Asian options are particular cases of options on average, and they were first traded in Tokyo in 1987. Given an underlying asset \( S_t \) with exercise date \( T \) and strike price \( K \), the payoff of the Asian call option is given by

\[
C := \left( \frac{1}{T} \int_0^T S_t dt - K \right)^+
\]

whereas the payoff of the Asian put option is

\[
C := \left( K - \frac{1}{T} \int_0^T S_t dt \right)^+.
\]

This chapter covers several probabilistic and PDE techniques for the pricing and hedging of Asian options. Due to their dependence on averaged asset prices, Asian options are less volatile than plain vanilla options whose payoffs depend only on the terminal value of the underlying asset.

13.1 Asian Call Options ............................. 413
13.2 Bounds on Asian Option Prices ................. 418
13.3 Pricing by the Hartman-Watson distribution ..... 420
13.4 Laplace Transform Method ..................... 422
13.5 Moment Matching Approximations.............. 423
13.6 PDE Method .................................. 429
Exercises ........................................... 440

13.1 Asian Call Options

In this case we can take

\[
C = \phi \left( \frac{1}{T} \int_0^T S_t dt \right)
\]
where
\[
\frac{1}{T} \int_0^T S_t \, dt
\]
represents the average of \((S_t)_{t \in \mathbb{R}^+}\) over the time interval \([0, T]\) and \(\phi : \mathbb{R} \rightarrow \mathbb{R}\) is a payoff function.

Figure 13.1 shows a graph of Brownian motion and its moving average process
\[
X_t := \frac{1}{t} \int_0^t B_s \, ds, \quad t > 0.
\]

**Arithmetic Asian options**

The payoff of the Asian call option on the underlying asset priced \(S_t\) with exercise date \(T\) and strike price \(K\) is given by
\[
C = \left(\frac{1}{T} \int_0^T S_t \, dt - K\right)^+.
\]

Similarly, the payoff of the Asian put option on the underlying asset priced \(S_t\) with exercise date \(T\) and strike price \(K\) is
\[
C = \left(K - \frac{1}{T} \int_0^T S_t \, dt\right)^+.
\]

Due to their dependence on averaged asset prices, Asian options are less volatile than plain vanilla options whose payoffs depend only on the terminal

* The animation works in Acrobat Reader on the entire pdf file.

This version: January 15, 2020
https://www.ntu.edu.sg/home/nprivault/index.html
Asian Options

value of the underlying asset. Asian options have become particularly popular in commodities trading.

Other types of exotic options include called Asian-American options, or Hawaiian options, that combine the Asian option payoff with American style exercise, and can be priced by variational PDEs, cf. §8.6.3.2 of Crépey (2013).

An option on average is an option whose payoff has the form

\[ C = \phi(\Lambda_T, S_T), \]

where

\[ \Lambda_T = S_0 \int_0^T e^{\sigma B_u + ru - \sigma^2 u/2} du = \int_0^T S_u du, \quad T \in \mathbb{R}_+. \]

- For example when \( \phi(y, x) = (y/T - K)^+ \) this yields the Asian call option with payoff

\[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ = \left( \frac{\Lambda_T}{T} - K \right)^+, \tag{13.1} \]

which is a path-dependent option whose price at time \( t \in [0, T] \) is given by

\[ e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \bigg| \mathcal{F}_t \right]. \tag{13.2} \]

- As another example, when \( \phi(y, x) := e^{-y} \) this yields the price

\[ P(0, T) = \mathbb{E}^* \left[ e^{-\int_0^T S_u du} \right] = \mathbb{E}^* \left[ e^{-\Lambda_T} \right] \]

at time 0 of a bond with underlying short-term rate process \( S_t \).

The option with payoff \( C = \phi(\Lambda_T, S_T) \) can be priced as

\[
e^{-(T-t)r} \mathbb{E}^* [\phi(\Lambda_T, S_T) \mid \mathcal{F}_t] = e^{-(T-t)r} \mathbb{E}^* \left[ \phi \left( \Lambda_t + \int_t^T S_u du, S_T \right) \bigg| \mathcal{F}_t \right] \\
e^{-(T-t)r} \mathbb{E}^* \left[ \phi \left( y + x \int_t^T \frac{S_u}{S_t} du, x \frac{S_T}{S_t} \right) \bigg| y=\Lambda_t, x=S_t \right] \\
e^{-(T-t)r} \mathbb{E}^* \left[ \phi \left( y + x \int_0^{T-t} \frac{S_u}{S_0} du, x \frac{S_{T-t}}{S_0} \right) \bigg| y=\Lambda_t, x=S_t \right]. \tag{13.3} \]

Using the Markov property of the process \((S_t, \Lambda_t)_{t \in \mathbb{R}_+}\), we can write down the option price as a function

\[ f(t, S_t, \Lambda_t) = e^{-(T-t)r} \mathbb{E}^* [\phi(\Lambda_T, S_T) \mid \mathcal{F}_t] \\
e^{-(T-t)r} \mathbb{E}^* [\phi(\Lambda_T, S_T) \mid S_t, \Lambda_t] \]
of \((t, S_t, \Lambda_t)\), where the function \(f(t, x, y)\) is given by

\[
f(t, x, y) = e^{-(T-t)r} \mathbb{E}^* \left[ \phi \left( y + x \int_0^{T-t} \frac{S_u}{S_0} du, x \frac{S_{T-t}}{S_0} \right) \right].
\]

As we will see below there exists no easily tractable closed-form solution for the price of an arithmetically averaged Asian option.

**Geometric Asian options**

On the other hand, replacing the arithmetic average

\[
\frac{1}{T} \sum_{k=1}^{n} S_{t_k} (t_k - t_{k-1}) \simeq \frac{1}{T} \int_0^T S_u du
\]

with the geometric average

\[
\prod_{k=1}^{n} S_{t_k}^{(t_k - t_{k-1})/T} = \exp \left( \log \prod_{k=1}^{n} S_{t_k}^{(t_k - t_{k-1})/T} \right) = \exp \left( \frac{1}{T} \sum_{k=1}^{n} \log S_{t_k}^{t_k - t_{k-1}} \right) = \exp \left( \frac{1}{T} \sum_{k=1}^{n} (t_k - t_{k-1}) \log S_{t_k} \right) \simeq \exp \left( \frac{1}{T} \int_0^T \log S_u du \right)
\]

leads to closed-form solutions using the Black Scholes formula, cf. Exercise 13.4.

**Pricing using probability density functions**

We note that the prices of option on averages can be estimated numerically using the joint probability density function \(\psi_{\Lambda_{T-t}, B_{T-t}}\) of \((\Lambda_{T-t}, B_{T-t})\), as follows:

\[
f(t, x, y) = e^{-(T-t)r} \mathbb{E}^* \left[ \phi \left( y + x \int_0^{T-t} \frac{S_u}{S_0} du, x \frac{S_{T-t}}{S_0} \right) \right]
\]

\[
e^{-(T-t)r} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi \left( y + x z, x e^{\sigma u+(T-t)r-(T-t)\sigma^2/2} \right) \psi_{\Lambda_{T-t}, B_{T-t}}(z, u) dz du.
\]

In Yor (1992), Proposition 2, the joint probability density function of

416
Asian Options

\[(A_t, B_t) = \left( \int_0^t e^{\sigma B_s - p \sigma^2 s / 2} ds, B_t - p \sigma t / 2 \right), \quad t > 0,\]

has been computed in the case \( \sigma = 2 \), cf. also Matsumoto and Yor (2005). In the next proposition we restate this result for an arbitrary variance parameter \( \sigma \) after rescaling. Let \( \theta(v, \tau) \) denote the function defined as

\[
\theta(v, \tau) = \frac{ve^{\pi^2/(2\tau)}}{\sqrt{2\pi^3 \tau}} \int_0^\infty e^{-\xi^2/(2\tau)} e^{-v \cosh \xi} \sinh(\xi) \sin(\pi \xi / \tau) d\xi, \quad v, \tau > 0.
\]

**Proposition 13.1.** For all \( t > 0 \) we have

\[
P\left( \int_0^t e^{\sigma B_s - p \sigma^2 s / 2} ds \in dy, \ B_t - p \frac{\sigma t}{2} \in dz \right)
= \frac{\sigma}{2} e^{-p\sigma z / 2 - p^2 \sigma^2 t / 8} \exp\left(-2 \frac{1 + e^{\sigma z}}{\sigma^2 y} \right) \theta \left( \frac{4 e^{\sigma z / 2}}{\sigma^2 y}, \frac{\sigma^2 t}{4} \right) dy dz,
\]

\( y > 0, \ z \in \mathbb{R}. \)

The expression of this probability density function can then be used for the pricing of options on average such as (13.3), as

\[
f(t, x, y) = e^{-(T-t)r} \mathbb{E}^* \left[ \phi \left( y + x \int_0^{T-t} \frac{S_v}{S_0} dv, x \frac{S_{T-t}}{S_0} \right) \right]
= e^{-(T-t)r} \times \int_0^\infty \phi \left( y + xz, xe^{\sigma u + (T-t)r - (T-t)\sigma^2 / 2} \right) P \left( \int_0^{T-t} \frac{S_v}{S_0} dv \in dz, B_{T-t} \in du \right)
= \frac{\sigma}{2} e^{-(T-t)r + (T-t)p^2 \sigma^2 / 8} \int_0^\infty \int_{-\infty}^{\infty} \phi \left( y + xz, xe^{\sigma u + (T-t)r - (T-t)(1+p)\sigma^2 / 2} \right)
\times \exp\left(-2 \frac{1 + e^{\sigma u - (T-t)p \sigma^2 / 2}}{\sigma^2 z} - \frac{p}{2} \sigma u \right) \theta \left( \frac{4 e^{\sigma u - (T-t)p \sigma^2 / 4}}{\sigma^2 z}, \frac{(T-t)\sigma^2}{4} \right) du dz
= e^{-(T-t)r - (T-t)p^2 \sigma^2 / 8} \int_0^\infty \int_{0}^{\infty} \phi \left( y + xz, xv e^{(T-t)r - (T-t)\sigma^2 / 2} \right)
\times v^{1-p} \exp\left(-2z \frac{1 + v^2}{\sigma^2} \right) \theta \left( \frac{4 vz}{\sigma^2}, \frac{(T-t)\sigma^2}{4} \right) dv dz,
\]

which actually stands as a triple integral due to the definition (13.4) of \( \theta(v, \tau) \). Note that here the order of integration between \( du \) and \( dz \) cannot be exchanged without particular precautions, at the risk of wrong computations.

