Appendix: Background on Probability Theory

In this appendix we review a number of basic probabilistic tools that are needed in option pricing and hedging. We refer to Jacod and Protter (2000), Devore (2003), Pitman (1999) for additional background on probability theory.

22.1 Probability Sample Space and Events

We will need the following notation coming from set theory. Given $A$ and $B$ to abstract sets, “$A \subset B$” means that $A$ is contained in $B$, and in this case, $B \setminus A$ denotes the set of elements of $B$ which do not belong to $A$. The property that the element $\omega$ belongs to the set $A$ is denoted by “$\omega \in A$”, and given two sets $A$ and $\Omega$ such that $A \subset \Omega$, we let $A^c = \Omega \setminus A$ denote the complement of $A$ in $\Omega$. The finite set made of $n$ elements $\omega_1, \ldots, \omega_n$ is denoted by \{\omega_1, \ldots, \omega_n\}, and we will usually distinguish between the element $\omega$ and its associated singleton set \{\omega\}.

A probability sample space is an abstract set $\Omega$ that contains the possible outcomes of a random experiment.

Examples

i) Coin tossing: $\Omega = \{H, T\}$.
ii) Rolling one die: $\Omega = \{1, 2, 3, 4, 5, 6\}$.

iii) Picking one card at random in a pack of 52: $\Omega = \{1, 2, 3, \ldots, 52\}$.

iv) An integer-valued random outcome: $\Omega = \mathbb{N} = \{0, 1, 2, \ldots\}$.

   In this case the outcome $\omega \in \mathbb{N}$ can be the random number of trials needed until some event occurs.

v) A nonnegative, real-valued outcome: $\Omega = \mathbb{R}^+$. 

   In this case the outcome $\omega \in \mathbb{R}^+$ may represent the (nonnegative) value of a continuous random time.

vi) A random continuous parameter (such as time, weather, price or wealth, temperature, ...): $\Omega = \mathbb{R}$.

vii) Random choice of a continuous path in the space $\Omega = \mathcal{C}(\mathbb{R}^+)$ of all continuous functions on $\mathbb{R}^+$.

   In this case, $\omega \in \Omega$ is a function $\omega : \mathbb{R}^+ \rightarrow \mathbb{R}$ and a typical example is the graph $t \mapsto \omega(t)$ of a stock price over time.

**Product spaces:**

Probability sample spaces can be built as product spaces and used for the modeling of repeated random experiments.

i) Rolling two dice: $\Omega = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$.

   In this case a typical element of $\Omega$ is written as $\omega = (k, l)$ with $k, l \in \{1, 2, 3, 4, 5, 6\}$.

ii) A finite number $n$ of real-valued samples: $\Omega = \mathbb{R}^n$.

   In this case the outcome $\omega$ is a vector $\omega = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with $n$ components.

Note that to some extent, the more complex $\Omega$ is, the better it fits a practical and useful situation, e.g. $\Omega = \{H, T\}$ corresponds to a simple coin tossing experiment while $\Omega = \mathcal{C}(\mathbb{R}^+)$ the space of continuous functions on $\mathbb{R}^+$ can be applied to the modeling of stock markets. On the other hand, in many cases and especially in the most complex situations, we will not attempt to specify $\Omega$ explicitly.

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Events

An event is a collection of outcomes, which is represented by a subset of Ω. The collections \( \mathcal{G} \) of events that we will consider are called \( \sigma \)-algebras, according to the following definition.

**Definition 22.1.** A collection \( \mathcal{G} \) of events is a \( \sigma \)-algebras provided that it satisfies the following conditions:

(i) \( \emptyset \in \mathcal{G} \),

(ii) For all countable sequences \((A_n)_{n \geq 1}\) such that \( A_n \in \mathcal{G}, \ n \geq 1 \), we have

\[
\bigcup_{n \geq 1} A_n \in \mathcal{G},
\]

(iii) \( A \in \mathcal{G} \implies (\Omega \setminus A) \in \mathcal{G}, \)

where \( \Omega \setminus A := \{ \omega \in \Omega : \omega \notin A \} \).

Note that Properties (ii) and (iii) above also imply

\[
\bigcap_{n \geq 1} A_n = \left( \bigcup_{n \geq 1} A_n^c \right)^c \in \mathcal{G}, \tag{22.9}
\]

for all countable sequences \( A_n \in \mathcal{G}, \ n \geq 1 \).

The collection of all events in \( \Omega \) will often be denoted by \( \mathcal{F} \). The empty set \( \emptyset \) and the full space \( \Omega \) are considered as events but they are of less importance because \( \Omega \) corresponds to “any outcome may occur” while \( \emptyset \) corresponds to an absence of outcome, or no experiment.

In the context of stochastic processes, two \( \sigma \)-algebras \( \mathcal{F} \) and \( \mathcal{G} \) such that \( \mathcal{F} \subset \mathcal{G} \) will refer to two different amounts of information, the amount of information associated to \( \mathcal{F} \) being here lower than the one associated to \( \mathcal{G} \).

The formalism of \( \sigma \)-algebras helps in describing events in a short and precise way.

**Examples**

i) Let \( \Omega = \{1, 2, 3, 4, 5, 6\} \).

The event \( A = \{2, 4, 6\} \) corresponds to

“the result of the experiment is an even number”.

ii) Taking again \( \Omega = \{1, 2, 3, 4, 5, 6\} \),

\[
\mathcal{F} := \{\Omega, \emptyset, \{2, 4, 6\}, \{1, 3, 5\}\}
\]
defines a $\sigma$-algebra on $\Omega$ which corresponds to the knowledge of parity of an integer picked at random from 1 to 6.

Note that in the set-theoretic notation, an event $A$ is a subset of $\Omega$, i.e. $A \subset \Omega$, while it is an element of $\mathcal{F}$, i.e. $A \in \mathcal{F}$. For example, we have $\Omega \supset \{2, 4, 6\} \in \mathcal{F}$, while $\{\{2, 4, 6\}, \{1, 3, 5\}\} \subset \mathcal{F}$.

iii) Taking
$$\mathcal{G} := \{\Omega, \emptyset, \{2, 4, 6\}, \{2, 4\}, \{6\}, \{1, 2, 3, 4, 5\}, \{1, 3, 5, 6\}, \{1, 3, 5\}\} \supset \mathcal{F},$$
defines a $\sigma$-algebra on $\Omega$ which is bigger than $\mathcal{F}$, and corresponds to the parity information contained in $\mathcal{F}$, completed by the knowledge of whether the outcome is equal to 6 or not.

iv) Take
$$\Omega = \{H, T\} \times \{H, T\} = \{(H, H), (H, T), (T, H), (T, T)\}.$$ 

In this case, the collection $\mathcal{F}$ of all possible events is given by
$$\mathcal{F} = \{\emptyset, \{(H, H)\}, \{(T, T)\}, \{(H, T)\}, \{(T, H)\},$$
$$\{(T, T), (H, H)\}, \{(H, T), (T, H)\}, \{(H, T), (H, H)\}, \{(T, H), (T, T)\}, \{(H, H), (T, T), (H, H)\}, \{(T, H), (T, T), (H, T)\}, \{(H, H), (T, T), (T, H)\}, \{(H, T), (T, H), (T, T)\}, \Omega\}. \tag{22.10}$$

Note that the set $\mathcal{F}$ of all events considered in (22.10) above has altogether

$$1 = \binom{n}{0} \text{ event of cardinality 0,}$$

$$4 = \binom{n}{1} \text{ events of cardinality 1,}$$

$$6 = \binom{n}{2} \text{ events of cardinality 2,}$$

$$4 = \binom{n}{3} \text{ events of cardinality 3,}$$

$$1 = \binom{n}{4} \text{ event of cardinality 4,}$$

with $n = 4$, for a total of

$$16 = 2^n = \sum_{k=0}^{4} \binom{4}{k} = 1 + 4 + 6 + 4 + 1.$$
events. The collection of events
\[ G := \{ \emptyset, \{(T,T),(H,H)\}, \{(H,T),(T,H)\}, \Omega \} \]
defines a sub \( \sigma \)-algebra of \( F \), associated to the information “the results of
two coin tossings are different”.

Exercise: Write down the set of all events on \( \Omega = \{H,T\} \).

Note also that \((H,T)\) is different from \((T,H)\), whereas \(\{(H,T),(T,H)\}\) is
equal to \(\{(T,H),(H,T)\}\).

In addition, we will distinguish between the outcome \( \omega \in \Omega \) and its associated
event \( \{\omega\} \in F \), which satisfies \( \{\omega\} \subset \Omega \).

### 22.2 Probability Measures

**Definition 22.2.** A probability measure is a mapping \( P : F \to [0,1] \) that
assigns a probability \( P(A) \in [0,1] \) to any event \( A \in F \), with the properties

\begin{itemize}
  \item[a)] \( P(\Omega) = 1 \), and
  \item[b)] \( P \left( \bigcup_{n \geq 1} A_n \right) = \sum_{n \geq 1} P(A_n) \), whenever \( A_k \cap A_l = \emptyset \), \( k \neq l \).
\end{itemize}

Property (b) above is named the law of total probability. It states in particular
that we have
\[ P(A_1 \cup \cdots \cup A_n) = P(A_1) + \cdots + P(A_n) \]
when the subsets \( A_1, \ldots, A_n \) of \( \Omega \) are disjoints, and
\[ P(A \cup B) = P(A) + P(B) \] (22.11)
if \( A \cap B = \emptyset \). We also have the complement rule
\[ P(A^c) = P(\Omega \setminus A) = P(\Omega) - P(A) = 1 - P(A). \]

When \( A \) and \( B \) are not necessarily disjoint we can write
\[ P(A \cup B) = P(A) + P(B) - P(A \cap B). \]

The triple
\[ (\Omega, F, P) \] (22.12)
is called a *probability space*, and was introduced by A.N. Kolmogorov (1903-1987). This setting is generally referred to as the Kolmogorov framework.

A property or event is said to hold $\mathbb{P}$-almost surely (also written $\mathbb{P}$-a.s.) if it holds with probability equal to one.

**Example**

Take 

$$\Omega = \{(T, T), (H, H), (H, T), (T, H)\}$$

and 

$$\mathcal{F} = \{\emptyset, \{(T, T), (H, H)\}, \{(H, T), (T, H)\}, \Omega\}.$$ 

The *uniform* probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is given by setting 

$$\mathbb{P}(\{(T, T), (H, H)\}) := \frac{1}{2} \quad \text{and} \quad \mathbb{P}(\{(H, T), (T, H)\}) := \frac{1}{2}.$$ 

In addition, we have the following convergence properties.

