Chapter 14
American Options

In contrast with European option which have fixed maturities, the holder of an American option is allowed to exercise at any given (random) time. This transforms the valuation problem into an optimization problem in which one has to find the optimal time to exercise in order to maximize the payoff of the option. As will be seen in the first section below, not all random times can be considered in this process, and we restrict ourselves to stopping times whose value at time $t$ can be decided based on the historical data available.

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14.1 Filtrations and Information Flow

Let $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ denote the filtration generated by a stochastic process $(X_t)_{t \in \mathbb{R}_+}$. In other words, $\mathcal{F}_t$ denotes the collection of all events possibly generated by $\{X_s : 0 \leq s \leq t\}$ up to time $t$. Examples of such events include the event

$$\{X_{t_0} \leq a_0, X_{t_1} \leq a_1, \ldots, X_{t_n} \leq a_n\}$$

for $a_0, a_1, \ldots, a_n$ a given fixed sequence of real numbers and $0 \leq t_1 < \cdots < t_n < t$, and $\mathcal{F}_t$ is said to represent the information generated by $(X_s)_{s \in [0,t]}$ up to time $t$.

By construction, $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is a non-decreasing family of $\sigma$-algebras in the sense that we have $\mathcal{F}_s \subset \mathcal{F}_t$ (information known at time $s$ is contained in the
information known at time \( t \) when \( 0 < s < t \).

One refers sometimes to \( (\mathcal{F}_t)_{t \in \mathbb{R}_+} \) as the non-decreasing flow of information generated by \( (X_t)_{t \in \mathbb{R}_+} \).

### 14.2 Submartingales and Supermartingales

Let us recall the definition of martingale (cf. Definition 5.4) and introduce in addition the definitions of supermartingale and submartingale.*

**Definition 14.1.** An integrable stochastic process \( (Z_t)_{t \in \mathbb{R}_+} \) is a martingale (resp. a supermartingale, resp. a submartingale) with respect to \( (\mathcal{F}_t)_{t \in \mathbb{R}_+} \) if it satisfies the property

\[
Z_s = \mathbb{E}[Z_t | \mathcal{F}_s], \quad 0 \leq s \leq t,
\]

resp.

\[
Z_s \geq \mathbb{E}[Z_t | \mathcal{F}_s], \quad 0 \leq s \leq t,
\]

resp.

\[
Z_s \leq \mathbb{E}[Z_t | \mathcal{F}_s], \quad 0 \leq s \leq t.
\]

Clearly, a stochastic process \( (Z_t)_{t \in \mathbb{R}_+} \) is a martingale if and only if it is both a supermartingale and a submartingale.

A particular property of martingales is that their expectation is constant.

**Proposition 14.2.** Let \( (Z_t)_{t \in \mathbb{R}_+} \) be a martingale. We have

\[
\mathbb{E}[Z_t] = \mathbb{E}[Z_s], \quad 0 \leq s \leq t.
\]

The above proposition follows from the “tower property” (22.38) of conditional expectations, which shows that

\[
\mathbb{E}[Z_t] = \mathbb{E}[\mathbb{E}[Z_t | \mathcal{F}_s]] = \mathbb{E}[Z_s], \quad 0 \leq s \leq t. \tag{14.1}
\]

Similarly, a supermartingale has a non-increasing expectation, while a submartingale has a non-decreasing expectation.

**Proposition 14.3.** Let \( (Z_t)_{t \in \mathbb{R}_+} \) be a supermartingale, resp. a submartingale. Then we have

\[
\mathbb{E}[Z_t] \leq \mathbb{E}[Z_s], \quad 0 \leq s \leq t,
\]

resp.

\[
\mathbb{E}[Z_t] \geq \mathbb{E}[Z_s], \quad 0 \leq s \leq t.
\]

* “This obviously inappropriate nomenclature was chosen under the malign influence of the noise level of radio’s SUPERman program, a favorite supper-time program of Doob’s son during the writing of Doob (1953)”, cf. Doob (1984), historical notes, page 808.
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Proof. As in (14.1) above we have
\[ \mathbb{E}[Z_t] = \mathbb{E}[\mathbb{E}[Z_t | F_s]] \leq \mathbb{E}[Z_s], \quad 0 \leq s \leq t. \]

The proof is similar in the submartingale case.

Independent increments processes whose increments have negative expectation give examples of supermartingales. For example, if \((X_t)_{t \in \mathbb{R}^+}\) is such a stochastic process then we have
\[
\mathbb{E}[X_t | F_s] = \mathbb{E}[X_s | F_s] + \mathbb{E}[X_t - X_s | F_s] \\
= \mathbb{E}[X_s | F_s] + \mathbb{E}[X_t - X_s] \\
\leq \mathbb{E}[X_s | F_s] \\
= X_s, \quad 0 \leq s \leq t.
\]

Similarly, a stochastic process with independent increments which have positive expectation will be a submartingale. Brownian motion \(B_t + \mu t\) with positive drift \(\mu > 0\) is such an example, as in Figure 14.1 below.