By repeating the argument of (13.3) for \( \phi(x, y) := (x - K)^+ \), the Asian call option can be priced as
Hence the Asian call option can be priced as

\[ f(t, S_t, \Lambda_t) = e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \right| \mathcal{F}_t \],

where the function \( f(t, x, y) \) is given by

\[ f(t, x, y) = e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \left( y + x \int_0^{T-t} \frac{S_u}{S_0} du \right) - K \right)^+ \right] 
= e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \left( y + x \frac{\Lambda_{T-t}}{S_0} \right) - K \right)^+ \right], \quad x, y > 0. \quad (13.5) \]

### 13.2 Bounds on Asian Option Prices

We note (see Lemma 1 of Kemna and Vorst (1990) and Exercise 13.6 below for the discrete-time version of that result), that the Asian call option price can be upper bounded by the corresponding European call option price using convexity arguments.

**Proposition 13.2.** Assume that \( r \geq 0 \). We have the bound

\[ e^{-rT} \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \right] \leq e^{-rT} \mathbb{E}^*[(S_T - K)^+]. \]

**Proof.** By Jensen’s inequality for the uniform measure with density \((1/T)1_{[0,T]}\) on \([0,T]\) and for the probability measure \(P^*\), we have

\[ e^{-rT} \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \right] = e^{-rT} \mathbb{E}^* \left[ \left( \int_0^T (S_u - K) \frac{du}{T} \right)^+ \right] \]

This version: January 15, 2020

https://www.ntu.edu.sg/home/nprivault/index.html
Asian Options

\[ \leq e^{-rT} \mathbb{E}^* \left[ \int_0^T (S_u - K)^+ \frac{du}{T} \right] \]

\[ = e^{-rT} \mathbb{E}^* \left[ \int_0^T (e^{-(T-u)r} \mathbb{E}^*[S_T | \mathcal{F}_u] - K)^+ \frac{du}{T} \right] \]

\[ \leq e^{-rT} \mathbb{E}^* \left[ \int_0^T \mathbb{E}^* \left[ (e^{-(T-u)r})^+ S_T - K \right] \frac{du}{T} \right] \]

\[ \leq e^{-rT} \int_0^T \mathbb{E}^* \left[ (S_T - K)^+ \right] \frac{du}{T} \]

\[ = e^{-rT} \int_0^T \mathbb{E}^* \left[ (S_T - K)^+ \right] \frac{du}{T} \]

where from (13.6) to (13.7) we used the fact that \( r \geq 0 \).

More generally, given that

\[ f(t, S_t, \Lambda_t) = e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \right], \]

where, from Proposition 13.2,

\[ f(t, x, y) = e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \left( y + x \int_0^{T-t} S_u du \right) - K \right)^+ \right] \]

\[ = e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \left( y + x \frac{\Lambda_{T-t} - T}{T} - K \right) \right)^+ \right] \]

\[ = e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{y}{T} - K + \frac{x}{TS_0} \Lambda_{T-t} \right)^+ \right] \]

\[ = \frac{(T-t)x}{TS_0} e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{yS_0}{(T-t)x} - \frac{KTS_0}{(T-t)x} + \frac{\Lambda_{T-t}}{T-t} \right)^+ \right] \]

\[ \leq \frac{(T-t)x}{TS_0} e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{yS_0}{(T-t)x} - \frac{KTS_0}{(T-t)x} + \frac{S_{T-t}}{T-t} \right)^+ \right] \]

\[ = e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{y}{T} - K + \frac{(T-t)xS_{T-t}}{TS_0} \right)^+ \right], \quad x, y > 0, \]

we find the bound

\[ f(t, S_t, \Lambda_t) = e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \right]. \]
\[
\leq e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^t S_u du + \frac{T-t}{T} S_T - K \right)^+ \left| \mathcal{F}_t \right. \right]
\]

at time \( t \in [0, T] \). See also Proposition 3.2-(ii) of Geman and Yor (1993) for lower bounds when \( r \) takes negative values. We also have the following bound which yields the behavior of Asian call option prices in large time.

**Proposition 13.3.** The Asian call option price satisfies the bound

\[
e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \left| \mathcal{F}_t \right. \right] \leq \frac{e^{-(T-t)r}}{T} \int_0^t S_u du + S_t \frac{1 - e^{-(T-t)r}}{rT},
\]

t \in [0, T], and tends to zero (almost surely) as time to maturity \( T \) tends to infinity:

\[
\lim_{T \to \infty} \left( e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \left| \mathcal{F}_t \right. \right] \right) = 0, \quad t \in \mathbb{R}_+.
\]

**Proof.** We have the bound

\[
0 \leq e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \left| \mathcal{F}_t \right. \right]
\]
\[
\leq e^{-(T-t)r} \mathbb{E}^* \left[ \frac{1}{T} \int_0^T S_u du \left| \mathcal{F}_t \right. \right]
\]
\[
= e^{-(T-t)r} \mathbb{E}^* \left[ \frac{1}{T} \int_0^T S_u du \left| \mathcal{F}_t \right. \right] + e^{-(T-t)r} \mathbb{E}^* \left[ \frac{1}{T} \int_t^T S_u du \left| \mathcal{F}_t \right. \right]
\]
\[
= e^{-(T-t)r} \frac{1}{T} \int_0^t \mathbb{E}^* \left[ S_u \left| \mathcal{F}_t \right. \right] du + \frac{1}{T} e^{-(T-t)r} \int_t^T \mathbb{E}^* \left[ S_u \left| \mathcal{F}_t \right. \right] du
\]
\[
= e^{-(T-t)r} \frac{1}{T} \int_0^t S_u du + \frac{1}{T} e^{-(T-t)r} \int_t^T e^{(u-t)r} S_u du
\]
\[
= \frac{1}{T} e^{-(T-t)r} \int_0^t S_u du + \frac{S_t}{T} \int_t^T e^{-(T-u)r} du
\]
\[
= \frac{1}{T} e^{-(T-t)r} \int_0^t S_u du + S_t \frac{1 - e^{-(T-t)r}}{rT}.
\]

\[\square\]

### 13.3 Pricing by the Hartman-Watson distribution

First we note that the numerical computation of Asian call option prices can be done using the probability density function of

This version: January 15, 2020

[https://www.ntu.edu.sg/home/nprivault/index.html](https://www.ntu.edu.sg/home/nprivault/index.html)
Asian Options

\[ \Lambda_T = \int_0^T S_t \, dt. \]

From Proposition 13.1, we deduce the marginal probability density function of

\[ \Lambda_T := \int_0^T e^{\sigma B_t - \rho \sigma^2 t^2/2} \, dt, \]

also called the Hartman-Watson distribution Barrieu et al. (2004), as follows:

\[
P \left( \int_0^T e^{\sigma B_t - \rho \sigma^2 t^2/2} \, dt \in du \right) = \frac{\sigma}{2u} e^{\rho^2 T/8} \int_{-\infty}^{\infty} \exp \left( -2 \frac{1 + e^{\sigma v - \rho \sigma^2 T/2}}{\sigma^2 u} - \frac{p}{2} \right) \theta \left( \frac{4 e^{\sigma v/2 - \rho \sigma^2 T/4}}{\sigma^2 u}, \frac{\sigma^2 T}{4} \right) \, dv \, du
\]

\[
e^{-p^2 \sigma^2 T/8} \int_0^\infty v^{-1 - p} \exp \left( -2 \frac{1 + v^2}{\sigma^2 u} \right) \theta \left( \frac{4v}{\sigma^2 u}, \frac{\sigma^2 T}{4} \right) \, dv \, du,
\]

\[ u > 0. \]

