

(b) With inflation adjustment.

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Fig. 5.2: Why apply discounting?

**Definition 5.2.** A portfolio strategy is a stochastic process \( (\xi_t)_{t \in \mathbb{R}_+} \subset \mathbb{R}^{d+1} \) where \( \xi_t^{(k)} \) denotes the (possibly fractional) quantity of asset \( n^k \) held at time \( t \in \mathbb{R}_+ \).

The value at time \( t \geq 0 \) of the portfolio strategy \( (\xi_t)_{t \in \mathbb{R}_+} \subset \mathbb{R}^{d+1} \) is defined by
\[ V_t := \xi_t \cdot S_t, \quad t \in \mathbb{R}_+. \]

The discounted value at time 0 of the portfolio is defined by
\[ \tilde{V}_t := e^{-rt} V_t, \quad t \in \mathbb{R}_+. \]

For \( t \in \mathbb{R}_+ \), we have
\[ \tilde{V}_t = e^{-rt} \xi_t \cdot S_t \]
\[ = e^{-rt} \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)} \]
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\[ d \tilde{X}_t = \xi_t \cdot S_t, \quad t \in \mathbb{R}_+. \]

The effect of discounting from time \( t \) to time 0 is to divide prices by \( e^{rt} \), making all prices comparable at time 0.

5.2 Arbitrage and Risk-Neutral Measures

In continuous-time, the definition of arbitrage follows the lines of its analogs in the one-step and discrete-time models. In the sequel we will only consider admissible portfolio strategies whose total value \( V_t \) remains nonnegative for all times \( t \in [0, T] \).

**Definition 5.3.** A portfolio strategy \( \left( \xi_t^{(k)} \right)_{t \in [0, T], k = 0, 1, \ldots, d} \) with value

\[ V_t = \tilde{\xi}_t \cdot \tilde{S}_t = \sum_{k=0}^{d} \xi_t^{(k)} \tilde{S}_t^{(k)}, \quad t \in \mathbb{R}_+, \]

constitutes an arbitrage opportunity if all three following conditions are satisfied:

i) \( V_0 \leq 0 \) at time \( t = 0 \), [start from a zero-cost portfolio or in debt]

ii) \( V_T \geq 0 \) at time \( t = T \), [finish with a nonnegative amount]

iii) \( P(V_T > 0) > 0 \) at time \( t = T \). [profit made with nonzero probability]

Roughly speaking, \( (ii) \) means that the investor wants no loss, \( (iii) \) means that he wishes to sometimes make a strictly positive gain, and \( (i) \) means that he starts with zero capital or even with a debt.

Next, we turn to the definition of risk-neutral probability measures (or martingale measures) in continuous time, which states that under a risk-neutral probability measure \( \mathbb{P}^* \), the return of the risky asset over the time interval \([u, t]\) equals the return of the riskless asset given by

\[ S_t^{(0)} = e^{(t-u)r} S_u^{(0)}, \quad 0 \leq u \leq t. \]

Recall that the filtration \( (\mathcal{F}_t)_{t \in \mathbb{R}_+} \) is generated by Brownian motion \( (B_t)_{t \in \mathbb{R}_+} \), i.e.

\[ \mathcal{F}_t = \sigma(B_u : 0 \leq u \leq t), \quad t \in \mathbb{R}_+. \]

**Definition 5.4.** A probability measure \( \mathbb{P}^* \) on \( \Omega \) is called a risk-neutral measure if it satisfies

\[ \mathbb{E}^* [S_t^{(k)} | \mathcal{F}_u] = e^{(t-u)r} S_u^{(k)}, \quad 0 \leq u \leq t, \quad k = 1, 2, \ldots, d. \quad (5.1) \]
where $\mathbb{E}^*$ denotes the expectation under $\mathbb{P}^*$.

As in the discrete-time case, $\mathbb{P}^\sharp$ would be called a risk premium measure if it satisfied

$$\mathbb{E}^\sharp [S_t^{(k)} \mid \mathcal{F}_u] > e^{(t-u)r} S_u^{(k)}, \quad 0 \leq u \leq t, \quad k = 1, 2, \ldots, d,$$

meaning that by taking risks in buying $S_t^{(i)}$, one could make an expected return higher than that of the riskless asset

$$S_t^{(0)} = e^{(t-u)r} S_u^{(0)}, \quad 0 \leq u \leq t.$$

Similarly, a negative risk premium measure $\mathbb{P}^\flat$ satisfies

$$\mathbb{E}^\flat [S_t^{(k)} \mid \mathcal{F}_u] < e^{(t-u)r} S_u^{(k)}, \quad 0 \leq u \leq t, \quad k = 1, 2, \ldots, d.$$

From the relation

$$S_t^{(0)} = e^{(t-u)r} S_u^{(0)}, \quad 0 \leq u \leq t,$$

we interpret (5.1) by saying that the expected return of the risky asset $S_t^{(k)}$ under $\mathbb{P}^*$ equals the return of the riskless asset $S_t^{(0)}$, $k = 1, 2, \ldots, d$. Recall that the discounted (in $ at time 0) price $\tilde{S}_t^{(k)}$ of the risky asset $n^o k$ is defined by

$$\tilde{S}_t^{(k)} := e^{-rt} S_t^{(k)} = \frac{S_t^{(k)}}{S_0^{(0)}}, \quad t \in \mathbb{R}_+, \quad k = 0, 1, \ldots, d,$$

i.e. $S_t^{(0)}/S_0^{(0)}$ plays the role of a numéraire in the sense of Chapter 15.