1. Let $(A_n)_{n \in \mathbb{N}}$ be a *non-decreasing* sequence of events, i.e. $A_n \subset A_{n+1}$, $n \in \mathbb{N}$. Then we have 

$$\mathbb{P}\left( \bigcup_{n \in \mathbb{N}} A_n \right) = \lim_{n \to \infty} \mathbb{P}(A_n). \quad (22.13)$$

2. Let $(A_n)_{n \in \mathbb{N}}$ be a *non-increasing* sequence of events, i.e. $A_{n+1} \subset A_n$, $n \in \mathbb{N}$. Then we have 

$$\mathbb{P}\left( \bigcap_{n \in \mathbb{N}} A_n \right) = \lim_{n \to \infty} \mathbb{P}(A_n). \quad (22.14)$$

**Proposition 22.3. (Fatou’s lemma)** Let $(F_n)_{n \in \mathbb{N}}$ be a sequence of nonnegative random variable. Then we have 

$$\mathbb{E}\left[ \liminf_{n \to \infty} F_n \right] \leq \liminf_{n \to \infty} \mathbb{E}[F_n].$$

In particular, Fatou’s lemma shows that if in addition the sequence $(F_n)_{n \in \mathbb{N}}$ converges with probability one and the sequence $(\mathbb{E}[F_n])_{n \in \mathbb{N}}$ converges in $\mathbb{R}$ then we have 

$$\mathbb{E}\left[ \lim_{n \to \infty} F_n \right] \leq \lim_{n \to \infty} \mathbb{E}[F_n].$$
22.3 Conditional Probabilities and Independence

We start with an example.

Consider a population $\Omega = M \cup W$ made of a set $M$ of men and a set $W$ of women. Here the $\sigma$-algebra $\mathcal{F} = \{\Omega, \emptyset, W, M\}$ corresponds to the information given by gender. After polling the population, e.g. for a market survey, it turns out that a proportion $p \in [0, 1]$ of the population declares to like apples, while a proportion $1 - p$ declares to dislike apples. Let $A \subset \Omega$ denote the subset of individuals who like apples, while $A^c \subset \Omega$ denotes the subset individuals who dislike apples, with

$$p = \mathbb{P}(A) \quad \text{and} \quad 1 - p = \mathbb{P}(A^c),$$

e.g. $p = 60\%$ of the population likes apples. It may be interesting to get a more precise information and to determine

- the relative proportion $\frac{\mathbb{P}(A \cap W)}{\mathbb{P}(W)}$ of women who like apples, and
- the relative proportion $\frac{\mathbb{P}(A \cap M)}{\mathbb{P}(M)}$ of men who like apples.

Here, $\mathbb{P}(A \cap W)/\mathbb{P}(W)$ represents the probability that a randomly chosen woman in $W$ likes apples, and $\mathbb{P}(A \cap M)/\mathbb{P}(M)$ represents the probability that a randomly chosen man in $M$ likes apples. Those two ratios are interpreted as conditional probabilities, for example $\mathbb{P}(A \cap M)/\mathbb{P}(M)$ denotes the probability that an individual likes apples given that he is a man.

For another example, suppose that the population $\Omega$ is split as $\Omega = Y \cup O$ into a set $Y$ of “young” people and another set $O$ of “old” people, and denote by $A \subset \Omega$ the set of people who voted for candidate $A$ in an election. Here it can be of interest to find out the relative proportion

$$\mathbb{P}(A \mid Y) = \frac{\mathbb{P}(Y \cap A)}{\mathbb{P}(Y)}$$

of young people who voted for candidate $A$.

More generally, given any two events $A, B \subset \Omega$ with $\mathbb{P}(B) \neq 0$, we call

$$\mathbb{P}(A \mid B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

the probability of $A$ given $B$, or conditionally to $B$.

Remark 22.4. We note that if $\mathbb{P}(B) = 1$ we have $\mathbb{P}(A \cap B^c) \leq \mathbb{P}(B^c) = 0$, hence $\mathbb{P}(A \cap B^c) = 0$, which implies

$\Box$
\[ \mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c) = \mathbb{P}(A \cap B), \]

and \( \mathbb{P}(A | B) = \mathbb{P}(A) \).

We also recall the following property:

\[
\mathbb{P}
\left( B \cap \bigcup_{n \geq 1} A_n \right) = \sum_{n \geq 1} \mathbb{P}(B \cap A_n) \\
= \sum_{n \geq 1} \mathbb{P}(B | A_n) \mathbb{P}(A_n) \\
= \sum_{n \geq 1} \mathbb{P}(A_n | B) \mathbb{P}(B),
\]

for any family of disjoint events \((A_n)_{n \geq 1}\) with \(A_i \cap A_j = \emptyset, i \neq j\), and \(\mathbb{P}(B) > 0, n \geq 1\). This also shows that conditional probability measures are probability measures, in the sense that whenever \(\mathbb{P}(B) > 0\) we have

a) \(\mathbb{P}(\Omega | B) = 1\), and

b) \(\mathbb{P}
\left( \bigcup_{n \geq 1} A_n \biggm| B \right) = \sum_{n \geq 1} \mathbb{P}(A_n | B), \) whenever \(A_k \cap A_l = \emptyset, k \neq l\).

In particular, if \(\bigcup_{n \geq 1} A_n = \Omega\), \((A_n)_{n \geq 1}\) becomes a partition of \(\Omega\) and we get the law of total probability

\[ \mathbb{P}(B) = \sum_{n \geq 1} \mathbb{P}(B \cap A_n) = \sum_{n \geq 1} \mathbb{P}(A_n | B) \mathbb{P}(B) = \sum_{n \geq 1} \mathbb{P}(B | A_n) \mathbb{P}(A_n), \]

provided that \(A_i \cap A_j = \emptyset, i \neq j\), and \(\mathbb{P}(B) > 0, n \geq 1\). However, in general we have

\[ \mathbb{P}
\left( A \biggm| \bigcup_{n \geq 1} B_n \right) \neq \sum_{n \geq 1} \mathbb{P}(A | B_n), \]

even when \(B_k \cap B_l = \emptyset, k \neq l\). Indeed, taking for example \(A = \Omega = B_1 \cup B_2\) with \(B_1 \cap B_2 = \emptyset\) and \(\mathbb{P}(B_1) = \mathbb{P}(B_2) = 1/2\), we have

\[ 1 = \mathbb{P}(\Omega | B_1 \cup B_2) \neq \mathbb{P}(\Omega | B_1) + \mathbb{P}(\Omega | B_2) = 2. \]

**Independent events**

Two events \(A\) and \(B\) such that \(\mathbb{P}(A), \mathbb{P}(B) > 0\) are said to be *independent* if

\[ \mathbb{P}(A | B) = \mathbb{P}(A), \]
which is equivalent to
\[ P(A \cap B) = P(A)P(B). \]

In this case we find
\[ P(A \mid B) = P(A). \]

### 22.4 Random Variables

A real-valued random variable is a mapping
\[ X : \Omega \longrightarrow \mathbb{R} \]
\[ \omega \longmapsto X(\omega) \]
from a probability sample space \( \Omega \) into the state space \( \mathbb{R} \). Given
\[ X : \Omega \longrightarrow \mathbb{R} \]
a random variable and \( A \) a (measurable)* subset of \( \mathbb{R} \), we denote by \( \{ X \in A \} \) the event
\[ \{ X \in A \} := \{ \omega \in \Omega : X(\omega) \in A \}. \]

**Examples**

i) Let \( \Omega := \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\} \), and consider the mapping
\[ X : \Omega \longrightarrow \mathbb{R} \]
\[ (k, l) \longmapsto k + l. \]

Then \( X \) is a random variable giving the sum of the two numbers appearing on each die.

ii) the time needed everyday to travel from home to work or school is a random variable, as the precise value of this time may change from day to day under unexpected circumstances.

iii) the price of a risky asset is modeled using a random variable.

In the sequel we will often use the notion of **indicator function** \( 1_A \) of an event \( A \subset \Omega \).

**Definition 22.5.** For any \( A \subset \Omega \), the **indicator function** \( 1_A \) is the random variable
\[ 1_A : \Omega \longrightarrow \{0, 1\} \]

* Measurability of subsets of \( \mathbb{R} \) refers to Borel measurability, a concept which will not be defined in this text.
ω \mapsto \mathbb{1}_A(\omega)

defined by

\begin{align*}
\mathbb{1}_A(\omega) &= \begin{cases} 
1 & \text{if } \omega \in A, \\
0 & \text{if } \omega \notin A.
\end{cases}
\end{align*}

Indicator functions satisfy the property

\begin{align}
\mathbb{1}_{A \cap B}(\omega) &= \mathbb{1}_A(\omega) \mathbb{1}_B(\omega), 
\end{align}

since

\begin{align*}
\omega \in A \cap B &\iff \{\omega \in A \text{ and } \omega \in B\} \\
&\iff \{\mathbb{1}_A(\omega) = 1 \text{ and } \mathbb{1}_B(\omega) = 1\} \\
&\iff \mathbb{1}_A(\omega) \mathbb{1}_B(\omega) = 1.
\end{align*}

We also have

\begin{align*}
\mathbb{1}_{A \cup B} &= \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_{A \cap B} = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_A \mathbb{1}_B,
\end{align*}

and

\begin{align}
\mathbb{1}_{A \cup B} &= \mathbb{1}_A + \mathbb{1}_B, 
\end{align}

if \( A \cap B = \emptyset \).

