Chapter 6
Black-Scholes Pricing and Hedging

In this chapter we review the notions of assets, self-financing portfolios, risk-neutral probability measures, and arbitrage in continuous time. We also derive the Black-Scholes partial differential equation (PDE) for self-financing portfolios, and we solve this equation using the heat kernel method.

6.1 The Black-Scholes PDE

In this chapter we work in a market based on a riskless asset with price \((A_t)_{t \in \mathbb{R}^+}\) given by

\[
\frac{A_{t+dt} - A_t}{A_t} = r dt, \quad \frac{dA_t}{A_t} = r dt, \quad A'_t = rA_t, \quad t \in \mathbb{R}^+.
\]

with

\[
A_t = A_0 e^{rt}, \quad t \in \mathbb{R}^+,
\]

and a risky asset with price \((S_t)_{t \in \mathbb{R}^+}\) modeled using a geometric Brownian motion defined from the equation

\[
\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \quad t \in \mathbb{R}^+, \tag{6.1}
\]

with solution

\[
\bigcirc
\]
\[ S_t = S_0 \exp \left( \sigma B_t + \left( \mu - \frac{1}{2} \sigma^2 \right) t \right), \quad t \in \mathbb{R}_+, \]

cf. Proposition 5.15. The next Figure 6.1 presents a graph of underlying asset market data, which can be compared to the geometric Brownian motion of Figure 5.3.

Fig. 6.1: Graphs of underlying asset prices.

``` R
install.packages("quantmod")
library(quantmod)
getSymbols("0005.HK", from="2016-02-15", to=Sys.Date(), src="yahoo")
stock=Ad(stock)
write(stock, file = "data_exp", sep="\n")
chartSeries(stock, up.col="blue", theme="white")
```

The adjusted close price \( \text{Ad}(\cdot) \) is the closing price after adjustments for applicable splits and dividend distributions.

We start by deriving the Black-Scholes Partial Differential Equation (PDE) for the price of a self-financing portfolio. Note that the drift parameter \( \mu \) in (6.1) is absent in the PDE (6.2), and it does not appear as well in the Black-Scholes formula (6.9).

**Proposition 6.1.** Let \( (\eta_t, \xi_t)_{t \in \mathbb{R}_+} \) be a portfolio process such that

(i) the portfolio strategy \( (\eta_t, \xi_t)_{t \in \mathbb{R}_+} \) is self-financing,

(ii) the portfolio price \( V_t := \eta_t A_t + \xi_t S_t \), takes the form

\[ V_t = g(t, S_t), \quad t \in \mathbb{R}_+, \]

for some function \( g \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}_+) \) of \( t \) and \( S_t \).

Then the function \( g(t, x) \) satisfies the Black-Scholes PDE
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\[ rg(t, x) = \frac{\partial g}{\partial t}(t, x) + rx \frac{\partial g}{\partial x}(t, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 g}{\partial x^2}(t, x), \quad x > 0, \quad (6.2) \]

and \( \xi_t = \xi_t(S_t) \) is given by the partial derivative

\[ \xi_t = \xi_t(S_t) = \frac{\partial g}{\partial x}(t, S_t), \quad t \in \mathbb{R}_+. \quad (6.3) \]

**Proof.** (i) First, we note that the self-financing condition (5.8) in Proposition 5.8 implies

\[ dV_t = \eta_t dA_t + \xi_t dS_t \]

\[ = r\eta_t A_t dt + \mu \xi_t S_t dt + \sigma \xi_t S_t dB_t \]

\[ = rV_t dt + (\mu - r)\xi_t S_t dt + \sigma \xi_t S_t dB_t \]

\[ = rg(t, S_t) dt + (\mu - r)\xi_t S_t dt + \sigma \xi_t S_t dB_t, \quad t \in \mathbb{R}_+. \quad (6.4) \]

We now rewrite (5.18) under the form of an Itô process

\[ S_t = S_0 + \int_0^t v_s ds + \int_0^t u_s dB_s, \quad t \in \mathbb{R}_+, \]

as in (4.21), by taking

\[ u_t = \sigma S_t, \quad \text{and} \quad v_t = \mu S_t, \quad t \in \mathbb{R}_+. \]

(ii) By (4.23), the application of Itô’s formula Theorem 4.19 to

\[ V_t = g(t, S_t) \]

leads to

\[ dV_t = dg(t, S_t) \]

\[ = \frac{\partial g}{\partial t}(t, S_t) dt + \frac{\partial g}{\partial x}(t, S_t) dS_t + \frac{1}{2}(dS_t)^2 \frac{\partial^2 g}{\partial x^2}(t, S_t) \]

\[ = \frac{\partial g}{\partial t}(t, S_t) dt + v_t \frac{\partial g}{\partial x}(t, S_t) dt + u_t \frac{\partial g}{\partial x}(t, S_t) dB_t + \frac{1}{2}|u_t|^2 \frac{\partial^2 g}{\partial x^2}(t, S_t) dt \]

\[ = \frac{\partial g}{\partial t}(t, S_t) dt + \mu S_t \frac{\partial g}{\partial x}(t, S_t) dt + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 g}{\partial x^2}(t, S_t) dt + \sigma S_t \frac{\partial g}{\partial x}(t, S_t) dB_t. \quad (6.5) \]

By respective identification of the terms in \( dB_t \) and \( dt \) in (6.4) and (6.5) we get

\[
\begin{cases}
rg(t, S_t) dt + (\mu - r)\xi_t S_t dt = \frac{\partial g}{\partial t}(t, S_t) dt + \mu S_t \frac{\partial g}{\partial x}(t, S_t) dt + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 g}{\partial x^2}(t, S_t) dt, \\
\xi_t S_t \sigma dB_t = S_t \sigma \frac{\partial g}{\partial x}(t, S_t) dB_t,
\end{cases}
\]

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hence

\[
\begin{cases}
  r g(t, S_t) = \frac{\partial g}{\partial t}(t, S_t) + r S_t \frac{\partial g}{\partial x}(t, S_t) + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 g}{\partial x^2}(t, S_t), \\
  \xi_t = \frac{\partial g}{\partial x}(t, S_t), \quad 0 \leq t \leq T.
\end{cases}
\]

The derivative giving $\xi_t$ in (6.3) is called the Delta of the option price, see Proposition 6.3 below. The amount invested on the riskless asset is

\[
\eta_t A_t = V_t - \xi_t S_t = g(t, S_t) - S_t \frac{\partial g}{\partial x}(t, S_t),
\]

and $\eta_t$ is given by

\[
\eta_t = \frac{V_t - \xi_t S_t}{A_t} = \frac{1}{A_t} \left( g(t, S_t) - S_t \frac{\partial g}{\partial x}(t, S_t) \right) = \frac{1}{A_0 e^{rt}} \left( g(t, S_t) - S_t \frac{\partial g}{\partial x}(t, S_t) \right).
\]

In the next proposition we add a terminal condition $g(T, x) = f(x)$ to the Black-Scholes PDE in order to price a claim payoff $C$ of the form $C = h(S_T)$. As in the discrete-time case, the arbitrage price $\pi_t(C)$ at time $t \in [0, T]$ of the claim payoff $C$ is defined to be the price $V_t$ of the self-financing portfolio hedging $C$.

**Proposition 6.2.** The arbitrage price $\pi_t(C)$ at time $t \in [0, T]$ of the (vanilla) option with payoff $C = h(S_T)$ is given by $\pi_t(C) = g(t, S_t)$ and the hedging allocation $\xi_t$ is given by the partial derivative (6.3), where the function $g(t, x)$ is solution of the following Black-Scholes PDE:

\[
\begin{cases}
  r g(t, x) = \frac{\partial g}{\partial t}(t, x) + r x \frac{\partial g}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 g}{\partial x^2}(t, x), \\
  g(T, x) = h(x), \quad x > 0.
\end{cases}
\]

**Example - forward contracts**

When $C = S_T - K$ is the (linear) payoff function of a long forward contract, i.e. $f(x) = x - K$, the Black-Scholes PDE (6.7) admits the easy solution
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\[ g(t, x) = x - K e^{-(T-t)r}, \quad x > 0, \quad t \in [0, T], \]  

(6.8)

showing that the price at time \( t \) of the forward contract with payoff \( C = S_T - K \) is

\[ S_t - K e^{-(T-t)r}, \quad x > 0, \quad t \in [0, T]. \]

In addition, the Delta of the option price is given by

\[ \xi_t = \frac{\partial g}{\partial x}(t, S_t) = 1, \quad t \in [0, T], \]

which leads to a static "hedge and forget" strategy, cf. Exercise 6.7. The forward contract can be realized by the option issuer as follows:

a) At time \( t \), receive the option premium \( V_t := S_t - e^{-(T-t)r}K \) from the option buyer.

b) Borrow \( e^{-(T-t)r}K \) from the bank, to be refunded at maturity.

c) Buy the risky asset using the amount \( S_t - e^{-(T-t)r}K + e^{-(T-t)r}K = S_t \).

d) Hold the risky asset until maturity (do nothing, constant portfolio strategy).

e) At maturity \( T \), hand in the asset to the option holder, who gives the price \( K \) in exchange.

f) Use the amount \( K = e^{(T-t)r}e^{-(T-t)r}K \) to refund the lender of \( e^{-(T-t)r}K \) borrowed at time \( t \).