**Definition 5.5.** A continuous-time process $(Z_t)_{t \in \mathbb{R}_+}$ of integrable random variables is a martingale under $\mathbb{P}$ and with respect to the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ if

$$\mathbb{E}[Z_t \mid \mathcal{F}_s] = Z_s, \quad 0 \leq s \leq t.$$

Note that when $(Z_t)_{t \in \mathbb{R}_+}$ is a martingale, $Z_t$ is in particular $\mathcal{F}_t$-measurable at all times $t \in \mathbb{R}_+$.

In continuous-time finance, the martingale property can be used to characterize risk-neutral probability measures, for the derivation of pricing partial differential equations (PDEs), and for the computation of conditional expectations.

As in the discrete-time case, the notion of martingale can be used to characterize risk-neutral probability measures as in the next proposition.
Proposition 5.6. The probability measure $P^*$ is risk-neutral if and only if the discounted risky asset price process $(\tilde{S}_t^{(k)})_{t \in \mathbb{R}_+}$ is a martingale under $P^*$, $k = 1, 2, \ldots, d$.

Proof. If $P^*$ is a risk-neutral probability measure, we have

$$E^*[\tilde{S}_t^{(i)} | F_u] = E^*[e^{-rt} S_t^{(i)} | F_u] = e^{-rt} E^*[S_t^{(i)} | F_u] = e^{-rt} e^{(t-u)r} S_u^{(i)} = e^{-ru} S_u^{(i)} = \tilde{S}_u^{(i)}, \quad 0 \leq u \leq t,$$

hence $(\tilde{S}_t^{(i)})_{t \in \mathbb{R}_+}$ is a martingale under $P^*$. Conversely, if $(\tilde{S}_t^{(i)})_{t \in \mathbb{R}_+}$ is a martingale under $P^*$ then

$$E^*[S_t^{(i)} | F_u] = E^*[e^{rt} \tilde{S}_t^{(i)} | F_u] = e^{rt} E^*[\tilde{S}_t^{(i)} | F_u] = e^{rt} \tilde{S}_u^{(i)} = e^{(t-u)r} S_u^{(i)}, \quad 0 \leq u \leq t, \quad i = 1, 2, \ldots, d,$$

hence the probability measure $P^*$ is risk-neutral according to Definition 5.4.

In the sequel we will only consider probability measures $P^*$ that are equivalent to $P$ in the sense that they have the same events of zero probability.

Definition 5.7. A probability measure $P^*$ on $(\Omega, \mathcal{F})$ is said to be equivalent to another probability measure $P$ when

$$P^*(A) = 0 \quad \text{if and only if} \quad P(A) = 0, \quad \text{for all} \quad A \in \mathcal{F}. \quad (5.2)$$

Next, we note that the first fundamental theorem of asset pricing also holds in continuous time, and can be used to check for the existence of arbitrage opportunities.

Theorem 5.8. A market is without arbitrage opportunity if and only if it admits at least one equivalent risk-neutral probability measure $P^*$.

5.3 Self-Financing Portfolio Strategies

Let \( \xi_t^{(i)} \) denote the (possibly fractional) quantity invested at time \( t \) over the time interval \([t, t + dt]\), in the asset \( S_t^{(k)} \), \( k = 0, 1, \ldots, d \), and let

\[
\xi_t = (\xi_t^{(k)})_{k=0,1,\ldots,d}, \quad S_t = (S_t^{(k)})_{k=0,1,\ldots,d}, \quad t \in \mathbb{R}^+,
\]
denote the associated portfolio value and asset price processes. The portfolio value \( V_t \) at time \( t \) is given by

\[
V_t = \xi_t \cdot S_t = \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)}, \quad t \in \mathbb{R}^+.
\]  
(5.3)

Our description of portfolio strategies proceeds in four equivalent formulations (5.4), (5.5) (5.7) and (5.8), which correspond to different interpretations of the self-financing condition.

Self-financing portfolio update

The portfolio strategy \( (\xi_t)_{t \in \mathbb{R}^+} \) is self-financing if the portfolio value remains constant after updating the portfolio from \( \xi_t \) to \( \xi_{t+dt} \), i.e.

\[
\xi_t \cdot S_{t+dt} = \sum_{k=0}^{d} \xi_t^{(k)} S_{t+dt}^{(k)} = \sum_{k=0}^{d} \xi_{t+dt}^{(k)} S_{t+dt}^{(k)} = \xi_{t+dt} \cdot S_{t+dt},
\]  
(5.4)

which is the continuous-time analog of the self-financing condition already encountered in the discrete setting of Chapter 2, see Definition 2.3. A major difference with the discrete-time case of Definition 2.3, however, is that the continuous-time differentials \( dS_t \) and \( d\xi_t \) do not make pathwise sense as continuous-time stochastic integrals are defined by \( L^2 \) limits, cf. Proposition 4.16, or by convergence in probability.

Equivalently, Condition (5.4) can be rewritten as

Fig. 5.3: Illustration of the self-financing condition (5.4).

