Chapter 11
Barrier Options

Exotic options, also called path-dependent options, are options whose payoff $C$ may depend on the whole path
\[ \{ S_t : 0 \leq t \leq T \} \]
of the underlying asset price process via a “complex” operation such as averaging or computing a maximum. They are opposed to vanilla options whose payoff
\[ C = \phi(S_T), \]
depend only using the terminal value $S_T$ of the price process via a payoff function $\phi$, and can be priced by the computation of path integrals, cf. Section 16.3. In this chapter we consider the pricing of barrier options, whose payoff depends on an extremum of the underlying asset price $S_t$, in addition to its terminal value.

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11.1 Options on Extrema

Vanilla options with payoff $C = \phi(S_T)$ can be priced as
\[ e^{-rT} \mathbb{E}^*[\phi(S_T)] = e^{-rT} \int_0^\infty \phi(y) \varphi_{S_T}(y) dy \]
where $\varphi_{S_T}(y)$ is the (one parameter) probability density function of $S_T$, which satisfies
\[ \mathbb{P}(S_T \leq y) = \int_0^y \varphi_{S_T}(v)dv, \quad y \in \mathbb{R}. \]

Recall that typically we have

\[ \phi(x) = (x - K)^+ = \begin{cases} x - K & \text{if } x \geq K, \\ 0 & \text{if } x < K, \end{cases} \]

for the European call option with strike price \( K \), and

\[ \phi(x) = \mathbb{1}_{[K,\infty)}(x) = \begin{cases} 1 & \text{if } x \geq K, \\ 0 & \text{if } x < K, \end{cases} \]

for the binary call option with strike price \( K \). On the other hand, the payoff of an option on extrema may take the form

\[ C := \phi(M^T_0, S_T), \]

where

\[ M^T_0 = \max_{t \in [0,T]} S_t \]

is the maximum of \( (S_t)_{t \in \mathbb{R}_+} \) over the time interval \([0, T]\). In such situations the option price at time \( t = 0 \) can be expressed as

\[ e^{-rT} \mathbb{E}^* \left[ \phi(M^T_0, S_T) \right] = e^{-rT} \int_0^\infty \int_0^\infty \phi(x, y) \varphi(M^T_0, S_T)(x, y) dx dy \]

where \( \varphi(M^T_0, S_T) \) is the joint probability density function of \((M^T_0, S_T)\), which satisfies

\[ \mathbb{P}(M^T_0 \leq x \text{ and } S_T \leq y) = \int_0^x \int_0^y \varphi(M^T_0, S_T)(u, v) du dv, \quad x, y \in \mathbb{R}_+. \]

Fig. 11.1: Probability \( \mathbb{P}((X, Y) \in [-0.5, 1] \times [-0.5, 1]) \) computed as a volume integral.
The density $\varphi(M_0^T,S_T)$ can be recovered from the joint cumulative distribution function

$$(x,y) \mapsto F_{(M_0^T,S_T)}(x,y) := \mathbb{P}(M_0^T \leq x \text{ and } S_T \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} \varphi(M_0^T,S_T)(s,t)dsdt,$$

and

$$(x,y) \mapsto \mathbb{P}(M_0^T \geq x \text{ and } S_T \geq y) = \int_{x}^{\infty} \int_{y}^{\infty} \varphi(M_0^T,S_T)(s,t)dsdt,$$

as

$$\varphi(M_0^T,S_T)(x,y) = \frac{\partial^2}{\partial x \partial y} F_{(M_0^T,S_T)}(x,y) = \frac{\partial^2}{\partial x \partial y} \int_{-\infty}^{x} \int_{-\infty}^{y} \varphi(M_0^T,S_T)(s,t)dsdt$$

$$= \frac{\partial^2}{\partial x \partial y} \int_{x}^{\infty} \int_{y}^{\infty} \varphi(M_0^T,S_T)(s,t)dsdt,$$

$x,y \in \mathbb{R}$. The probability densities $\varphi_{M_0^T} : \mathbb{R} \to \mathbb{R}_+$ and $\varphi_{S_T} : \mathbb{R} \to \mathbb{R}_+$ of $M_0^T$ and $S_T$ are called the marginal densities of $(M_0^T,S_T)$, and are given by

$$\varphi_{M_0^T}(x) = \int_{-\infty}^{\infty} \varphi(M_0^T,S_T)(x,y)dy, \quad x \in \mathbb{R},$$

and

$$\varphi_{S_T}(y) = \int_{-\infty}^{\infty} \varphi(M_0^T,S_T)(x,y)dx, \quad y \in \mathbb{R}.$$

**General case**

Using the joint probability density function of $\tilde{W}_T = W_T + \mu T$ and

$$\tilde{X}_0^T = \text{Max}_{t \in [0,T]} \tilde{W} = \text{Max}_{t \in [0,T]} (W_t + \mu t),$$

we are able to price any exotic option with payoff $\phi(\tilde{W}_T,\tilde{X}_0^T)$, as

$$e^{-r(T-t)} \mathbb{E}^* [\phi(\tilde{X}_0^T,\tilde{W}_T) \mid \mathcal{F}_t],$$

with in particular, letting $a \lor b := \text{Max}(a,b)$,

$$e^{-rT} \mathbb{E}^* [\phi(\tilde{X}_0^T,\tilde{W}_T)] = e^{-rT} \int_{-\infty}^{\infty} \int_{y \lor 0}^{\infty} \phi(x,y)d\mathbb{P}^*(\tilde{X}_0^T \leq x, \tilde{W}_T \leq y).$$
In this chapter we work in a (continuous) geometric Brownian model in which
the asset price $S_t$ has the dynamics
$$dS_t = rS_t dt + \sigma S_t dW_t, \quad t \in \mathbb{R}_+,$$
where $(W_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$. In particular, by Lemma 5.15 the value $V_t$ of a self-financing portfolio satisfies
$$V_T e^{-rT} = V_0 + \sigma \int_0^T \xi_t S_t e^{-rt} dW_t, \quad t \in [0, T].$$
In order to price barrier options by the above probabilistic method, we will use the probability density function of the maximum
$$M^T_0 = \max_{t \in [0, T]} S_t$$
of geometric Brownian motion $(S_t)_{t \in \mathbb{R}_+}$ over a given time interval $[0, T]$ and the joint probability density function $\varphi_{(M^T_0, S_T)}(u, v)$ derived in Chapter 10 by the reflection principle.

**Proposition 11.1.** An exotic option with integrable claim payoff of the form
$$C = \phi(M^T_0, S_T) = \phi(\max_{t \in [0, T]} S_t, S_T)$$
can be priced at time 0 as
$$e^{-rT} \mathbb{E}^*[C] = \frac{1}{T} e^{-rT} \sqrt{\frac{2}{\pi T}} \int_0^\infty \int_0^\infty \phi(S_0 e^{\sigma y}, S_0 e^{\sigma x}) (2x - y) e^{-\frac{\mu^2 T}{2} + \mu y - (2x-y)^2/(2T)} dxdy$$
$$+ \frac{1}{T} e^{-rT} \sqrt{\frac{2}{\pi T}} \int_0^\infty \int_{-\infty}^0 \phi(S_0 e^{\sigma y}, S_0 e^{\sigma x}) (2x - y) e^{-\frac{\mu^2 T}{2} + \mu y - (2x-y)^2/(2T)} dxdy.$$

**Proof.** We have
$$S_T = S_0 e^{\sigma W_T - \sigma^2 T/2 + r T} = S_0 e^{(W_T + \mu T)\sigma} = S_0 e^{\sigma \bar{W}_T},$$
with
$$\mu := -\frac{\sigma}{2} + \frac{r}{\sigma}$$
and
$$\bar{W}_T = W_T + \mu T,$$
and