Another way to compute the option premium \( V_t \) is to state that the amount \( V_t - S_t \) has to be borrowed at time \( t \) in order to purchase the asset, and that the asset price \( K \) received at maturity \( T \) should be used to refund the loan, which yields

\[ (V_t - S_t)e^{-(T-t)r} = K, \quad 0 \leq t \leq T. \]

Forward contracts can be used for physical delivery, e.g. for live cattle. In the case of European options, the basic "hedge and forget" constant strategy

\[ \xi_t = 1, \quad \eta_t = \eta_0, \quad t \in [0, T], \]

will hedge the option only if

\[ S_T + \eta_0 A_T \geq (S_T - K)^+, \]

d.\ e. if \(-\eta_0 A_T \leq K \leq S_T\).

**Future contracts**

For a future contract expiring at time \( T \) we take \( K = S_0 e^{rT} \) and the contract is usually quoted at time \( t \) in terms of the forward price

\[ e^{(T-t)r}(S_t - K e^{-(T-t)r}) = e^{(T-t)r}S_t - K = e^{(T-t)r}S_t - S_0 e^{rT}, \]
discounted at time $T$, or simply using $e^{(T-t)r}S_t$. Future contracts are non-deliverable forward contracts which are “marked to market” at each time step via a cash flow exchange between the two parties, ensuring that the absolute difference $|e^{(T-t)r}S_t - K|$ is being credited to the buyer’s account if $e^{(T-t)r}S_t > K$, or to the seller’s account if $e^{(T-t)r}S_t < K$.

### 6.2 European Call Options

Recall that in the case of a European call option with strike price $K$ the payoff function is given by $f(x) = (x - K)^+$ and the Black-Scholes PDE (6.7) reads

$$
\begin{align*}
rg_c(t, x) &= \frac{\partial g_c}{\partial t}(t, x) + rx\frac{\partial g_c}{\partial x}(t, x) + \frac{1}{2}\sigma^2x^2\frac{\partial^2 g_c}{\partial x^2}(t, x) \\
g_c(T, x) &= (x - K)^+.
\end{align*}
$$

In Sections 6.5 and 6.6 we will prove that the solution of this PDE is given by the Black-Scholes formula

$$
g_c(t, x) = \text{Bl}(K, x, \sigma, r, T-t) = x\Phi(d_+(T-t)) - K e^{-(T-t)r}\Phi(d_-(T-t)),
$$

with

$$
\begin{align*}
d_+(T-t) &:= \frac{\log(x/K) + (r + \sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}, \\
d_-(T-t) &:= \frac{\log(x/K) + (r - \sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}, 
\end{align*}
$$

where “log” denotes the natural logarithm “ln”, see Proposition 6.8 below, where “log” denotes the natural logarithm “ln” and we have the relation

$$
d_+(T-t) = d_-(T-t) + |\sigma|\sqrt{T-t}.
$$

Here,

$$
\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy, \quad x \in \mathbb{R},
$$

denotes the standard Gaussian Cumulative Distribution Function (CDF), which satisfies the relation

$$
\Phi(x) = 1 - \Phi(x), \quad x \in \mathbb{R}.
$$
In other words, a European call option with strike price $K$ and maturity $T$ is priced at time $t \in [0, T]$ as

$$g_c(t, S_t) = Bl(K, S_t, \sigma, r, T - t)$$
$$= S_t \Phi(d_+(T - t)) - K e^{-(T-t)r} \Phi(d_-(T - t)), \quad t \in [0, T].$$

The following R script is an implementation of the Black-Scholes formula for European call options in R.*

```r
BSCall <- function(S, K, r, T, sigma) {
  d1 <- (log(S/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T))
  d2 <- d1 - sigma * sqrt(T)
  BSCall = S*pnorm(d1) - K*exp(-r*T)*pnorm(d2)
  BSCall
}
```

The interest in the formula (6.9) in comparison with the Cox-Ross-Rubinstein (CRR) model of Section 2.6 is that provides an analytical solution that can be evaluated in one single step, which is therefore computationally much more efficient.

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* Download the corresponding IPython notebook that can be run here.
† Right-click on the figure for interaction and “Full Screen Multimedia” view.
Figure 6.3 presents an interactive graph of the Black-Scholes call price map, \( i.e. \) the solution 
\[
(t, x) \mapsto g_c(t, x) = x\Phi(d_+(T-t)) - K e^{-(T-t)r}\Phi(d_-(T-t))
\]
of the Black-Scholes PDE (6.7) for a call option.

Proposition 6.3. The Black-Scholes Delta of the European call option is given by
\[
\xi_t = \xi_t(S_t) = \frac{\partial g_c}{\partial x}(t, S_t) = \Phi\left( d_+(T-t) \right) \in [0, 1],
\]
where \( d_+(T-t) \) is given by (6.10).

Proof. By (6.9) we have
\[
\frac{\partial g_c}{\partial x}(t, x) = \frac{\partial}{\partial x}\left(x\Phi\left(\frac{\log(x/K) + (r + \sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)\right)
\]
\[
- Ke^{-(T-t)r} \frac{\partial}{\partial x}\left(\Phi\left(\frac{\log(x/K) + (r - \sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)\right)
\]
\[
= \Phi\left(\frac{\log(x/K) + (r + \sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)
\]
\[
+ x\frac{\partial}{\partial x}\Phi\left(\frac{\log(x/K) + (r + \sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)
\]
\[
- K e^{-(T-t)r} \frac{\partial}{\partial x}\Phi\left(\frac{\log(x/K) + (r - \sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)
\]

* The animation works in Acrobat Reader on the entire pdf file.

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= \Phi \left( \frac{\log(x/K) + (r + \sigma^2/2)(T - t)}{|\sigma|\sqrt{T - t}} \right) \\
+ \frac{x}{|\sigma|\sqrt{2(T - t)\pi}} \exp\left( -\frac{1}{2} \left( \frac{\log(x/K) + (r + \sigma^2/2)(T - t)}{|\sigma|\sqrt{T - t}} \right)^2 \right) \\
- \frac{K e^{-(T-t)r}}{|\sigma|\sqrt{2(T - t)\pi}} \exp\left( -\frac{1}{2} \left( \frac{\log(x/K) + (r - \sigma^2/2)(T - t)}{|\sigma|\sqrt{T - t}} \right)^2 \right) \\
+ (T - t)r + \log \frac{x}{K} \right) \\
= \Phi \left( \frac{\log(x/K) + (r + \sigma^2/2)(T - t)}{|\sigma|\sqrt{T - t}} \right). \tag{6.14}

As a consequence of Proposition 6.3, the Black-Scholes price splits into a risky component $S_t \Phi(d_+(T - t))$ and a riskless component, as follows:

$$g_c(t, S_t) = S_t \Phi(d_+(T - t)) - K e^{-(T-t)r} \Phi(d_-(T - t)), \quad t \in [0, T].$$

See Exercise 6.4 for a computation of the boundary values of $g_c(t, x)$, $t \in [0, T], x > 0$. The following R script is an implementation of the Black-Scholes Delta for European call options in R.

```r
Delta <- function(S, K, r, T, sigma)
{d1 <- (log(S/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T))
Delta = pnorm(d1);Delta}
```

In Figure 6.5 we plot the Delta of the European call option as a function of the underlying asset price and of time to maturity.
The Gamma of the European call option is defined as the second derivative of the option price with respect to the underlying asset price, which gives

\[
\gamma_t = \frac{1}{S_t|\sigma|\sqrt{T-t}} \Phi'(d_+(T-t)) \\
= \frac{1}{S_t|\sigma|\sqrt{2(T-t)}\pi} \exp\left(-\frac{1}{2} \left( \log\left(\frac{S_t}{K}\right) + (r + \sigma^2/2)(T-t) \right)^2 \right) \\
\geq 0.
\]

In particular, a positive value of \(\gamma_t\) implies that the Delta \(\xi_t = \xi_t(S_t)\) should increase when the underlying asset price \(S_t\) increases.