For example, if \( \Omega = \mathbb{N} \) and \( A = \{k\} \), for all \( l \in \mathbb{N} \) we have

\begin{align*}
\mathbb{1}_{\{k\}}(l) &= \begin{cases} 
1 & \text{if } k = l, \\
0 & \text{if } k \neq l.
\end{cases}
\end{align*}

Given \( X \) a random variable, we also let

\begin{align*}
\mathbb{1}_{\{X=n\}} &= \begin{cases} 
1 & \text{if } X = n, \\
0 & \text{if } X \neq n,
\end{cases}
\end{align*}

and

\begin{align*}
\mathbb{1}_{\{X<n\}} &= \begin{cases} 
1 & \text{if } X < n, \\
0 & \text{if } X \geq n.
\end{cases}
\end{align*}

### 22.5 Probability Distributions

The probability distribution of a random variable \( X : \Omega \rightarrow \mathbb{R} \) is the collection

\[ \{ \mathbb{P}(X \in A) : A \text{ is a measurable subset of } \mathbb{R} \}. \]
As the collection of measurable subsets of \( \mathbb{R} \) coincides with the \( \sigma \)-algebra generated by the intervals in \( \mathbb{R} \), the distribution of \( X \) can be reduced to the knowledge of either

\[
\{ \mathbb{P}(a < X \leq b) = \mathbb{P}(X \leq b) - \mathbb{P}(X \leq a) : a < b \in \mathbb{R} \},
\]

or

\[
\{ \mathbb{P}(X \leq a) : a \in \mathbb{R} \}, \text{ or } \{ \mathbb{P}(X \geq a) : a \in \mathbb{R} \},
\]

see e.g. Corollary 3.8 in Çinlar (2011).

Two random variables \( X \) and \( Y \) are said to be independent under the probability \( \mathbb{P} \) if their probability distributions satisfy

\[
\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A) \mathbb{P}(Y \in B)
\]

for all (measurable) subsets \( A \) and \( B \) of \( \mathbb{R} \).

### Distributions admitting a density

We say that the distribution of \( X \) admits a probability density distribution function \( \varphi_X : \mathbb{R} \to \mathbb{R}_+ \) if, for all \( a \leq b \), the probability \( \mathbb{P}(a \leq X \leq b) \) can be written as

\[
\mathbb{P}(a \leq X \leq b) = \int_a^b \varphi_X(x) \, dx.
\]

We also say that the distribution of \( X \) is absolutely continuous, or that \( X \) is an absolutely continuous random variable. This, however, does not imply that the density function \( \varphi_X : \mathbb{R} \to \mathbb{R}_+ \) is continuous.

In particular, we always have

\[
\int_{-\infty}^{\infty} \varphi_X(x) \, dx = \mathbb{P}(-\infty \leq X \leq \infty) = 1
\]

for all probability density functions \( \varphi_X : \mathbb{R} \to \mathbb{R}_+ \).

**Remark 22.6.** Note that if the distribution of \( X \) admits a density function \( \varphi_X \) then for all \( a \in \mathbb{R} \), we have

\[
\mathbb{P}(X = a) = \int_a^a \varphi_X(x) \, dx = 0,
\]

and this is not a contradiction.

In particular, Remark 22.6 shows that

\[
\mathbb{P}(a \leq X \leq b) = \mathbb{P}(X = a) + \mathbb{P}(a < X \leq b) = \mathbb{P}(a < X \leq b) = \mathbb{P}(a < X < b),
\]
for $a \leq b$. Property (22.18) appears for example in the framework of lottery games with a large number of participants, in which a given number “$a$” selected in advance has a very low (almost zero) probability to be chosen.

The density $\varphi_X$ can be recovered from the cumulative distribution functions

$$x \mapsto F_X(x) := \mathbb{P}(X \leq x) = \int_{-\infty}^{x} \varphi_X(s)ds,$$

and

$$x \mapsto 1 - F_X(x) = \mathbb{P}(X \geq x) = \int_{x}^{\infty} \varphi_X(s)ds,$$

as

$$\varphi_X(x) = \frac{\partial F_X}{\partial x}(x) = \frac{\partial}{\partial x} \int_{-\infty}^{x} \varphi_X(s)ds = -\frac{\partial}{\partial x} \int_{x}^{\infty} \varphi_X(s)ds, \quad x \in \mathbb{R}.$$ 

Examples

i) The uniform distribution on an interval.

The probability density function of the uniform distribution on the interval $[a, b]$, $a < b$, is given by

$$\varphi(x) = \frac{1}{b-a} \mathbb{1}_{[a,b]}(x), \quad x \in \mathbb{R}.$$ 

ii) The Gaussian distribution.

The probability density function of the standard normal distribution is given by

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad x \in \mathbb{R}.$$ 

More generally, the probability density function of the Gaussian distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$ is given by

$$\varphi(x) := \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)}, \quad x \in \mathbb{R}.$$ 

In this case, we write $X \sim \mathcal{N}(\mu, \sigma^2)$.

iii) The exponential distribution.

The probability density function of the exponential distribution with parameter $\lambda > 0$ is given by
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\[ \varphi(x) := \lambda \mathbb{1}_{[0,\infty)}(x) e^{-\lambda x} = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0. \end{cases} \]  

(22.19)

We also have

\[ P(X > t) = e^{-\lambda t}, \quad t \in \mathbb{R}_+. \]  

(22.20)

iv) The \textit{gamma} distribution.

The probability density function of the gamma distribution is given by

\[ \varphi(x) := \frac{a^\lambda}{\Gamma(\lambda)} \mathbb{1}_{[0,\infty)}(x) x^{\lambda-1} e^{-ax} = \begin{cases} \frac{a^\lambda}{\Gamma(\lambda)} x^{\lambda-1} e^{-ax}, & x \geq 0 \\ 0, & x < 0, \end{cases} \]

where \( a > 0 \) and \( \lambda > 0 \) are parameters, and

\[ \Gamma(\lambda) := \int_0^\infty x^{\lambda-1} e^{-x} dx, \quad \lambda > 0, \]

is the gamma function.

v) The \textit{Cauchy} distribution.

The probability density function of the Cauchy distribution is given by

\[ \varphi(x) := \frac{1}{\pi(1 + x^2)}, \quad x \in \mathbb{R}. \]

vi) The \textit{lognormal} distribution.

The probability density function of the lognormal distribution is given by

\[ \varphi(x) := \mathbb{1}_{[0,\infty)}(x) \frac{1}{x\sigma \sqrt{2\pi}} e^{-(\mu - \log x)^2/(2\sigma^2)} = \begin{cases} \frac{1}{x\sigma \sqrt{2\pi}} e^{-(\mu - \log x)^2/(2\sigma^2)}, & x \geq 0 \\ 0, & x < 0. \end{cases} \]

Exercise: For each of the above probability density functions \( \varphi \), check that the condition

\[ \int_{-\infty}^{\infty} \varphi(x) dx = 1 \]

is satisfied.
Joint densities

Given two absolutely continuous random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ we can form the $\mathbb{R}^2$-valued random variable $(X, Y)$ defined by

$$(X, Y) : \Omega \to \mathbb{R}^2$$

$$\omega \mapsto (X(\omega), Y(\omega)).$$

We say that $(X, Y)$ admits a joint probability density

$$\varphi_{(X,Y)} : \mathbb{R}^2 \to \mathbb{R}_+$$

when

$$\mathbb{P}((X, Y) \in A \times B) = \int_B \int_A \varphi_{(X,Y)}(x, y) dx dy$$

for all measurable subsets $A, B$ of $\mathbb{R}$, cf. Figure 22.3.

Fig. 22.3: Probability $\mathbb{P}((X, Y) \in [-0.5, 1] \times [-0.5, 1])$ computed as a volume integral.

The density $\varphi_{(X,Y)}$ can be recovered from the joint cumulative distribution function

$$(x, y) \mapsto F_{(X,Y)}(x, y) := \mathbb{P}(X \leq x \text{ and } Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} \varphi_{(X,Y)}(s, t) ds dt,$$

and

$$(x, y) \mapsto \mathbb{P}(X \geq x \text{ and } Y \geq y) = \int_{x}^{\infty} \int_{y}^{\infty} \varphi_{(X,Y)}(s, t) ds dt,$$

as

$$\varphi_{(X,Y)}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{(X,Y)}(x, y)$$

$$= \frac{\partial^2}{\partial x \partial y} \int_{-\infty}^{x} \int_{-\infty}^{y} \varphi_{(X,Y)}(s, t) ds dt$$

(22.21) (22.22)
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\[
\frac{\partial^2}{\partial x \partial y} \int_{x}^{\infty} \int_{y}^{\infty} \varphi(X,Y)(s,t)dsdt,
\]

\(x, y \in \mathbb{R}.
\)

The probability densities \(\varphi_X : \mathbb{R} \rightarrow \mathbb{R}_+\) and \(\varphi_Y : \mathbb{R} \rightarrow \mathbb{R}_+\) of \(X : \Omega \rightarrow \mathbb{R}\) and \(Y : \Omega \rightarrow \mathbb{R}\) are called the marginal densities of \((X, Y)\), and are given by

\[
\varphi_X(x) = \int_{-\infty}^{\infty} \varphi(X,Y)(x,y)dy, \quad x \in \mathbb{R}, \quad (22.23)
\]

and

\[
\varphi_Y(y) = \int_{-\infty}^{\infty} \varphi(X,Y)(x,y)dx, \quad y \in \mathbb{R}.
\]

The conditional density \(\varphi_{X|Y=y} : \mathbb{R} \rightarrow \mathbb{R}_+\) of \(X\) given \(Y = y\) is defined by

\[
\varphi_{X|Y=y}(x) := \frac{\varphi(X,Y)(x,y)}{\varphi_Y(y)}, \quad x, y \in \mathbb{R}, \quad (22.24)
\]

provided that \(\varphi_Y(y) > 0\). In particular, \(X\) and \(Y\) are independent if and only if \(\varphi_{X|Y=y}(x) = \varphi_X(x), x, y \in \mathbb{R\), i.e.,

\[
\varphi(X,Y)(x,y) = \varphi_X(x)\varphi_Y(y), \quad x, y \in \mathbb{R}.
\]

Example

If \(X_1, \ldots, X_n\) are independent exponentially distributed random variables with parameters \(\lambda_1, \ldots, \lambda_n\) we have

\[
\mathbb{P}(\min(X_1, \ldots, X_n) > t) = \mathbb{P}(X_1 > t, \ldots, X_n > t) = \mathbb{P}(X_1 > t) \cdots \mathbb{P}(X_n > t) = e^{-(\lambda_1 + \cdots + \lambda_n)t}, \quad t \in \mathbb{R}_+, \quad (22.25)
\]

hence \(\min(X_1, \ldots, X_n)\) is an exponentially distributed random variable with parameter \(\lambda_1 + \cdots + \lambda_n\).