*A former MBA student in finance told me on March 26, 2004, that she did not understand why I covered barrier options until she started working in a bank*. Prof. Yuh-Dauh Lyuu, NTU, 2012.
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\[ M_T^0 = \max_{t \in [0,T]} S_t = S_0 \max_{t \in [0,T]} e^{\sigma W_t - \sigma^2 t/2 + rt} \]

\[ = S_0 \max_{t \in [0,T]} e^{\sigma \tilde{W}_t} = S_0 e^{\sigma \max_{t \in [0,T]} \tilde{W}_t} \]

we have

\[ C = \phi(S_T, M_T^0) = \phi(S_0 e^{\sigma W_T - \sigma^2 T/2 + rT}, M_T^0) = \phi(S_0 e^{\sigma \tilde{W}_T}, S_0 e^{\sigma \tilde{X}_0^T}), \]

hence

\[ e^{-rT} E^*[C] = e^{-rT} E^* [\phi(S_0 e^{\sigma \tilde{W}_T}, S_0 e^{\sigma \tilde{X}_0^T})] \]

\[ = e^{-rT} \int_{-\infty}^{\infty} \int_{y \geq 0} \phi(S_0 e^{\sigma y}, S_0 e^{\sigma x}) dP(\tilde{X}_0^T \leq x, \tilde{W}_T \leq y) \]

\[ = \frac{1}{T} \sqrt{\frac{2}{\pi T}} e^{-rT} \int_{-\infty}^{\infty} \int_{y \geq 0} \phi(S_0 e^{\sigma y}, S_0 e^{\sigma x}) (2x - y) e^{-\mu^2 T/2 + \mu y - (2x - y)^2/(2T)} dxdy \]

\[ = \frac{1}{T} e^{-rT} \sqrt{\frac{2}{\pi T}} \int_{0}^{\infty} \int_{y \geq 0} \phi(S_0 e^{\sigma y}, S_0 e^{\sigma x}) (2x - y) e^{-\mu^2 T/2 + \mu y - (2x - y)^2/(2T)} dxdy \]

\[ + \frac{1}{T} e^{-rT} \sqrt{\frac{2}{\pi T}} \int_{-\infty}^{0} \int_{y \geq 0} \phi(S_0 e^{\sigma y}, S_0 e^{\sigma x}) (2x - y) e^{-\mu^2 T/2 + \mu y - (2x - y)^2/(2T)} dxdy. \]

\[ \square \]

Pricing Barrier Options

The payoff of an up-and-out barrier put option on the underlying asset price \( S_t \) with exercise date \( T \), strike price \( K \) and barrier level (or call level) \( B \) is

\[ C = (K - S_T)^+ \mathbb{1}_{\left\{ \max_{0 \leq t \leq T} S_t < B \right\}} = \begin{cases} (K - S_T)^+ & \text{if } \max_{0 \leq t \leq T} S_t < B, \\ 0 & \text{if } \max_{0 \leq t \leq T} S_t \geq B. \end{cases} \]

This option is also called a Callable Bear Contract (with no rebate or residual value), or turbo warrant, in which the call level \( B \) usually satisfies \( B \leq K \).

The payoff of a down-and-out barrier call option on the underlying asset price \( S_t \) with exercise date \( T \), strike price \( K \) and barrier level \( B \) is

\[ C = (S_T - K)^+ \mathbb{1}_{\left\{ \min_{0 \leq t \leq T} S_t > B \right\}} = \begin{cases} (S_T - K)^+ & \text{if } \min_{0 \leq t \leq T} S_t > B, \\ 0 & \text{if } \min_{0 \leq t \leq T} S_t \leq B. \end{cases} \]
This option is also called a Callable Bull Contract (with no rebate or residual value), or turbo warrant, in which $B$ denotes the call level $B \geq K$.*

See Eriksson and Persson (2006) and Wong and Chan (2008) for the pricing of type R Callable Bull/Bear Contracts, or CBBCs, also called turbo warrants, which involve a rebate or residual value computed as the payoff of a down-and-in lookback option.

We can distinguish eight different variations on barrier options, according to Table 11.1.

<table>
<thead>
<tr>
<th>Option type</th>
<th>CBBC Behavior</th>
<th>Payoff</th>
<th>Fig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Barrier call</td>
<td>Bull</td>
<td>down-and-out (knock-out)</td>
<td>$(S_T - K)^+ \min_{0 \leq t \leq T} S_t &gt; B$</td>
</tr>
<tr>
<td></td>
<td>Bull</td>
<td>down-and-in (knock-in)</td>
<td>$(S_T - K)^+ \min_{0 \leq t \leq T} S_t &lt; B$</td>
</tr>
<tr>
<td></td>
<td>Bull</td>
<td>up-and-out (knock-out)</td>
<td>$(S_T - K)^+ \max_{0 \leq t \leq T} S_t &lt; B$</td>
</tr>
<tr>
<td></td>
<td>Bull</td>
<td>up-and-in (knock-in)</td>
<td>$(S_T - K)^+ \max_{0 \leq t \leq T} S_t &gt; B$</td>
</tr>
<tr>
<td></td>
<td>Bear</td>
<td>down-and-out (knock-out)</td>
<td>$(K - S_T)^+ \min_{0 \leq t \leq T} S_t &gt; B$</td>
</tr>
<tr>
<td></td>
<td>Bear</td>
<td>down-and-in (knock-in)</td>
<td>$(K - S_T)^+ \min_{0 \leq t \leq T} S_t &lt; B$</td>
</tr>
<tr>
<td></td>
<td>Bear</td>
<td>up-and-out (knock-out)</td>
<td>$(K - S_T)^+ \max_{0 \leq t \leq T} S_t &lt; B$</td>
</tr>
<tr>
<td></td>
<td>Bear</td>
<td>up-and-in (knock-in)</td>
<td>$(K - S_T)^+ \max_{0 \leq t \leq T} S_t &gt; B$</td>
</tr>
</tbody>
</table>

Table 11.1: Barrier option types.

**In-out parity**

We have the following parity relations between the prices of barrier options and vanilla call and put options:

* Download the corresponding R code for the pricing of CBBCs.
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\[
\begin{align*}
C_{\text{up-in}}(t) + C_{\text{up-out}}(t) &= e^{-r(T-t)} \mathbb{E}^*[ (S_T - K)^+ \mid \mathcal{F}_t], \quad (11.3) \\
C_{\text{down-in}}(t) + C_{\text{down-out}}(t) &= e^{-r(T-t)} \mathbb{E}^*[ (S_T - K)^+ \mid \mathcal{F}_t], \quad (11.4) \\
P_{\text{up-in}}(t) + P_{\text{up-out}}(t) &= e^{-r(T-t)} \mathbb{E}^*[ (K - S_T)^+ \mid \mathcal{F}_t], \quad (11.5) \\
P_{\text{down-in}}(t) + P_{\text{down-out}}(t) &= e^{-r(T-t)} \mathbb{E}^*[ (K - S_T)^+ \mid \mathcal{F}_t], \quad (11.6)
\end{align*}
\]

where the price of the European call, resp. put option with strike price \( K \) are obtained from the Black-Scholes formula. Consequently, in the sequel we will only compute the prices of the up-and-out barrier call and put options and of the down-and-out barrier call and put options.

Note that all knock-out barrier option prices vanish when \( M_0^t > B \) or \( m_0^t < B \), while the barrier up-and-out call, resp. the down-and-out barrier put option prices require \( B > K \), resp. \( B < K \), in order not to vanish.

11.2 Knock-Out Barrier Options

Up-and-out barrier call option

Let us consider an up-and-out barrier call option with maturity \( T \), strike price \( K \), barrier (or call level) \( B \), and payoff

\[
C = (S_T - K)^+ \mathbb{1}\left\{ \max_{0 \leq t \leq T} S_t < B \right\} = \begin{cases} S_T - K & \text{if } \max_{0 \leq t \leq T} S_t \leq B, \\ 0 & \text{if } \max_{0 \leq t \leq T} S_t > B, \end{cases}
\]

with \( B \geq K \).

**Proposition 11.2.** When \( K \leq B \), the price

\[
e^{-r(T-t)} \mathbb{1}\{M_0^t < B\} \mathbb{E}^*\left[ \left( x \frac{S_T - t}{S_0} - K \right)^+ \mathbb{1}\left\{ \max_{0 \leq u \leq T-t} \frac{S_u}{S_0} < B \right\} \right]_{x=S_t}
\]