In Figure 6.6 we plot the (truncated) value of the Gamma of a European call option as a function of the underlying asset price and of time to maturity.

As Gamma is always nonnegative, the Black-Scholes hedging strategy is to keep buying the risky underlying asset when its price increases, and to sell it when its price decreases, as can be checked from Figure 6.6.
Numerical example - hedging of a call option

In Figure 6.7 we consider the historical stock price of HSBC Holdings (0005.HK) over one year:

![Graph of the stock price of HSBC Holdings.](image)

Fig. 6.7: Graph of the stock price of HSBC Holdings.

Consider the call option issued by Societe Generale on 31 December 2008 with strike price $K=63.704$, maturity $T =$ October 05, 2009, and an entitlement ratio of 100, meaning that one option contract is divided into 100 *warrants*, cf. page 8. The next graph gives the time evolution of the Black-Scholes portfolio price

$$ t \mapsto g_c(t, S_t) $$

driven by the market price $t \mapsto S_t$ of the risky underlying asset as given in Figure 6.7, in which the number of days is counted from the origin and not from maturity.

![Path of the Black-Scholes price for a call option on HSBC.](image)

Fig. 6.8: Path of the Black-Scholes price for a call option on HSBC.

As a consequence of Proposition 6.3, in the Black-Scholes call option hedging model, the amount invested in the risky asset is
\[ S_t \xi_t = S_t \Phi(d_+(T - t)) = S_t \Phi \left( \frac{\log(S_t/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \right) \geq 0, \]

which is always nonnegative, \textit{i.e.} there is no short selling, and the amount invested on the riskless asset is

\[ \eta_t A_t = -K e^{-(T-t)r} \Phi \left( \frac{\log(S_t/K) + (r - \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \right) \leq 0, \]

which is always negative, \textit{i.e.} we are constantly borrowing money, as noted in Figure 6.9.

![Fig. 6.9: Time evolution of a hedging portfolio for a call option on HSBC.](https://www.ntu.edu.sg/home/nprivault/indext.html)

A comparison of Figure 6.9 with market data can be found in Figures 9.11 and 9.12 below.

\textit{Cash settlement.} In the case of a cash settlement, the option issuer will satisfy the option contract by selling \( \xi_T = 1 \) stock at the price \( S_T = \$83 \), refund the \( K = \$63 \) risk-free investment, and hand in the remaining amount \( C = (S_T - K)^+ = 83 - 63 = \$20 \) to the option holder.

\textit{Physical delivery.} In the case of physical delivery of the underlying asset, the option issuer will deliver \( \xi_T = 1 \) stock to the option holder in exchange for \( K = \$63 \), which will be used together with the portfolio value to refund the risk-free loan.

### 6.3 European Put Options

Similarly, in the case of a European put option with strike price \( K \) the payoff function is given by \( f(x) = (K - x)^+ \) and the Black-Scholes PDE (6.7) reads
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\[
\begin{align*}
rg_p(t, x) &= \frac{\partial g_p}{\partial t}(t, x) + rx \frac{\partial g_p}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 g_p}{\partial x^2}(t, x), \\
g_p(T, x) &= (K - x)^+, 
\end{align*}
\]

with explicit solution

\[
g_p(t, x) = Ke^{-(T-t)r} \Phi (-d_-(T-t)) - x \Phi (-d_+(T-t)), \tag{6.15}
\]

with

\[
d_+(T-t) = \frac{\log(x/K) + (r + \sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}, \tag{6.16}
\]

\[
d_-(T-t) = \frac{\log(x/K) + (r - \sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}, \tag{6.17}
\]

as illustrated in Figure 6.10.

\[
\begin{array}{c}
\text{Fig. 6.10: Graph of the Black-Scholes put price function with strike price } K = 100. \)
\end{array}
\]

In other words, a European put option with strike price \(K\) and maturity \(T\) is priced at time \(t \in [0, T]\) as

\[
g_p(t, S_t) = Ke^{-(T-t)r} \Phi (-d_-(T-t)) - S_t \Phi (-d_+(T-t)), 
\]

\[
t \in [0, T].
\]

* Right-click on the figure for interaction and “Full Screen Multimedia” view.
The following R script is an implementation of the Black-Scholes formula for European put options in R.

```r
BSPut <- function(S, K, r, T, sigma)
    {d1 = (log(S/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T))
     d2 = d1 - sigma * sqrt(T)
     BSPut = K*exp(-r*T) * pnorm(-d2) - S*pnorm(-d1)
     BSPut}
```

**Call-put parity**

**Proposition 6.4.** Call-put parity. We have the call-put parity relation

$$g_c(t, S_t) - g_p(t, S_t) = S_t - K e^{-(T-t)r}, \quad 0 \leq t \leq T,$$

between the Black-Scholes prices of call and put options, in terms of the forward contract price $S_t - K e^{-(T-t)r}$.

**Proof.** The call-put parity (6.18) is a consequence of the relation

$$x - K = (x - K)^+ - (K - x)^+$$

satisfied by the terminal call and put payoff functions in the Black-Scholes PDE (6.7). It can also be verified directly from (6.9) and (6.15) as

$$g_c(t, x) - g_p(t, x) = x\Phi(d_+(T-t)) - K e^{-(T-t)r}\Phi(d_-(T-t))$$

$$- (K e^{-(T-t)r}\Phi(-d_-(T-t)) - x\Phi(-d_+(T-t)))$$

$$= x\Phi(d_+(T-t)) - K e^{-(T-t)r}\Phi(d_-(T-t))$$

$$- K e^{-(T-t)r}(1 - \Phi(d_-(T-t))) + x(1 - \Phi(d_+(T-t)))$$

* The animation works in Acrobat Reader on the entire pdf file.
The \textit{Delta} of the Black-Scholes put option can be obtained by differentiation of the call-put parity relation (6.18) and Proposition 6.3.

\begin{proposition}
The Delta of the Black-Scholes put option is given by
\begin{equation*}
\xi_t = -(1 - \Phi(d_+(T - t))) = -\Phi(-d_+(T - t)) \in [-1, 0], \quad 0 \leq t \leq T.
\end{equation*}
\end{proposition}

Numerical example - hedging of a put option

For one more example, we consider a put option issued by BNP Paribas on 04 November 2008 with strike price $K=\$77.667$, maturity $T = \text{October 05, 2009}$, and entitlement ratio $92.593$, cf. page 8. In the next Figure 6.12, the number of days is counted from the origin and not from maturity.

As a consequence of Proposition 6.5, the amount invested on the risky asset for the hedging of a put option is
\begin{equation*}
-S_t \Phi(d_+(T - t)) = -S_t \Phi\left(-\frac{\log(S_t/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}\right) \leq 0,
\end{equation*}
\textit{i.e.} there is always short selling, and the amount invested on the riskless asset priced $A_t = e^{rt}$, $t \in [0, T]$, is
\begin{equation*}
K e^{-(T-t)r} \Phi\left(-\frac{\log(S_t/K) + (r - \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}\right) \geq 0,
\end{equation*}
which is always positive, *i.e.* we are constantly investing on the riskless asset.

In the above example the put option finished out of the money (OTM), so that no cash settlement or physical delivery occurs. A comparison of Figure 6.9 with market data can be found in Figures 9.13 and 9.14 below.