From this expression we get

\[
P(\Lambda_T / S_0 \in du) = P \left( \int_0^T S_t \, dt \in du \right) = e^{-p^2 \sigma^2 T/8} \int_0^\infty v^{-1 - p} \exp \left( -2 \frac{1 + v^2}{\sigma^2 u} \right) \theta \left( \frac{4v}{\sigma^2 u}, \frac{\sigma^2 T}{4} \right) \, dv \, du,
\]

where \( S_t = S_0 e^{\sigma B_t - \rho \sigma^2 t^2/2} \) and \( p = 1 - 2r/\sigma^2 \). By (13.5), this probability density function can then be used for the pricing of Asian options, as

\[
f(t, x, y) = e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \left( \frac{y + x}{S_0} \Lambda_{T-t} \right) - K \right)^+ \right] = e^{-(T-t)r} \int_0^\infty \left( \frac{y + xz}{T} - K \right)^+ \mathbb{P}(\Lambda_{T-t} / S_0 \in dz)
\]

\[= e^{-(T-t)r} \frac{\sigma}{2} e^{-(T-t)p^2 \sigma^2/8} \int_0^\infty \int_0^\infty \left( \frac{y + xz}{T} - K \right)^+ \times v^{-1 - p} \exp \left( -2 \frac{1 + v^2}{\sigma^2 z} \right) \theta \left( \frac{4v}{\sigma^2 z}, \frac{(T-t)\sigma^2}{4z} \right) \, dv \, dz
\]

\[= \frac{1}{T} e^{-(T-t)r-(T-t)p^2 \sigma^2/8} \int_0^{\infty} \left( \frac{y + xz}{T} - K \right)^+ \times \exp \left( -2 \frac{1 + v^2}{\sigma^2 z} \right) \theta \left( \frac{4v}{\sigma^2 z}, \frac{(T-t)\sigma^2}{4z} \right) \, dv \, dz
\]

\[= \frac{4x}{\sigma^2 T} e^{-(T-t)r-(T-t)p^2 \sigma^2/8} \int_0^\infty \int_0^\infty \left( \frac{1}{z} - \frac{(KT-y)\sigma^2}{4x} \right)^+ \, dz \, dv.
\]
\[ \times v^{-1-p} \exp \left( -z \frac{1 + v^2}{2} \right) \theta \left( vz, \frac{(T - t)\sigma^2}{4} \right) dv \frac{dz}{z}, \]

cf. Theorem in § 5 of Carr and Schröder (2004), which is actually a triple integral due to the definition (13.4) of \( \theta(v, t) \). Note that since the integrals are not absolutely convergent, here the order of integration between \( dv \) and \( dz \) cannot be exchanged without particular precautions, at the risk of wrong computations.

13.4 Laplace Transform Method

The time Laplace transform of the rescaled option price

\[ C(t) := \mathbb{E}^* \left[ \left( \frac{1}{t} \int_0^t S_u du - \kappa \right)^+ \right], \quad t \in \mathbb{R}_+, \]

as

\[ \int_0^\infty e^{-\lambda t} C(t) dt = \frac{\int_0^{K/2} e^{-x x^2 + (1 - 2 Kx)^2 + (\sqrt{2(\lambda + p^2)} - p)/2} dx}{\lambda (\lambda - 2 + 2p) \Gamma(-1 + (p + \sqrt{2\lambda + p^2})/2)}, \]

with here \( \sigma := 2 \), and \( \Gamma(z) \) denotes the gamma function, see Relation (3.10) in Geman and Yor (1993). This expression can be used for pricing by numerical inversion of the Laplace transform using e.g. the Widder method, the Gaver-Stehfest method, the Durbin-Crump method, or the Papoulis method. The following Figure 13.2 represents Asian call option prices computed by the Geman and Yor (1993) method.

![Figure 13.2: Graph of Asian call option prices with \( \sigma = 1 \), \( r = 0.1 \) and \( K = 90 \).](https://www.ntu.edu.sg/home/nprivault/indext.html)

We refer to e.g. Carr and Schröder (2004), Dufresne (2000), and references therein for more results on Asian option pricing using the probability density

422
function of the averaged geometric Brownian motion.

Figure 9.6 presents a graph of implied volatility surface for Asian options on light sweet crude oil futures.

### 13.5 Moment Matching Approximations

#### Lognormal approximation

Other numerical approaches to the pricing of Asian options include Levy (1992), Turnbull and Wakeman (1992) which rely on approximations of the average price distribution based on the lognormal distribution. The lognormal distribution has the probability density function

$$g(x) = \frac{1}{\eta \sqrt{2\pi}} e^{-(\mu - \log x)^2/(2\eta^2)} \frac{dx}{x}, \quad x > 0,$$

where $\mu \in \mathbb{R}$, $\eta > 0$, with moments

$$\mathbb{E}[X] = e^{\mu + \eta^2/2} \quad \text{and} \quad \mathbb{E}[X^2] = e^{2\mu + 2\eta^2}. \quad (13.10)$$

The approximation is implemented by matching the above first two moments to those of time integral

$$\Lambda_T := \int_0^T S_t dt$$

of geometric Brownian motion

$$S_t = S_0 e^{\sigma B_t + (r - \sigma^2/2)t}, \quad 0 \leq t \leq T,$$

as computed in the next proposition, cf. also (7) and (8) page 480 of Levy (1992).

**Proposition 13.4.** We have

$$\mathbb{E}^*[\Lambda_T] = S_0 \frac{e^{rT} - 1}{r},$$

and

$$\mathbb{E}^*[(\Lambda_T)^2] = 2S_0^2 r e^{(\sigma^2 + 2r)T} - (\sigma^2 + 2r) e^{rT} + (\sigma^2 + r) \frac{(\sigma^2 + r)(\sigma^2 + 2r)}{(\sigma^2 + r)(\sigma^2 + 2r)r}.$$

**Proof.** The computation of the first moment is straightforward: we have

$$\mathbb{E}^*[\Lambda_T] = \mathbb{E}^* \left[ \int_0^T S_u du \right]$$

$$= \int_0^T \mathbb{E}^*[S_u] du$$

\[\Box\]
Under this approximation, the probability density function of the lognormal distribution with the moments of Proposition 13.4 we estimate as

\[ \Phi_{\Lambda_T}(x) \approx \frac{1}{x\sigma_{\Lambda_T}\sqrt{2(T-t)\pi}} \exp \left( -\frac{(\mu_T - \log x)^2}{2(T-t)\eta_T^2} \right), \quad x > 0. \]

For the second moment we have, letting \( p := 1 - 2r/\sigma^2 \),

\[
\mathbb{E}^* \left[ (\Lambda_T)^2 \right] = S_0^2 \int_0^T \int_0^T e^{-p \sigma^2 a/2 - \sigma^2 b^2} \mathbb{E}^* \left[ e^{\sigma B_a} e^{\sigma B_b} \right] dbda \\
= 2S_0^2 \int_0^T \int_0^a e^{-p \sigma^2 a/2 - \sigma^2 b^2} e^{(a+b)\sigma^2/2} e^{\sigma^2} dbda \\
= 2S_0^2 \int_0^T e^{-(p-1)\sigma^2 a/2} \int_0^a e^{-(p-3)\sigma^2 b^2/2} dbda \\
= \frac{4S_0^2}{(p-3)\sigma^2} \int_0^T e^{-(p-1)\sigma^2 a/2} (1 - e^{-(p-3)\sigma^2 a/2}) da \\
= \frac{4S_0^2}{(p-3)\sigma^2} \int_0^T e^{-(p-1)\sigma^2 a/2} da - \frac{4S_0^2}{(p-3)\sigma^2} \int_0^T e^{-(p-1)\sigma^2 a/2} e^{-(p-3)\sigma^2 a/2} da \\
= \frac{8S_0^2}{(p-3)(p-1)\sigma^4} (1 - e^{-(p-1)\sigma^2 T/2}) - \frac{4S_0^2}{(p-3)\sigma^2} \int_0^T e^{-(2p-4)\sigma^2 a/2} da \\
= \frac{8S_0^2}{(p-3)(p-1)\sigma^4} (1 - e^{-(p-1)\sigma^2 T/2}) - \frac{4S_0^2}{(p-3)(p-2)\sigma^4} (1 - e^{-(p-2)\sigma^2 T}) \\
= 2S_0 \frac{e^{(\sigma^2 + 2r)T} - (\sigma^2 + 2r)e^{\sigma^2 T} + (\sigma^2 + r)}{(\sigma^2 + r)(\sigma^2 + 2r)r},
\]

since \( r - \sigma^2/2 = -p\sigma^2/2 \).

By matching the first and second moments

\[ \mathbb{E}[\Lambda_T] \approx e^{\hat{\mu}_T + \hat{\eta}_T^2 T/2} \quad \text{and} \quad \mathbb{E} \left[ \Lambda_T^2 \right] \approx e^{2(\hat{\mu}_T + \hat{\eta}_T^2 T)} \]

of the lognormal distribution with the moments of Proposition 13.4 we estimate \( \hat{\mu}_T \) and \( \hat{\eta}_T \) as

\[ \hat{\eta}_T^2 = \frac{1}{T} \log \left( \frac{\mathbb{E}[\Lambda_T^2]}{(\mathbb{E}[\Lambda_T])^2} \right) \quad \text{and} \quad \hat{\mu}_T = \frac{1}{T} \log \mathbb{E}^* [\Lambda_T] - \frac{1}{2} \hat{\eta}_T^2. \]