of the up-and-out barrier call option with maturity \( T \), strike price \( K \) and barrier level \( B \) is given by
\[
\begin{align*}
&e^{-r(T-t)} \mathbb{E}^* \left[ (S_T - K)^+ \mathbb{1}_{\{M_0^T < B\}} \right | \mathcal{F}_t] \\
&= S_t \mathbb{1}_{\{M_0^t < B\}} \left\{ \Phi \left( \delta^{T-t} \left( \frac{S_t}{K} \right) \right) - \Phi \left( \delta^{T-t} \left( \frac{S_t}{B} \right) \right) \right\} \\
&\quad - \left( \frac{B}{S_t} \right)^{1+2r/\sigma^2} \left\{ \Phi \left( \delta^{T-t} \left( \frac{B^2}{K S_t} \right) \right) - \Phi \left( \delta^{T-t} \left( \frac{B}{S_t} \right) \right) \right\} \\
&\quad - e^{-r(T-t)} K \mathbb{1}_{\{M_0^t < B\}} \left\{ \Phi \left( \delta^{T-t} \left( \frac{S_t}{K} \right) \right) - \Phi \left( \delta^{T-t} \left( \frac{S_t}{B} \right) \right) \right\} \\
&\quad - \left( \frac{S_t}{B} \right)^{1-2r/\sigma^2} \left\{ \Phi \left( \delta^{T-t} \left( \frac{B^2}{K S_t} \right) \right) - \Phi \left( \delta^{T-t} \left( \frac{B}{S_t} \right) \right) \right\} \\
&= \mathbb{1}_{\{M_0^t < B\}} B \left( S_t, r, T - t, \sigma, K \right) - S_t \mathbb{1}_{\{M_0^t < B\}} \Phi \left( \delta^{T-t} \left( \frac{S_t}{K} \right) \right) \\
&\quad - B \left( \frac{B}{S_t} \right)^{2r/\sigma^2} \left\{ \Phi \left( \delta^{T-t} \left( \frac{B^2}{K S_t} \right) \right) - \Phi \left( \delta^{T-t} \left( \frac{B}{S_t} \right) \right) \right\} \\
&\quad + e^{-r(T-t)} K \mathbb{1}_{\{M_0^t < B\}} \Phi \left( \delta^{T-t} \left( \frac{S_t}{B} \right) \right) \\
&\quad + e^{-r(T-t)} K \left( \frac{S_t}{B} \right)^{1-2r/\sigma^2} \left\{ \Phi \left( \delta^{T-t} \left( \frac{B^2}{K S_t} \right) \right) - \Phi \left( \delta^{T-t} \left( \frac{B}{S_t} \right) \right) \right\},
\end{align*}
\]

where
\[
\delta^\tau_\pm (s) = \frac{1}{\sigma \sqrt{\tau}} \left( \log s + \left( r \pm \frac{\sigma^2}{2} \right) \tau \right), \quad s > 0.
\] (11.8)

The price of the up-and-out barrier call option is zero when \( B \leq K \).

Note that taking \( B = +\infty \) in the above identity (11.7) recovers the Black-Scholes formula

\[
e^{-r(T-t)} \mathbb{E}^* [(S_T - K)^+ | \mathcal{F}_t] = S_t \Phi \left( \delta^{T-t} \left( \frac{S_t}{K} \right) \right) - e^{-r(T-t)} K \Phi \left( \delta^{T-t} \left( \frac{S_t}{K} \right) \right)
\]

for the price of European call options.

The graph of Figure 11.2 represents the up-and-out barrier call option price given the value \( S_t \) of the underlying asset and the time \( t \in [0, T] \) with \( T = 220 \) days.
Fig. 11.2: Graph of the up-and-out call option price with $B = 80 > K = 65$.

Proof of Proposition 11.2. We have $C = \phi(S_T, M^T_0)$ with

$$\phi(x, y) = (x - K)^+ \mathbb{1}_{y < B} = \begin{cases} (x - K)^+ & \text{if } y < B, \\ 0 & \text{if } y \geq B, \end{cases}$$

hence

$$e^{-r(T-t)} \mathbb{E}^* \left[ (S_T - K)^+ \mathbb{1}_{M^T_0 < B} \right]_{\mathcal{F}_t}$$

$$= e^{-r(T-t)} \mathbb{E}^* \left[ (S_T - K)^+ \mathbb{1}_{M_0^T < B} \mathbb{1}_{M^T_t < B} \right]_{\mathcal{F}_t}$$

$$= e^{-r(T-t)} \mathbb{E}^* \left[ (S_T - K)^+ \mathbb{1}_{\max_{t \leq r \leq T} S_r < B} \right]_{\mathcal{F}_t}$$

$$= e^{-r(T-t)} \mathbb{E}^* \left[ \left( \frac{S_T}{S_t} - K \right)^+ \mathbb{1}_{x \max_{t \leq r \leq T} \frac{S_r}{S_t} > B} \right]_{x=S_t}$$

$$= e^{-r(T-t)} \mathbb{E}^* \left[ \left( \frac{S_{T-t}}{S_0} - K \right)^+ \mathbb{1}_{x \max_{0 \leq r \leq T-t} \frac{S_r}{S_0} < B} \right]_{x=S_t}.$$

It then suffices to compute, using (10.8),

$$\mathbb{E}^* \left[ (S_T - K)^+ \mathbb{1}_{M^T_0 < B} \right]$$

$$= \mathbb{E}^* \left[ (S_0 e^{\sigma \tilde{W}_T} - K) \mathbb{1}_{S_0 e^{\sigma \tilde{W}_T} > K} \mathbb{1}_{S_0 e^{\sigma \tilde{X}_0^T > B}} \right]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (S_0 e^{\sigma y} - K) \mathbb{1}_{S_0 e^{\sigma y} > K} \mathbb{1}_{S_0 e^{\sigma x} < B} d\mathbb{P}(\tilde{X}_0^T \leq x, \tilde{W}_T \leq y).$$

* Right-click on the figure for interaction and “Full Screen Multimedia” view.
\[ \mathbb{E}^* \left[ (S_T - K)^+ \mathbb{I}_{M_0^T < B} \right] \]

\[ = \frac{e^{-\mu^2 T/2}}{\sqrt{2\pi T}} \int_c^b \exp\left( \frac{y(\sigma + \mu) - y^2}{2(2T)} \right) \left( 1 - e^{2b(y-b)/T} \right) dy \]

\[ = S_0 e^{-\mu^2 T/2} \frac{1}{\sqrt{2\pi T}} \int_c^b e^{y(\sigma + \mu) - y^2/2(2T)} \left( 1 - e^{2b(y-b)/T} \right) dy \]

\[ - K e^{-\mu^2 T/2} \frac{1}{\sqrt{2\pi T}} \int_c^b e^{y(\sigma + \mu) - y^2/2(2T)} \left( 1 - e^{2b(y-b)/T} \right) dy \]

\[ = S_0 e^{-\mu^2 T/2} \frac{1}{\sqrt{2\pi T}} \int_c^b e^{y(\sigma + \mu) - y^2/2(2T)} dy \]

\[ - S_0 e^{-\mu^2 T/2 - 2b^2/T} \frac{1}{\sqrt{2\pi T}} \int_c^b e^{y(\sigma + 2b/T) - y^2/2(2T)} dy \]

\[ - K e^{-\mu^2 T/2} \frac{1}{\sqrt{2\pi T}} \int_c^b e^{y^2/2(2T)} dy \]
Using Relation (10.18), we find

\[
e^{-rT} \mathbb{E}^* \left[ (S_T - K)^+ \mathbb{1}_{\{M_0^T < B\}} \right]
\]

\[
= S_0 e^{-r(\mu + \sigma T/2 + b^2/2)} T^{1/2} \mathbb{E}^* \left[ e^{y(\mu + 2b/T) - y^2/2T} \right] dy
\]

\[
= S_0 \left( \Phi \left( \delta^T_+ \left( \frac{S_0}{K} \right) \right) - \Phi \left( \delta^T_- \left( \frac{S_0}{B} \right) \right) \right)
\]

\[
= S_0 \left( \Phi \left( \delta^T_+ \left( \frac{S_0}{K} \right) \right) - \Phi \left( \delta^T_- \left( \frac{S_0}{B} \right) \right) \right)
\]

\[
0 \leq x \leq B, \text{ where } \delta^T_{\pm}(s) \text{ is defined in (11.8). Given the relations}
\]

\[
-T \left( r + \frac{\mu^2}{2} \right) - 2 \frac{b^2}{T} + T \left( \sigma + \mu + \frac{2b}{T} \right)^2 = 2b \left( \frac{r}{\sigma} + \frac{\sigma}{2} \right) = \left( 1 + \frac{2r}{\sigma^2} \right) \log \frac{B}{S_0},
\]

and

\[
-T \left( r + \frac{\mu^2}{2} \right) - 2 \frac{b^2}{T} + T \left( \mu + \frac{2b}{T} \right)^2 = -rT + 2\mu b = -rT + \left( -1 + \frac{2r}{\sigma^2} \right) \log \frac{B}{S_0}.
\]

this yields

\[
e^{-rT} \mathbb{E}^* \left[ (S_T - K)^+ \mathbb{1}_{\{M_0^T < B\}} \right] = S_0 \left( \Phi \left( \delta^T_+ \left( \frac{S_0}{K} \right) \right) - \Phi \left( \delta^T_- \left( \frac{S_0}{B} \right) \right) \right)
\]

\[
\text{This version: January 15, 2020}
\]