Given the joint density of \((X_1, X_2)\) given by

\[
\varphi(X_1, X_2)(x,y) = \varphi_X(x)\varphi_Y(y) = \lambda_1 \lambda_2 e^{-\lambda_1 x - \lambda_2 y}, \quad x, y \geq 0,
\]

we can write

\[
\mathbb{P}(X_1 < X_2) = \mathbb{P}(X_1 \leq X_2)
\]

\[
= \int_{0}^{\infty} \int_{0}^{y} \varphi(X_1, X_2)(x,y)dxdy
\]

\[
= \lambda_1 \lambda_2 \int_{0}^{\infty} \int_{0}^{y} e^{-\lambda_1 x - \lambda_2 y}dxdy
\]
and we note that
\[ \mathbb{P}(X_1 = X_2) = \lambda_1 \lambda_2 \int_{\{(x,y) \in \mathbb{R}_+^2 : x=y\}} e^{-\lambda_1 x - \lambda_2 y} dx dy = 0. \]

**Discrete distributions**

We only consider integer-valued random variables, *i.e.* the distribution of $X$ is given by the values of $\mathbb{P}(X = k), k \in \mathbb{N}$.

**Examples**

i) The *Bernoulli* distribution.

We have
\[ \mathbb{P}(X = 1) = p \quad \text{and} \quad \mathbb{P}(X = 0) = 1 - p, \]
where $p \in [0, 1]$ is a parameter.

Note that any Bernoulli random variable $X : \Omega \rightarrow \{0, 1\}$ can be written as the indicator function
\[ X = 1_A \]
on $\Omega$ with $A = \{X = 1\} = \{\omega \in \Omega : X(\omega) = 1\}$.

ii) The *binomial* distribution.

We have
\[ \mathbb{P}(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0, 1, \ldots, n, \]
where $n \geq 1$ and $p \in [0, 1]$ are parameters.

iii) The *geometric* distribution.

In this case, we have
\[ \mathbb{P}(X = k) = (1 - p)p^k, \quad k \in \mathbb{N}, \]
where $p \in (0, 1)$ is a parameter. For example, if $(X_k)_{k \in \mathbb{N}}$ is a sequence of independent Bernoulli random variables with distribution (22.27), then
the random variable,*

\[ T_0 := \inf\{k \in \mathbb{N} : X_k = 0\} \]

can denote the duration of a game until the time that the wealth \( X_k \) of a player reaches 0. The random variable \( T_0 \) has the geometric distribution (22.28) with parameter \( p \in (0, 1) \).

iv) The **negative binomial** (or **Pascal**) distribution.

We have

\[ P(X = k) = \binom{k + r - 1}{r - 1} (1 - p)^r p^k, \quad k \in \mathbb{N}, \quad (22.29) \]

where \( p \in (0, 1) \) and \( r \geq 1 \) are parameters. Note that the sum of \( r \geq 1 \) independent geometric random variables with parameter \( p \) has a negative binomial distribution with parameter \( (r, p) \). In particular, the negative binomial distribution recovers the geometric distribution when \( r = 1 \).

v) The **Poisson** distribution.

We have

\[ P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k \in \mathbb{N}, \]

where \( \lambda > 0 \) is a parameter.

The probability that a discrete nonnegative random variable \( X : \Omega \rightarrow \mathbb{N} \cup \{+\infty\} \) is finite is given by

\[ P(X < \infty) = \sum_{k \geq 0} P(X = k), \quad (22.30) \]

and we have

\[ 1 = P(X = \infty) + P(X < \infty) = P(X = \infty) + \sum_{k \geq 0} P(X = k). \]

**Remark 22.7.** The distribution of a discrete random variable cannot admit a density. If this were the case, by Remark 22.6 we would have \( P(X = k) = 0 \) for all \( k \in \mathbb{N} \) and

\[ 1 = P(X \in \mathbb{R}) = P(X \in \mathbb{N}) = \sum_{k \geq 0} P(X = k) = 0, \]

* The notation “\( \inf \)” stands for “infimum”, meaning the smallest \( n \geq 0 \) such that \( X_n = 0 \), if such an \( n \) exists.
which is a contradiction.

Given two discrete random variables $X$ and $Y$, the conditional distribution of $X$ given $Y = k$ is given by

$$
P(X = n | Y = k) = \frac{P(X = n \text{ and } Y = k)}{P(Y = k)}, \quad n \in \mathbb{N},
$$

provided that $P(Y = k) > 0$, $k \in \mathbb{N}$.

### 22.6 Expectation of Random Variables

The expectation, or expected value, of a random variable $X$ is the mean, or average value, of $X$. In practice, expectations can be even more useful than probabilities. For example, knowing that a given equipment (such as a bridge) has a failure probability of 1.78493 out of a billion can be of less practical use than knowing the expected lifetime (e.g. 200000 years) of that equipment.

For example, the time $T(\omega)$ to travel from home to work/school can be a random variable with a new outcome and value every day, however we usually refer to its expectation $\mathbb{E}[T]$ rather than to its sample values that may change from day to day.

#### Expected value of a Bernoulli random variable

Any Bernoulli random variable $X : \Omega \rightarrow \{0, 1\}$ can be written as the indicator function $X := 1_A$ where $A$ is the event $A = \{X = 1\}$, and the parameter $p \in [0, 1]$ of $X$ is given by

$$p = P(X = 1) = P(A) = \mathbb{E}[1_A] = \mathbb{E}[X].$$

The expectation of a Bernoulli random variable with parameter $p$ is defined as

$$\mathbb{E}[1_A] := 1 \times P(A) + 0 \times P(A^c) = P(A). \quad (22.31)$$

#### Expected value of a discrete random variable

Next, let $X : \Omega \rightarrow \mathbb{N}$ be a discrete random variable. The expectation $\mathbb{E}[X]$ of $X$ is defined as the sum

$$\mathbb{E}[X] = \sum_{k \geq 0} kP(X = k), \quad (22.32)$$
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in which the possible values \( k \in \mathbb{N} \) of \( X \) are weighted by their probabilities. More generally we have

\[
\mathbb{E}[\phi(X)] = \sum_{k \geq 0} \phi(k) \mathbb{P}(X = k),
\]

for all sufficiently summable functions \( \phi : \mathbb{N} \to \mathbb{R} \).

The expectation of the indicator function \( X = 1_A = 1_{\{X=1\}} \) can be recovered from (22.32) as

\[
\mathbb{E}[X] = \mathbb{E}[1_A] = 0 \times \mathbb{P}(\omega \setminus A) + 1 \times \mathbb{P}(A) = 0 \times \mathbb{P}(\omega \setminus A) + 1 \times \mathbb{P}(A) = \mathbb{P}(A).
\]

Note that the expectation is a linear operation, \emph{i.e.} we have

\[
\mathbb{E}[aX + bY] = a \mathbb{E}[X] + b \mathbb{E}[Y], \quad a, b \in \mathbb{R},
\]

provided that

\[
\mathbb{E}[|X|] + \mathbb{E}[|Y|] < \infty.
\]

**Examples**

i) Expected value of a Poisson random variable with parameter \( \lambda > 0 \):

\[
\mathbb{E}[X] = \sum_{k \geq 0} k \mathbb{P}(X = k) = e^{-\lambda} \sum_{k \geq 1} \frac{\lambda^k}{k!} = \lambda e^{-\lambda} \sum_{k \geq 0} \frac{\lambda^k}{k!} = \lambda. \quad (22.34)
\]

ii) Estimating the expected value of a Poisson random variable using R:

Taking \( \lambda := 2 \), we can use the following R code:

```r
poisson_samples <- rpois(100000, lambda = 2)
poisson_samples
mean(poisson_samples)
```

Given \( X : \Omega \to \mathbb{N} \cup \{+\infty\} \) a discrete nonnegative random variable \( X \), we have

\[
\mathbb{P}(X < \infty) = \sum_{k \geq 0} \mathbb{P}(X = k),
\]

and

\[
1 = \mathbb{P}(X = \infty) + \mathbb{P}(X < \infty) = \mathbb{P}(X = \infty) + \sum_{k \geq 0} \mathbb{P}(X = k),
\]

and in general

\[
\mathbb{E}[X] = +\infty \times \mathbb{P}(X = \infty) + \sum_{k \geq 0} k \mathbb{P}(X = k).
\]
In particular, $\mathbb{P}(X = \infty) > 0$ implies $\mathbb{E}[X] = \infty$, and the finiteness $\mathbb{E}[X] < \infty$ condition implies $\mathbb{P}(X < \infty) = 1$, however the converse is not true.

**Examples**

a) Assume that $X$ has the geometric distribution

$\mathbb{P}(X = k) := \frac{1}{2^{k+1}}, \quad k \geq 0,$

(22.35)

with parameter $p = 1/2$, and

$$\mathbb{E}[X] = \sum_{k \geq 0} \frac{k}{2^{k+1}} = \frac{1}{4} \sum_{k \geq 1} \frac{k}{2^{k-1}} = \frac{1}{4} \frac{1}{(1-1/2)^2} = 1 < \infty.$$  

Letting $\phi(X) := 2^X$, we have

$$\mathbb{P}(\phi(X) < \infty) = \mathbb{P}(X < \infty) = \sum_{k \geq 0} \frac{1}{2^{k+1}} = 1,$$

and

$$\mathbb{E}[\phi(X)] = \sum_{k \geq 0} \phi(k) \mathbb{P}(X = k) = \sum_{k \geq 0} \frac{2^k}{2^{k+1}} = \sum_{k \geq 0} \frac{1}{2} = +\infty,$$

hence the expectation $\mathbb{E}[\phi(X)]$ is infinite although $\phi(X)$ is finite with probability one.*

b) The uniform random variable $U$ on $[0, 1]$ satisfies $\mathbb{E}[U] = 1/2 < \infty$ and

$$\mathbb{P}(1/U < \infty) = \mathbb{P}(U > 0) = \mathbb{P}(U \in (0, 1]) = 1,$$

however we have

$$\mathbb{E}[1/U] = \int_0^1 \frac{dx}{x} = +\infty,$$

and $\mathbb{P}(1/U = +\infty) = \mathbb{P}(U = 0) = 0$.

c) If the random variable $X$ has an exponential distribution with parameter $\mu > 0$ we have

$$\mathbb{E}[e^{\lambda X}] = \mu \int_0^\infty e^{\lambda x} e^{-\mu x} dx = \begin{cases} \frac{\mu}{\mu - \lambda} < \infty & \text{if } \mu > \lambda, \\ +\infty, & \text{if } \mu \leq \lambda. \end{cases}$$

* This is the St. Petersburg paradox.
**Conditional expectation**

The notion of expectation takes its full meaning under conditioning. For example, the expected return of a random asset usually depends on information such as economic data, location, etc. In this case, replacing the expectation by a conditional expectation will provide a better estimate of the expected value.