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\[ \sum_{k=0}^{d} S_{t+dt}^{(k)} d\xi_t^{(k)} = 0, \]  

(5.5)

where

\[ d\xi_t^{(k)} := \xi_t^{(k)} - \xi_t^{(k)}, \quad k = 0, 1, \ldots, d, \]

denote the respective changes in portfolio allocations. In other words, (5.5) rewrites as

\[ \sum_{k=0}^{d} S_{t+dt}^{(k)} (\xi_t^{(k)} - \xi_t^{(k)}) = 0. \]  

(5.6)

Condition (5.6) can be rewritten as

\[
S_t^{(k)} (\xi_{t+dt}^{(k)} - \xi_t^{(k)}) + \sum_{k=0}^{d} S_{t+dt}^{(k)} - S_t^{(k)} (\xi_{t+dt}^{(k)} - \xi_t^{(k)}) = 0,
\]

which shows that (5.4) and (5.5) are equivalent to

\[
\bar{S}_t d\bar{\xi}_t + d\bar{S}_t \cdot d\bar{\xi}_t = \sum_{k=0}^{d} S_t^{(k)} d\xi_t^{(k)} + \sum_{k=0}^{d} dS_t^{(k)} \cdot d\xi_t^{(k)} = 0
\]

(5.7)

in differential notation.

**Portfolio differential**

In practice, the self-financing portfolio property will be characterized by the following proposition.

**Proposition 5.9.** A portfolio strategy \((\xi_t^{(k)})_{t \in [0,T], k = 0, 1, \ldots, d}\) with value

\[ V_t = \bar{\xi}_t \cdot \bar{S}_t = \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)}, \quad t \in \mathbb{R}_+, \]

is self-financing according to (5.4) if and only if the relation

\[ dV_t = \sum_{k=0}^{d} \xi_t^{(k)} dS_t^{(k)} \text{ for asset } n^o_i \]

holds.

**Proof.** By Itô’s calculus we have
\[
dV_t = \sum_{k=0}^{d} \xi^{(k)}_t dS^{(k)}_t + \sum_{k=0}^{d} S^{(k)}_t d\xi^{(k)}_t + \sum_{k=0}^{d} dS^{(k)}_t \cdot d\xi^{(k)}_t,
\]
which shows that (5.7) is equivalent to (5.8).

\section*{Market Completeness}

\begin{definition}
A contingent claim with payoff \( C \) is said to be attainable if there exists a (self-financing) portfolio strategy \( (\xi^{(k)}_t)_{t \in [0,T], k=0,1,\ldots,d} \) such that at the maturity time \( T \) the equality
\[
V_T = \xi_T \cdot S_T = \sum_{k=0}^{d} \xi^{(k)}_T S^{(k)}_T = C
\]
holds (almost surely) between random variables.
\end{definition}

When a claim with payoff \( C \) is attainable, its price at time \( t \) will be given by the value \( V_t \) of a self-financing portfolio hedging \( C \).

\begin{definition}
A market model is said to be complete if every contingent claim payoff \( C \) is attainable.
\end{definition}

The next result is the continuous-time statement of the second fundamental theorem of asset pricing.

\begin{theorem}
A market model without arbitrage opportunities is complete if and only if it admits only one equivalent risk-neutral probability measure \( \mathbb{P}^* \).
\end{theorem}

\begin{proof}
\end{proof}

In the Black and Scholes (1973) model, one can show the existence of a unique risk-neutral probability measure, hence the model is without arbitrage and complete.

\section*{5.4 Black-Scholes Market Model}

From now one we work with \( d = 1 \), i.e. with a market based on a riskless asset with price \( (A_t)_{t \in \mathbb{R}^+_+} \) and a risky asset with price \( (S_t)_{t \in \mathbb{R}^+_+} \).

The riskless asset price process \( (A_t)_{t \in \mathbb{R}^+_+} \) admits the following equivalent constructions:
\[
\frac{A_{t+dt} - A_t}{A_t} = rdt,
\frac{dA_t}{A_t} = rdt,
A'_t = rA_t,
\]
\( t \in \mathbb{R}^+_+ \).

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with the solution
\[ A_t = A_0 e^{rt}, \quad t \in \mathbb{R}_+, \quad (5.9) \]
where \( r > 0 \) is the risk-free interest rate.*

**Self-Financing Portfolio Strategies**

Let \( \xi_t \) and \( \eta_t \) denote the (possibly fractional) quantities invested at time \( t \) over the time interval \([t, t + dt)\), respectively in the assets \( S_t \) and \( A_t \), and let
\[ \bar{\xi}_t = (\eta_t, \xi_t), \quad \bar{S}_t = (A_t, S_t), \quad t \in \mathbb{R}_+, \]
denote the associated portfolio value and asset price processes. The portfolio value \( V_t \) at time \( t \) is given by
\[ V_t = \bar{\xi}_t \cdot \bar{S}_t = \eta_t A_t + \xi_t S_t, \quad t \in \mathbb{R}_+. \]

Our description of portfolio strategies proceeds in four equivalent formulations presented below in Equations (5.10), (5.11), (5.13) and (5.14), which correspond to different interpretations of the self-financing condition.

**Self-financing portfolio update**

The portfolio strategy \((\eta_t, \xi_t)_{t \in \mathbb{R}_+}\) is self-financing if the portfolio value remains constant after updating the portfolio from \((\eta_t, \xi_t)\) to \((\eta_{t+dt}, \xi_{t+dt})\), i.e.
\[ \bar{\xi}_t \cdot \bar{S}_{t+dt} = \eta_t A_{t+dt} + \xi_t S_{t+dt} = \eta_{t+dt} A_{t+dt} + \xi_{t+dt} S_{t+dt} = \bar{\xi}_{t+dt} \cdot \bar{S}_{t+dt}. \quad (5.10) \]

[Fig. 5.4: Illustration of the self-financing condition (5.10).]