https://www.ntu.edu.sg/home/nprivault/index.html
\[-e^{-rTK} \left( \Phi \left( \delta_T^+ \left( \frac{S_0}{K} \right) \right) - \Phi \left( \delta_T^+ \left( \frac{S_0}{B} \right) \right) \right) \]
\[ -B \left( \frac{B}{S_0} \right)^{2r/\sigma^2} \left( \Phi \left( \delta_T^+ \left( \frac{B^2}{KS_0} \right) \right) - \Phi \left( \delta_T^+ \left( \frac{B}{S_0} \right) \right) \right) \]
\[ + e^{-rTK} \left( \frac{S_0}{B} \right)^{1-2r/\sigma^2} \left( \Phi \left( \delta_T^+ \left( \frac{B^2}{KS_0} \right) \right) - \Phi \left( \delta_T^+ \left( \frac{B}{S_0} \right) \right) \right) \]
\[ = S_0 \left( \Phi \left( \delta_T^+ \left( \frac{S_0}{K} \right) \right) - \Phi \left( \delta_T^+ \left( \frac{S_0}{B} \right) \right) \right) \]
\[ - S_0 \left( \frac{B}{S_0} \right)^{1+2r/\sigma^2} \left( \Phi \left( \delta_T^+ \left( \frac{B^2}{KS_0} \right) \right) - \Phi \left( \delta_T^+ \left( \frac{B}{S_0} \right) \right) \right) \]
\[ - e^{-rTK} \left( \Phi \left( \delta_T^+ \left( \frac{S_0}{K} \right) \right) - \Phi \left( \delta_T^+ \left( \frac{S_0}{B} \right) \right) \right) \]
\[ + e^{-rTK} \left( \frac{S_0}{B} \right)^{1-2r/\sigma^2} \left( \Phi \left( \delta_T^+ \left( \frac{B^2}{KS_0} \right) \right) - \Phi \left( \delta_T^+ \left( \frac{B}{S_0} \right) \right) \right) , \]

and this yields the result of Proposition 11.2, cf. § 7.3.3 pages 304-307 of Shreve (2004) for a different calculation. This concludes the proof of Proposition 11.2. □

The attached R code performs an implied volatility calculation for Up-and-out barrier option prices based on this CBBC market data.

**Up-and-out barrier put option**

The price

\[ e^{-r(T-t)} \mathbb{1}_{\{M_0^t < B\}} \mathbb{E}^* \left[ \left( K - x \frac{S_{T-t}}{S_0} \right)^+ \mathbb{1}_{\left\{ x \max_{0 \leq r \leq T-t} \frac{S_r}{S_0} < B \right\}} \right] \bigg|_{x=S_t} \]

of the up-and-out barrier put option with maturity \( T \), strike price \( K \) and barrier level \( B \) is given, if \( B \leq K \), by

\[ e^{-r(T-t)} \mathbb{E}^* \left[ (K - S_T)^+ \mathbb{1}_{\{M_0^T < B\}} \bigg| \mathcal{F}_t \right] \]
\[ = S_t \mathbb{1}_{\{M_0^t < B\}} \left( \Phi \left( \delta_{T-t}^+ \left( \frac{S_t}{B} \right) \right) - 1 - \left( \frac{B}{S_t} \right)^{1+2r/\sigma^2} \left( \Phi \left( \delta_{T-t}^+ \left( \frac{B}{S_t} \right) - 1 \right) \right) \)
\[ - e^{-r(T-t)} K \mathbb{1}_{\{M_0^t < B\}} \left( \Phi \left( \delta_{T-t}^+ \left( \frac{S_t}{B} \right) \right) - 1 - \left( \frac{S_t}{B} \right)^{1-2r/\sigma^2} \left( \Phi \left( \delta_{T-t}^+ \left( \frac{B}{S_t} \right) - 1 \right) \right) \right) \]
and if $B \geq K$, by

\begin{align}
&= S_t \mathbb{I}_{\{M^t_0 < B\}} \left( -\Phi \left( -\delta_{-}^{T-t} \left( \frac{S_t}{B} \right) \right) + \left( \frac{S_t}{B} \right)^{1+2r/\sigma^2} \Phi \left( -\delta_{+}^{T-t} \left( \frac{B^2}{K} \right) \right) \right) \\
&- K e^{-r(T-t)} \\
&\times \mathbb{I}_{\{M^t_0 < B\}} \left( -\Phi \left( -\delta_{-}^{T-t} \left( \frac{S_t}{K} \right) \right) + \left( \frac{S_t}{K} \right)^{1-2r/\sigma^2} \Phi \left( -\delta_{+}^{T-t} \left( \frac{B^2}{KS_t} \right) \right) \right),
\end{align}

(11.10)

and if $B \geq K$, by

\begin{align}
e^{-r(T-t)} \mathbb{E}^* \left[ (K - S_T)^+ \mathbb{I}_{\{M^T_0 < B\}} | \mathcal{F}_t \right] \\
&= S_t \mathbb{I}_{\{M^t_0 < B\}} \left( \Phi \left( \delta_{-}^{T-t} \left( \frac{S_t}{K} \right) \right) - 1 - \left( \frac{S_t}{B} \right)^{1+2r/\sigma^2} \Phi \left( \delta_{-}^{T-t} \left( \frac{B^2}{KS_t} \right) \right) - 1 \right) \\
&- e^{-r(T-t)} K \\
&\times \mathbb{I}_{\{M^t_0 < B\}} \left( \Phi \left( \delta_{-}^{T-t} \left( \frac{S_t}{K} \right) \right) - 1 - \left( \frac{S_t}{B} \right)^{1-2r/\sigma^2} \Phi \left( \delta_{-}^{T-t} \left( \frac{B^2}{KS_t} \right) \right) - 1 \right) \\
&= S_t \mathbb{I}_{\{M^t_0 < B\}} \left( -\Phi \left( -\delta_{-}^{T-t} \left( \frac{S_t}{K} \right) \right) + \left( \frac{S_t}{B} \right)^{1+2r/\sigma^2} \Phi \left( -\delta_{-}^{T-t} \left( \frac{B^2}{KS_t} \right) \right) \right) \\
&- K e^{-r(T-t)} \\
&\times \mathbb{I}_{\{M^t_0 < B\}} \left( -\Phi \left( -\delta_{-}^{T-t} \left( \frac{S_t}{K} \right) \right) + \left( \frac{S_t}{B} \right)^{1-2r/\sigma^2} \Phi \left( -\delta_{-}^{T-t} \left( \frac{B^2}{KS_t} \right) \right) \right),
\end{align}

(11.11)
Fig. 11.3: Graph of the up-and-out put option price (11.10) with $K = 100 > B = 80$.

Fig. 11.4: Graph of the up-and-out put option price (11.11) with $B = 80 > K = 60$.

The following Figure 11.5 shows the market pricing data of an up-and-out barrier put option on BHP Billiton Limited ASX:BHP with $B = K = $28 for half a share, priced at 1.79.
Barrier Options

Fig. 11.5: Pricing data for an up-and-out put option with $B = K = 28$.

Down-and-out barrier call option

Let us now consider a down-and-out barrier call option on the underlying asset price $S_t$ with exercise date $T$, strike price $K$, barrier level $B$, and payoff

$$C = (S_T - K)^+ \mathbb{1}_{\min_{0 \leq t \leq T} S_t > B} = \begin{cases} S_T - K & \text{if } \min_{0 \leq t \leq T} S_t > B, \\ 0 & \text{if } \min_{0 \leq t \leq T} S_t \leq B, \end{cases}$$

with $0 \leq B \leq K$. This option is also called a Callable Bull Contract with no residual value, or a turbo warrant with no rebate, in which $B$ denotes the call level.* When $B \leq K$ we have

$$e^{-(T-t)r} \mathbb{E}^* \left[ (S_T - K)^+ \mathbb{1}_{\min_{0 \leq t \leq T} S_t > B} \right] F_t = S_t \mathbb{1}_{\{m_0^t > B\}} \Phi \left( \delta_{+}^{T-t} \left( \frac{S_t}{K} \right) \right) - e^{-r(T-t)} K \mathbb{1}_{\{m_0^t > B\}} \Phi \left( \delta_{-}^{T-t} \left( \frac{S_t}{K} \right) \right) - \left( \frac{B}{S_t} \right)^{2r/\sigma^2} \Phi \left( \delta_{+}^{T-t} \left( \frac{B^2}{KS_t} \right) \right) + e^{-r(T-t)} K \mathbb{1}_{\{m_0^t > B\}} \left( \frac{S_t}{B} \right)^{1-2r/\sigma^2} \Phi \left( \delta_{-}^{T-t} \left( \frac{B^2}{KS_t} \right) \right),$$