Under this approximation, the probability density function \( \varphi_{\Lambda_T} \) of \( \Lambda_T = \int_0^T S_t dt \) is approximated by the lognormal density

\[ \varphi_{\Lambda_T}(x) \approx \frac{1}{x\sigma_{\Lambda_T}\sqrt{2(T-t)\pi}} \exp \left( -\frac{(\hat{\mu}_T - \log x)^2}{2(T-t)\hat{\eta}_T^2} \right), \quad x > 0. \]
As a consequence, from Lemma 7.8 we find the approximation

\[ e^{-rT} \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_t dt - K \right)^+ \right] = e^{-rT} \int_0^\infty \left( \frac{x}{T} - K \right)^+ \varphi_{\Lambda_T}(x) dx \]

\[ = \frac{e^{-rT}}{\sigma_t \sqrt{T} \sqrt{2(T-t)\pi}} \int_0^\infty (x/T - K)^+ \exp \left( -\frac{(\hat{\mu}_T - \log x)^2}{2(T-t)\hat{\eta}_T^2} \right) \frac{dx}{x} \]

\[ \simeq \frac{1}{T} e^{(\hat{\mu} + \hat{\eta}^2/2)/T} \Phi(d_1) - K \Phi(d_2), \quad (13.11) \]

where

\[ d_1 = \frac{\log(\mathbb{E}^*[\Lambda_T]/(KT))}{\hat{\eta} \sqrt{T}} + \frac{\hat{\eta} \sqrt{T}}{2} = \frac{\hat{\mu} T + \hat{\eta}^2 T - \log(KT)}{\hat{\eta} \sqrt{T}} \]

and

\[ d_2 = d_1 - \hat{\eta} \sqrt{T} = \frac{\log(\mathbb{E}^*[\Lambda_T]/(KT))}{\hat{\eta} \sqrt{T}} - \frac{\hat{\eta} \sqrt{T}}{2}. \]

The next Figure 13.4 compares the lognormal approximation to a Monte Carlo estimate of Asian call option prices with \( \sigma = 0.5, r = 0.05 \) and \( K/S_t = 1.1 \).
Fig. 13.4: Lognormal approximation to the Asian call option price.

Figure 13.4 also includes the stratified approximation

\[
e^{-rT} \mathbb{E} \left[ \left( \frac{1}{T} \int_0^T S_t \, dt - K \right)^+ \right]
\]

\[
= e^{-rT} \int_0^\infty \mathbb{E} \left[ \left( \frac{x}{T} - K \right)^+ \right] \varphi_{\Lambda_T|S_T\leq y}(x) \, d\mathbb{P}(S_T \leq y) \, dx
\]

\[
\simeq \frac{e^{-rT}}{T} \int_0^\infty \left( e^{-p(y/x)\sigma^2(y/x)T/2} + \sigma^2(y/x)T/2 \Phi(d_+(K, y, x)) - KT \Phi(d_-(K, y, x)) \right)
\times d\mathbb{P}(S_T \leq y) \, dx,
\]

cf. Privault and Yu (2016), where

\[
d_{\pm}(K, y, x) := \frac{1}{2\sigma(y/x)\sqrt{T}} \log \left( \frac{2x(b_T(y/x) - (1 + y/x)a_T(y/x))}{\sigma^2 K^2 T^2} \right) \pm \frac{\sigma(y/x)\sqrt{T}}{2}
\]

and

\[
\sigma^2(z) := \frac{1}{T} \log \left( \frac{2}{\sigma^2 a_T(z)} \left( \frac{b_T(z)}{a_T(z)} - 1 - z \right) \right),
\]

\[
a_T(z) := \frac{1}{\sigma^2 p(z)} \left( \Phi \left( \frac{\log z}{\sqrt{\sigma^2 T}} + \frac{1}{2} \sqrt{\sigma^2 T} \right) - \Phi \left( \frac{\log z}{\sqrt{\sigma^2 T}} - \frac{1}{2} \sqrt{\sigma^2 T} \right) \right),
\]

\[
b_T(z) := \frac{1}{\sigma^2 q(z)} \left( \Phi \left( \frac{\log z}{\sqrt{\sigma^2 T}} + \frac{1}{2} \sqrt{\sigma^2 T} \right) - \Phi \left( \frac{\log z}{\sqrt{\sigma^2 T}} - \frac{1}{2} \sqrt{\sigma^2 T} \right) \right),
\]

and

\[
p(z) := \frac{1}{\sqrt{2\pi \sigma^2 T}} e^{-\left(\sigma^2 T/2 + \log z\right)^2/(2\sigma^2 T)}, \quad q(z) := \frac{1}{\sqrt{2\pi \sigma^2 T}} e^{-\left(\sigma^2 T + \log z\right)^2/(2\sigma^2 T)}.
\]
Asian Options

Conditioning on the geometric mean price

Asian options on the arithmetic average

\[ \frac{1}{T} \int_0^T S \, dt \]

have been priced by conditioning on the geometric mean price

\[ G := \exp \left( \frac{1}{T} \int_0^T \log S \, dt \right) \leq \exp \left( \log \left( \frac{1}{T} \int_0^T S \, dt \right) \right) = \frac{1}{T} \int_0^T S \, dt \]

in Curran (1994), as

\[ e^{-rT} \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S \, du - K \right)^+ \right] \]

\[ = e^{-rT} \int_0^\infty \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S \, du - K \right)^+ \mid G = x \right] \, d\mathbb{P}(G \leq x) \]

\[ = e^{-rT} \int_0^K \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S \, du - K \right)^+ \mid G = x \right] \, d\mathbb{P}(G \leq x) \]

\[ + e^{-rT} \int_K^\infty \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S \, du - K \right)^+ \mid G = x \right] \, d\mathbb{P}(G \leq x) \]

\[ = C_1 + C_2, \]

where

\[ C_1 := e^{-rT} \int_0^K \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S \, du - K \right)^+ \mid G = x \right] \, d\mathbb{P}(G \leq x), \]

and

\[ C_2 := e^{-rT} \int_K^\infty \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S \, du - K \right)^+ \mid G = x \right] \, d\mathbb{P}(G \leq x) \]

\[ = e^{-rT} \int_K^\infty \mathbb{E}^* \left[ \frac{1}{T} \int_0^T S \, du - K \mid G = x \right] \, d\mathbb{P}(G \leq x) \]

\[ = e^{-rT} \int_K^\infty \mathbb{E}^* \left[ \int_0^T S \, du \mid G = x \right] \, d\mathbb{P}(G \leq x) - K e^{-rT} \int_K^\infty d\mathbb{P}(G \leq x) \]

\[ = \frac{e^{-rT}}{T} \mathbb{E}^* \left[ \int_0^T S \, du \mathbb{1}_{\{G \geq K\}} \right] - K e^{-rT} \mathbb{P}(G \geq K). \]

The term \( C_1 \) can be estimated by a lognormal approximation given that \( G = x \). As for \( C_2 \), we note that
\[ G = \exp \left( \frac{1}{T} \int_0^T \log S_t dt \right) \]
\[ = \exp \left( \frac{1}{T} \int_0^T \left( \mu t + \sigma B_t - \frac{\sigma^2 t}{2} \right) dt \right) \]
\[ = \exp \left( \frac{T}{2} \left( \mu - \frac{\sigma^2}{2} \right) + \frac{\sigma}{T} \int_0^T B_t dt \right), \]

hence

\[ \log G = \frac{T}{2} (\mu - \sigma^2 / 2) + \frac{\sigma}{T} \int_0^T B_t dt \]

is Gaussian \( N \left( (\mu - \sigma^2 / 2)T / 2, \sigma^2 T / 3 \right) \) with mean \((\mu - \sigma^2 / 2)T / 2\), and variance

\[
\mathbb{E} \left[ \left( \int_0^T B_t dt \right)^2 \right] = \mathbb{E} \left[ \int_0^T \int_0^T B_s B_t dsdt \right] \\
= \int_0^T \int_0^T \mathbb{E}[B_s B_t] dsdt \\
= 2 \int_0^T \int_0^t s dsdt \\
= \int_0^T t^2 dt \\
= \frac{T^3}{3}.
\]

Hence we have

\[
\mathbb{P}(G \geq K) = \mathbb{P}(\log G \geq \log K) \\
= \mathbb{P} \left( \frac{T}{2} \left( \mu - \frac{\sigma^2}{2} \right) + \frac{\sigma}{T} \int_0^T B_t dt \geq \log K \right) \\
= \mathbb{P} \left( \int_0^T B_t dt \geq \frac{T}{\sigma} \left( \frac{T}{2} \left( \mu - \frac{\sigma^2}{2} \right) + \log K \right) \right) \\
= \Phi \left( \frac{\sqrt{3}}{\sigma \sqrt{T}} \left( \frac{T}{2} \left( \mu - \frac{\sigma^2}{2} \right) - \log K \right) \right).
\]

**Basket options**

Basket options on the portfolio

\[ A_T := \sum_{k=1}^N \alpha_k S_T^{(k)} \]
Asian Options

have also been priced in Milevsky (1998) by approximating $A_T$ by a lognormal or a reciprocal gamma random variable, see also Deelstra et al. (2004) for additional conditioning on the geometric average of asset prices.

Asian basket options

Moment matching techniques combined with conditioning have been applied to Asian basket options in Deelstra et al. (2010). See also Dahl and Benth (2002) for the pricing of Asian basket options using quasi Monte Carlo simulation.

13.6 PDE Method

Two variables

The price at time $t$ of the Asian call option with payoff (13.1) can be written as

$$f(t, S_t, \Lambda_t) = e^{-(T-t)r} \mathbb{E}^{*} \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \bigg| \mathcal{F}_t \right], \quad 0 \leq t \leq T.$$  

(13.13)

Next, we derive the Black-Scholes partial differential equation (PDE) for the value of a self-financing portfolio. Until the end of this chapter we model the asset price $(S_t)_{t \in [0,T]}$ as

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad t \in \mathbb{R}^+,$$

where $(B_t)_{t \in \mathbb{R}^+}$ is a standard Brownian motion under the historical probability measure $\mathbb{P}$.