Equivalently, Condition (5.10) can be rewritten as
\[ A_{t+dt} d\eta_t + S_{t+dt} d\xi_t = 0, \quad (5.11) \]

*“Anyone who believes exponential growth can go on forever in a finite world is either a madman or an economist”, Kenneth E. Boulding, Boulding (1973), page 248.
where \( d\eta_t := \eta_{t+dt} - \eta_t \) and \( d\xi_t := \xi_{t+dt} - \xi_t \) denote the respective changes in portfolio allocations. In other words, we have

\[
A_{t+dt}(\eta_t - \eta_{t+dt}) = S_{t+dt}(\xi_{t+dt} - \xi_t).
\]  

(5.12)

In other words, when one sells a (possibly fractional) quantity \( \eta_t - \eta_{t+dt} > 0 \) of the riskless asset valued \( A_{t+dt} \) at the end of the time interval \([t, t+dt]\) for the total amount \( A_{t+dt}(\eta_t - \eta_{t+dt}) \), one should entirely spend this income to buy the corresponding quantity \( \xi_{t+dt} - \xi_t > 0 \) of the risky asset for the same amount \( S_{t+dt}(\xi_{t+dt} - \xi_t) > 0 \).

Similarly, if one sells a quantity \(-d\xi_t > 0\) of the risky asset \( S_{t+dt} \) between the time intervals \([t, t+dt]\) and \([t+dt, t+2dt]\) for a total amount \(-S_{t+dt}d\xi_t\), one should entirely use this income to buy a quantity \( d\eta_t > 0 \) of the riskless asset for an amount \( A_{t+dt}d\eta_t > 0 \), i.e.

\[
A_{t+dt}d\eta_t = -S_{t+dt}d\xi_t.
\]

Condition (5.12) can be rewritten as

\[
S_t(\xi_{t+dt} - \xi_t) + A_t(\eta_{t+dt} - \eta_t) + (S_{t+dt} - S_t)(\xi_{t+dt} - \xi_t) + (A_{t+dt} - A_t)(\eta_{t+dt} - \eta_t) = 0,
\]

which shows that (5.10) and (5.11) are equivalent to

\[
S_t d\xi_t + A_t d\eta_t + dS_t \cdot d\xi_t + dA_t \cdot d\eta_t = 0
\]  

(5.13)

in differential notation, with

\[
dA_t \cdot d\eta_t \simeq (A_{t+dt} - A_t) \cdot (\eta_{t+dt} - \eta_t) = r A_t (dt \cdot d\eta_t) = 0
\]

in the sense of the Itô calculus by the Itô Table 4.1. The following proposition is consequence of Proposition 5.9.

**Proposition 5.13.** A portfolio allocation \((\xi_t, \eta_t)_{t \in \mathbb{R}^+}\) with value

\[
V_t = \eta_t A_t + \xi_t S_t,
\]

is self-financing according to (5.10) if and only if the relation

\[
dV_t = \underbrace{\eta_t dA_t}_{\text{risk-free } P/L} + \underbrace{\xi_t dS_t}_{\text{risky } P/L}
\]

(5.14)

holds.

**Proof.** By Itô’s calculus we have
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\[ dV_t = [\eta_t dA_t + \xi_t dS_t] + [S_t d\xi_t + A_t d\eta_t + dS_t \cdot d\xi_t + dA_t \cdot d\eta_t], \]

which shows that (5.13) is equivalent to (5.14). \qed

Let

\[ \tilde{V}_t := e^{-rt}V_t \quad \text{and} \quad \tilde{S}_t := e^{-rt}S_t, \quad t \in \mathbb{R}_+, \]

respectively denote the discounted portfolio value and discounted risky asset price at time \( t \geq 0 \).

**Geometric Brownian motion**

The risky asset price process \( (S_t)_{t \in \mathbb{R}_+} \) will be modeled using a geometric Brownian motion defined from the equation

\[ \frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \quad t \in \mathbb{R}_+, \tag{5.15} \]

see Section 5.5.

```r
N=2000; t <- 0:N; dt <- 1.0/N; mu=0.5; sigma=0.2; nsim <- 10
X <- matrix(rnorm(nsim*N,mean=0,sd=sqrt(dt)), nsim, N)
X <- cbind(rep(0, nsim), t(apply(X, 1, cumsum)))
for (i in 1:nsim){X[i,] <- exp(mu*t*dt+sigma*X[i,]-sigma*sigma*t*dt/2)}
plot(t*dt, rep(0, N+1), xlab = "time", ylab = "Geometric Brownian motion", lwd=2, ylim = c(min(X),max(X)), type = "l", col = 0)
for (i in 1:nsim){lines(t*dt, X[i,], lwd=2, type = "l", col = i)}
```

By Proposition 5.16 below, we have

\[ S_t = S_0 \exp \left( \sigma B_t + \left( \mu - \frac{1}{2} \sigma^2 \right) t \right), \quad t \in \mathbb{R}_+. \]

The next Figure 5.5 presents sample paths of geometric Brownian motion.
Lemma 5.14. Discounting lemma. Consider an asset price process \((S_t)_{t \in \mathbb{R}^+}\) be as in (5.15), i.e.
\[
    dS_t = \mu S_t dt + \sigma S_t dB_t, \quad t \in \mathbb{R}^+.
\]
Then the discounted asset price process \((\tilde{S}_t)_{t \in \mathbb{R}^+}\) satisfies the equation
\[
    d\tilde{S}_t = (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t dB_t.
\]

Proof. We have
\[
    d\tilde{S}_t = d(e^{-rt}S_t) \\
    = S_t d(e^{-rt}) + e^{-rt} dS_t + (de^{-rt}) \cdot dS_t \\
    = -re^{-rt}S_t dt + e^{-rt} dS_t + (-r e^{-rt} S_t dt) \cdot dS_t \\
    = -re^{-rt}S_t dt + \mu e^{-rt} S_t dt + \sigma e^{-rt} S_t dB_t \\
    = (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t dB_t.
\]

In the next Lemma 5.15, which is the continuous-time analog of Lemma 3.2, we show that when a portfolio is self-financing, its discounted value is a gain process given by the sum over time of discounted trading profits and losses (number of risky assets \(\xi_t\) times discounted price variation \(d\tilde{S}_t\)).