* Download the corresponding R code for CBBC pricing.
\[= \mathbb{1}_{\{m_0^t > B\}} \mathcal{B}_t(S_t, r, T-t, \sigma, K)\]
\[-B \mathbb{1}_{\{m_0^t > B\}} \left(\frac{B}{S_t}\right)^{2r/\sigma^2} \Phi\left(\delta_{t-t} \left(\frac{B^2}{K S_t}\right)\right)\]
\[+ e^{-r(T-t)} K \mathbb{1}_{\{m_0^t > B\}} \left(\frac{S_t}{B}\right)^{1-2r/\sigma^2} \Phi\left(\delta_{t-t} \left(\frac{B^2}{K S_t}\right)\right)\]
\[= \mathbb{1}_{\{m_0^t > B\}} \mathcal{B}_t(S_t, r, T-t, \sigma, K)\]
\[-S_t \mathbb{1}_{\{m_0^t > B\}} \left(\frac{B}{S_t}\right)^{2r/\sigma^2} \mathcal{B}_t\left(\frac{B}{S_t}, r, T-t, \frac{K}{B}\right),\]
\[0 \leq t \leq T.\] When \(B \geq K\), we find
\[e^{-r(T-t)} \mathbb{E}^*\left[(S_T - K) + \mathbb{1}_{\left\{\min_{0 \leq t \leq T} S_t > B\right\}} \mid \mathcal{F}_t\right].\]
\[= S_t \mathbb{1}_{\{m_0^t > B\}} \Phi\left(\delta_{t-t} \left(\frac{S_t}{B}\right)\right) - e^{-r(T-t)} K \mathbb{1}_{\{m_0^t > B\}} \Phi\left(\delta_{t-t} \left(\frac{S_t}{B}\right)\right)\]
\[-B \mathbb{1}_{\{m_0^t > B\}} \left(\frac{B}{S_t}\right)^{2r/\sigma^2} \Phi\left(\delta_{t-t} \left(\frac{B}{S_t}\right)\right)\]
\[+ e^{-r(T-t)} K \mathbb{1}_{\{m_0^t > B\}} \left(\frac{S_t}{B}\right)^{1-2r/\sigma^2} \Phi\left(\delta_{t-t} \left(\frac{B}{S_t}\right)\right),\]
\[S_t > B, 0 \leq t \leq T, \text{ cf. Exercise 11.1 below.}\]

Fig. 11.6: Graph of the down-and-out call option price (11.12) with \(B = 60 < K = 80\).
Fig. 11.7: Graph of the down-and-out call option price (11.13) with $K = 40 < B = 60$.

In the next Figure 11.8 we plot the down-and-out barrier call option price (11.13) as a function of volatility with $B = 349.2 > K = 346.4$, $r = 0.03$, $T = 99/365$, and $S_0 = 360$.

Fig. 11.8: Down-and-out call option price as a function of $\sigma$.

We note that with those parameters, the down-and-out barrier call option price (11.13) is upper bounded by the forward contract price $S_t - Ke^{-(T-t)r}$ in the limit as $\sigma$ tends to zero, and that it converges to $S_t - B$ in the limit as $\sigma$ tends to infinity.

**Down-and-out barrier put option**

When $K \geq B$, the price

$$e^{-r(T-t)} \mathbb{1}_{\{m^*_0 > B\}} \mathbb{E}^\ast \left[ \left( K - x \frac{S_{T-t}}{S_0} \right)^+ \mathbb{1}_{\left\{ x \min_{0 \leq t \leq T} S_r / S_0 > B \right\}} \right]_{x=S_t}$$

of the down-and-out barrier put option with maturity $T$, strike price $K$ and barrier level $B$ is given by
\[ e^{-r(T-t)} \mathbb{E}^* \left[ (K - S_T)^+ \mathbb{I}_{\{m_0^T > B\}} \right] \]

\[ = S_t \mathbb{I}_{\{m_0^t > B\}} \left\{ \Phi \left( \delta_{+}^{T-t} \left( \frac{S_t}{K} \right) \right) - \Phi \left( \delta_{+}^{T-t} \left( \frac{S_t}{B} \right) \right) \right. \]

\[ - \left( \frac{B}{S_t} \right)^{1+2r/\sigma^2} \left\{ \Phi \left( \delta_{+}^{T-t} \left( \frac{B^2}{KS_t} \right) \right) - \Phi \left( \delta_{+}^{T-t} \left( \frac{S_t}{S_t} \right) \right) \right\} \]

\[ - e^{-r(T-t)} K \mathbb{I}_{\{m_0^t > B\}} \left\{ \Phi \left( \delta_{-}^{T-t} \left( \frac{S_t}{K} \right) \right) - \Phi \left( \delta_{-}^{T-t} \left( \frac{S_t}{B} \right) \right) \right. \]

\[ - \left( \frac{S_t}{B} \right)^{1-2r/\sigma^2} \left\{ \Phi \left( \delta_{-}^{T-t} \left( \frac{B^2}{KS_t} \right) \right) - \Phi \left( \delta_{-}^{T-t} \left( \frac{B}{S_t} \right) \right) \right\} \]

\[ = S_t \mathbb{I}_{\{m_0^t > B\}} \left\{ \Phi \left( -\delta_{+}^{T-t} \left( \frac{S_t}{B} \right) \right) - \Phi \left( -\delta_{+}^{T-t} \left( \frac{S_t}{K} \right) \right) \right. \]

\[ - \left( \frac{B}{S_t} \right)^{1+2r/\sigma^2} \left\{ \Phi \left( \delta_{+}^{T-t} \left( \frac{B^2}{KS_t} \right) \right) - \Phi \left( \delta_{+}^{T-t} \left( \frac{S_t}{S_t} \right) \right) \right\} \]

\[ - e^{-r(T-t)} K \mathbb{I}_{\{m_0^t > B\}} \left\{ \Phi \left( -\delta_{-}^{T-t} \left( \frac{S_t}{B} \right) \right) - \Phi \left( -\delta_{-}^{T-t} \left( \frac{S_t}{K} \right) \right) \right. \]

\[ - \left( \frac{S_t}{B} \right)^{1-2r/\sigma^2} \left\{ \Phi \left( \delta_{-}^{T-t} \left( \frac{B^2}{KS_t} \right) \right) - \Phi \left( \delta_{-}^{T-t} \left( \frac{B}{S_t} \right) \right) \right\} \]

(11.14)

\[ = \mathbb{I}_{\{m_0^t > B\}} B \mathbb{I}_{\text{put}} \left( S_t, r, T-t, \sigma, K \right) + S_t \mathbb{I}_{\{m_0^t > B\}} \Phi \left( -\delta_{+}^{T-t} \left( \frac{S_t}{B} \right) \right) \]

\[ - B \mathbb{I}_{\{m_0^t > B\}} \left( \frac{B}{S_t} \right)^{2r/\sigma^2} \left\{ \Phi \left( \delta_{+}^{T-t} \left( \frac{B^2}{KS_t} \right) \right) - \Phi \left( \delta_{+}^{T-t} \left( \frac{B}{S_t} \right) \right) \right\} \]

\[ - e^{-r(T-t)} K \mathbb{I}_{\{m_0^t > B\}} \Phi \left( -\delta_{-}^{T-t} \left( \frac{S_t}{B} \right) \right) \]

\[ + e^{-r(T-t)} K \mathbb{I}_{\{m_0^t > B\}} \left( \frac{S_t}{B} \right)^{1-2r/\sigma^2} \left\{ \Phi \left( \delta_{-}^{T-t} \left( \frac{B^2}{KS_t} \right) \right) - \Phi \left( \delta_{-}^{T-t} \left( \frac{B}{S_t} \right) \right) \right\}, \]

while the corresponding price vanishes when \( K \leq B. \)
Fig. 11.9: Graph of the down-and-out put option price (11.14) with $K = 80 > B = 65$.

Note that although Figures 11.4 and 11.6, resp. 11.3 and 11.7, appear to share some symmetry property, the functions themselves are not exactly symmetric. Regarding Figures 11.2 and 11.9, the pricing function is actually the same, but the conditions $B < K$ and $B > K$ play opposite roles.