**Proposition 13.5.** Let $(\eta_t, \xi_t)_{t \in \mathbb{R}^+}$ be a self-financing portfolio strategy whose value $V_t := \eta_t A_t + \xi_t S_t, t \in \mathbb{R}^+$, takes the form

$$V_t = f(t, S_t, \Lambda_t), \quad t \in \mathbb{R}^+,$$

where $f \in C^{1,2,1}((0, T) \times (0, \infty)^2)$ is given by (13.13). Then $f(t, x, y)$ satisfies the PDE
rf(t, x, y) = \frac{\partial f}{\partial t}(t, x, y) + x \frac{\partial f}{\partial y}(t, x, y) + rx \frac{\partial f}{\partial x}(t, x, y) + \frac{1}{2} x^2 \sigma^2 \frac{\partial^2 f}{\partial x^2}(t, x, y),

0 \leq t \leq T, x > 0, under the boundary conditions

\begin{align}
&f(t, 0^+, y) = \lim_{x \searrow 0} f(t, x, y) = e^{-(T-t)r} \left( \frac{y}{T} - K \right)^+, \quad (13.14a) \\
&f(t, x, 0^+) = \lim_{y \searrow 0} f(t, x, y) = 0, \quad (13.14b) \\
&f(T, x, y) = \left( \frac{y}{T} - K \right)^+, \quad (13.14c)
\end{align}

0 \leq t \leq T, x, y \in \mathbb{R}_+,

and \( \xi_t \) is given by

\[ \xi_t = \frac{\partial f}{\partial x}(t, S_t, \Lambda_t), \quad 0 \leq t \leq T. \]  

(13.15)

**Proof.** We note that the self-financing condition (5.8) implies

\[ dV_t = \eta_t dA_t + \xi_t dS_t = r\eta_t A_t dt + \mu \xi_t S_t dt + \sigma \xi_t S_t dB_t \quad (13.16) \]

Since \( dA_t = S_t dt \), an application of Itô’s formula to \( f(t, x, y) \) leads to

\[ dV_t = f(t, S_t, \Lambda_t) = \frac{\partial f}{\partial t}(t, S_t, \Lambda_t) dt + \frac{\partial f}{\partial y}(t, S_t, \Lambda_t) d\Lambda_t \\
+ \mu S_t \frac{\partial f}{\partial x}(t, S_t, \Lambda_t) dt + \frac{1}{2} S_t^2 \sigma^2 \frac{\partial^2 f}{\partial x^2}(t, S_t, \Lambda_t) dt + \sigma S_t \frac{\partial f}{\partial x}(t, S_t, \Lambda_t) dB_t \\
= \frac{\partial f}{\partial t}(t, S_t, \Lambda_t) dt + S_t \frac{\partial f}{\partial y}(t, S_t, \Lambda_t) dt \\
+ \mu S_t \frac{\partial f}{\partial x}(t, S_t, \Lambda_t) dt + \frac{1}{2} S_t^2 \sigma^2 \frac{\partial^2 f}{\partial x^2}(t, S_t, \Lambda_t) dt + \sigma S_t \frac{\partial f}{\partial x}(t, S_t, \Lambda_t) dB_t. \]

(13.17)

By respective identification of the terms in \( dB_t \) and \( dt \) in (13.16) and (13.17) we get
\[
\begin{align*}
& r_t A_t dt + \mu_t S_t dt = \frac{\partial f}{\partial t}(t, S_t, \Lambda_t) dt + S_t \frac{\partial f}{\partial y}(t, S_t, \Lambda_t) dt + \mu_t S_t \frac{\partial f}{\partial x}(t, S_t, \Lambda_t) dt \\
& \quad + \frac{1}{2} S_t^2 \sigma^2 \frac{\partial^2 f}{\partial x^2}(t, S_t, \Lambda_t) dt , \\
& \xi_t S_t \sigma d B_t = S_t \sigma \frac{\partial f}{\partial x}(t, S_t, \Lambda_t) dB_t ,
\end{align*}
\]

hence
\[
\begin{align*}
& r V_t - r \xi_t S_t = \frac{\partial f}{\partial t}(t, S_t, \Lambda_t) + S_t \frac{\partial f}{\partial y}(t, S_t, \Lambda_t) dt + \frac{1}{2} S_t^2 \sigma^2 \frac{\partial^2 f}{\partial x^2}(t, S_t, \Lambda_t) , \\
& \xi_t = \frac{\partial f}{\partial x}(t, S_t, \Lambda_t) ,
\end{align*}
\]
i.e.
\[
\begin{align*}
& rf(t, S_t, \Lambda_t) = \frac{\partial f}{\partial t}(t, S_t, \Lambda_t) + S_t \frac{\partial f}{\partial y}(t, S_t, \Lambda_t) + r S_t \frac{\partial f}{\partial x}(t, S_t, \Lambda_t) \\
& \quad + \frac{1}{2} S_t^2 \sigma^2 \frac{\partial^2 f}{\partial x^2}(t, S_t, \Lambda_t) , \\
& \xi_t = \frac{\partial f}{\partial x}(t, S_t, \Lambda_t) .
\end{align*}
\]

When \( \Lambda_t / T \geq K \), by Exercise 13.7 we have
\[
\begin{align*}
& f(t, S_t, \Lambda_t) = e^{-(T-t)r} \mathbb{E}^* \left[ (1 \bigg/ T \int_0^T S_u du - K)^+ \bigg| \mathcal{F}_t \right] \\
& = e^{-(T-t)r} \left( \frac{\Lambda_t}{T} - K \right) + S_t \frac{1 - e^{-(T-t)r}}{r T} ,
\end{align*}
\]
t \in [0, T], hence in this case the Delta \( \xi_t \) is given by
\[
\xi_t = \frac{\partial f}{\partial x}(t, S_t, \Lambda_t) = \frac{1 - e^{-(T-t)r}}{r T} , \quad 0 \leq t \leq T .
\]

In addition, we have \( \xi_T = 0 \) at maturity from (13.14c) and (13.15). This is consistent with intuition that, close to maturity, the Asian payoff \( (\Lambda_T / T - K)^+ \) fluctuates less than the European payoff \( (S_T - K) \).

Next, we examine two methods which allow one to reduce the Asian option pricing PDE from three variables \((t, x, y)\) to two variables \((t, z)\). Reduction
\[
\square
\]
of dimensionality can be of crucial importance when applying discretization scheme whose complexity are of the form $N^d$ where $N$ is the number of discretization steps and $d$ is the dimension of the problem (curse of dimensionality).

(1) One variable with time-independent coefficients

Following Lamberton and Lapeyre (1996), page 91, we define the auxiliary process

$$Z_t = \frac{1}{S_t} \left( \frac{1}{T} \int_0^t S_u du - K \right) = \frac{1}{S_t} \left( \frac{\Lambda_t}{T} - K \right), \quad 0 \leq t \leq T.$$  

With this notation, the price of the Asian call option at time $t$ becomes

$$e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \ | \mathcal{F}_t \right] = e^{-(T-t)r} \mathbb{E}^* \left[ S_T (Z_T)^+ \ | \mathcal{F}_t \right].$$

**Lemma 13.6.** The price $(13.2)$ at time $t$ of the Asian call option with payoff $(13.1)$ can be written as

$$f(t, S_t, \Lambda_t) = S_t g(t, Z_t) = e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \ | \mathcal{F}_t \right],$$

$t \in [0, T]$, with the relation

$$f(t, x, y) = x g\left(t, \frac{1}{x} \left( \frac{y}{T} - K \right) \right), \quad x, y \in \mathbb{R}^+, \quad 0 \leq t \leq T,$$

where

$$g(t, z) = e^{-(T-t)r} \mathbb{E}^* \left[ \left( z + \frac{1}{T} \int_0^{T-t} S_u du \right)^+ \right] \quad (13.21)$$

$$= e^{-(T-t)r} \mathbb{E}^* \left[ \left( z + \frac{\Lambda_{T-t}}{S_0 T} \right)^+ \right],$$

with the boundary condition

$$g(T, z) = z^+, \quad z \in \mathbb{R}.$$

**Proof.** For $0 \leq s \leq t \leq T$, we have

$$d(S_t Z_t) = \frac{1}{T} d \left( \int_0^t S_u du - K \right) = \frac{S_t}{T} dt,$$

hence

432
Asian Options

\[ \frac{S_T Z_t}{S_s} = Z_s + \frac{1}{T} \int_s^t \frac{S_u}{S_s} du, \quad t \geq s. \]

Since for any \( t \in [0, T] \), \( S_t \) is positive and \( \mathcal{F}_t \)-measurable, and \( S_u/S_t \) is independent of \( \mathcal{F}_t \), \( u \geq t \), we have:

\[
e^{-(T-t)r} \mathbb{E}^* \left[ S_T (Z_T)^+ | \mathcal{F}_t \right] = e^{-(T-t)r} S_t \mathbb{E}^* \left[ \left( \frac{S_T Z_T}{S_t} \right)^+ | \mathcal{F}_t \right]
\]

\[
e^{-(T-t)r} S_t \mathbb{E}^* \left[ \left( Z_t + \frac{1}{T} \int_t^T \frac{S_u}{S_t} du \right)^+ | \mathcal{F}_t \right]
\]

\[
e^{-(T-t)r} S_t \mathbb{E}^* \left[ \left( z + \frac{1}{T} \int_0^T \frac{S_u}{S_t} du \right)^+ \right]_{z=Z_t}
\]

\[
e^{-(T-t)r} S_t \mathbb{E}^* \left[ \left( z + \frac{\Lambda_T - t}{S_0 T} \right)^+ \right]_{z=Z_t}
\]

\[ = S_t g(t, Z_t), \]

which proves (13.21).