Note that in Equation (5.16) below, no profit or loss arises from trading the discounted riskless asset \(\tilde{A}_t := e^{-rt}A_t = A_0\), because its value is constant over time.

Lemma 5.15. Let \((\eta_t, \xi_t)_{t \in \mathbb{R}^+}\) be a portfolio strategy with value
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\[ V_t = \eta_t A_t + \xi_t S_t, \quad t \in \mathbb{R}_+. \]

The following statements are equivalent:

(i) the portfolio strategy \((\eta_t, \xi_t)_{t \in \mathbb{R}_+}\) is self-financing,

(ii) the discounted portfolio value \(\tilde{V}_t\) can be written as the stochastic integral sum

\[
\tilde{V}_t = \tilde{V}_0 + \int_0^t \xi_u d\tilde{S}_u, \quad t \in \mathbb{R}_+, \tag{5.16}
\]

of discounted profits and losses.

Proof. Assuming that (i) holds, the self-financing condition and (5.9)-(5.15) show that

\[
dV_t = \eta_t dA_t + \xi_t dS_t
\]

\[
= r \eta_t A_t dt + \mu \xi_t S_t dt + \sigma \xi_t S_t dB_t
\]

\[
= rV_t dt + (\mu - r) \xi_t S_t dt + \sigma \xi_t S_t dB_t, \quad t \in \mathbb{R}_+,
\]

hence

\[
e^{-rt} dV_t = r e^{-rt} V_t dt + (\mu - r) e^{-rt} \xi_t S_t dt + \sigma e^{-rt} \xi_t S_t dB_t, \quad t \in \mathbb{R}_+,
\]

and

\[
d\tilde{V}_t = d(e^{-rt} V_t)
\]

\[
= -r e^{-rt} V_t dt + e^{-rt} dV_t
\]

\[
= (\mu - r) \xi_t e^{-rt} S_t dt + \sigma \xi_t e^{-rt} S_t dB_t
\]

\[
= (\mu - r) \xi_t \tilde{S}_t dt + \sigma \xi_t \tilde{S}_t dB_t
\]

\[
= \xi_t d\tilde{S}_t, \quad t \in \mathbb{R}_+,
\]

i.e. (5.16) holds by integrating on both sides as

\[
\tilde{V}_t - \tilde{V}_0 = \int_0^t d\tilde{V}_u = \int_0^t \xi_u d\tilde{S}_u, \quad t \in \mathbb{R}_+.
\]

(ii) Conversely, if (5.16) is satisfied we have

\[
dV_t = d(e^{rt} \tilde{V}_t)
\]

\[
= r e^{rt} \tilde{V}_t dt + e^{rt} d\tilde{V}_t
\]

\[
= r e^{rt} \tilde{V}_t dt + e^{rt} \xi_t d\tilde{S}_t
\]

\[
= rV_t dt + e^{rt} \xi_t d\tilde{S}_t
\]

\[
= rV_t dt + e^{rt} \xi_t \tilde{S}_t ((\mu - r) dt + \sigma dB_t)
\]
\[
= rV_t dt + \xi_t S_t ((\mu - r) dt + \sigma dB_t)
\]
\[
= r\eta A_t dt + \mu \xi_t S_t dt + \sigma \xi_t S_t dB_t
\]
\[
= \eta_t dA_t + \xi_t dS_t,
\]
hence the portfolio is self-financing according to Definition 5.9. \(\square\)

As a consequence of Relation (5.16), the problem of hedging a claim payoff \(C\) with maturity \(T\) also reduces to that of finding the process \((\xi_t)_{t \in [0, T]}\) appearing in the decomposition of the discounted claim payoff \(\tilde{C} = e^{-rT}C\) as a stochastic integral:

\[
\tilde{C} = \tilde{V}_T = \tilde{V}_0 + \int_0^T \xi_t d\tilde{S}_t,
\]
see Section 7.5 on hedging by the martingale method.

**Example. Power options in the Bachelier model.**

In the Bachelier model, the underlying asset price can be modeled by Brownian motion \((B_t)_{t \in \mathbb{R}_+}\). The claim payoff \(C = (B_T)^2\) of a power option with at maturity \(T > 0\) admits the stochastic integral decomposition

\[
(B_T)^2 = T + 2 \int_0^T B_t dB_t
\]
which shows that the claim can be hedged using \(\xi_t = 2B_t\) units of the underlying asset at time \(t \in [0, T]\).

Similarly, in the case of power claim payoff \(C = (B_T)^3\) we have

\[
(B_T)^3 = 3 \int_0^T (T - t + (B_t)^2) dB_t,
\]

cf. Exercise 4.5.