For instance, **life expectancy** is a natural example of a conditional expectation since it typically depends on location, gender, and other parameters.

The conditional expectation of a finite discrete random variable $X : \Omega \to \mathbb{N}$ given an event $A$ is defined by

$$
\mathbb{E}[X \mid A] = \sum_{k \geq 0} k \mathbb{P}(X = k \mid A) = \sum_{k \geq 0} \frac{k \mathbb{P}(X = k \text{ and } A)}{\mathbb{P}(A)}.
$$

**Lemma 22.8.** Given an event $A$ such that $\mathbb{P}(A) > 0$, we have

$$
\mathbb{E}[X \mid A] = \frac{1}{\mathbb{P}(A)} \mathbb{E} \left[ X \mathbb{1}_A \right].
$$

(*Proof.*) The proof is done only for $X : \Omega \to \mathbb{N}$ a discrete random variable, however (22.36) is valid for general real-valued random variables. By Relation (22.16) we have

$$
\mathbb{E}[X \mid A] = \sum_{k \geq 0} k \mathbb{P}(X = k \mid A)
$$

$$
= \frac{1}{\mathbb{P}(A)} \sum_{k \geq 0} k \mathbb{P}(\{X = k\} \cap A) = \frac{1}{\mathbb{P}(A)} \sum_{k \geq 0} k \mathbb{E} \left[ \mathbb{1}_{\{X = k\} \cap A} \right]
$$

$$
= \frac{1}{\mathbb{P}(A)} \sum_{k \geq 0} k \mathbb{E} \left[ \mathbb{1}_{X = k} \mathbb{1}_A \right] = \frac{1}{\mathbb{P}(A)} \mathbb{E} \left[ \mathbb{1}_A \sum_{k \geq 0} k \mathbb{1}_{\{X = k\}} \right]
$$

$$
= \frac{1}{\mathbb{P}(A)} \mathbb{E} \left[ \mathbb{1}_A X \right],
$$

where we used the relation

$$
X = \sum_{k \geq 0} k \mathbb{1}_{\{X = k\}}
$$

which holds since $X$ takes only integer values. 

**Example**

i) Consider $\Omega = \{1, 3, -1, -2, 5, 7\}$ with the uniform probability measure given by
\( \mathbb{P}(\{k\}) = 1/6, \quad k = 1, 3, -1, -2, 5, 7, \)

and the random variable
\[ X : \Omega \longrightarrow \mathbb{Z} \]
given by
\[ X(k) = k, \quad k = 1, 3, -1, -2, 5, 7. \]

Then \( \mathbb{E}[X \mid X > 0] \) denotes the expected value of \( X \) given \( \{X > 0\} = \{1, 3, 5, 7\} \subset \Omega, \)
\( i.e. \) the mean value of \( X \) given that \( X \) is strictly positive. This conditional expectation can be computed as
\[
\mathbb{E}[X \mid X > 0] = \frac{1 + 3 + 5 + 7}{4} = \frac{16}{4} = 4/6 = \frac{1}{\mathbb{P}(X > 0)} \mathbb{E}[X \mathbb{1}_{\{X > 0\}}],
\]
where \( \mathbb{P}(X > 0) = 4/6 \) and the truncated expectation \( \mathbb{E}[X \mathbb{1}_{\{X > 0\}}] \)
is given by \( \mathbb{E}[X \mathbb{1}_{\{X > 0\}}] = (1 + 3 + 5 + 7)/6. \)

ii) Estimating a conditional expectation using \( R: \)

```
1 geo_samples <- rgeom(100000, prob = 1/4)
2 mean(geo_samples)
3 mean(geo_samples[geo_samples<10])
```

Taking \( p := 3/4 \), we have
\[
\mathbb{E}[X] = (1 - p) \sum_{k \geq 1} kp^k = \frac{p}{1 - p} = 3,
\]
and
\[
\mathbb{E}[X \mid X < 10] = \frac{1}{\mathbb{P}(X < 10)} \mathbb{E}[X \mathbb{1}_{\{X < 10\}}] = \frac{1}{\mathbb{P}(X < 10)} \sum_{k=0}^{9} k\mathbb{P}(X = k)
= \frac{1}{9} \sum_{k=0}^{9} kp^k \sum_{k=1}^{9} p^k
\]
\[
\begin{align*}
&= \frac{p(1-p)}{1-p^{10}} \frac{\partial}{\partial p} \sum_{k=0}^{9} p^k \\
&= \frac{p(1-p)}{1-p^{10}} \frac{\partial}{\partial p} \left( \frac{1-p^{10}}{1-p} \right) \\
&= \frac{p(1-p^{10}) - 10(1-p)p^9}{(1-p)(1-p^{10})} \\
&\simeq 2.4032603455.
\end{align*}
\]

If the random variable \( X : \Omega \to \mathbb{N} \) is independent\(^*\) of the event \( A \), we have

\[
\mathbb{E}[X \mathbb{1}_A] = \mathbb{E}[X] \mathbb{E}[\mathbb{1}_A] = \mathbb{E}[X] \mathbb{P}(A),
\]
and we naturally find

\[
\mathbb{E}[X | A] = \mathbb{E}[X].
\]  
(22.37)

Taking \( X = \mathbb{1}_A \) with

\[
\mathbb{1}_A : \Omega \to \{0, 1\} \\
\omega \mapsto \mathbb{1}_A := \begin{cases} 
1 & \text{if } \omega \in A, \\
0 & \text{if } \omega \notin A,
\end{cases}
\]

shows that, in particular,

\[
\mathbb{E}[\mathbb{1}_A | A] = 0 \times \mathbb{P}(X = 0 | A) + 1 \times \mathbb{P}(X = 1 | A) \\
= \mathbb{P}(X = 1 | A) \\
= \mathbb{P}(A | A) \\
= 1.
\]

One can also define the conditional expectation of \( X \) given \( A = \{Y = k\} \), as

\[
\mathbb{E}[X | Y = k] = \sum_{n \geq 0} n \mathbb{P}(X = n | Y = k),
\]

where \( Y : \Omega \to \mathbb{N} \) is a discrete random variable.

**Proposition 22.9.** Given \( X \) a discrete random variable such that \( \mathbb{E}[|X|] < \infty \), we have the relation

\[
\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X | Y]],
\]  
(22.38)

which is sometimes referred to as the tower property.

**Proof.** We have

\(^*\) i.e., \( \mathbb{P}({X = k} \cap A) = \mathbb{P}({X = k}) \mathbb{P}(A) \) for all \( k \in \mathbb{N} \).

\( \Box \)

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\[ \mathbb{E}[\mathbb{E}[X \mid Y]] = \sum_{k \geq 0} \mathbb{E}[X \mid Y = k] \mathbb{P}(Y = k) \]
\[ = \sum_{k \geq 0} \sum_{n \geq 0} n \mathbb{P}(X = n \mid Y = k) \mathbb{P}(Y = k) \]
\[ = \sum_{n \geq 0} \sum_{k \geq 0} \mathbb{P}(X = n \text{ and } Y = k) \]
\[ = \sum_{n \geq 0} n \mathbb{P}(X = n) = \mathbb{E}[X], \]

where we used the marginal distribution
\[ \mathbb{P}(X = n) = \sum_{k \geq 0} \mathbb{P}(X = n \text{ and } Y = k), \quad n \in \mathbb{N}, \]

that follows from the law of total probability (22.15) with \( A_k = \{Y = k\}, k \geq 0. \]

Taking
\[ Y = \sum_{k \geq 0} k \mathbf{1}_{A_k}, \]

with \( A_k := \{Y = k\}, k \in \mathbb{N}, \) from (22.38) we also get the law of total expectation
\[ \mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]] \]
\[ = \sum_{k \geq 0} \mathbb{E}[X \mid Y = k] \mathbb{P}(Y = k) \]
\[ = \sum_{k \geq 0} \mathbb{E}[X \mid A_k] \mathbb{P}(A_k). \]

**Example**

Life expectancy in Singapore is \( \mathbb{E}[T] = 80 \) years overall, where \( T \) denotes the lifetime of a given individual chosen at random. Let \( G \in \{m, w\} \) denote the gender of that individual. The statistics show that
\[ \mathbb{E}[T \mid G = m] = 78 \quad \text{and} \quad \mathbb{E}[T \mid G = w] = 81.9, \]

and we have
\[ 80 = \mathbb{E}[T] \]
\[ = \mathbb{E}[\mathbb{E}[T \mid G]] \]
\[ = \mathbb{P}(G = w) \mathbb{E}[T \mid G = w] + \mathbb{P}(G = m) \mathbb{E}[T \mid G = m] \]
\[ = 81.9 \times \mathbb{P}(G = w) + 78 \times \mathbb{P}(G = m) \]
\[ = 81.9 \times (1 - \mathbb{P}(G = m)) + 78 \times \mathbb{P}(G = m), \]

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showing that

\[ 80 = 81.9 \times (1 - \mathbb{P}(G = m)) + 78 \times \mathbb{P}(G = m), \]

\[ i.e. \quad \mathbb{P}(G = m) = \frac{81.9 - 80}{81.9 - 78} = 1.9 \quad \frac{3.9}{3.9} = 0.487. \]