6.4 Market Terms and Data

The following Table 6.1 provides a summary of formulas for the computation of Black-Scholes sensitivities, also called *Greeks.*

<table>
<thead>
<tr>
<th></th>
<th>Call option</th>
<th>Put option</th>
</tr>
</thead>
<tbody>
<tr>
<td>Option price</td>
<td>( g(t, S_t) )</td>
<td>( K e^{-(T-t)r} \Phi(-d_-(T-t)) - S_t \Phi(-d_+(T-t)) )</td>
</tr>
<tr>
<td>Delta (( \Delta ))</td>
<td>( \frac{\partial g}{\partial x}(t, S_t) ) ( \Phi(d_+(T-t)) )</td>
<td>( -\Phi(-d_+(T-t)) )</td>
</tr>
<tr>
<td>Gamma (( \Gamma ))</td>
<td>( \frac{\partial^2 g}{\partial x^2}(t, S_t) ) ( \frac{\Phi'(d_+(T-t))}{S_t \sigma \sqrt{T-t}} )</td>
<td></td>
</tr>
<tr>
<td>Vega</td>
<td>( \frac{\partial g}{\partial \sigma}(t, S_t) ) ( S_t \sqrt{T-t} \Phi'(d_+(T-t)) )</td>
<td></td>
</tr>
<tr>
<td>Theta (( \Theta ))</td>
<td>( \frac{\partial g}{\partial t}(t, S_t) ) ( -\frac{S_t \sigma \Phi'(d_+(T-t))}{2\sqrt{T-t}} - rK e^{-(T-t)r} \Phi(d_-(T-t)) )</td>
<td>( -\frac{S_t \sigma \Phi'(d_+(T-t))}{2\sqrt{T-t}} + rK e^{-(T-t)r} \Phi(-d_-(T-t)) )</td>
</tr>
<tr>
<td>Rho (( \rho ))</td>
<td>( \frac{\partial g}{\partial r}(t, S_t) ) ( K(T-t) e^{-(T-t)r} \Phi(d_-(T-t)) )</td>
<td>( -K(T-t) e^{-(T-t)r} \Phi(-d_-(T-t)) )</td>
</tr>
</tbody>
</table>

Table 6.1: Black-Scholes Greeks (Wikipedia).

From Table 6.1 we can conclude that call option prices are increasing functions of the underlying asset price \( S_t \), of the interest rate \( r \), and of the volatility parameter \( \sigma \). Similarly, put option prices are decreasing functions of the

---

*“Every class feels like attending a Greek lesson” (AY2018-2019 student feedback).*
Black-Scholes Pricing and Hedging

underlying asset price $S_t$, of the interest rate $r$, and increasing functions of the volatility parameter $\sigma$.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Variation of call option prices</th>
<th>Variation of put option prices</th>
</tr>
</thead>
<tbody>
<tr>
<td>Underlying $S_t$</td>
<td>increasing ↗</td>
<td>decreasing ↘</td>
</tr>
<tr>
<td>Volatility $\sigma$</td>
<td>increasing ↗</td>
<td>increasing ↗</td>
</tr>
<tr>
<td>Time $t$</td>
<td>decreasing ↘</td>
<td>depends on the underlying price level</td>
</tr>
<tr>
<td>Interest rate $r$</td>
<td>increasing ↗</td>
<td>decreasing ↘</td>
</tr>
</tbody>
</table>

Table 6.2: Variations of Black-Scholes prices.

**Intrinsic value.** The *intrinsic value* at time $t \in [0, T]$ of the option with payoff $C = h(S_T^{(1)})$ is given by the immediate exercise payoff $h(S_t^{(1)})$. The *extrinsic value* at time $t \in [0, T]$ of the option is the remaining difference $\pi_t(C) - h(S_t^{(1)})$ between the option price $\pi_t(C)$ and the immediate exercise payoff $h(S_t^{(1)})$. In general, the option price $\pi_t(C)$ decomposes as

$$\pi_t(C) = h(S_T^{(1)}) + \pi_t(C) - h(S_t^{(1)}) = \underbrace{\text{intrinsic value}}_{\text{extrinsic value}}, \quad t \in [0, T].$$

**Gearing.** The *gearing* at time $t \in [0, T]$ of the option with payoff $C = h(S_T)$ is defined as the ratio

$$G_t := \frac{S_t}{\pi_t(C)} = \frac{S_t}{g(t, S_t)}, \quad t \in [0, T].$$

**Effective gearing.** The *effective gearing* at time $t \in [0, T]$ of the option with payoff $C = h(S_T)$ is defined as the ratio

$$G_t^e := G_t \xi_t = \frac{\xi_t S_t}{\pi_t(C)} = \frac{S_t}{\pi_t(C)} \frac{\partial g}{\partial x}(t, S_t)$$
\[ \frac{S_t}{g(t, S_t)} \frac{\partial g}{\partial x}(t, S_t) = S_t \frac{\partial \log g}{\partial x}(t, S_t), \quad t \in [0, T]. \]

The effective gearing \( G^e_t = \xi_t S_t / \pi_t(C) \) can be interpreted as the *hedge ratio*, i.e. the percentage of the portfolio which is invested on the risky asset. The ratio \( G^e_t = S_t \frac{\partial \log g(t, S_t)}{\partial x} \) can also be interpreted as an *elasticity coefficient*.

**Break-even price.** The *break-even* price \( \text{BEP}_t \) of the underlying asset is the value of \( S \) for which the intrinsic option value \( h(S) \) equals the option price \( \pi_t(C) \) at time \( t \in [0, T] \). For European call options it is given by

\[ \text{BEP}_t := K + \pi_t(C) = K + g(t, S_t), \quad t = 0, 1, \ldots, N. \]

whereas for European put options it is given by

\[ \text{BEP}_t := K - \pi_t(C) = K - g(t, S_t), \quad 0 \leq t \leq T. \]

**Premium.** The option *premium* \( \text{OP}_t \) can be defined as the variation required from the underlying asset price in order to reach the break-even price, i.e. we have

\[ \text{OP}_t := \frac{\text{BEP}_t - S_t}{S_t} = \frac{K + g(t, S_t) - S_t}{S_t}, \quad 0 \leq t \leq T, \]

for European call options, and

\[ \text{OP}_t := \frac{S_t - \text{BEP}_t}{S_t} = \frac{S_t + g(t, S_t) - K}{S_t}, \quad 0 \leq t \leq T, \]

for European put options, see Figure 6.14 below. The term “premium” is sometimes also used to denote the arbitrage price \( g(t, S_t) \) of the option.
In the next proposition we notice that the solution \( f(t, x) \) of the Black-Scholes PDE (6.7) can be transformed into a solution \( g(t, y) \) of the simpler heat equation by a change of variable and a time inversion \( t \mapsto -T - t \) on the interval \([0, T]\), so that the terminal condition at time \( T \) in the Black-Scholes equation (6.19) becomes an initial condition at time \( t = 0 \) in the heat equation.
heat equation (6.22). See also here for a related discussion on changes of variables for the Black-Scholes PDE.

**Proposition 6.6.** Assume that $f(t, x)$ solves the Black-Scholes PDE

$$
\begin{align*}
rf(t, x) &= \frac{\partial f}{\partial t}(t, x) + rx \frac{\partial f}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}(t, x), \\
(T, x) &= (x - K)^+,
\end{align*}
$$

with terminal condition $h(x) = (x - K)^+, x > 0$. Then the function $g(t, y)$ defined by

$$
g(t, y) = e^{rt} f(T - t, e^{\sigma|y|+(\sigma^2/2-r)t})
$$

solves the heat equation (6.24) with initial condition

$$
g(0, y) = h(e^{\sigma|y|}), \quad y \in \mathbb{R},
$$

i.e. we have

$$
\begin{align*}
\frac{\partial g}{\partial t}(t, y) &= \frac{1}{2} \frac{\partial^2 g}{\partial y^2}(t, y) \\
g(0, y) &= h(e^{\sigma|y|}).
\end{align*}
$$

Proposition 6.6 will be proved in Section 6.6.
In this section we focus on the heat equation

\[
\frac{\partial \varphi}{\partial t}(t, y) = \frac{1}{2} \frac{\partial^2 \varphi}{\partial y^2}(t, y)
\]  

(6.23)

which is used to model the diffusion of heat over time through solids. Here, the data of \( g(x, t) \) represents the temperature measured at time \( t \) and point \( x \). We refer the reader to Widder (1975) for a complete treatment of this topic.

We can check by a direct calculation that the Gaussian probability density function

\[
\varphi(t, y) := \frac{1}{\sqrt{2\pi t}} e^{-y^2/(2t)}, \quad t > 0, \ y \in \mathbb{R},
\]
solves the heat equation (6.23), as follows:

\[
\frac{\partial \varphi}{\partial t}(t, y) = \frac{\partial}{\partial t} \left( \frac{e^{-y^2/(2t)}}{\sqrt{2\pi t}} \right)
= -\frac{e^{-y^2/(2t)}}{2t^{3/2}\sqrt{2\pi}} + \frac{y^2}{2t^2} \frac{e^{-y^2/(2t)}}{\sqrt{2\pi t}}
= \left( -\frac{1}{2t} + \frac{y^2}{2t^2} \right) \varphi(t, y),
\]

and

\[
\frac{1}{2} \frac{\partial^2 \varphi}{\partial y^2}(t, y) = -\frac{1}{2} \frac{\partial}{\partial y} \left( \frac{y e^{-y^2/(2t)}}{t \sqrt{2\pi t}} \right)
\]

* The animation works in Acrobat Reader on the entire pdf file.
\[= -\frac{e^{-y^2/(2t)}}{2t\sqrt{2\pi t}} + \frac{y^2}{2t^2} \frac{e^{-y^2/(2t)}}{\sqrt{2\pi t}}\]

\[= \left(-\frac{1}{2t} + \frac{y^2}{2t^2}\right) \varphi(t,y), \quad t \in \mathbb{R}_+, \ y \in \mathbb{R}.\]

Fig. 6.17: Time-dependent solution of the heat equation.*

In Section 6.6 the heat equation (6.23) will be shown to be equivalent to the Black-Scholes PDE after a change of variables. In particular this will lead to the explicit solution of the Black-Scholes PDE.