![Drifted Brownian path](https://www.ntu.edu.sg/home/nprivault/indext.html)

**Fig. 14.1:** Drifted Brownian path.

The following example comes from gambling.
A natural way to construct submartingales is to take convex functions of martingales.

**Proposition 14.4.** Given \((M_t)_{t \in \mathbb{R}^+}\) a martingale and \(\phi : \mathbb{R} \to \mathbb{R}\) a convex function we have

\[
\phi(M_s) \leq \mathbb{E}[\phi(M_t) \mid \mathcal{F}_s], \quad 0 \leq s \leq t,
\]

i.e. \((\phi(M_t))_{t \in \mathbb{R}^+}\) is a submartingale.

**Proof.** By Jensen’s inequality we have

\[
\phi(\mathbb{E}[M_t \mid \mathcal{F}_s]) \leq \mathbb{E}[\phi(M_t) \mid \mathcal{F}_s], \quad 0 \leq s \leq t, \quad (14.2)
\]

which shows that

\[
\phi(M_s) = \phi(\mathbb{E}[M_t \mid \mathcal{F}_s]) \leq \mathbb{E}[\phi(M_t) \mid \mathcal{F}_s], \quad 0 \leq s \leq t.
\]

\[\square\]

More generally, the proof of Proposition 14.4 shows that \(\phi(M_t)_{t \in \mathbb{R}^+}\) remains a submartingale when \(\phi\) is convex non-decreasing and \((M_t)_{t \in \mathbb{R}^+}\) is a submartingale. Similarly, \((\phi(M_t))_{t \in \mathbb{R}^+}\) will be a supermartingale when \((M_t)_{t \in \mathbb{R}^+}\) is a martingale and the function \(\phi\) is concave.

Other examples of (super, sub)-martingales include geometric Brownian motion

\[
S_t = S_0 e^{rt + \sigma B_t - \sigma^2 t/2}, \quad t \in \mathbb{R}^+,
\]

which is a martingale for \(r = 0\), a supermartingale for \(r \leq 0\), and a submartingale for \(r \geq 0\).

### 14.3 Stopping Times

Next, we turn to the definition of *stopping time*. 

---

Fig. 14.2: Evolution of the fortune of a poker player vs number of games played.
Definition 14.5. An \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\)-stopping time is a random variable \(\tau : \Omega \rightarrow \mathbb{R}_+ \cup \{+\infty\}\) such that
\[
\{\tau > t\} \in \mathcal{F}_t, \quad t \in \mathbb{R}_+. \quad (14.3)
\]
The meaning of Relation (14.3) is that the knowledge of the event \(\{\tau > t\}\) depends only on the information present in \(\mathcal{F}_t\) up to time \(t\), i.e. on the knowledge of \(X_s\) for \(0 \leq s \leq t\).

In other words, an event occurs at a stopping time \(\tau\) if at any time \(t\) it can be decided whether the event has already occurred \((\tau \leq t)\) or not \((\tau > t)\) based on the information \(\mathcal{F}_t\) generated by \((X_s)_{s \in \mathbb{R}_+}\) up to time \(t\).

For example, the day you bought your first car is a stopping time (one can always answer the question “did I ever buy a car”), whereas the day you will buy your last car may not be a stopping time (one may not be able to answer the question “will I ever buy another car”).

Proposition 14.6. Let \(\tau\) and \(\theta\) be stopping times.

i) Every constant time is a stopping time.
ii) The minimum \(\tau \wedge \theta := \min(\tau, \theta)\) of \(\tau\) and \(\theta\) is also a stopping time.
iii) The maximum \(\tau \vee \theta := \max(\tau, \theta)\) of \(\tau\) and \(\theta\) is also a stopping time.

Proof. Point (i) is easily checked. Regarding (ii), we have
\[
\{\tau \wedge \theta > t\} = \{\tau > t\ \text{and} \ \theta > t\} = \{\tau > t\} \cap \{\theta > t\} \in \mathcal{F}_t, \quad t \in \mathbb{R}_+.
\]
On the other hand, we have
\[
\{\tau \vee \theta \leq t\} = \{\tau \leq t\ \text{and} \ \theta \leq t\} = \{\tau > t\}^c \cap \{\theta > t\}^c \in \mathcal{F}_t, \quad t \in \mathbb{R}_+,
\]
which implies
\[
\{\tau \vee \theta > t\} = \{\tau \vee \theta \leq t\}^c \in \mathcal{F}_t, \quad t \in \mathbb{R}_+.
\]