When \( \Lambda_t/T \geq K \) we have \( Z_t \geq 0 \), hence by (13.18) and (13.20) we find

\[
g(t, Z_t) = e^{-(T-t)r} Z_t + \frac{1 - e^{-(T-t)r}}{r T}, \quad 0 \leq t \leq T. \quad (13.22)
\]

Note that as in (13.9), \( g(t, z) \) can be computed from the probability density function (13.8) of \( \Lambda_{T-t} \), as

\[
g(t, z) = \mathbb{E}^* \left[ \left( z + \frac{\Lambda_{T-t}}{S_0 T} \right)^+ \right]
\]

\[
= \int_0^\infty \left( z + \frac{u}{T} \right)^+ d\mathbb{P} \left( \frac{\Lambda_t}{S_0} \leq u \right)
\]

\[
= e^{-p^2 \sigma^2 t/8} \times \int_0^\infty \left( z + \frac{u}{T} \right)^+ \int_0^\infty v^{-1-p} \exp \left( -2 \frac{1 + v^2}{\sigma^2} \right) \theta \left( \frac{4 v}{\sigma^2 u}, \frac{(T-t) \sigma^2}{4} \right) dv du
\]

\[
= e^{-p^2 \sigma^2 t/8} \times \int_{(zT)_0}^\infty \left( z + \frac{u}{T} \right)^+ \int_0^\infty v^{-1-p} \exp \left( -2 \frac{1 + v^2}{\sigma^2} \right) \theta \left( \frac{4 v}{\sigma^2 u}, \frac{(T-t) \sigma^2}{4} \right) dv du
\]

\[
= z e^{-p^2 \sigma^2 t/8} \int_{(zT)_0}^\infty \int_0^\infty v^{-1-p} \exp \left( -2 \frac{1 + v^2}{\sigma^2} \right) \theta \left( \frac{4 v}{\sigma^2 u}, \frac{(T-t) \sigma^2}{4} \right) dv du
\]

\[ \diamondsuit \]

This version: January 15, 2020

https://www.ntu.edu.sg/home/nprivault/index.html
\[ + \frac{1}{T} e^{-p^2 \sigma^2 t/8} \int_0^\infty \int_0^\infty v^{-1-p} \exp \left( -\frac{1}{2} \frac{v^2}{\sigma^2} \right) \theta \left( \frac{4v}{\sigma^2 u}, \frac{(T-t)\sigma^2}{4} \right) \, dv \, du. \]


**Proposition 13.7.** Let \((\eta_t, \xi_t)_{t \in \mathbb{R}^+}\) be a self-financing portfolio strategy whose value \(V_t := \eta_t A_t + \xi_t S_t, t \in \mathbb{R}^+\), is given by

\[ V_t = S_t g(t, Z_t), \quad t \in \mathbb{R}^+, \]

where \(g \in C^{1,2}((0, T) \times (0, \infty))\) is given by (13.21). Then \(g(t, z)\) satisfies the PDE

\[
\frac{\partial g}{\partial t}(t, z) + \left( \frac{1}{T} - rz \right) \frac{\partial g}{\partial z}(t, z) + \frac{1}{2} \sigma^2 z^2 \frac{\partial^2 g}{\partial z^2}(t, z) = 0, \tag{13.23}
\]

under the terminal condition

\[ g(T, z) = z^+, \quad z \in \mathbb{R}, \tag{13.24} \]

and the corresponding replicating portfolio Delta is given by

\[ \xi_t = g(t, Z_t) - Z_t \frac{\partial g}{\partial z}(t, Z_t), \quad 0 \leq t \leq T. \tag{13.25} \]

**Proof.** By the Itô formula applied to \(1/S_t\) we have

\[
\begin{align*}
\frac{1}{S_t} & = \frac{1}{S_t} \left( (\mu + \sigma^2) \, dt - \sigma dB_t \right),
\end{align*}
\]

hence

\[
\begin{align*}
dZ_t & = d\left( \frac{1}{S_t} \left( \frac{\Lambda_t}{T} - K \right) \right) \\
& = d\left( \frac{\Lambda_t}{TS_t} - \frac{K}{S_t} \right) \\
& = \frac{1}{T} d\left( \frac{\Lambda_t}{S_t} \right) - Kd\left( \frac{1}{S_t} \right) \\
& = \frac{1}{T} d\Lambda_t + \left( \frac{\Lambda_t}{T} - K \right) d\left( \frac{1}{S_t} \right) \\
& = \frac{dt}{T} + S_tZ_t d\left( \frac{1}{S_t} \right) \\
& = \frac{dt}{T} + Z_t (\mu + \sigma^2) dt - Z_t \sigma dB_t.
\end{align*}
\]

By the self-financing condition (5.8) we have

434

This version: January 15, 2020

https://www.ntu.edu.sg/home/nprivault/index.html
Asian Options

\[ dV_t = \eta_t dA_t + \xi_t dS_t \]
\[ = r \eta_t A_t dt + \mu \xi_t S_t dt + \sigma \xi_t S_t dB_t, \quad (13.26) \]

\( t \in \mathbb{R}_+ \). An application of Itô’s formula to \( f(t, x, y) \) leads to

\[
d(S_t g(t, Z_t)) = g(t, Z_t) dS_t + S_t dg(t, Z_t) + dS_t \cdot dg(t, Z_t)
\]
\[
= g(t, Z_t) dS_t + S_t \frac{\partial g}{\partial t} (t, Z_t) dt + S_t \frac{\partial g}{\partial z} (t, Z_t) dZ_t
\]
\[
+ \frac{1}{2} S_t \frac{\partial^2 g}{\partial z^2} (t, Z_t) (dZ_t)^2 + dS_t \cdot dg(t, Z_t)
\]
\[
= \mu S_t g(t, Z_t) dt + \sigma S_t g(t, Z_t) dB_t + S_t \frac{\partial g}{\partial t} (t, Z_t) dt
\]
\[
+ S_t Z_t (\mu + \sigma^2) \frac{\partial g}{\partial z} (t, Z_t) dt + \frac{1}{T} S_t \frac{\partial g}{\partial z} (t, Z_t) dt - \sigma S_t Z_t \frac{\partial g}{\partial z} (t, Z_t) dB_t
\]
\[
+ \frac{1}{2} \sigma^2 Z_t^2 S_t \frac{\partial^2 g}{\partial z^2} (t, Z_t) dt - \sigma^2 S_t Z_t \frac{\partial g}{\partial z} (t, Z_t) dt
\]
\[
= \mu S_t g(t, Z_t) dt + S_t \frac{\partial g}{\partial t} (t, Z_t) dt + S_t Z_t (\mu + \sigma^2) \frac{\partial g}{\partial z} (t, Z_t) dt + \frac{1}{T} S_t \frac{\partial g}{\partial z} (t, Z_t) dt
\]
\[
+ \frac{1}{2} \sigma^2 Z_t^2 S_t \frac{\partial^2 g}{\partial z^2} (t, Z_t) dt - \sigma^2 S_t Z_t \frac{\partial g}{\partial z} (t, Z_t) dt
\]
\[
+ \sigma S_t g(t, Z_t) dB_t - \sigma S_t Z_t \frac{\partial g}{\partial z} (t, Z_t) dB_t.
\]

By respective identification of the terms in \( dB_t \) and \( dt \) in (13.26) and (13.17) we get

\[
\begin{cases}
  r \eta_t A_t + \mu \xi_t S_t = \mu S_t g(t, Z_t) + S_t \frac{\partial g}{\partial t} (t, Z_t) - \mu S_t Z_t \frac{\partial g}{\partial z} (t, Z_t) \\
  \quad + \frac{1}{T} S_t \frac{\partial g}{\partial z} (t, Z_t) + \frac{1}{2} \sigma^2 Z_t^2 S_t \frac{\partial^2 g}{\partial z^2} (t, Z_t), \\
  \xi_t S_t \sigma = \sigma S_t g(t, Z_t) - \sigma S_t Z_t \frac{\partial g}{\partial z} (t, Z_t),
\end{cases}
\]

hence

\[
\begin{cases}
  r V_t - r \xi_t S_t = S_t \frac{\partial g}{\partial t} (t, Z_t) + \frac{1}{T} S_t \frac{\partial g}{\partial z} (t, Z_t) + \frac{1}{2} \sigma^2 Z_t^2 S_t \frac{\partial^2 g}{\partial z^2} (t, Z_t), \\
  \xi_t = g(t, Z_t) - Z_t \frac{\partial g}{\partial z} (t, Z_t),
\end{cases}
\]

i.e.
\[
\left\{ \begin{array}{l}
\frac{\partial g}{\partial t}(t,z) + \left( \frac{1}{T} - rz \right) \frac{\partial g}{\partial z}(t,z) + \frac{1}{2} \sigma^2 z^2 \frac{\partial^2 g}{\partial z^2}(t,z) = 0, \\
\xi_t = g(t,Z_t) - Z_t \frac{\partial g}{\partial z}(t,Z_t),
\end{array} \right.
\]

under the terminal condition \( g(T,z) = z^+, z \in \mathbb{R} \), which follows from (13.21).