Variance

The variance of a random variable \( X \) is defined in general by

\[ \text{Var} [X] := \mathbb{E} [X^2] - (\mathbb{E}[X])^2, \]

provided that \( \mathbb{E} [X^2] < \infty \). If \( (X_k)_{k \in \mathbb{N}} \) is a sequence of independent random variables we have

\[
\text{Var} \left[ \sum_{k=1}^{n} X_k \right] = \mathbb{E} \left[ \left( \sum_{k=1}^{n} X_k \right)^2 \right] - \left( \mathbb{E} \left[ \sum_{k=1}^{n} X_k \right] \right)^2 \\
= \mathbb{E} \left[ \sum_{k=1}^{n} X_k \sum_{l=1}^{n} X_l \right] - \mathbb{E} \left[ \sum_{k=1}^{n} X_k \right] \mathbb{E} \left[ \sum_{l=1}^{n} X_l \right] \\
= \mathbb{E} \left[ \sum_{k=1}^{n} \sum_{l=1}^{n} X_k X_l \right] - \sum_{k=1}^{n} \sum_{l=1}^{n} \mathbb{E}[X_k] \mathbb{E}[X_l] \\
= \sum_{k=1}^{n} \mathbb{E} [X_k^2] + \sum_{1 \leq k \neq l \leq n} \mathbb{E}[X_k X_l] - \sum_{k=1}^{n} (\mathbb{E}[X_k])^2 - \sum_{1 \leq k \neq l \leq n} \mathbb{E}[X_k] \mathbb{E}[X_l] \\
= \sum_{k=1}^{n} (\mathbb{E} [X_k^2] - (\mathbb{E} [X_k])^2) \\
= \sum_{k=1}^{n} \text{Var} [X_k]. \quad (22.40)
\]

Random sums

In the sequel we consider \( Y : \Omega \rightarrow \mathbb{N} \) an a.s. finite, integer-valued random variable, \( i.e. \) we have \( \mathbb{P}(Y < \infty) = 1 \) and \( \mathbb{P}(Y = \infty) = 0 \). Based on the tower property of conditional expectations (22.38) or ordinary conditioning, the expectation of a random sum \( \sum_{k=1}^{Y} X_k \), where \( (X_k)_{k \in \mathbb{N}} \) is a sequence of random variables, can be computed from the tower property (22.38) or from the law of total expectation (22.39) as
\[ \mathbb{E} \left[ \sum_{k=1}^{Y} X_k \right] = \mathbb{E} \left[ \mathbb{E} \left[ \sum_{k=1}^{Y} X_k \mid Y \right] \right] \]
\[ = \sum_{n \geq 0} \mathbb{E} \left[ \sum_{k=1}^{Y} X_k \mid Y = n \right] \mathbb{P}(Y = n) \]
\[ = \sum_{n \geq 0} \mathbb{E} \left[ \sum_{k=1}^{n} X_k \mid Y = n \right] \mathbb{P}(Y = n), \]
and if the sequence \((X_k)_{k \in \mathbb{N}}\) is (mutually) independent of \(Y\), this yields
\[ \mathbb{E} \left[ \sum_{k=1}^{Y} X_k \right] = \sum_{n \geq 0} \mathbb{E} \left[ \sum_{k=1}^{n} X_k \right] \mathbb{P}(Y = n) \]
\[ = \sum_{n \geq 0} \mathbb{P}(Y = n) \sum_{k=1}^{n} \mathbb{E}[X_k]. \]

Random products

Similarly, for a random product we will have, using the independence of \(Y\) with \((X_k)_{k \in \mathbb{N}}\),
\[ \mathbb{E} \left[ \prod_{k=1}^{Y} X_k \right] = \sum_{n \geq 0} \mathbb{E} \left[ \prod_{k=1}^{n} X_k \right] \mathbb{P}(Y = n) \]
\[ = \sum_{n \geq 0} \mathbb{P}(Y = n) \prod_{k=1}^{n} \mathbb{E}[X_k], \quad (22.41) \]
where the last equality requires the (mutual) independence of the random variables in the sequence \((X_k)_{k \geq 1}\).

Distributions admitting a density

Given a random variable \(X\) whose distribution admits a density \(\varphi_X : \mathbb{R} \rightarrow \mathbb{R}_+\) we have
\[ \mathbb{E}[X] = \int_{-\infty}^{\infty} x \varphi_X(x) dx, \]
and more generally,
\[ \mathbb{E}[\phi(X)] = \int_{-\infty}^{\infty} \phi(x) \varphi_X(x) dx, \quad (22.42) \]
for all sufficiently integrable function \(\phi\) on \(\mathbb{R}\). For example, if \(X\) has a standard normal distribution we have
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\[
\mathbb{E}[\phi(X)] = \int_{-\infty}^{\infty} \phi(x) e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}.
\]

In case \(X\) has a Gaussian distribution with mean \(\mu \in \mathbb{R}\) and variance \(\sigma^2 > 0\) we get

\[
\mathbb{E}[\phi(X)] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \phi(x) e^{-(x-\mu)^2/(2\sigma^2)} dx. \tag{22.43}
\]

Exercise: In case \(X \sim \mathcal{N}(\mu, \sigma^2)\) has a Gaussian distribution with mean \(\mu \in \mathbb{R}\) and variance \(\sigma^2 > 0\), check that

\[
\mu = \mathbb{E}[X] \quad \text{and} \quad \sigma^2 = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.
\]

When \((X, Y) : \Omega \rightarrow \mathbb{R}^2\) is a \(\mathbb{R}^2\)-valued couple of random variables whose distribution admits a density \(\varphi_{X,Y} : \mathbb{R}^2 \rightarrow \mathbb{R}_+\) we have

\[
\mathbb{E}[\phi(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x, y) \varphi_{X,Y}(x, y) dx dy,
\]

for all sufficiently integrable function \(\phi\) on \(\mathbb{R}^2\).

The expectation of an absolutely continuous random variable satisfies the same linearity property (22.33) as in the discrete case.

The conditional expectation of an absolutely continuous random variable can be defined as

\[
\mathbb{E}[X \mid Y = y] = \int_{-\infty}^{\infty} x \varphi_{X \mid Y = y}(x) dx
\]

where the conditional density \(\varphi_{X \mid Y = y}(x)\) is defined in (22.24), with the relation

\[
\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]] \tag{22.44}
\]

which is called the tower property and holds as in the discrete case, since

\[
\mathbb{E}[\mathbb{E}[X \mid Y]] = \int_{-\infty}^{\infty} \mathbb{E}[X \mid Y = y] \varphi_Y(y) dy
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \varphi_{X \mid Y = y}(x) \varphi_Y(y) dx dy
= \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} \varphi_{X,Y}(x, y) dy dx
= \int_{-\infty}^{\infty} x \varphi_X(x) dx = \mathbb{E}[X],
\]

where we used Relation (22.23) between the density of \((X, Y)\) and its marginal \(X\).
For example, an exponentially distributed random variable $X$ with probability density function \((22.19)\) has the expected value
\[
\mathbb{E}[X] = \lambda \int_0^\infty x e^{-\lambda x} \, dx = \frac{1}{\lambda}.
\]

**Conditional Expectation Revisited**

The construction of conditional expectations of the form $\mathbb{E}[X \mid Y]$ given above for discrete and absolutely continuous random variables can be generalized to $\sigma$-algebras.

**Definition 22.10.** Given $\mathcal{F}$ a $\sigma$-algebra on $\Omega$, a random variable $F : \Omega \rightarrow \mathbb{R}$ is said to be $\mathcal{F}$-measurable if
\[
\{F \leq x\} := \{\omega \in \Omega : F(\omega) \leq x\} \in \mathcal{F},
\]
for all $x \in \mathbb{R}$.

Intuitively, when $F$ is $\mathcal{F}$-measurable, the knowledge of the values of $F$ depends only on the information contained in $\mathcal{F}$. For example, when $\mathcal{F} = \sigma(A_1, \ldots, A_n)$ where $(A_n)_{n \geq 1}$ is a partition of $\Omega$ with $\bigcup_{n \geq 1} A_n = \Omega$, any $\mathcal{F}$-measurable random variable $F$ can be written as
\[
F(\omega) = \sum_{k=1}^n c_k 1_{A_k}(\omega), \quad \omega \in \Omega,
\]
for some $c_1, \ldots, c_n \in \mathbb{R}$.

**Definition 22.11.** Given $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space we let $L^2(\Omega, \mathcal{F})$ denote the space of $\mathcal{F}$-measurable and square-integrable random variables, i.e.
\[
L^2(\Omega, \mathcal{F}) := \{F : \Omega \rightarrow \mathbb{R} : \mathbb{E}[|F|^2] < \infty\}.
\]

More generally, for $p \geq 1$ one can define the space $L^p(\Omega, \mathcal{F})$ of $\mathcal{F}$-measurable and $p$-integrable random variables as
\[
L^p(\Omega, \mathcal{F}) := \{F : \Omega \rightarrow \mathbb{R} : \mathbb{E}[|F|^p] < \infty\}.
\]

We define a scalar product $\langle \cdot, \cdot \rangle_{L^2(\Omega, \mathcal{F})}$ between elements of $L^2(\Omega, \mathcal{F})$, as
\[
\langle F, G \rangle_{L^2(\Omega, \mathcal{F})} := \mathbb{E}[FG], \quad F, G \in L^2(\Omega, \mathcal{F}).
\]

This scalar product is associated to the norm $\| \cdot \|_{L^2(\Omega)}$ by the relation
\[
\|F\|_{L^2(\Omega)} = \sqrt{\mathbb{E}[F^2]} = \sqrt{\langle F, F \rangle_{L^2(\Omega, \mathcal{F})}}; \quad F \in L^2(\Omega, \mathcal{F}).
\]
The norm \( \| \cdot \|_{L^2(\Omega)} \) also defines the mean square distance
\[
\| F - G \|_{L^2(\Omega)} = \sqrt{\mathbb{E}[(F - G)^2]}
\]
between random variables \( F, G \in L^2(\Omega, \mathcal{F}) \), and it induces a notion of orthogonality, namely \( F \) is orthogonal to \( G \) in \( L^2(\Omega, \mathcal{F}) \) if and only if
\[
\langle F, G \rangle_{L^2(\Omega, \mathcal{F})} = 0.
\]

**Proposition 22.12.** The ordinary expectation \( \mathbb{E}[F] \) achieves the minimum distance
\[
\| F - \mathbb{E}[F] \|_{L^2(\Omega)}^2 = \min_{c \in \mathbb{R}} \| F - c \|_{L^2(\Omega)}^2.
\]  \hspace{1cm} (22.45)

**Proof.** It suffices to differentiate
\[
\frac{\partial}{\partial c} \mathbb{E}[(F - c)^2] = 2 \mathbb{E}[F - c] = 0,
\]
showing that the minimum in (22.45) is reached when \( \mathbb{E}[F - c] = 0 \), i.e. \( c = \mathbb{E}[F] \). \( \square \)

Similarly to Proposition 22.12, the conditional expectation will be defined by a distance minimizing procedure.