**Proposition 6.7.** The heat equation

\[
\left\{ \begin{array}{l}
\frac{\partial g}{\partial t}(t,y) = \frac{1}{2} \frac{\partial^2 g}{\partial y^2}(t,y) \\
g(0,y) = \psi(y)
\end{array} \right. \quad (6.24)
\]

with continuous initial condition

\[g(0,y) = \psi(y)\]

has the solution

\[g(t,y) = \int_{-\infty}^{\infty} \psi(z) e^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}}, \quad y \in \mathbb{R}, \quad t > 0. \quad (6.25)\]

**Proof.** We have

\[
\frac{\partial g}{\partial t}(t,y) = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} \psi(z) e^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}}
\]

* The animation works in Acrobat Reader on the entire pdf file.
\begin{align*}
&= \int_{-\infty}^{\infty} \psi(z) \frac{\partial}{\partial t} \left( \frac{e^{-(y-z)^2/(2t)}}{\sqrt{2\pi t}} \right) \, dz \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \psi(z) \left( \frac{(y-z)^2}{t} - \frac{1}{t} \right) e^{-(y-z)^2/(2t)} \, dz \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \psi(z) \frac{\partial^2}{\partial z^2} e^{-(y-z)^2/(2t)} \, dz \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \psi(z) \frac{\partial^2}{\partial y^2} e^{-(y-z)^2/(2t)} \, dz \\
&= \frac{1}{2} \frac{\partial^2}{\partial y^2} \int_{-\infty}^{\infty} \psi(z) e^{-(y-z)^2/(2t)} \, dz \\
&= \frac{1}{2} \frac{\partial^2}{\partial y^2} g(t, y).
\end{align*}

On the other hand, it can be checked that at time \( t = 0 \) we have
\[
\lim_{t \to 0} \int_{-\infty}^{\infty} \psi(z) e^{-(y-z)^2/(2t)} \, dz = \frac{\psi(y)}{\sqrt{2\pi t}}
\]
\[
= \psi(y), \quad y \in \mathbb{R}.
\]

The next Figure 6.18 shows the evolution of \( g(t, x) \) with initial condition based on the European call payoff function, as
\[
g(0, y) = (e^{\sigma |y - K|})^+, \quad y \in \mathbb{R}.
\]

Fig. 6.18: Time-dependent solution of the heat equation.*

Let us provide a second proof of Proposition 6.7, this time using Brownian motion and stochastic calculus. First, note that under the change of variable

* The animation works in Acrobat Reader on the entire pdf file.
\[ x = z - y \] we have
\[
g(t, y) = \int_{-\infty}^{\infty} \psi(z) e^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}}
\]
\[
= \int_{-\infty}^{\infty} \psi(y + x) e^{-x^2/(2t)} \frac{dx}{\sqrt{2\pi t}}
\]
\[
= \mathbb{E}[\psi(y + B_t)]
\]
\[
= \mathbb{E}[\psi(y - B_t)],
\]

where \((B_t)_{t \in \mathbb{R}^+}\) is a standard Brownian motion and \(B_t \sim \mathcal{N}(0, t), \ t \in \mathbb{R}^+\).

Applying Itô’s formula and using the fact that the expectation of the stochastic integral with respect to Brownian motion is zero, see Relation (4.16) in Proposition 4.16, we find
\[
g(t, y) = \mathbb{E}[\psi(y - B_t)]
\]
\[
= \psi(y) - \mathbb{E}\left[ \int_0^t \psi'(y - B_s) dB_s \right] + \frac{1}{2} \mathbb{E}\left[ \int_0^t \psi''(y - B_s) ds \right]
\]
\[
= \psi(y) + \frac{1}{2} \int_0^t \mathbb{E}[\psi''(y - B_s)] \ ds
\]
\[
= \psi(y) + \frac{1}{2} \int_0^t \frac{\partial^2}{\partial y^2} \mathbb{E}[\psi(y - B_s)] \ ds
\]
\[
= \psi(y) + \frac{1}{2} \int_0^t \frac{\partial^2 g}{\partial y^2}(s, y) ds.
\]

Hence we have
\[
\frac{\partial g}{\partial t}(t, y) = \frac{\partial}{\partial t} \mathbb{E}[\psi(y - B_t)]
\]
\[
= \frac{1}{2} \frac{\partial^2}{\partial y^2} \mathbb{E}[\psi(y - B_t)]
\]
\[
= \frac{1}{2} \frac{\partial^2 g}{\partial y^2}(t, y).
\]

Concerning the initial condition we check that
\[
g(0, y) = \mathbb{E}[\psi(y - B_0)] = \mathbb{E}[\psi(y)] = \psi(y).
\]

The expression \(g(t, y) = \mathbb{E}[\psi(y - B_t)]\) provides a probabilistic interpretation of the heat diffusion phenomenon based on Brownian motion. Namely, when \(\psi_\varepsilon(y) := 1_{[-\varepsilon, \varepsilon]}(y)\), we find that
\[
g_\varepsilon(t, y) = \mathbb{E}[\psi_\varepsilon(y - B_t)]
\]
\[
= \mathbb{E}[1_{[-\varepsilon, \varepsilon]}(y - B_t)]
\]
\[
= \mathbb{P}(y - B_t \in [-\varepsilon, \varepsilon]).
\]
= \mathbb{P}(y - \varepsilon \leq B_t \leq y + \varepsilon)

represents the probability of finding $B_t$ within a neighborhood $[y - \varepsilon, y + \varepsilon]$ of the point $y \in \mathbb{R}$.

## 6.6 Solution of the Black-Scholes PDE

In this section we solve the Black-Scholes PDE by the kernel method of Section 6.5 and a change of variables. This solution method uses the change of variables (6.20) of Proposition 6.6 and a time inversion from which the terminal condition at time $T$ in the Black-Scholes equation becomes an initial condition at time $t = 0$ in the heat equation.

Next, we state the proof Proposition 6.6.

**Proof.** Letting $s = T - t$ and $x = e^{\sigma|y+(\sigma^2/2-r)t}}$ and using Relation (6.20), i.e.
\[
g(t, y) = e^{rt} f(T - t, e^{\sigma|y+(\sigma^2/2-r)t}}),
\]
we have
\[
\frac{\partial g}{\partial t}(t, y) = r e^{rt} f(T - t, e^{\sigma|y+(\sigma^2/2-r)t}}) - e^{rt} \frac{\partial f}{\partial s}(T - t, e^{\sigma|y+(\sigma^2/2-r)t}}) \\
+ \left( \frac{\sigma^2}{2} - r \right) e^{rt} e^{\sigma|y+(\sigma^2/2-r)t}} \frac{\partial f}{\partial x}(T - t, e^{\sigma|y+(\sigma^2/2-r)t}})
\]
\[
= r e^{rt} f(T - t, x) - e^{rt} \frac{\partial f}{\partial s}(T - t, x) + \left( \frac{\sigma^2}{2} - r \right) e^{rt} x \frac{\partial f}{\partial x}(T - t, x)
\]
\[
= \frac{1}{2} e^{rt} x^2 \sigma^2 \frac{\partial^2 f}{\partial x^2}(T - t, x) + \frac{\sigma^2}{2} e^{rt} x \frac{\partial^2 f}{\partial x^2}(T - t, x),
\]
(6.26)

where on the last step we used the Black-Scholes PDE. On the other hand we have
\[
\frac{\partial g}{\partial y}(t, y) = \sigma|e^{rt} e^{\sigma|y+(\sigma^2/2-r)t}} \frac{\partial f}{\partial x}(T - t, e^{\sigma|y+(\sigma^2/2-r)t}})
\]
and
\[
\frac{1}{2} \frac{\partial^2 g}{\partial y^2}(t, y) = \frac{\sigma^2}{2} e^{rt} e^{\sigma|y+(\sigma^2/2-r)t}} \frac{\partial f}{\partial x}(T - t, e^{\sigma|y+(\sigma^2/2-r)t}}) \\
+ \frac{\sigma^2}{2} e^{rt} e^{2|\sigma|y+2(\sigma^2/2-r)t}} \frac{\partial^2 f}{\partial x^2}(T - t, e^{\sigma|y+(\sigma^2/2-r)t}})
\]
\[
= \frac{\sigma^2}{2} e^{rt} x \frac{\partial f}{\partial x}(T - t, x) + \frac{\sigma^2}{2} e^{rt} x^2 \frac{\partial^2 f}{\partial x^2}(T - t, x).
\]
(6.27)
We conclude by comparing (6.26) with (6.27), which shows that \( g(t, x) \) solves the heat equation (6.24) with initial condition

\[
g(0, y) = f(T, e^{\sigma |y|}) = h(e^{\sigma |y|}).
\]

\[
\square
\]

In the next proposition we recover the Black-Scholes formula (6.9) by solving the PDE (6.19). The Black-Scholes formula will also be recovered by a probabilistic argument via the computation of an expected value in Proposition 7.7.