Hitting times provide natural examples of stopping times. The hitting time of level \(x\) by the process \((X_t)_{t \in \mathbb{R}_+}\), defined as
\[
\tau_x = \inf\{t \in \mathbb{R}_+ : X_t = x\},
\]
is a stopping time,\(^*\) as we have (here in discrete time)
\[
\{\tau_x > t\} = \{X_s \neq x \text{ for all } s \in [0, t]\}
= \{X_0 \neq x\} \cap \{X_1 \neq x\} \cap \cdots \cap \{X_t \neq x\} \in \mathcal{F}_t, \quad t \in \mathbb{N}.
\]

\(^*\) As a convention we let \(\tau = +\infty\) in case there exists no \(t \in \mathbb{R}_+\) such that \(X_t = x\).
In gambling, a hitting time can be used as an exit strategy from the game. For example, letting
\[
\tau_{x,y} := \inf\{t \in \mathbb{R}^+ : X_t = x \text{ or } X_t = y\}
\] defines a hitting time (hence a stopping time) which allows a gambler to exit the game as soon as losses become equal to \(x = -10\), or gains become equal to \(y = +100\), whichever comes first.

However, not every \(\mathbb{R}^+\)-valued random variable is a stopping time. For example the random time
\[
\tau = \inf \left\{ t \in [0,T] : X_t = \sup_{s \in [0,T]} X_s \right\},
\] which represents the first time the process \((X_t)_{t \in [0,T]}\) reaches its maximum over \([0,T]\), is not a stopping time with respect to the filtration generated by \((X_t)_{t \in [0,T]}\). Indeed, the information known at time \(t \in (0,T)\) is not sufficient to determine whether \(\{\tau > t\}\).

**Stopped process**

Given \((Z_t)_{t \in \mathbb{R}^+}\) a stochastic process and \(\tau : \Omega \rightarrow \mathbb{R}^+ \cup \{+\infty\}\) a stopping time, the *stopped process* \((Z_{t \wedge \tau})_{t \in \mathbb{R}^+}\) is defined as
\[
Z_{t \wedge \tau} = \left\{ \begin{array}{ll}
Z_t & \text{if } t < \tau, \\
Z_\tau & \text{if } t \geq \tau,
\end{array} \right.
\]

Using indicator functions we may also write
\[
Z_{t \wedge \tau} = Z_t \mathbb{1}_{\{t < \tau\}} + Z_\tau \mathbb{1}_{\{t \geq \tau\}}, \quad t \in \mathbb{R}^+.
\]

The following Figure 14.3 is an illustration of the path of a stopped process.
Theorem 14.7 below is called the Stopping Time (or Optional Sampling, or Optional Stopping) Theorem, it is due to the mathematician J.L. Doob (1910-2004). It is also used in Exercise 14.5 below.

**Theorem 14.7.** Assume that \((M_t)_{t \in \mathbb{R}^+}\) is a martingale with respect to \((\mathcal{F}_t)_{t \in \mathbb{R}^+}\), and that \(\tau\) is an \((\mathcal{F}_t)_{t \in \mathbb{R}^+}\)-stopping time. Then the stopped process \((M_{t \wedge \tau})_{t \in \mathbb{R}^+}\) is also a martingale with respect to \((\mathcal{F}_t)_{t \in \mathbb{R}^+}\).

**Proof.** We only give the proof in discrete time by applying the martingale transform argument of Proposition 2.10. Writing the telescoping sum

\[
M_n = M_0 + \sum_{l=1}^{n} (M_l - M_{l-1}),
\]

we have

\[
M_{\tau \wedge n} = M_0 + \sum_{1 \leq l \leq \tau} (M_l - M_{l-1}) = M_0 + \sum_{l=1}^{n} \mathbb{1}_{\{l \leq \tau\}} (M_l - M_{l-1}),
\]

and for \(k \leq n\),

\[
\mathbb{E}[M_{\tau \wedge n} \mid \mathcal{F}_k] = \mathbb{E} \left[ M_0 + \sum_{l=1}^{n} \mathbb{1}_{\{l \leq \tau\}} (M_l - M_{l-1}) \mid \mathcal{F}_k \right]
= M_0 + \sum_{l=1}^{k} \mathbb{E}[\mathbb{1}_{\{l \leq \tau\}} (M_l - M_{l-1}) \mid \mathcal{F}_k]
+ \sum_{l=k+1}^{n} \mathbb{E}[\mathbb{1}_{\{l \leq \tau\}} (M_l - M_{l-1}) \mid \mathcal{F}_k].
\]
More generally, if $(M_t)_{t \in \mathbb{R}_+}$ is a supermartingale with respect to $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$, then the stopped process $(M_{t \wedge \tau})_{t \in \mathbb{R}_+}$ remains a supermartingale with respect to $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$.

b) Since by Theorem 14.7 the stopped process $(M_{t \wedge \tau})_{t \in \mathbb{R}_+}$ is a martingale, we find that its expected value $\mathbb{E}[M_{t \wedge \tau}]$ is constant over time $t \in \mathbb{R}_+$ by Proposition 14.2.