Note that according to (5.16), the (non-discounted) self-financing portfolio value \(V_t\) can be written as

\[
V_t = e^{rt}V_0 + (\mu - r) \int_0^t e^{(t-u)r} \xi_u S_u du + \sigma \int_0^t e^{(t-u)r} \xi_u S_u dB_u, \quad t \in \mathbb{R}_+.
\]

(5.17)

5.5 Geometric Brownian Motion

In this section we solve the stochastic differential equation

\[
dS_t = \mu S_t dt + \sigma S_t dB_t
\]
which is used to model the $S_t$ the risky asset price at time $t$, where $\mu \in \mathbb{R}$ and $\sigma > 0$. This equation is rewritten in integral form as

$$S_t = S_0 + \mu \int_0^t S_u du + \sigma \int_0^t S_u dB_u, \quad t \in \mathbb{R}_+. \quad (5.18)$$

It can be solved by applying Itô’s formula to the Itô process $(S_t)_{t \in \mathbb{R}_+}$ as in (4.21) with $v_t = \mu S_t$ and $u_t = \sigma S_t$, and by taking $f(S_t) = \log S_t$ with $f(x) = \log x$, from which we derive the log-return dynamics

$$d \log S_t = \mu S_t f'(S_t) dt + \sigma S_t f'(S_t) dB_t + \frac{1}{2} \sigma^2 S_t^2 f''(S_t) dt$$

$$= \mu dt + \sigma dB_t - \frac{1}{2} \sigma^2 dt,$$

hence

$$\log S_t - \log S_0 = \int_0^t d \log S_r$$

$$= \left( \mu - \frac{1}{2} \sigma^2 \right) \int_0^t ds + \sigma \int_0^t dB_s$$

$$= \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma B_t, \quad t \in \mathbb{R}_+, \quad \text{and} \quad S_t = S_0 \exp \left( \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma B_t \right), \quad t \in \mathbb{R}_+. \quad \text{The above calculation provides a proof for the next proposition.}$$

**Proposition 5.16.** The solution of the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t \quad (5.19)$$

is given by

$$S_t = S_0 \exp \left( \sigma B_t + \left( \mu - \frac{1}{2} \sigma^2 \right) t \right), \quad t \in \mathbb{R}_+. \quad \text{Proof. Let us provide an alternative proof by searching for a solution of the form}$$

$$S_t = f(t, B_t)$$

where $f(t, x)$ is a function to be determined. By Itô’s formula (4.24) we have

$$dS_t = df(t, B_t) = \frac{\partial f}{\partial t} (t, B_t) dt + \frac{\partial f}{\partial x} (t, B_t) dB_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (t, B_t) dt.$$
\[
\begin{aligned}
\frac{\partial f}{\partial x}(t, B_t) &= \sigma S_t, \\
\frac{\partial f}{\partial t}(t, B_t) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, B_t) &= \mu S_t.
\end{aligned}
\]

Using the relation \( S_t = f(t, B_t) \), these two equations rewrite as
\[
\begin{aligned}
\frac{\partial f}{\partial x}(t, B_t) &= \sigma f(t, B_t), \\
\frac{\partial f}{\partial t}(t, B_t) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, B_t) &= \mu f(t, B_t).
\end{aligned}
\]

Since \( B_t \) is a Gaussian random variable taking all possible values in \( \mathbb{R} \), the equations should hold for all \( x \in \mathbb{R} \), as follows:
\[
\begin{aligned}
\frac{\partial f}{\partial x}(t, x) &= \sigma f(t, x), \\
\frac{\partial f}{\partial t}(t, x) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, x) &= \mu f(t, x).
\end{aligned}
\] (5.22a) (5.22b)

To find the solution \( f(t, x) = f(t, 0) e^{\sigma x} \) of (5.22a) we let \( g(t, x) = \log f(t, x) \) and rewrite (5.22a) as
\[
\frac{\partial g}{\partial x}(t, x) = \frac{\partial \log f}{\partial x}(t, x) = \frac{1}{f(t, x)} \frac{\partial f}{\partial x}(t, x) = \sigma,
\]
i.e.
\[
\frac{\partial g}{\partial x}(t, x) = \sigma,
\]
which is solved as
\( g(t, x) = g(t, 0) + \sigma x \),

hence
\[
f(t, x) = e^{g(t, 0)} e^{\sigma x} = f(t, 0) e^{\sigma x}.
\]

Plugging back this expression into the second equation (5.22b) yields
\[
e^{\sigma x} \frac{\partial f}{\partial t}(t, 0) + \frac{1}{2} \sigma^2 e^{\sigma x} f(t, 0) = \mu f(t, 0) e^{\sigma x},
\]
i.e.
\[
\frac{\partial f}{\partial t}(t, 0) = \left( \mu - \frac{\sigma^2}{2} \right) f(t, 0).
\]
In other words, we have \( \frac{\partial g}{\partial t}(t, 0) = \mu - \sigma^2/2 \), which yields

\[
g(t, 0) = g(0, 0) + \left( \mu - \frac{\sigma^2}{2} \right) t,
\]

i.e.

\[
f(t, x) = e^{g(t, x)} = e^{g(t, 0) + \sigma x}
= e^{g(0, 0) + \sigma x + (\mu - \sigma^2/2) t}
= f(0, 0) e^{\sigma x + (\mu - \sigma^2/2) t}, \quad t \in \mathbb{R}_+.
\]

We conclude that

\[
S_t = f(t, B_t) = f(0, 0) e^{\sigma B_t + (\mu - \sigma^2/2) t},
\]

and the solution to (5.19) is given by

\[
S_t = S_0 e^{\sigma B_t + (\mu - \sigma^2/2) t}, \quad t \in \mathbb{R}_+.
\]

The next Figure 5.6 presents an illustration of the geometric Brownian process of Proposition 5.16.