**Proposition 6.8.** When \( h(x) = (x - K)^+ \), the solution of the Black-Scholes PDE (6.19) is given by

\[
f(t, x) = x\Phi(d_+(T - t)) - K e^{-(T-t)r} \Phi(d_-(T - t)), \quad x > 0,
\]

where

\[
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy, \quad x \in \mathbb{R},
\]

and

\[
\begin{align*}
d_+(T - t) &:= \frac{\log(x/K) + (r + \sigma^2/2)(T - t)}{|\sigma|\sqrt{T - t}}, \\
d_-(T - t) &:= \frac{\log(x/K) + (r - \sigma^2/2)(T - t)}{|\sigma|\sqrt{T - t}},
\end{align*}
\]

\( x > 0, \; t \in [0,T). \)

**Proof.** By inversion of Relation (6.20) with \( s = T - t \) and \( x = e^{\sigma |y| + (\sigma^2/2 - r)t} \) we get

\[
f(s, x) = e^{-(T-s)r} g \left( T - s, \frac{-\sigma^2/2 - r(T - s) + \log x}{|\sigma|} \right)
\]

and

\[
h(x) = \psi \left( \frac{\log x}{|\sigma|} \right), \quad x > 0, \quad \text{or} \quad \psi(y) = h \left( e^{\sigma |y|} \right), \quad y \in \mathbb{R}.
\]

Hence, using the solution (6.25) and Relation (6.21) we get

\[
f(t, x) = e^{-(T-t)r} g \left( T - t, \frac{-\sigma^2/2 - r(T - t) + \log x}{|\sigma|} \right)
\]

\[= e^{-(T-t)r} \int_{-\infty}^{\infty} \psi \left( \frac{-\sigma^2/2 - r(T - t) + \log x}{|\sigma|} + z \right) e^{-z^2/(2(T-t))} \frac{dz}{\sqrt{2(T-t)} \pi}\]
Exercise 6.1 Bachelier model. Consider a market made of a riskless asset valued \( A_t = A_0 \) with zero interest rate, \( t \in \mathbb{R}_+ \), and a risky asset whose price

\[
= e^{-(T-t)} \int_{-\infty}^{\infty} h\left(x e^{\sigma z-(\sigma^2/2-r)(T-t)} \right) e^{-x^2/(2(T-t))} \frac{dz}{\sqrt{2(T-t)\pi}}
\]

\[
= e^{-(T-t)} \int_{-\infty}^{\infty} (x e^{\sigma z-(\sigma^2/2-r)(T-t)} - K) + e^{-z^2/(2(T-t))} \frac{dz}{\sqrt{2(T-t)\pi}}
\]

\[
= e^{-(T-t)} \times \int_{\log(x)-\sigma^2/4(T-t)\log(K-x)}^{\infty} (x e^{\sigma z-(\sigma^2/2-r)(T-t)} - K) e^{-z^2/(2(T-t))} \frac{dz}{\sqrt{2(T-t)\pi}}
\]

\[
= x e^{-(T-t)} \int_{-\infty}^{d_-(T-t)\sqrt{T-t}} e^{\sigma z-(\sigma^2/2-r)(T-t)} e^{-z^2/(2(T-t))} \frac{dz}{\sqrt{2(T-t)\pi}}
\]

\[
= x \int_{-\infty}^{d_-(T-t)\sqrt{T-t}} e^{\sigma z-(\sigma^2/2-r)(T-t)} e^{-z^2/(2(T-t))} \frac{dz}{\sqrt{2(T-t)\pi}}
\]

\[
= x \int_{-\infty}^{d_-(T-t)\sqrt{T-t}} e^{-z-(T-t)\sigma^2/2-z^2/(2T-t)} \frac{dz}{\sqrt{2(T-t)\pi}}
\]

\[
= x \int_{-\infty}^{d_-(T-t)\sqrt{T-t}} e^{-z-(T-t)|\sigma|\sigma^2/2T-t} \frac{dz}{\sqrt{2(T-t)\pi}}
\]

\[
= x \int_{-\infty}^{d_-(T-t)\sqrt{T-t}} e^{-z-(T-t)|\sigma|\sigma^2/2T-t} \frac{dz}{\sqrt{2(T-t)\pi}}
\]

\[
= x \int_{-\infty}^{d_-(T-t)\sqrt{T-t}-(T-t)|\sigma|\sigma^2/2(T-t)} e^{-z^2/(2(T-t))} \frac{dz}{\sqrt{2(T-t)\pi}}
\]

\[
= x \int_{-\infty}^{d_-(T-t)\sqrt{T-t}-(T-t)|\sigma|\sigma^2/2(T-t)} e^{-z^2/(2(T-t))} \frac{dz}{\sqrt{2(T-t)\pi}}
\]

\[
= x \left(1 - \Phi(-d+(T-t)) \right) - K e^{-(T-t)r} \int_{-\infty}^{d_-(T-t)} e^{-z^2/2} \frac{dz}{\sqrt{2\pi}}
\]

where we used the relation

\[
1 - \Phi(a) = \Phi(-a), \quad a \in \mathbb{R}.
\]

\[
\Box
\]

Exercises

Exercise 6.1 Bachelier model. Consider a market made of a riskless asset valued \( A_t = A_0 \) with zero interest rate, \( t \in \mathbb{R}_+ \), and a risky asset whose price
$S_t$ is modeled by a standard Brownian motion as $S_t = B_t$, $t \in \mathbb{R}_+$. Show that the price $g(t, B_t)$ of the option with payoff $C = B_T^2$ satisfies the heat equation (6.23) with terminal condition $g(T, x) = x^2$. See Exercises 6.11, 7.11 and 7.12 for extensions to nonzero interest rates.

Exercise 6.2 Consider a risky asset price $(S_t)_{t \in \mathbb{R}}$ modeled in the Cox et al. (1985) (CIR) model as

$$dS_t = \beta(\alpha - S_t)dt + \sigma \sqrt{S_t}dB_t, \quad \alpha, \beta, \sigma > 0, \quad (6.28)$$

and let $(\eta_t, \xi_t)_{t \in \mathbb{R}_+}$ be a portfolio strategy whose price $V_t := \eta_t A_t + \xi_t S_t$, takes the form $V_t = g(t, S_t)$, $t \in \mathbb{R}_+$. Figure 6.19 presents a random simulation of the solution to (6.28) with $\alpha = 0.025$, $\beta = 1$, and $\sigma = 1.3$.

![Graph of the CIR short rate](https://www.ntu.edu.sg/home/nprivault/indext.html)

Fig. 6.19: Graph of the CIR short rate $t \mapsto r_t$ with $\alpha = 2.5\%$, $\beta = 1$, and $\sigma = 1.3$.

Based on the self-financing condition written as

$$dV_t = rV_tdt - r\xi_tS_tdt + \beta(\alpha - S_t)\xi_tS_tdt + \sigma\xi_tS_tdB_t, \quad t \in \mathbb{R}_+, \quad (6.29)$$

derive the PDE satisfied by the function $g(t, x)$ using the Itô formula.