**Definition 22.13.** Given \( \mathcal{G} \subseteq \mathcal{F} \) a sub \( \sigma \)-algebra of \( \mathcal{F} \) and \( F \in L^2(\Omega, \mathcal{F}) \), the conditional expectation of \( F \) given \( \mathcal{G} \), and denoted
\[
\mathbb{E}[F \mid \mathcal{G}],
\]
is defined as the orthogonal projection of \( F \) onto \( L^2(\Omega, \mathcal{G}) \).

As a consequence of the uniqueness of the orthogonal projection onto the subspace \( L^2(\Omega, \mathcal{G}) \) of \( L^2(\Omega, \mathcal{F}) \), the conditional expectation \( \mathbb{E}[F \mid \mathcal{G}] \) is characterized by the relation
\[
\langle G, F - \mathbb{E}[F \mid \mathcal{G}] \rangle_{L^2(\Omega, \mathcal{F})} = 0,
\]
which rewrites as
\[
\mathbb{E}[G(F - \mathbb{E}[F \mid \mathcal{G}])] = 0,
\]
\[ \mathbb{E}[GF] = \mathbb{E}[G \mathbb{E}[F | G]], \]

for all bounded and \( \mathcal{G} \)-measurable random variables \( G \), where \( \langle \cdot, \cdot \rangle_{L^2(\Omega, \mathcal{F})} \) denotes the inner product in \( L^2(\Omega, \mathcal{F}) \). The next proposition extends Proposition 22.12 as a consequence of Definition 22.13, see Theorem 5.1.4 page 197 of Stroock (2011) for an extension of the construction of conditional expectation to the space \( L^1(\Omega, \mathcal{F}) \) of integrable random variable.

**Proposition 22.14.** The conditional expectation \( \mathbb{E}[F | G] \) realizes the minimum in mean square distance between \( F \in L^2(\Omega, \mathcal{F}) \) and \( L^2(\Omega, \mathcal{G}) \), i.e. we have

\[ \| F - \mathbb{E}[F | G] \|_{L^2(\Omega)} = \inf_{G \in L^2(\Omega, \mathcal{G})} \| F - G \|_{L^2(\Omega)}. \quad (22.46) \]

The following proposition will often be used as a characterization of \( \mathbb{E}[F | \mathcal{G}] \).

**Proposition 22.15.** Given \( F \in L^2(\Omega, \mathcal{F}) \), \( X := \mathbb{E}[F | \mathcal{G}] \) is the unique random variable \( X \) in \( L^2(\Omega, \mathcal{G}) \) that satisfies the relation

\[ \mathbb{E}[GF] = \mathbb{E}[GX] \quad (22.47) \]

for all bounded and \( \mathcal{G} \)-measurable random variables \( G \).

We note that taking \( G = 1 \) in (22.47) yields

\[ \mathbb{E}[F] = \mathbb{E}[\mathbb{E}[F | \mathcal{G}]]. \quad (22.48) \]

In particular, when \( \mathcal{G} = \{\emptyset, \Omega\} \) we have

\[ \mathbb{E}[F] = \mathbb{E}[\mathbb{E}[F | \mathcal{G}]] = \mathbb{E}[F | \mathcal{G}] \]

because the random variables in \( L^2(\Omega, \{\emptyset, \Omega\}) \) are a.s. constant. In addition, the conditional expectation operator has the following properties.

i) \( \mathbb{E}[FG | \mathcal{G}] = G \mathbb{E}[F | \mathcal{G}] \) if \( G \) depends only on the information contained in \( \mathcal{G} \).

**Proof.** By the characterization (22.47) it suffices to show that

\[ \mathbb{E}[H(GF)] = \mathbb{E}[H(G \mathbb{E}[F|\mathcal{G}])], \quad (22.49) \]

for all bounded and \( \mathcal{G} \)-measurable random variables \( H \), which implies \( \mathbb{E}[FG | \mathcal{G}] = G \mathbb{E}[F | \mathcal{G}] \).

Relation (22.49) holds from (22.47) because the product \( HG \) is \( \mathcal{G} \)-measurable hence \( G \) in (22.47) can be replaced with \( HG \).
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ii) \( \mathbb{E}[G | G] = G \) when \( G \) depends only on the information contained in \( G \).

Proof. This is a consequence of point \((i)\) above by taking \( F := 1 \).

iii) \( \mathbb{E}[\mathbb{E}[F | G] | \mathcal{H}] = \mathbb{E}[F | \mathcal{H}] \) if \( \mathcal{H} \subset \mathcal{G} \), called the tower property.

Proof. First, we note that by (22.48), \((iii)\) holds when \( \mathcal{H} = \{\emptyset, \Omega\} \). Next, by the characterization (22.47) it suffices to show that

\[
\mathbb{E}[H \mathbb{E}[F | G]] = \mathbb{E}[H \mathbb{E}[F | \mathcal{H}]],
\]

for all bounded and \( G \)-measurable random variables \( H \), which will imply \((iii)\) from (22.47).

In order to prove (22.50) we check that by point \((i)\) above and (22.48) we have

\[
\mathbb{E}[H \mathbb{E}[F | G]] = \mathbb{E}[\mathbb{E}[HF | G]] = \mathbb{E}[HF]
\]

\[
= \mathbb{E}[\mathbb{E}[HF | \mathcal{H}]] = \mathbb{E}[H \mathbb{E}[F | \mathcal{H}]],
\]

and we conclude by the characterization (22.47).

iv) \( \mathbb{E}[F | G] = \mathbb{E}[F] \) when \( F \) “does not depend” on the information contained in \( G \) or, more precisely stated, when the random variable \( F \) is independent of the \( \sigma \)-algebra \( G \).

Proof. It suffices to note that for all bounded \( G \)-measurable \( G \) we have

\[
\mathbb{E}[FG] = \mathbb{E}[F] \mathbb{E}[G] = \mathbb{E}[G \mathbb{E}[F]],
\]

and we conclude again by (22.47).

v) If \( G \) depends only on \( G \) and \( F \) is independent of \( G \), then

\[
\mathbb{E}[h(F, G) | G] = \mathbb{E}[h(F, x)]_{x=G}.
\]

Proof. This relation can be proved using the tower property, by noting that for any \( K \in L^2(\Omega, \mathcal{G}) \) we have

\[
\mathbb{E}[K \mathbb{E}[h(x, F)]_{x=G}] = \mathbb{E}[K \mathbb{E}[h(x, F) | G]_{x=G}]
\]

\[
= \mathbb{E}[K \mathbb{E}[h(G, F) | G]]
\]

\[
= \mathbb{E}[K \mathbb{E}[Kh(G, F) | G]]
\]

\[
= \mathbb{E}[Kh(G, F)],
\]

which yields (22.51) by the characterization (22.47).
The notion of conditional expectation can be extended from square-integrable random variables in $L^2(\Omega, \mathcal{F})$ to integrable random variables in $L^1(\Omega, \mathcal{F})$, cf. e.g. Theorem 5.1 in Kallenberg (2002).

**Proposition 22.16.** When the $\sigma$-algebra $\mathcal{G} := \sigma(A_1, A_2, \ldots, A_n)$ is generated by $n$ disjoint events $A_1, A_2, \ldots, A_n \in \mathcal{F}$, we have

$$
\mathbb{E}[F \mid \mathcal{G}] = \sum_{k=1}^{n} \mathbb{1}_{A_k} \mathbb{E}[F \mid A_k] = \sum_{k=1}^{n} \mathbb{1}_{A_k} \frac{\mathbb{E}[F \mathbb{1}_{A_k}]}{\mathbb{P}(A_k)}.
$$

**Proof.** It suffices to note that the $\mathcal{G}$-measurable random variables can be generated by indicators of the form $\mathbb{1}_{A_l}$, and that

$$
\mathbb{E} \left[ \mathbb{1}_{A_l} \sum_{k=1}^{n} \mathbb{1}_{A_k} \frac{\mathbb{E}[F \mathbb{1}_{A_k}]}{\mathbb{P}(A_k)} \right] = \mathbb{E} \left[ \mathbb{1}_{A_l} \frac{\mathbb{E}[F \mathbb{1}_{A_l}]}{\mathbb{P}(A_l)} \right] \\
= \frac{\mathbb{E}[F \mathbb{1}_{A_l}]}{\mathbb{P}(A_l)} \mathbb{E}[\mathbb{1}_{A_l}] \\
= \mathbb{E}[F \mathbb{1}_{A_l}], \quad l = 1, 2, \ldots, n,
$$

showing (22.47). The relation

$$
\mathbb{E}[F \mid A_k] = \frac{\mathbb{E}[F \mathbb{1}_{A_k}]}{\mathbb{P}(A_k)}, \quad k = 1, 2, \ldots, n,
$$

follows from Lemma 22.8. □

For example, in case $\Omega = \{a, b, c, d\}$ and $\mathcal{G} = \{\emptyset, \Omega, \{a, b\}, \{c\}, \{d\}\}$, we have

$$
\mathbb{E}[F \mid \mathcal{G}] = \mathbb{1}_{\{a, b\}} \mathbb{E}[F \mid \{a, b\}] + \mathbb{1}_{\{c\}} \mathbb{E}[F \mid \{c\}] + \mathbb{1}_{\{d\}} \mathbb{E}[F \mid \{d\}] \\
= \mathbb{1}_{\{a, b\}} \frac{\mathbb{E}[F \mathbb{1}_{\{a, b\}}]}{\mathbb{P}(\{a, b\})} + \mathbb{1}_{\{c\}} \frac{\mathbb{E}[F \mathbb{1}_{\{c\}}]}{\mathbb{P}(\{c\})} + \mathbb{1}_{\{d\}} \frac{\mathbb{E}[F \mathbb{1}_{\{d\}}]}{\mathbb{P}(\{d\})}.
$$

Regarding conditional probabilities we have similarly, for $A \subset \Omega = \{a, b, c, d\}$,

$$
\mathbb{P}(A \mid \mathcal{G}) = \mathbb{1}_{\{a, b\}} \frac{\mathbb{P}(A \cap \{a, b\})}{\mathbb{P}(\{a, b\})} + \mathbb{1}_{\{c\}} \frac{\mathbb{P}(A \cap \{c\})}{\mathbb{P}(\{c\})} + \mathbb{1}_{\{d\}} \frac{\mathbb{P}(A \cap \{d\})}{\mathbb{P}(\{d\})} \\
= \mathbb{1}_{\{a, b\}} \mathbb{P}(A \mid \{a, b\}) + \mathbb{1}_{\{c\}} \mathbb{P}(A \mid \{c\}) + \mathbb{1}_{\{d\}} \mathbb{P}(A \mid \{d\}).
$$

In particular, if $A = \{a\} \subset \Omega = \{a, b, c, d\}$ we find

$$
\mathbb{P}(\{a\} \mid \mathcal{G}) = \mathbb{1}_{\{a, b\}} \mathbb{P}(\{a\} \mid \{a, b\}) \\
= \mathbb{1}_{\{a, b\}} \frac{\mathbb{P}(\{a\} \cap \{a, b\})}{\mathbb{P}(\{a, b\})} \\
= \mathbb{1}_{\{a, b\}} \frac{\mathbb{P}(\{a\})}{\mathbb{P}(\{a, b\})}.
$$

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In other words, the probability of getting the outcome $a$ is $\mathbb{P}(\{a\})/\mathbb{P}(\{a, b\})$ knowing that the outcome is either $a$ or $b$, otherwise it is zero.