Fig. 5.6: Geometric Brownian motion started at \( S_0 = 1 \), with \( \mu = r = 1 \) and \( \sigma^2 = 0.5 \).∗

∗ The animation works in Acrobat Reader on the entire pdf file.
Conversely, taking $S_t = f(t, B_t)$ with $f(t, x) = S_0 e^{\sigma x - \sigma^2 t/2 + \mu t}$ we may apply Itô’s formula to check that

$$dS_t = df(t, B_t) = \frac{\partial f}{\partial t} (t, B_t) dt + \frac{\partial f}{\partial x} (t, B_t) dB_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (t, B_t) dt$$

$$= (\mu - \sigma^2/2) S_0 e^{\sigma B_t + (\mu - \sigma^2/2) t} dt + \sigma S_0 e^{\sigma B_t + (\mu - \sigma^2/2) t} dB_t$$

$$+ \frac{1}{2} \sigma^2 S_0 e^{\sigma B_t + (\mu - \sigma^2/2) t} dt$$

$$= \mu S_0 e^{\sigma B_t + (\mu - \sigma^2/2) t} dt + \sigma S_0 e^{\sigma B_t + (\mu - \sigma^2/2) t} dB_t$$

$$= \mu S_t dt + \sigma S_t dB_t.$$

**Exercises**

**Exercise 5.1** Show that at any time $T > 0$, the random variable $S_T := S_0 e^{\sigma B_T + (\mu - \sigma^2/2) T}$ has the lognormal distribution with probability density function

$$f(x) = \frac{1}{x \sigma \sqrt{2\pi T}} e^{-(\mu - \sigma^2/2) T + \log(x/S_0)^2/(2\sigma^2 T)}, \quad x > 0,$$

and log-variance $\sigma^2$.

**Exercise 5.2** Consider the price process $(S_t)_{t \in \mathbb{R}_+}$ given by the stochastic differential equation

$$dS_t = r S_t dt + \sigma S_t dB_t.$$

Find the stochastic integral decomposition of the random variable $S_T$, i.e., find the constant $C(S_0, r, T) \in \mathbb{R}$ and the process $(\zeta_t, T)_{t \in [0, T]}$ such that

$$S_T = C(S_0, r, T) + \int_0^T \zeta_{t, T} dB_t. \quad (5.23)$$

**Exercise 5.3** Consider $(B_t)_{t \in \mathbb{R}_+}$ a standard Brownian motion generating the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ and the process $(S_t)_{t \in \mathbb{R}_+}$ defined by

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\[ S_t = S_0 \exp \left( \int_0^t \sigma_s dB_s + \int_0^t u_s ds \right), \quad t \in \mathbb{R}_+, \]

where \((\sigma_t)_{t \in \mathbb{R}_+}\) and \((u_t)_{t \in \mathbb{R}_+}\) are \((\mathcal{F}_t)_{t \in [0,T]}\)-adapted processes.

a) Compute \(dS_t\) using Itô calculus.

b) Show that \(S_t\) satisfies a stochastic differential equation to be determined.

**Exercise 5.4** Consider \((B_t)_{t \in \mathbb{R}_+}\) a standard Brownian motion generating the filtration \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\), and let \(\sigma > 0\).

a) Compute the mean and variance of the random variable \(S_t\) defined as

\[ S_t := 1 + \sigma \int_0^t e^{\sigma B_s - \sigma^2 s/2} dB_s, \quad t \in \mathbb{R}_+. \]

b) Express \(d \log (S_t)\) using the Itô formula.

c) Show that \(S_t = e^{\sigma B_t - \sigma^2 t/2}\) for \(t \in \mathbb{R}_+\).

**Exercise 5.5**

a) Solve the ordinary differential equation \(df(t) = cf(t) dt\) and the stochastic differential equation \(dS_t = rS_t dt + \sigma S_t dB_t,\ t \in \mathbb{R}_+,\) where \(r, \sigma \in \mathbb{R}\) are constants and \((B_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion.

b) Show that

\[ \mathbb{E}[S_t] = S_0 e^{rt} \quad \text{and} \quad \text{Var}[S_t] = S_0^2 e^{2rt} (e^{\sigma^2 t} - 1), \quad t \in \mathbb{R}_+. \]

c) Compute \(d \log S_t\) using the Itô formula.

d) Assume that \((W_t)_{t \in \mathbb{R}_+}\) is another standard Brownian motion, correlated to \((B_t)_{t \in \mathbb{R}_+}\) according to the Itô rule \(dW_t \cdot dB_t = \rho dt,\) for \(\rho \in [-1, 2],\) and consider the solution \((Y_t)_{t \in \mathbb{R}_+}\) of the stochastic differential equation

\[ dY_t = \mu Y_t dt + \eta Y_t dW_t, \quad t \in \mathbb{R}_+, \]

where \(\mu, \eta \in \mathbb{R}\) are constants. Compute \(f(S_t, Y_t),\) for \(f\) a \(C^2\) function of \(\mathbb{R}^2.\)

**Exercise 5.6** We consider a leveraged fund with factor \(\beta : 1\) on an index \((S_t)_{t \in \mathbb{R}_+}\) modeled as the geometric Brownian motion

\[ dS_t = rS_t dt + \sigma S_t dB_t, \quad t \in \mathbb{R}_+, \]

under the risk-neutral probability measure \(\mathbb{P}^*.\)

a) Find the portfolio allocation \((\xi_t, \eta_t)\) of the leveraged fund value

\[ F_t = \xi_t S_t + \eta_t A_t, \quad t \in \mathbb{R}_+, \]

where \(A_t := A_0 e^{rt}\) is the risk-free money market account.
b) Find the stochastic differential equation satisfied by \((F_t)_{t \in \mathbb{R}_+}\) under the self-financing condition \(dF_t = \xi_t dS_t + \eta_t dA_t\).

c) Find the relation between the fund value \(F_t\) and the index \(S_t\) by solving the stochastic differential equation obtained for \(F_t\) in Question (b). For simplicity we take \(F_0 := S_0^\alpha\).