Exercise 6.3 Black-Scholes PDE with dividends. Consider a riskless asset with price $A_t = A_0 e^{rt}$, $t \in \mathbb{R}_+$, and an underlying asset price process $(S_t)_{t \in \mathbb{R}_+}$ modeled as

$$dS_t = (\mu - \delta)S_tdt + \sigma S_tdB_t,$$

where $(B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion and $\delta > 0$ is a continuous-time dividend rate. By absence of arbitrage, the payment of a dividend entails

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a drop in the stock price by the same amount occurring generally on the *ex-dividend date*, on which the purchase of the security no longer entitles the investor to the dividend amount. The list of investors entitled to dividend payment is consolidated on the *date of record*, and payment is made on the *payable date*.

a) Assuming that the portfolio with value $V_t = \xi_t S_t + \eta_t A_t$ at time $t$ is self-financing and that dividends are continuously reinvested, write down the portfolio variation $dV_t$.

b) Assuming that the portfolio value $V_t$ takes the form $V_t = g(t, S_t)$ at time $t$, derive the Black-Scholes PDE for the function $g(t, x)$ with its terminal condition.

c) Compute the price at time $t \in [0, T]$ of a European call option with strike price $K$ by solving the corresponding Black-Scholes PDE.

Exercise 6.4

a) Check that the Black-Scholes formula (6.9) for European call options

$$g_c(t, x) = x \Phi(d_+(T-t)) - K e^{-r(T-t)} \Phi(d_-(T-t)),$$

satisfies the boundary conditions $g_c(t, 0) = 0$ for all $t \in [0, T)$, with

$$g_c(T, x) = (x - K)^+ = \begin{cases} 
  x - K, & x > K \\
  0, & x \leq K,
\end{cases}$$

at $t = T$, and

$$\lim_{T \to \infty} B(t, x) = x, \quad t \in \mathbb{R}_+. $$

b) Check that the Black-Scholes formula (6.15) for European put options

$$g_p(t, x) = K e^{-r(T-t)} \Phi(-d_-(T-t)) - x \Phi(-d_+(T-t))$$

satisfies the boundary conditions $g_p(t, 0) = K e^{-r(T-t)}$ and $g_p(t, \infty) = 0$ for all $t \in [0, T)$, with

$$g_p(T, x) = (K - x)^+ = \begin{cases} 
  0, & x > K \\
  K - x, & x \leq K,
\end{cases}$$
Exercise 6.5 Power option. (Exercise 3.14 continued).

a) Solve the Black-Scholes PDE

\[ rg(x,t) = \frac{\partial g}{\partial t}(x,t) + rx \frac{\partial g}{\partial x}(x,t) + \frac{\sigma^2}{2} x^2 \frac{\partial^2 g}{\partial x^2}(x,t) \tag{6.30} \]

with terminal condition \( g(x,T) = x^2, \quad x > 0 \).

**Hint:** Try a solution of the form \( g(x,t) = x^2 f(t) \), and find \( f(t) \).

b) Find the respective quantities \( \xi_t \) and \( \eta_t \) of the risky asset \( S_t \) and riskless asset \( A_t = e^{rt} \) in the portfolio with value

\[ V_t = g(S_t,t) = \xi_t S_t + \eta_t A_t \]

hedging the contract with payoff \( S_t^2 \) at maturity.

Exercise 6.6 On December 18, 2007, a call warrant has been issued by Fortis Bank on the stock price \( S \) of the MTR Corporation with maturity \( T = 23/12/2008 \), strike price \( K = \text{HK$36.08} \) and entitlement ratio=10.

Recall that in the Black-Scholes model, the price at time \( t \) of a European claim on the underlying asset priced \( S_t \), with strike price \( K \), maturity \( T \), interest rate \( r \) and volatility \( \sigma > 0 \) is given by the Black-Scholes formula as

\[ f(t,S_t) = S_t \Phi(d_+(T-t)) - K e^{-(T-t)r} \Phi(d_-((T-t))) \]

where

\[
\begin{align*}
    d_-(T-t) &= \frac{(r - \sigma^2/2)(T-t) + \log(S_t/K)}{|\sigma|\sqrt{T-t}}, \\
    d_+(T-t) &= d_-(T-t) + |\sigma|\sqrt{T-t} = \frac{(r + \sigma^2/2)(T-t) + \log(S_t/K)}{|\sigma|\sqrt{T-t}}.
\end{align*}
\]

Recall that by Proposition 6.3 we have

\[ \frac{\partial f}{\partial x}(t,S_t) = \Phi(d_+(T-t)), \quad 0 \leq t \leq T. \]

a) Using the values of the Gaussian cumulative distribution function, compute the Black-Scholes price of the corresponding call option at time \( t = \text{November 07, 2008} \) with \( S_t = \text{HK$17.200} \), assuming a volatility \( \sigma = 90\% = 0.90 \) and an annual risk-free interest rate \( r = 4.377\% = 0.04377 \),
b) Still using the Gaussian cumulative distribution function, compute the quantity of the risky asset required in your portfolio at time $t = \text{November 07, 2008}$ in order to hedge one such option at maturity $T = 23/12/2008$.

c) Figure 1 represents the Black-Scholes price of the call option as a function of $\sigma \in [0.5, 1.5] = [50\%, 150\%]$.

Knowing that the closing price of the warrant on November 07, 2008 was HK$ 0.023, which value can you infer for the implied volatility $\sigma$ at this date?*

Exercise 6.7 Forward contracts. Recall that the price $\pi_t(C)$ of a claim payoff $C = h(S_T)$ of maturity $T$ can be written as $\pi_t(C) = g(t, S_t)$, where the function $g(t, x)$ satisfies the Black-Scholes PDE

$$
\begin{cases}
rg(t, x) = \frac{\partial g}{\partial t}(t, x) + rx \frac{\partial g}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 g}{\partial x^2}(t, x), \\
g(T, x) = h(x),
\end{cases}
$$

with terminal condition $g(T, x) = h(x), \ x > 0$.

a) Assume that $C$ is a forward contract with payoff

$$
C = S_T - K,
$$

at time $T$. Find the function $h(x)$ in (1).

b) Find the solution $g(t, x)$ of the above PDE and compute the price $\pi_t(C)$ at time $t \in [0, T]$.

*Hint: search for a solution of the form $g(t, x) = x - \alpha(t)$ where $\alpha(t)$ is a function of $t$ to be determined.*

* Download the corresponding **R code** or the **IPython notebook** that can be run here.
c) Compute the quantity
\[ \xi_t = \frac{\partial g}{\partial x}(t, S_t) \]
of risky assets in a self-financing portfolio hedging \( C \).
d) Repeat the above questions with the terminal condition \( g(T, x) = x \).

Exercise 6.8

a) Solve the Black-Scholes PDE
\[
gr(t, x) = \frac{\partial g}{\partial t}(t, x) + rx \frac{\partial g}{\partial x}(t, x) + \frac{\sigma^2}{2} x^2 \frac{\partial^2 g}{\partial x^2}(t, x) \quad (6.31)
\]
with terminal condition \( g(T, x) = 1, x > 0 \).

*Hint:* Try a solution of the form \( g(t, x) = f(t) \) and find \( f(t) \).

b) Find the respective quantities \( \xi_t \) and \( \eta_t \) of the risky asset \( S_t \) and riskless asset \( A_t = e^{rt} \) in the portfolio with value
\[ V_t = g(t, S_t) = \xi_t S_t + \eta_t A_t \]
hedging the contract with payoff $1 at maturity.

Exercise 6.9 Log-contracts, see also Exercise 8.3.

a) Solve the PDE
\[
0 = \frac{\partial g}{\partial t}(x, t) + rx \frac{\partial g}{\partial x}(x, t) + \frac{\sigma^2}{2} x^2 \frac{\partial^2 g}{\partial x^2}(x, t)
\]
with the terminal condition \( g(x, T) := \log x, x > 0 \).

*Hint:* Try a solution of the form \( g(x, t) = f(t) + \log x \), and find \( f(t) \).

b) Solve the Black-Scholes PDE
\[
rh(x, t) = \frac{\partial h}{\partial t}(x, t) + rx \frac{\partial h}{\partial x}(x, t) + \frac{\sigma^2}{2} x^2 \frac{\partial^2 h}{\partial x^2}(x, t) \quad (6.32)
\]
with the terminal condition \( h(x, T) := \log x, x > 0 \).

*Hint:* Try a solution of the form \( h(x, t) = u(t)g(x, t) \), and find \( u(t) \).

c) Find the respective quantities \( \xi_t \) and \( \eta_t \) of the risky asset \( S_t \) and riskless asset \( A_t = e^{rt} \) in the portfolio with value
\[ V_t = g(S_t, t) = \xi_t S_t + \eta_t A_t \]
hedge a log-contract with payoff \( \log S_T \) at maturity.