When \( \Lambda_t/T \geq K \) we have \( Z_t \geq 0 \) and (13.22) and (13.25) show that

\[
\xi_t = g(t,Z_t) - Z_t \frac{\partial g}{\partial z}(t,Z_t)
\]

\[
= e^{-(T-t)r} Z_t + \frac{1 - e^{-(T-t)r}}{rT} - e^{-(T-t)r} Z_t
\]

\[
= \frac{1 - e^{-(T-t)r}}{rT}, \quad 0 \leq t \leq T,
\]

which recovers (13.19). Similarly, from (13.24) we recover

\[
\xi_T = g(T,Z_T) - Z_T \frac{\partial g}{\partial z}(T,Z_T) = Z_T \mathbb{1}_{\{Z_T \geq 0\}} - Z_T \mathbb{1}_{\{Z_T \geq 0\}} = 0
\]
at maturity.

We also check that

\[
\xi_t = e^{-(T-t)r} \sigma S_t \frac{\partial f}{\partial x}(t,S_t,Z_t) - \sigma Z_t \frac{\partial f}{\partial z}(t,S_t,Z_t)
\]

\[
= e^{-(T-t)r} \left( -Z_t \frac{\partial g}{\partial z}(t,Z_t) + g(t,Z_t) \right)
\]

\[
= e^{-(T-t)r} \left( S_t \frac{\partial g}{\partial z} \left( t, \frac{1}{x} \left( \frac{1}{T} \int_0^t S_u du - K \right) \right) \right|_{x=S_t} + g(t,Z_t) \right)
\]

\[
= \left. \frac{\partial}{\partial x} \left( x e^{-(T-t)r} g \left( t, \frac{1}{x} \left( \frac{1}{T} \int_0^t S_u du - K \right) \right) \right) \right|_{x=S_t}, \quad 0 \leq t \leq T.
\]

We also find that the amount invested on the riskless asset is given by

\[
\eta_t A_t = Z_t S_t \frac{\partial g}{\partial z}(t,Z_t).
\]

Next we note that a PDE with no first order derivative term can be obtained using time-dependent coefficients.

(2) One variable with - Time-dependent coefficients

Define now the auxiliary process
Asian Options

\[ U_t := \frac{1 - e^{-(T-t)r}}{rT} + e^{-(T-t)r} \frac{1}{S_t} \left( \frac{1}{T} \int_0^T S_u du - K \right) \]
\[ = \frac{1}{rT} (1 - e^{-(T-t)r}) + e^{-(T-t)r} Z_t, \quad 0 \leq t \leq T, \]

i.e.

\[ Z_t = e^{(T-t)r} U_t + \frac{e^{(T-t)r} - 1}{rT}, \quad 0 \leq t \leq T. \]

We have

\[ dU_t = -\frac{1}{T} e^{-(T-t)r} dt + r e^{-(T-t)r} Z_t dt + e^{-(T-t)r} dZ_t \]
\[ = e^{-(T-t)r} \sigma^2 Z_t dt - e^{-(T-t)r} \sigma Z_t dB_t - (\mu - r) e^{-(T-t)r} Z_t dt \]
\[ = -e^{-(T-t)r} \sigma Z_t dB_t, \quad t \in \mathbb{R}^+. \]

where

\[ d\hat{B}_t = dB_t - \sigma dt + \frac{\mu - r}{\sigma} dt = d\hat{B}_t - \sigma dt \]

is a standard Brownian motion under

\[ d\hat{P} = e^{\sigma B_T - \sigma^2 t/2} d\mathbb{P}^* = e^{-rT S_T} S_0 d\mathbb{P}^*. \]

**Lemma 13.8.** The Asian call option price can be written as

\[ S_t h(t, U_t) = e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ | F_t \right], \]

where the function \( h(t, y) \) is given by

\[ h(t, y) = \hat{\mathbb{E}}[(U_T)^+ | U_t = y], \quad 0 \leq t \leq T. \] (13.27)

**Proof.** We have

\[ U_T = \frac{1}{S_T} \left( \frac{1}{T} \int_0^T S_u du - K \right) = Z_T, \]

and

\[ \frac{d\hat{P}}{d\mathbb{P}^*}_{|F_t} = e^{(B_T - B_t)\sigma - (T-t)\sigma^2/2} = e^{-rT S_T} S_t, \]

hence the price of the Asian call option is

\[ e^{-(T-t)r} \mathbb{E}^* [S_T (Z_T)^+ | F_t] = e^{-(T-t)r} \mathbb{E}^* [S_T (U_T)^+ | F_t] \]
\[ = S_t \mathbb{E}^* \left[ \frac{e^{-rT S_T}}{e^{-rT S_t}} (U_T)^+ | F_t \right]. \]
The next proposition gives a replicating hedging strategy for Asian options. See § 7.5.3 of Shreve (2004) and references therein for a different derivation of the PDE (13.28).

**Proposition 13.9.** Let \((\eta_t, \xi_t)_{t \in \mathbb{R}_+}\) be a self-financing portfolio strategy whose value \(V_t := \eta_t A_t + \xi_t S_t, t \in \mathbb{R}_+,\) is given by

\[ V_t = S_t h(t, U_t), \quad t \in \mathbb{R}_+, \]

where \(h \in C^{1,2}((0, T) \times (0, \infty))\) is given by (13.27). Then \(h(t, z)\) satisfies the PDE

\[
\frac{\partial h}{\partial t}(t, y) + \frac{\sigma^2}{2} \left( \frac{1 - e^{-(T-t)r}}{rT} - y \right)^2 \frac{\partial^2 h}{\partial y^2}(t, y) = 0,
\]

under the terminal condition

\[ h(T, z) = z^+, \]

and the corresponding replicating portfolio is given by

\[ \xi_t = h(t, U_t) - Z_t \frac{\partial h}{\partial y}(t, U_t), \quad 0 \leq t \leq T. \]

**Proof.** By the self-financing condition (13.16) we have

\[
dV_t = rV_t dt + (\mu - r)\xi_t S_t dt + \sigma \xi_t S_t dB_t,
\]

\(t \in \mathbb{R}_+.\) By Itô’s formula we get

\[
d(S_t h(t, U_t)) = h(t, U_t) dS_t + S_t dh(t, U_t) + dS_t \cdot dh(t, U_t)
\]

\[= \mu S_t h(t, U_t) dt + \sigma S_t h(t, U_t) dB_t + \frac{\partial h}{\partial t}(t, U_t) dt + \frac{\partial h}{\partial y}(t, U_t) dU_t + \frac{1}{2} \frac{\partial^2 h}{\partial y^2}(t, U_t) (dU_t)^2 + \frac{\partial h}{\partial y}(t, U_t) dS_t \cdot dU_t
\]

\[= \mu S_t h(t, U_t) dt + \sigma S_t h(t, U_t) dB_t - S_t (\mu - r) \frac{\partial h}{\partial y}(t, U_t) Z_t dt.
\]
Asian Options

\[ + S_t \left( \frac{\partial h}{\partial t}(t, U_t) dt - \sigma \frac{\partial h}{\partial y}(t, U_t) Z_t d\tilde{B}_t + \frac{\sigma^2}{2} Z_t^2 \frac{\partial^2 h}{\partial y^2}(t, U_t) dt \right) \]

\[ - \sigma^2 S_t \frac{\partial h}{\partial y}(t, U_t) Z_t dt \]

\[ = \mu S_t h(t, U_t) dt + \sigma S_t h(t, U_t) dB_t - \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 h}{\partial y^2}(t, U_t) \]

By respective identification of the terms in \( dB_t \) and \( dt \) in (13.29) and (13.17) we get

\[ \begin{cases} 
 r \eta A_t + \mu \xi_t S_t = \mu S_t h(t, U_t) - (\mu - r) S_t Z_t \frac{\partial h}{\partial y}(t, U_t) dt + S_t \frac{\partial h}{\partial t}(t, U_t) \\
 + \frac{\sigma^2}{2} S_t Z_t^2 \frac{\partial^2 h}{\partial y^2}(t, U_t),
\end{cases} \]

\[ \xi_t = h(t, U_t) - Z_t \frac{\partial h}{\partial y}(t, U_t), \]

hence

\[ \begin{cases} 
 r \eta A_t = -r S_t (\xi_t - h(t, U_t)) + S_t \frac{\partial h}{\partial t}(t, U_t) + \frac{\sigma^2}{2} S_t Z_t^2 \frac{\partial^2 h}{\partial y^2}(t, U_t),
\end{cases} \]

\[ \xi_t = h(t, U_t) - Z_t \frac{\partial h}{\partial y}(t, U_t), \]

and

\[ \begin{cases} 
 \frac{\partial h}{\partial t}(t, y) + \frac{\sigma^2}{2} \left( \frac{1 - e^{-(T-t)r}}{rT} - y \right)^2 \frac{\partial^2 h}{\partial y^2}(t, y) = 0,
\end{cases} \]

\[ \xi_t = h(t, U_t) + \left( \frac{1 - e^{-(T-t)r}}{rT} - U_t \right) \frac{\partial h}{\partial y}(t, U_t), \]

under the terminal condition

\[ h(T, z) = z^+. \]

\[ \square \]

We also find the riskless portfolio allocation

\[ \diamond \]
\[ \eta_t A_t = e^{(T-t)r} S_t \left( U_t - \frac{1 - e^{-(T-t)r}}{r T} \right) \frac{\partial h}{\partial y}(t, U_t) = S_t Z_t \frac{\partial h}{\partial y}(t, U_t). \]

**Exercises**

**Exercise 13.1** Compute the first and second moments of the time integral \( \int_\tau^T S_t dt \) for \( \tau \in [0, T) \), where \((S_t)_{t \in \mathbb{R}^+}\) is the geometric Brownian motion \( S_t := S_0 e^{\sigma B_t + rt - \sigma^2 t/2}, \ t \in \mathbb{R}^+ \).