**Moment Generating Functions**

**Characteristic functions**

The characteristic function of a random variable $X$ is the function

$$\Psi_X : \mathbb{R} \rightarrow \mathbb{C}$$

defined by

$$\Psi_X(t) = \mathbb{E} \left[ e^{itX} \right], \quad t \in \mathbb{R}. $$

The characteristic function $\Psi_X$ of a random variable $X$ with density $f : \mathbb{R} \rightarrow \mathbb{R}^+$ satisfies

$$\Psi_X(t) = \int_{-\infty}^{\infty} e^{ixt} \phi(x) dx, \quad t \in \mathbb{R}. $$

On the other hand, if $X : \Omega \rightarrow \mathbb{N}$ is a discrete random variable we have

$$\Psi_X(t) = \sum_{n \geq 0} e^{itn} \mathbb{P}(X = n), \quad t \in \mathbb{R}. $$

The main applications of characteristic functions lie in the following theorems:

**Theorem 22.17.** Two random variables $X : \Omega \rightarrow \mathbb{R}$ and $Y : \Omega \rightarrow \mathbb{R}$ have same distribution if and only if

$$\Psi_X(t) = \Psi_Y(t), \quad t \in \mathbb{R}. $$

Theorem 22.17 is used to identify or to determine the probability distribution of a random variable $X$, by comparison with the characteristic function $\Psi_Y$ of a random variable $Y$ whose distribution is known.

The characteristic function of a random vector $(X, Y)$ is the function $\Psi_{X,Y} : \mathbb{R}^2 \rightarrow \mathbb{C}$ defined by

$$\Psi_{X,Y}(s, t) = \mathbb{E} \left[ e^{isX + itY} \right], \quad s, t \in \mathbb{R}. $$

**Theorem 22.18.** The random variables $X : \Omega \rightarrow \mathbb{R}$ and $Y : \Omega \rightarrow \mathbb{R}$ are independent if and only if

$$\Psi_{X,Y}(s, t) = \Psi_X(s)\Psi_Y(t), \quad s, t \in \mathbb{R}. $$

A random variable $X$ is Gaussian with mean $\mu$ and variance $\sigma^2$ if and only if its characteristic function satisfies

\[ \mathbb{E} \left[ e^{itX} \right] = e^{i\mu t - \frac{1}{2} \sigma^2 t^2}, \quad t \in \mathbb{R}. \]
In terms of moment generating functions we have, replacing \( i\alpha \) by \( \alpha \),

\[
\mathbb{E} \left[ e^{\alpha X} \right] = e^{\alpha \mu + \alpha^2 \sigma^2/2}, \quad \alpha \in \mathbb{R}. \tag{22.53}
\]

From Theorems 22.17 and 22.18 we deduce the following proposition.

**Proposition 22.19.** Let \( X \simeq \mathcal{N}(\mu, \sigma_X^2) \) and \( Y \simeq \mathcal{N}(\nu, \sigma_Y^2) \) be independent Gaussian random variables. Then \( X + Y \) also has a Gaussian distribution \( X + Y \simeq \mathcal{N}(\mu + \nu, \sigma_X^2 + \sigma_Y^2) \).

**Proof.** Since \( X \) and \( Y \) are independent, by Theorem 22.18 the characteristic function \( \Psi_{X+Y} \) of \( X + Y \) is given by

\[
\Phi_{X+Y}(t) = \Phi_X(t)\Phi_Y(t) = e^{it\mu - t^2\sigma_X^2/2}e^{it\nu - t^2\sigma_Y^2/2} = e^{it(\mu + \nu) - t^2(\sigma_X^2 + \sigma_Y^2)/2}, \quad t \in \mathbb{R},
\]

where we used (22.52). Consequently, the characteristic function of \( X + Y \) is that of a Gaussian random variable with mean \( \mu + \nu \) and variance \( \sigma_X^2 + \sigma_Y^2 \) and we conclude by Theorem 22.17. \( \square \)

**Moment generating functions**

The moment generating function of a random variable \( X \) is the function \( \Phi_X : \mathbb{R} \to \mathbb{R} \) defined by

\[
\Phi_X(t) = \mathbb{E} \left[ e^{tX} \right], \quad t \in \mathbb{R},
\]

provided that the expectation is finite. In particular, we have

\[
\mathbb{E}[X^n] = \frac{\partial^n}{\partial t^n} \Phi_X(0), \quad n \geq 1,
\]

provided that \( \mathbb{E}[|X|^n] < \infty \), and

\[
\Phi_X(t) = \mathbb{E} \left[ e^{tX} \right] = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathbb{E}[X^n],
\]

provided that \( \mathbb{E} \left[ e^{t|X|} \right] < \infty, \ t \in \mathbb{R}, \) and for this reason the moment generating function \( G_X \) characterizes the moments \( \mathbb{E}[X^n] \) of \( X : \Omega \to \mathbb{N}, \ n \geq 0 \).

The moment generating function \( \Phi_X \) of a random variable \( X \) with density \( f : \mathbb{R} \to \mathbb{R}_+ \) satisfies

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\[ \Phi_X(t) = \int_{-\infty}^{\infty} e^{xt} \varphi(x) dx, \quad t \in \mathbb{R}. \]

Note that in probability, the moment generating function is written as a bilateral transform defined using an integral from \(-\infty\) to \(+\infty\).

Exercises

Exercise A.1 Compute the expected value \( \mathbb{E}[X] \) of a Poisson random variable \( X \) with parameter \( \lambda > 0 \).

Exercise A.2 Let \( X \) denote a centered Gaussian random variable with variance \( \eta^2, \eta > 0 \). Show that the probability \( P(e^X > c) \) is given by

\[ P(e^X > c) = \Phi(- \left( \log c \right)/\eta), \]

where \( \log = \ln \) denotes the natural logarithm and

\[ \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy, \quad x \in \mathbb{R}, \]

denotes the Gaussian cumulative distribution function.

Exercise A.3 Let \( X \sim \mathcal{N}(\mu, \sigma^2) \) be a Gaussian random variable with parameters \( \mu > 0 \) and \( \sigma^2 > 0 \), and probability density function

\[ f(x) = \frac{1}{\sqrt{2\pi} \sigma^2} e^{-\left( x-\mu \right)^2/(2\sigma^2)}, \quad x \in \mathbb{R}. \]

a) Write down \( \mathbb{E}[X] \) as an integral and show that

\[ \mu = \mathbb{E}[X]. \]

b) Write down \( \mathbb{E}[X^2] \) as an integral and show that

\[ \sigma^2 = \mathbb{E}[(X - \mathbb{E}[X])^2]. \]

c) Consider the function \( x \mapsto (x - K)^+ \) from \( \mathbb{R} \) to \( \mathbb{R}_+ \), defined as

\[ (x - K)^+ = \begin{cases} x - K & \text{if } x \geq K, \\ 0 & \text{if } x \leq K, \end{cases} \]

where \( K \in \mathbb{R} \) be a fixed real number. Write down \( \mathbb{E}[(X - K)^+] \) as an integral and compute this integral.
Hints: \((x - K)^+\) is zero when \(x < K\), and when \(\mu = 0\) and \(\sigma = 1\) the result is
\[
\mathbb{E}[(X - K)^+] = \frac{1}{\sqrt{2\pi}} e^{-K^2/2} - K \Phi(-K),
\]
where
\[
\Phi(x) := \int_{-\infty}^{x} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}, \quad x \in \mathbb{R}.
\]

d) Write down \(\mathbb{E}[e^X]\) as an integral, and compute \(\mathbb{E}[e^X]\).

Exercise A.4 Let \(X\) be a centered Gaussian random variable with variance \(\alpha^2 > 0\) and probability density function \(x \mapsto \frac{1}{\sqrt{2\pi\alpha^2}} e^{-x^2/(2\alpha^2)}\) and let \(\beta \in \mathbb{R}\).

a) Write down \(\mathbb{E}[(\beta - X)^+]\) as an integral. Hint: \((\beta - x)^+\) is zero when \(x > \beta\).

b) Compute this integral to show that
\[
\mathbb{E}[(\beta - X)^+] = \frac{\alpha}{\sqrt{2\pi}} e^{-\beta^2/(2\alpha^2)} + \beta \Phi(\beta/\alpha),
\]
where \(\Phi(x) = \int_{-\infty}^{x} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}, \quad x \in \mathbb{R}.
\)

Exercise A.5 Let \(X\) be a centered Gaussian random variable with variance \(v^2\).

a) Compute
\[
\mathbb{E}\left[e^{\sigma X} \mathbb{1}_{[K,\infty)}(x e^{\sigma X})\right] = \frac{1}{\sqrt{2\pi v^2}} \int_{\sigma^{-1} \log(K/x)}^{\infty} e^{\sigma y - y^2/(2v^2)} dy.
\]

Hint. Use the identity
\[
\sigma y - \frac{y^2}{v^2} = \frac{v^2 \sigma^2}{4} - \left(\frac{y}{v} - \frac{v \sigma}{2}\right)^2.
\]

b) Compute
\[
\mathbb{E}[(e^{m+X} - K)^+] = \frac{1}{\sqrt{2\pi v^2}} \int_{-\infty}^{\infty} (e^{m+x} - K)^+ e^{-x^2/(2v^2)} dx.
\]

c) Compute the expectation (22.53) above.