11.3 Knock-In Barrier Options

Down-and-in barrier call option

When $B \leq K$ the price of the down-and-in barrier call option is given from the down-and-out barrier call option price (11.12) and the down-in-out call parity relation (11.4) as

\[
e^{-r(T-t)} \mathbb{E}^* \left[ (S_T - K)^+ \mathbb{1}_{\{m_t^0 < B\}} \right] \mathcal{F}_t \]

\[
= \mathbb{1}_{\{m_t^0 \leq B\}} B^l(S_t, r, T - t, \sigma, K) \]

\[
+ S_t \mathbb{1}_{\{m_t^0 > B\}} \left( \frac{B}{S_t} \right)^{1+2r/\sigma^2} \Phi \left( \frac{\delta^T-t}{K S_t} \left( \frac{B^2}{K S_t} \right) \right) \]

\[
- e^{-r(T-t)} K \mathbb{1}_{\{m_t^0 > B\}} \left( \frac{S_t}{B} \right)^{1-2r/\sigma^2} \Phi \left( \frac{\delta^T-t}{K S_t} \left( \frac{B^2}{K S_t} \right) \right). \]
Fig. 11.10: Graph of the down-and-in call option price (11.15) with $K = 80 > B = 65$.

When $B \geq K$, the price of the down-and-in barrier call option is given from
the down-and-out barrier call option price (11.13) and the down-in-out call
parity relation (11.4) as

$$
e^{-r(T-t)} \mathbb{E}^* \left[ (S_T - K) \mathbb{1}_{\{m_0^T < B\}} \right]_{\mathcal{F}_t}$$  

$$= B_l(S_t, r, T - t, \sigma, K)$$

$$= -S_t \mathbb{1}_{\{m_0^T > B\}} \Phi \left( \frac{S_T - K}{\sigma} \right) + e^{-r(T-t)} K \mathbb{1}_{\{m_0^T > B\}} \Phi \left( \frac{S_T - K}{\sigma} \right)$$

$$+ \mathbb{1}_{\{m_0^T > B\}} S_t \left( \frac{B}{S_t} \right)^{1+2r/\sigma^2} \Phi \left( \frac{S_T - B}{\sigma} \right)$$

$$- e^{-r(T-t)} K \mathbb{1}_{\{m_0^T > B\}} \left( \frac{S_T}{B} \right)^{1-2r/\sigma^2} \Phi \left( \frac{S_T - B}{\sigma} \right), \quad 0 \leq t \leq T.$$  

Fig. 11.11: Graph of the down-and-in call option price (11.11) with $K = 40 < B = 60$.  

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Up-and-in barrier call option

When \( B \geq K \) the price of the up-and-in barrier call option is given from (11.7) and the up-in-out call parity relation (11.3) as

\[
e^{-r(T-t)} \mathbb{E}^* \left[ (S_T - K)^{+} \mathbb{1}_{\{M_0^T > B\}} \right| \mathcal{F}_t \right) = \mathbb{1}_{\{M_0^t > B\}} \text{Bl}(S_t, r, T - t, \sigma, K) + S_t \mathbb{1}_{\{M_0^t < B\}} \Phi \left( \delta_+^{T-t} \left( \frac{S_t}{B} \right) \right) + B \mathbb{1}_{\{M_0^t < B\}} \left( \frac{B}{S_t} \right)^{2r/\sigma^2} \left( \Phi \left( \delta_+^{T-t} \left( \frac{B^2}{KS_t} \right) \right) - \Phi \left( \delta_+^{T-t} \left( \frac{B}{S_t} \right) \right) \right)
\]

\[
+ e^{-r(T-t)} K \mathbb{1}_{\{M_0^t < B\}} \Phi \left( \delta_-^{T-t} \left( \frac{S_t}{B} \right) \right) - e^{-r(T-t)} K \left( \frac{S_t}{B} \right)^{1-2r/\sigma^2} \left( \Phi \left( \delta_-^{T-t} \left( \frac{B^2}{KS_t} \right) \right) - \Phi \left( \delta_-^{T-t} \left( \frac{B}{S_t} \right) \right) \right).
\]

Fig. 11.12: Graph of the up-and-in call option price (11.17) with \( B = 80 > K = 65 \).

When \( B \leq K \) the price of the up-and-in barrier call option is given from the Black-Scholes formula and the up-in-out call parity relation (11.3) as

\[
e^{-r(T-t)} \mathbb{E}^* \left[ (S_T - K)^{+} \mathbb{1}_{\{M_0^T > B\}} \right| \mathcal{F}_t \right) = \text{Bl}(S_t, r, T - t, \sigma, K).
\]

Down-and-in barrier put option

When \( B \leq K \) the price of the down-and-in barrier put option is given from (11.14) and the down-in-out put parity relation (11.6) as

\[
e^{-r(T-t)} \mathbb{E}^* \left[ (K - S_T)^{+} \mathbb{1}_{\{M_0^T < B\}} \right| \mathcal{F}_t \right) = \text{Bl}(S_t, r, T - t, \sigma, K).
\]
Fig. 11.13: Graph of the down-and-in put option price (11.18) with $K = 80 > B = 65$.

When $B \geq K$ the price of the down-and-in barrier put option is given from the Black-Scholes put function and the down-in-out put parity relation (11.6) as

$$e^{-(T-t) r} \mathbb{E}^* \left[(K - S_T)^+ \mathbb{1}_{\{m_T^0 < B\}} \bigg| \mathcal{F}_t\right] = B \text{bl}_{\text{put}}(S_t, r, T - t, \sigma, K),$$

$0 \leq t \leq T$.

**Up-and-in barrier put option**

When $B \leq K$ the price of the down-and-in barrier put option is given from (11.10) and the up-in-out put parity relation (11.5) as

$$e^{-(T-t) r} \mathbb{E}^* \left[(K - S_T)^+ \mathbb{1}_{\{m_T^0 > B\}} \bigg| \mathcal{F}_t\right] = B \text{bl}_{\text{put}}(S_t, r, T - t, \sigma, K)$$

$0 \leq t \leq T$.

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$$+K e^{-r(T-t)}$$

$$\times \mathbb{1}_{\{M_t^B < B\}} \left( -\Phi \left( -\delta_{-}^{T-t} \left( \frac{S_t}{B} \right) \right) + \left( \frac{S_t}{B} \right)^{1-2r/\sigma^2} \Phi \left( -\delta_{-}^{T-t} \left( \frac{B}{S_t} \right) \right) \right).$$

$$0 \leq t \leq T.$$ 

Fig. 11.14: Graph of the up-and-in put option price (11.19) with $K = 80 > B = 70$.

By (11.11) and the up-in-out put parity relation (11.5), the price of the up-and-in barrier put option is given when $B \geq K$ by

$$e^{-(T-t)r} \mathbb{E}^* \left[ (K - S_T) + \mathbb{1}_{\{M_t^B > B\}} \right] \bigg| \mathcal{F}_t \right]$$

$$= \mathbb{1}_{\{M_0^B \geq B\}} B_{\text{put}}(S_t, r, T-t, \sigma, K)$$

$$-S_t \mathbb{1}_{\{M_0^B < B\}} \left( \frac{B}{S_t} \right)^{1+2r/\sigma^2} \Phi \left( -\delta_{+}^{T-t} \left( \frac{B^2}{KS_t} \right) \right)$$

$$+K \mathbb{1}_{\{M_0^B < B\}} e^{-r(T-t)} \left( \frac{S_t}{B} \right)^{1-2r/\sigma^2} \Phi \left( -\delta_{-}^{T-t} \left( \frac{B^2}{KS_t} \right) \right).$$
11.4 PDE Method

The up-and-out barrier call option price has been evaluated by probabilistic arguments in the previous sections. In this section we complement this approach with the derivation of a Partial Differential Equation (PDE) for this option price function.

The up-and-out barrier call option price can be written as

\[
  e^{-r(T-t)} \mathbb{E}^* \left[ (S_T - K)^+ \mathbb{1}_{\{M_0^T < B\}} \bigg| \mathcal{F}_t \right] \\
  = e^{-r(T-t)} \mathbb{1}_{\{\max_{0 \leq r \leq t} S_r < B\}} \mathbb{E}^* \left[ (S_T - K)^+ \mathbb{1}_{\{\max_{t \leq r \leq T} S_r < B\}} \bigg| \mathcal{F}_t \right] \\
  = \mathbb{1}_{\{M_0^t < B\}} g(t, S_t),
\]

where the function \( g(t, x) \) of \( t \) and \( S_t \) is given by

\[
  g(t, x) = e^{-r(T-t)} \mathbb{E}^* \left[ (S_T - K)^+ \mathbb{1}_{\{\max_{t \leq r \leq T} S_r < B\}} \bigg| S_t = x \right]. \quad (11.21)
\]

Next, by the same argument as in the proof of Proposition 6.1 we derive the Black-Scholes partial differential equation (PDE) satisfied by \( g(t, x) \), and written for the value of a self-financing portfolio.