Exercise 5.7 Solve the stochastic differential equation

\[
dX_t = h(t)X_t dt + \sigma X_t dB_t,
\]

where \(\sigma > 0\) and \(h(t)\) is a deterministic, integrable function of \(t \in \mathbb{R}_+\).

*Hint:* Look for a solution of the form \(X_t = f(t) e^{\sigma B_t - \sigma^2 t/2}\), where \(f(t)\) is a function to be determined, \(t \in \mathbb{R}_+\).

Exercise 5.8 Let \((B_t)_{t \in \mathbb{R}_+}\) denote a standard Brownian motion generating the filtration \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\).

a) Consider the Itô formula

\[
f(X_t) = f(X_0) + \int_0^t u_s \frac{\partial f}{\partial x}(X_s) dB_s + \int_0^t v_s \frac{\partial f}{\partial x}(X_s) ds + \frac{1}{2} \int_0^t u_s^2 \frac{\partial^2 f}{\partial x^2}(X_s) ds,
\]

(5.24)

where \(X_t = X_0 + \int_0^t u_s dB_s + \int_0^t v_s ds\).

Compute \(S_t := e^{X_t}\) by the Itô formula (5.24) applied to \(f(x) = e^x\) and \(X_t = \sigma B_t + \nu t\), \(\sigma > 0\), \(\nu \in \mathbb{R}\).

b) Let \(r > 0\). For which value of \(\nu\) does \((S_t)_{t \in \mathbb{R}_+}\) satisfy the stochastic differential equation

\[
dS_t = rS_t dt + \sigma S_t dB_t?
\]

c) Given \(\sigma > 0\), let \(X_t := (B_T - B_t)\sigma\), and compute \(\text{Var}[X_t], t \in [0, T]\).

d) Let the process \((S_t)_{t \in \mathbb{R}_+}\) be defined by \(S_t = S_0 e^{\sigma B_t + \nu t}, t \in \mathbb{R}_+\). Using the result of Exercise A.2, show that the conditional probability that \(S_T > K\) given \(S_t = x\) can be computed as

\[
\mathbb{P}(S_T > K \mid S_t = x) = \Phi\left(\frac{\log(x/K) + (T-t)\nu}{\sigma \sqrt{T-t}}\right), \quad t \in [0, T).
\]

*Hint:* Use the time splitting decomposition

\[
S_T = S_t \frac{S_T}{S_t} = S_t e^{(B_T - B_t)\sigma + (T-t)\nu}, \quad t \in [0, T].
\]
Problem 5.9  Stop-loss start-gain strategy (Lipton (2001) § 8.3.3., Exercise 4.14 continued). Let \((B_t)_{t \in \mathbb{R}_+}\) be a standard Brownian motion started at \(B_0 \in \mathbb{R}\).

a) We consider a simplified foreign exchange model in which the AUD is a risky asset and the AUD/SGD exchange rate at time \(t\) is modeled by \(B_t\), i.e. AU$1 equals SG$\(B_t\) at time \(t\). A foreign exchange (FX) European call option gives to its holder the right (but not the obligation) to receive AU$1 in exchange for \(K = \text{SG}1\) at maturity \(T\). Give the option payoff at maturity, quoted in SGD.

In the sequel, for simplicity we assume no time value of money \((r = 0)\), i.e. the (riskless) SGD account is priced \(A_t = A_0 = 1\), \(t \in [0, T]\).

b) Consider the following hedging strategy for the European call option of Question (a):
   i) If \(B_0 > 1\), charge the premium \(B_0 - 1\) at time 0, borrow SG$1 and purchase AU$1.
   ii) If \(B_0 < 1\), issue the option for free.
   iii) From time 0 to time \(T\), purchase* AU$1 every time \(B_t\) crosses \(K = 1\) from below, and sell† AU$1 each time \(B_t\) crosses \(K = 1\) from above.

Show that this strategy effectively hedges the foreign exchange European call option at maturity \(T\).

* Hint. Note that it suffices to consider four scenarios based on \(B_0 < 1\) vs \(B_0 < 1\) and \(B_T > 1\) vs \(B_T < 1\).

b) Determine the quantities \(\eta_t\) of SGD cash and \(\xi_t\) of (risky) AUDs to be held in the portfolio and express the portfolio value

\[ V_t = \eta_t + \xi_t B_t \]

at all times \(t \in [0, T]\).

d) Compute the integral summation

\[ \int_0^t \eta_s dA_s + \int_0^t \xi_s dB_s \]

of portfolio profits and losses at any time \(t \in [0, T]\).

* Hint. Apply the result of Question (e).

e) Is the portfolio strategy \((\eta_t, \xi_t)_{t \in [0, T]}\) self-financing? How to interpret the answer in practice?

* We need to borrow SG$1 if this is the first AUD purchase.
† We use the SG$1 product of the sale to refund the loan.