**Exercise 6.10** Binary options. Consider a price process \((S_t)_{t \in \mathbb{R}_+}\) given by

\[
\frac{dS_t}{S_t} = r dt + \sigma dB_t, \quad S_0 = 1,
\]

under the risk-neutral probability measure \( \mathbb{P}^* \). A binary (or digital) *call* option is a contract with maturity \( T \), strike price \( K \), and payoff

\[
C_d := 1_{[K, \infty)}(S_T) = \begin{cases} 
1 & \text{if } S_T \geq K, \\
0 & \text{if } S_T < K.
\end{cases}
\]

a) Derive the Black-Scholes PDE satisfied by the pricing function \( C_d(t, S_t) \) of the binary call option, together with its terminal condition.

b) Show that the solution \( C_d(t, x) \) of the Black-Scholes PDE of Question (a) is given by

\[
C_d(t, x) = e^{-(T-t)r} \Phi \left( \frac{(r - \sigma^2/2)(T - t) + \log(x/K)}{\sigma \sqrt{T-t}} \right) \\
= e^{-(T-t)r} \Phi(d_-(T-t)),
\]

where

\[
d_-(T-t) := \frac{(r - \sigma^2/2)(T - t) + \log(S_t/K)}{\sigma \sqrt{T-t}}, \quad 0 \leq t < T.
\]

**Exercise 6.11**

a) Bachelier model. Solve the stochastic differential equation

\[
dS_t = \alpha S_t dt + \sigma dB_t
\]

in terms of \( \alpha, \sigma \in \mathbb{R} \), and the initial condition \( S_0 \).

b) Write down the Black-Scholes PDE satisfied by the function \( C(t, x) \), where \( C(t, S_t) \) is the price at time \( t \in [0, T] \) of the contingent claim with payoff \( \phi(S_T) = \exp(S_T) \), and identify the process Delta \( (\xi_t)_{t \in [0, T]} \) that hedges this claim.

c) Solve the Black-Scholes PDE of Question (b) with the terminal condition \( \phi(x) = e^x, x \in \mathbb{R} \).

*Hint:* Search for a solution of the form

\[
\Phi(d_{-}(T-t)).
\]
\[ C(t, x) = \exp \left( -(T-t)r + xh(t) + \frac{\sigma^2}{4r}(h^2(t) - 1) \right), \quad (6.34) \]

where \( h(t) \) is a function to be determined, with \( h(T) = 1 \).

d) Compute the portfolio strategy \((\xi_t, \eta_t)_{t \in [0,T]}\) that hedges the contingent claim with payoff \( \exp(S_T) \).

Exercise 6.12

a) Show that for every fixed value of \( S \), the function
\[ d \mapsto h(S, d) := S \Phi(d + |\sigma|\sqrt{T}) - K e^{-rT} \Phi(d), \]
reaches its maximum at \( d_*(S) := \frac{\log(S/K) + (r - \sigma^2/2)T}{|\sigma|\sqrt{T}} \).

b) By the differentiation rule
\[ \frac{d}{dS} h(S, d_*(S)) = \frac{\partial h}{\partial S}(S, d_*(S)) + d'_*(S) \frac{\partial h}{\partial d}(S, d_*(S)), \]
recover the value of the Black-Scholes Delta.

Exercise 6.13 Compute the Black-Scholes Vega by differentiation of the Black-Scholes function
\[ g_c(t, x) = Bl(K, x, \sigma, r, T-t) = x \Phi(d_+(T-t)) - K e^{-(T-t)r} \Phi(d_-(T-t)), \]
knowing that
\[ -\frac{1}{2} (d_-(T-t))^2 = -\frac{1}{2} \left( \frac{\log(x/K) + (r - \sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}} \right)^2 \]
\[ = -\frac{1}{2} (d_+(T-t))^2 + (T-t)r + \log \frac{x}{K}. \quad (6.35) \]

Exercise 6.14 Consider the backward induction relation (3.13), i.e.
\[ \tilde{v}(t, x) = (1 - p_N^x) \tilde{v}(t+1, x(1+a_N)) + p_N^x \tilde{v}(t+1, x(1+b_N)), \]
using the renormalizations \( r_N := rT/N \) and
\[ a_N := (1 + r_N)(1 - |\sigma|\sqrt{T/N}) - 1, \quad b_N := (1 + r_N)(1 + |\sigma|\sqrt{T/N}) - 1, \]
of Section 3.6, \( N \geq 1 \), with

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\[ p_N^* = \frac{r_N - a_N}{b_N - a_N} \quad \text{and} \quad p_N^* = \frac{b_N - r_N}{b_N - a_N}. \]

a) Show that the Black-Scholes PDE (6.2) of Proposition 6.1 can be recovered from the induction relation (3.13) when the number \( N \) of time steps tends to infinity.

b) Show that the expression of the Delta \( \xi_t = \frac{\partial g_c}{\partial x}(t, S_t) \) can be similarly recovered from the finite difference relation (3.19), i.e.

\[ \xi_t^{(1)}(S_{t-1}) = \frac{v(t, (1 + b_N)S_{t-1}) - v(t, (1 + a_N)S_{t-1})}{S_{t-1}(b_N - a_N)} \]

as \( N \) tends to infinity.

Problem 6.15 (Leung and Sircar (2015)) ProShares Ultra S&P500 and ProShares UltraShort S&P500 are leveraged investment funds that seek daily investment results, before fees and expenses, that correspond to \( \beta \) times \((\beta x)\) the daily performance of the S&P500\(^\circledR\) with respectively \( \beta = 2 \) for ProShares Ultra and \( \beta = -2 \) for ProShares UltraShort. Here, leveraging with a factor \( \beta : 1 \) aims at multiplying the potential return of an investment by a factor \( \beta \). The following 10 questions are interdependent and should be treated in sequence.

a) Consider a risky asset priced \( S_0 := \$4 \) at time \( t = 0 \) and taking two possible values \( S_1 = \$5 \) and \( S_1 = \$2 \) at time \( t = 1 \). Compute the two possible returns (in \%) achieved when investing \$4 in one share of the asset \( S \), and the expected return under the risk-neutral probability measure, assuming that the risk-free interest rate is zero.

b) Leveraging. Still based on an initial \$4 investment, we decide to leverage by a factor \( \beta = 3 \) by borrowing another \((\beta - 1) \times \$4 = 2 \times \$4\) at rate zero to purchase a total of \( \beta = 3 \) shares of the asset \( S \). Compute the two returns (in \%) possibly achieved in this case, and the expected return under the risk-neutral probability measure, assuming that the risk-free interest rate is zero.

c) Denoting by \( F_t \) the ProShares value at time \( t \), how much should the fund invest in the underlying asset priced \( S_t \), and how much \$ should it borrow or save on the risk-free market at any time \( t \) in order to leverage with a factor \( \beta : 1 \)?

d) Find the portfolio allocation \((\xi_t, \eta_t)\) for the fund value

\[ F_t = \xi_t S_t + \eta_t A_t, \quad t \in \mathbb{R}_+, \]

according to Question (c), where \( A_t := A_0 e^{rt} \) is the riskless money market account.

e) We choose to model the S&P500 index \( S_t \) as the geometric Brownian motion

\[ \circ \]

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\[ dS_t = rS_t dt + \sigma S_t dB_t, \quad t \in \mathbb{R}_+, \]

under the risk-neutral probability measure \( \mathbb{P}^* \). Find the stochastic differential equation satisfied by \((F_t)_{t \in \mathbb{R}_+}\) under the self-financing condition \(dF_t = \xi_t dS_t + \eta_t dA_t\).

f) Is the discounted fund value \((e^{-rt}F_t)_{t \in \mathbb{R}_+}\) a martingale under the risk-neutral probability measure \( \mathbb{P}^* \)?

g) Find the relation between the fund value \(F_t\) and the index \(S_t\) by solving the stochastic differential equation obtained for \(F_t\) in Question (e). For simplicity we normalize \(F_0 := S_0^\beta\).

h) Write the price at time \(t = 0\) of the call option with payoff \((F_T - K)^+\) on the ProShares index using the Black-Scholes formula.

i) Show that when \(\beta > 0\), the Delta at time \(t \in [0, T]\) of the call option with payoff \((F_T - K)^+\) on ProShares Ultra is equal to the Delta of the call option with payoff \((S_T - K_{\beta(t)})^+\) on the S&P500, for a certain strike price \(K_{\beta(t)}\) to be determined explicitly.

j) When \(\beta < 0\), find the relation between the Delta at time \(t \in [0, T]\) of the call option with payoff \((F_T - K)^+\) on ProShares UltraShort and the Delta of the put option with payoff \((K_{\beta(t)} - S_T)^+\) on the S&P500.