**Proposition 11.3.** Let \((\eta_t, \xi_t)_{t \in \mathbb{R}_+}\) be a portfolio strategy such that

(i) \((\eta_t, \xi_t)_{t \in \mathbb{R}_+}\) is self-financing,

(ii) the portfolio value \( V_t := \eta_t A_t + \xi_t S_t, t \in \mathbb{R}_+ \), is given as in (11.21) by
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\[ V_t = \mathbb{1}_{\{M^t_0 < B\}} g(t, S_t), \quad t \in \mathbb{R}_+. \]

Then the function \( g(t, x) \) satisfies the Black-Scholes PDE

\[ rg(t, x) = \frac{\partial g}{\partial t} (t, x) + rx \frac{\partial g}{\partial x} (t, x) + \frac{1}{2} x^2 \sigma^2 \frac{\partial^2 g}{\partial x^2} (t, x), \quad (11.22) \]

\( t > 0, \; x > 0, \; 0 < y < B, \) and \( \xi_t \) is given by

\[ \xi_t = \frac{\partial g}{\partial x} (t, S_t), \quad t \in [0, T], \quad (11.23) \]

provided that \( M^t_0 < B. \)

**Proof.** By (11.21) the price at time \( t \) of the down-and-out barrier call option discounted to time 0 is given by

\[ e^{-rt} \mathbb{1}_{\{M^t_0 < B\}} g(t, S_t) \]

\[ = e^{-rT} \mathbb{1}_{\{M^t_0 < B\}} \mathbb{E}^* \left[ (S_T - K)^+ \mathbb{1}_{\{\text{Max}_{0 \leq r \leq T} S_r < B\}} \mathbb{1}_{\{M^t_0 < B\}} \right| \mathcal{F}_t \]

\[ = e^{-rT} \mathbb{E}^* \left[ (S_T - K)^+ \mathbb{1}_{\{M^t_0 < B\}} \mathbb{1}_{\{\text{Max}_{t \leq r \leq T} S_r < B\}} \mathbb{1}_{\{M^t_0 < B\}} \right| \mathcal{F}_t \]

\[ = e^{-rT} \mathbb{E}^* \left[ (S_T - K)^+ \mathbb{1}_{\{\text{Max}_{0 \leq r \leq T} S_r < B\}} \right| S_t \], \]

which is a martingale indexed by \( t \in \mathbb{R}_+. \) Next, applying the Itô formula to \( t \mapsto e^{-rt} g(t, S_t) \) “on \( \{M^t_0 \leq B, \; 0 \leq t \leq T\} \”, we have

\[ d(e^{-rt} g(t, S_t)) = -r e^{-rt} g(t, S_t) dt + e^{-rt} dg(t, S_t) \]

\[ = -r e^{-rt} g(t, S_t) dt + e^{-rt} \frac{\partial g}{\partial t} (t, S_t) dt \]

\[ + r e^{-rt} S_t \frac{\partial g}{\partial x} (t, S_t) dt + \frac{1}{2} e^{-rt} \sigma^2 S_t^2 \frac{\partial^2 g}{\partial x^2} (t, S_t) dt \]

\[ + e^{-rt} \sigma S_t \frac{\partial g}{\partial x} (t, S_t) dW_t. \quad (11.24) \]

In order to derive (11.23) we note that, as in the proof of Proposition 6.1, the self-financing condition (5.8) implies

\[ d(e^{-rt} V_t) = -r e^{-rt} V_t dt + e^{-rt} dV_t \]

\[ = -r e^{-rt} V_t dt + \eta_t e^{-rt} dA_t + \xi_t e^{-rt} dS_t \]

\[ \Diamond \]
\[
N. \text{ Privault}
\]
\[
= -r(\eta_t A_t + \xi_t S_t) e^{-rt} dt + r\eta_t A_t e^{-rt} dt + r\xi_t S_t e^{-rt} dt + \sigma \xi_t S_t e^{-rt} dW_t
\]
\[
= \sigma \xi_t S_t e^{-rt} dW_t, \quad t \in \mathbb{R}_+, \tag{11.25}
\]

and (11.23) follows by identification of (11.24) with (11.25) which shows that the sum of components in factor of \( dt \) have to vanish, hence
\[
-r g(t, S_t) + \frac{\partial g}{\partial t}(t, S_t) + r S_t \frac{\partial g}{\partial S_t}(t, S_t) + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 g}{\partial S_t^2}(t, S_t) = 0.
\]

□

In the next proposition we add a boundary condition to the Black-Scholes PDE (11.22) in order to hedge the up-and-out barrier call option with maturity \( T \), strike price \( K \), barrier (or call level) \( B \), and payoff
\[
C = (S_T - K)^+ \mathbb{1}_{\{\text{Max } S_t < B\}} = \begin{cases}
S_T - K & \text{ if } \text{Max } S_t \leq B, \\
0 & \text{ if } \text{Max } S_t > B,
\end{cases}
\]
with \( B \geq K \).

**Proposition 11.4.** The value \( V_t = \mathbb{1}_{\{M_t < B\}} g(t, S_t) \) of the self-financing portfolio hedging the up-and-out barrier call option satisfies the Black-Scholes PDE
\[
rg(t, x) = \frac{\partial g}{\partial t}(t, x) + r x \frac{\partial g}{\partial x}(t, x) + \frac{1}{2} x^2 \sigma^2 \frac{\partial^2 g}{\partial x^2}(t, x), \tag{11.26a}
\]
\[
g(t, x) = 0, \quad x \geq B, \quad t \in [0, T],
\]
\[
g(T, x) = (x - K)^+ \mathbb{1}_{\{x < B\}},
\]
on the time-space domain \([0, T] \times [0, B]\) with terminal condition
\[
g(T, x) = (x - K)^+ \mathbb{1}_{\{x < B\}},
\]
and additional boundary condition
\[
g(t, x) = 0, \quad x \geq B. \tag{11.27}
\]

Condition (11.27) holds since the price of the claim at time \( t \) is 0 whenever \( S_t = B \), cf. e.g. Eriksson and Persson (2006). The closed-form solution for the PDE (11.26a) is given by (11.9), as

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\[ g(t, x) = x \left( \Phi \left( \delta_+^{T-t} \left( \frac{x}{K} \right) \right) - \Phi \left( \delta_+^{T-t} \left( \frac{x}{B} \right) \right) \right) \tag{11.28} \]

\[ -x \left( \frac{x}{B} \right)^{-1-2r/\sigma^2} \left( \Phi \left( \delta_+^{T-t} \left( \frac{B^2}{Kx} \right) \right) - \Phi \left( \delta_+^{T-t} \left( \frac{B}{x} \right) \right) \right) \]

\[ -K e^{-r(T-t)} \left( \Phi \left( \delta_-^{T-t} \left( \frac{x}{K} \right) \right) - \Phi \left( \delta_-^{T-t} \left( \frac{x}{B} \right) \right) \right) \]

\[ + K e^{-r(T-t)} \left( \frac{x}{B} \right)^{1-2r/\sigma^2} \left( \Phi \left( \delta_-^{T-t} \left( \frac{B^2}{Kx} \right) \right) - \Phi \left( \delta_-^{T-t} \left( \frac{B}{x} \right) \right) \right) , \]

\[ 0 < x \leq B, \, 0 \leq t \leq T. \]

We note that the expression (11.28) can be rewritten using the standard Black-Scholes formula

\[ \text{Bl}(S, r, T, \sigma, K) = S \Phi \left( \delta_+^{T} \left( \frac{S}{K} \right) \right) - K e^{-rT} \Phi \left( \delta_+^{T} \left( \frac{S}{K} \right) \right) \]

for the price of the European call option, as

\[ g(t, x) = \text{Bl}(x, r, T-t, \sigma, K) - x \Phi \left( \delta_+^{T-t} \left( \frac{x}{B} \right) \right) + e^{-r(T-t)} K \Phi \left( \delta_+^{T-t} \left( \frac{x}{B} \right) \right) \]

\[ -B \left( \frac{B}{x} \right)^{2r/\sigma^2} \left( \Phi \left( \delta_+^{T-t} \left( \frac{B^2}{Kx} \right) \right) - \Phi \left( \delta_+^{T-t} \left( \frac{B}{x} \right) \right) \right) \]

\[ + e^{-r(T-t)} K \left( \frac{x}{B} \right)^{1-2r/\sigma^2} \left( \Phi \left( \delta_-^{T-t} \left( \frac{B^2}{Kx} \right) \right) - \Phi \left( \delta_-^{T-t} \left( \frac{B}{x} \right) \right) \right) , \]

\[ 0 < x \leq B, \, 0 \leq t \leq T. \] Table 11.2 below summarizes the boundary conditions satisfied for barrier option pricing in the Black-Scholes PDE.