**Exercise 13.2** Consider the short rate process \( r_t = \sigma B_t \), where \((B_t)_{t \in \mathbb{R}^+}\) is a standard Brownian motion.

a) Find the probability distribution of the time integral \( \int_0^T r_s ds \).

b) Compute the price \( e^{-r T} \mathbb{E}^* \left[ \left( \int_0^T r_u du - \kappa \right)^+ \right] \)

of a caplet on the forward rate \( \int_0^T r_s ds \).

**Exercise 13.3** Asian call options with *negative* strike price. Consider the asset price process

\[ S_t = S_0 e^{rt + \sigma B_t - \sigma^2 t/2}, \ t \in \mathbb{R}^+, \]

where \((B_t)_{t \in \mathbb{R}^+}\) is a standard Brownian motion. Assuming that \( \kappa \leq 0 \), compute the price

\[ e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_u du - \kappa \right)^+ \mid \mathcal{F}_t \right] \]

of the Asian option at time \( t \in [0, T] \).

**Exercise 13.4** Compute the price

\[ e^{-(T-t)r} \mathbb{E}^* \left[ \left( \exp \left( \frac{1}{T} \int_0^T \log S_u du \right) - K \right)^+ \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T. \]

at time \( t \) of the geometric Asian option with maturity \( T \), where \( S_t = S_0 e^{rt + \sigma B_t - \sigma^2 t/2}, t \in [0, T] \).

440

This version: January 15, 2020

https://www.ntu.edu.sg/home/nprivault/index.html
Asian Options

Hint: When \( X \approx \mathcal{N}(0, v^2) \) we have

\[
\mathbb{E}^*[(e^{m + X} - K)^+] = e^{m + v^2/2} \Phi(v + (m - \log K)/v) - K \Phi((m - \log K)/v).
\]

Exercise 13.5 Consider a CIR process \((r_t)_{t \in \mathbb{R}^+}\) given by

\[
dr_t = -\lambda (r_t - m) dt + \sigma \sqrt{r_t} dB_t,
\]

where \((B_t)_{t \in \mathbb{R}^+}\) is a standard Brownian motion under the risk-neutral probability measure \(\mathbb{P}^*\), and let

\[
\Lambda_t := \frac{1}{T - \tau} \int_\tau^t r_s ds, \quad t \in [\tau, T].
\]

Compute the price at time \( t \in [\tau, T] \) of the Asian option with payoff \((\Lambda_T - K)^+\), under the condition \(\Lambda_t \geq K\).

Exercise 13.6 Consider an asset price \((S_t)_{t \in \mathbb{R}^+}\) which is a submartingale under the risk-neutral probability measure \(\mathbb{P}^*\), in a market with risk-free interest rate \(r > 0\), and let \(\phi(x) = (x - K)^+\) be the (convex) payoff function of the European call option.

Show that, for any sequence \(0 < T_1 < \cdots < T_n\), the price of the option on average with payoff

\[
\phi \left( \frac{S_{T_1} + \cdots + S_{T_n}}{n} \right)
\]

can be upper bounded by the price of the European call option with maturity \(T_n\), i.e.

\[
\mathbb{E}^* \left[ \phi \left( \frac{S_{T_1} + \cdots + S_{T_n}}{n} \right) \right] \leq \mathbb{E}^*[\phi(S_{T_n})].
\]

Exercise 13.7 Let \((S_t)_{t \in \mathbb{R}^+}\) denote a risky asset whose price \(S_t\) is given by

\[
dS_t = \mu S_t dt + \sigma S_t dB_t,
\]

where \((B_t)_{t \in \mathbb{R}^+}\) is a standard Brownian motion under the risk-neutral probability measure \(\mathbb{P}^*\). Compute the price at time \( t \in [\tau, T] \) of the Asian option with payoff

\[
\left( \frac{1}{T - \tau} \int_\tau^T S_u du - K \right)^+,
\]

under the condition that

\[
A_t := \frac{1}{T - \tau} \int_\tau^t S_u du \geq K.
\]
Exercise 13.8 Pricing Asian options by PDEs. Show that the functions $g(t, z)$ and $h(t, y)$ are linked by the relation

$$g(t, z) = h\left(t, \frac{1 - e^{-(T-t)r}}{rT} + e^{-(T-t)r}z\right), \quad 0 \leq t \leq T, \quad z > 0,$$

and that the PDE (1.35) for $h(t, y)$ can be derived from the PDE (1.33) for $g(t, z)$ and the above relation.

Exercise 13.9 Hedging Asian options (Yang et al. (2011)).

a) Compute the Asian option price $f(t, S_t, \Lambda_t)$ when $\Lambda_t / T \geq K$.

b) Compute the hedging portfolio allocation $(\xi_t, \eta_t)$ when $\Lambda_t / T \geq K$. When $\Lambda_t / T \geq K$ we have

$$\xi_t = \frac{1 - e^{-(T-t)r}}{rT} \quad \text{and} \quad \eta_t A_t = e^{(T-t)r} \left(\frac{\Lambda_t}{T} - K\right), \quad 0 \leq t \leq T.$$

c) At maturity we have $f(T, S_T, \Lambda_T) = (\Lambda_T / T - K)^+$, hence $\xi_T = 0$ and

$$\eta_T A_T = A_T \frac{e^{-rT}}{A_0} \left(\frac{\Lambda_T}{T} - K\right) 1_{\{\Lambda_T > KT\}} = \left(\frac{\Lambda_T}{T} - K\right)^+.$$  

d) Show that the Asian option with payoff $(\Lambda_T - K)^+$ can be hedged by the self-financing portfolio

$$\xi_t = \frac{1}{S_t} \left(f(t, S_t, \Lambda_t) - e^{-(T-t)r} \left(\frac{\Lambda_t}{T} - K\right) h\left(t, \frac{1}{S_t} \left(\frac{\Lambda_t}{T} - K\right)\right)\right)$$

in the asset $S_t$ and

$$\eta_t = \frac{e^{-rT}}{A_0} \left(\frac{\Lambda_t}{T} - K\right) h\left(t, \frac{1}{S_t} \left(\frac{\Lambda_t}{T} - K\right)\right), \quad 0 \leq t \leq T,$$

in the riskless asset $A_t = A_0 e^{rt}$, where $h(t, z)$ is solution to a partial differential equation to be written explicitly.

Exercise 13.10 Asian options with dividends. Consider an underlying asset price process $(S_t)_{t \in \mathbb{R}_+}$ modeled as $dS_t = (\mu - \delta)S_t dt + \sigma S_t dB_t$, where $(B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion and $\delta > 0$ is a continuous-time dividend rate.

a) Write down the self-financing condition for the portfolio value $V_t = \xi_t S_t + \eta_t A_t$ with $A_t = A_0 e^{rt}$, assuming that all dividends are reinvested.

b) Derive the Black-Scholes PDE for the function $g_\delta(t, x, y)$ such that $V_t = g_\delta(t, S_t, \Lambda_t)$ at time $t \in [0, T]$.  

This version: January 15, 2020  
https://www.ntu.edu.sg/home/nprivault/indext.html
Asian Options

install.packages("quantmod")
library(quantmod)
getDividends("Z74.SI",from="2018-01-01",to="2018-12-31",src="yahoo")
getSymbols("Z74.SI",from="2018-11-16",to="2018-12-19",src="yahoo")
T <- chart_theme(); T$col$line.col <- "black"
chart_Series(Op(Z74.SI),name="Opening prices (black) - Closing prices (blue)",lty=4,theme=T)
add_TA(Cl(Z74.SI),lwd=2,lty=5,legend='Difference',col="blue",on = 1)

Z74.SI.div
2018-07-26 0.107
2018-12-17 0.068
2018-12-18 0.068

Fig. 13.5: SGD0.068 dividend detached on 18 Dec 2018 on Z74.SI.

The difference between the closing price on Dec 17 ($3.06) and the opening price on Dec 18 ($2.99) is $3.06 − $2.99 = $0.07. The adjusted price on Dec 17 ($2.992) is the closing price ($3.06) minus the dividend ($0.068).

<table>
<thead>
<tr>
<th>Z74.SI</th>
<th>Open</th>
<th>High</th>
<th>Low</th>
<th>Close</th>
<th>Volume</th>
<th>Adjusted (ex-dividend)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2018-12-17</td>
<td>3.05</td>
<td>3.08</td>
<td>3.05</td>
<td>3.06</td>
<td>17441000</td>
<td>2.992</td>
</tr>
<tr>
<td>2018-12-18</td>
<td>2.99</td>
<td>2.99</td>
<td>2.96</td>
<td>2.96</td>
<td>28456400</td>
<td>2.960</td>
</tr>
</tbody>
</table>

The dividend rate $\alpha$ is given by $\alpha = 0.068/3.06 = 2.22\%$. 