**Hedging barrier options**

Figure 11.16 represents the value of Delta obtained from (11.23) for the up-and-out barrier call option in Exercise 11.1-(a).
<table>
<thead>
<tr>
<th>Option type</th>
<th>CBBC</th>
<th>Behavior</th>
<th>Boundary conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Maturity $T$</td>
</tr>
<tr>
<td><strong>Barrier call</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bull</td>
<td>down-and-out</td>
<td>$B \leq K$ (knock-out)</td>
<td>$(x - K)^+$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$B \geq K$ (knock-out)</td>
<td>$(x - K)^+_{x &gt; B}$</td>
</tr>
<tr>
<td></td>
<td>down-and-in</td>
<td>$B \leq K$ (knock-in)</td>
<td>0</td>
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<tr>
<td></td>
<td></td>
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<tr>
<td></td>
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<td></td>
<td></td>
<td>$B \geq K$ (knock-in)</td>
<td>$(x - K)^+_{x &gt; B}$</td>
</tr>
<tr>
<td><strong>Barrier put</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bear</td>
<td>down-and-out</td>
<td>$B \leq K$ (knock-out)</td>
<td>$(K - x)^+_{x &gt; B}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$B \geq K$ (knock-out)</td>
<td>0</td>
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<td></td>
<td></td>
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<td>0</td>
</tr>
</tbody>
</table>

Table 11.2: Boundary conditions for barrier option prices.

Fig. 11.16: Delta of the up-and-out barrier call with $B = 80$ and $K = 55$.*

* The animation works in Acrobat Reader on the entire pdf file.
Barrier Options

**Down-and-out barrier call option**

Similarly the price $g(t, S_t)$ at time $t$ of the down-and-out barrier call option satisfies the Black-Scholes PDE

$$
\begin{cases}
rg(t, x) = \frac{\partial g}{\partial t}(t, x) + rx \frac{\partial g}{\partial x}(t, x) + \frac{1}{2}x^2\sigma^2 \frac{\partial^2 g}{\partial x^2}(t, x), \\
g(t, B) = 0, \quad t \in [0, T], \\
g(T, x) = (x - K)^+ \mathbb{1}_{\{x > B\}},
\end{cases}
$$

on the time-space domain $[0, T] \times [0, B]$ with terminal condition $g(T, x) = (x - K)^+ \mathbb{1}_{\{x > B\}}$ and the additional boundary condition

$$g(t, x) = 0, \quad x \leq B,$$

since the price of the claim at time $t$ is 0 whenever $S_t \leq B$.

**Exercises**

**Exercise 11.1** Barrier options.

a) Compute the hedging strategy of the up-and-out barrier call option on the underlying asset price $S_t$ with exercise date $T$, strike price $K$ and barrier level $B$, with $B \geq K$.

b) Compute the joint probability density function

$$
\varphi_{Y_T, W_T}(a, b) = \frac{dP(Y_T \leq a \text{ and } W_T \leq b)}{da \, db}, \quad a, b \in \mathbb{R},
$$

of standard Brownian motion $W_T$ and its minimum

$$
Y_T = \min_{t \in [0, T]} W_t.
$$

c) Compute the joint probability density function

$$
\varphi_{\tilde{Y}_T, \tilde{W}_T}(a, b) = \frac{dP(\tilde{Y}_T \leq a \text{ and } \tilde{W}_T \leq b)}{da \, db}, \quad a, b \in \mathbb{R},
$$

of drifted Brownian motion $\tilde{W}_T = W_T + \mu T$ and its minimum

$$
\tilde{Y}_T = \min_{t \in [0, T]} \tilde{W}_t = \min_{t \in [0, T]} (W_t + \mu t).
$$
Exercise 11.2 Barrier forward contracts. Compute the price at time $t$ of the following barrier forward contracts on the underlying asset price $S_t$ with exercise date $T$, strike price $K$, barrier level $B$, and the following payoffs. In addition, compute the corresponding hedging strategies.

a) **Up-and-in barrier long forward contract.** Take

$$C = (S_T - K)_{\mathbb{1}} \left\{ \max_{0 \leq t \leq T} S_t > B \right\} = \begin{cases} S_T - K & \text{if } \max_{0 \leq t \leq T} S_t > B, \\ 0 & \text{if } \max_{0 \leq t \leq T} S_t \leq B. \end{cases}$$

b) **Up-and-out barrier long forward contract.** Take

$$C = (S_T - K)_{\mathbb{1}} \left\{ \max_{0 \leq t \leq T} S_t < B \right\} = \begin{cases} S_T - K & \text{if } \max_{0 \leq t \leq T} S_t < B, \\ 0 & \text{if } \max_{0 \leq t \leq T} S_t \geq B. \end{cases}$$

c) **Down-and-in barrier long forward contract.** Take

$$C = (S_T - K)_{\mathbb{1}} \left\{ \min_{0 \leq t \leq T} S_t < B \right\} = \begin{cases} S_T - K & \text{if } \min_{0 \leq t \leq T} S_t < B, \\ 0 & \text{if } \min_{0 \leq t \leq T} S_t \geq B. \end{cases}$$

d) **Down-and-out barrier long forward contract.** Take

$$C = (S_T - K)_{\mathbb{1}} \left\{ \min_{0 \leq t \leq T} S_t > B \right\} = \begin{cases} S_T - K & \text{if } \min_{0 \leq t \leq T} S_t > B, \\ 0 & \text{if } \min_{0 \leq t \leq T} S_t \leq B. \end{cases}$$

e) **Up-and-in barrier short forward contract.** Take

$$C = (S_T - K)_{\mathbb{1}} \left\{ \min_{0 \leq t \leq T} S_t > B \right\} = \begin{cases} S_T - K & \text{if } \min_{0 \leq t \leq T} S_t > B, \\ 0 & \text{if } \min_{0 \leq t \leq T} S_t \leq B. \end{cases}$$
Barrier Options

\[ C = (K - S_T) I \{ \max_{0 \leq t \leq T} S_t > B \} = \begin{cases} K - S_T & \text{if } \max_{0 \leq t \leq T} S_t > B, \\ 0 & \text{if } \max_{0 \leq t \leq T} S_t \leq B. \end{cases} \]

f) **Up-and-out barrier short forward contract.** Take

\[ C = (K - S_T) I \{ \max_{0 \leq t \leq T} S_t < B \} = \begin{cases} K - S_T & \text{if } \max_{0 \leq t \leq T} S_t < B, \\ 0 & \text{if } \max_{0 \leq t \leq T} S_t \geq B. \end{cases} \]

g) **Down-and-in barrier short forward contract.** Take

\[ C = (K - S_T) I \{ \min_{0 \leq t \leq T} S_t < B \} = \begin{cases} K - S_T & \text{if } \min_{0 \leq t \leq T} S_t < B, \\ 0 & \text{if } \min_{0 \leq t \leq T} S_t \geq B. \end{cases} \]

h) **Down-and-out barrier short forward contract.** Take

\[ C = (K - S_T) I \{ \min_{0 \leq t \leq T} S_t > B \} = \begin{cases} K - S_T & \text{if } \min_{0 \leq t \leq T} S_t > B, \\ 0 & \text{if } \min_{0 \leq t \leq T} S_t \leq B. \end{cases} \]

Exercise 11.3 Compute the Vega of the down-and-out and down-and-in barrier call option prices, i.e. compute the sensitivity of down-and-out and down-and-in barrier option prices with respect to the volatility parameter \( \sigma \).

Exercise 11.4 Stability warrants. Price the up-and-out binary barrier option with payoff

\[ C := I \{ S_T > K \} I \{ M_0^T < B \} = I \{ S_T > K \text{ and } M_0^T < B \} \]

at time \( t = 0 \), with \( K \leq B \).

Exercise 11.5 Check that the function \( g(t, x) \) in (11.28) satisfies the boundary conditions
\[
g(t, B) = 0, \quad t \in [0, T], \\
g(T, x) = 0, \quad x < K < B, \\
g(T, x) = x - K, \quad K < x < B, \\
g(T, x) = 0, \quad x > B.
\]

Exercise 11.6 European knock-in/knock-out options. Price the following vanilla options by computing their conditional discounted expected payoffs:

a) European knock-out call option with payoff \((S_T - K)^+ \mathbb{1}_{\{S_T \leq B\}}\),

b) European knock-in put option with payoff \((K - S_T)^+ \mathbb{1}_{\{S_T \leq B\}}\),

c) European knock-in call option with payoff \((S_T - K)^+ \mathbb{1}_{\{S_T \geq B\}}\),

d) European knock-out put option with payoff \((K - S_T)^+ \mathbb{1}_{\{S_T \geq B\}}\).