As a consequence, if $(M_t)_{t \in \mathbb{R}_+}$ is an $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$-martingale and $\tau$ is a stopping time bounded by a constant $T > 0$, i.e. $\tau \leq T$ almost surely,* we have

$$\mathbb{E}[M_{\tau}] = \mathbb{E}[M_{\tau \wedge T}] = \mathbb{E}[M_{\tau \wedge 0}] = \mathbb{E}[M_0] = \mathbb{E}[M_T]. \quad (14.5)$$

* “$\tau \leq T$ almost surely” means $\mathbb{P}(\tau \leq T) = 1$, i.e. $\mathbb{P}(\tau > T) = 0$. 

Remarks.

a) More generally, if $(M_t)_{t \in \mathbb{N}}$ is a martingale with respect to $(\mathcal{F}_t)_{t \in \mathbb{N}}$, then the stopped process $(M_{t \wedge \tau})_{t \in \mathbb{N}}$ remains a martingale, with respect to $(\mathcal{F}_t)_{t \in \mathbb{N}}$.

As by the martingale property of $(M_t)_{t \in \mathbb{N}}$, we have

$$\mathbb{E}[(M_t - M_{t-1}) | \mathcal{F}_{t-1}] = \mathbb{E}[M_t | \mathcal{F}_{t-1}] - \mathbb{E}[M_{t-1} | \mathcal{F}_{t-1}] = \mathbb{E}[M_t | \mathcal{F}_{t-1}] - M_{t-1} = 0, \quad l \geq 1.$$
c) From (14.5), if $\tau$ and $\nu$ are two bounded stopping times and $(M_t)_{t \in \mathbb{R}_+}$ is a martingale, we have
\[
\mathbb{E}[M_\tau] = \mathbb{E}[M_\nu]. \tag{14.6}
\]

d) If $\tau$ and $\nu$ are two a.s. bounded stopping times such that $\tau \leq \nu$ a.s.,

(i) if $(M_t)_{t \in \mathbb{R}_+}$ is a supermartingale, we have
\[
\mathbb{E}[M_0] \geq \mathbb{E}[M_\tau] \geq \mathbb{E}[M_\nu] \tag{14.7}
\]

(ii) if $(M_t)_{t \in \mathbb{R}_+}$ is a submartingale, we have
\[
\mathbb{E}[M_\tau] \leq \mathbb{E}[M_\nu] \leq \mathbb{E}[M_0], \tag{14.8}
\]

see Exercise 14.5 below for a proof in discrete time.

e) In case $\tau$ is finite with probability one (but not bounded) we may also write
\[
\mathbb{E}[M_\tau] = \mathbb{E}\left[\lim_{t \to \infty} M_{\tau \wedge t}\right] = \lim_{t \to \infty} \mathbb{E}[M_{\tau \wedge t}] = \mathbb{E}[M_0], \tag{14.9}
\]
provided that
\[
|M_{\tau \wedge t}| \leq C, \quad a.s., \quad t \in \mathbb{R}_+. \tag{14.10}
\]

More generally, (14.9) holds provided that the limit and expectation signs can be exchanged, and this can be done using e.g. the Dominated Convergence Theorem.

In case $\mathbb{P}(\tau = +\infty) > 0$, (14.9) holds under the above conditions, provided that
\[
M_\infty := \lim_{t \to \infty} M_t \tag{14.11}
\]
exists with probability one.

Relations (14.7), (14.8) and (14.6) can be extended to unbounded stopping times along the same lines and conditions as (14.9), such as (14.10) applied to both $\tau$ and $\nu$. Dealing with unbounded stopping times can be necessary in the case of hitting times.

f) In general, for all a.s. finite (bounded or unbounded) stopping times $\tau$ it remains true that
\[
\mathbb{E}[M_\tau] = \mathbb{E}\left[\lim_{t \to \infty} M_{\tau \wedge t}\right] \leq \lim_{t \to \infty} \mathbb{E}[M_{\tau \wedge t}] \leq \lim_{t \to \infty} \mathbb{E}[M_0] = \mathbb{E}[M_0], \tag{14.12}
\]
provided that \((M_t)_{t \in \mathbb{R}_+}\) is a nonnegative supermartingale, where we used Fatou’s Lemma 22.3. As in (14.9), the limit (14.11) is required to exist with probability one if \(P(\tau = +\infty) > 0\).

g) As a counterexample to (14.6), the random time
\[
\tau := \inf \left\{ t \in [0, T] : M_t = \sup_{s \in [0, T]} M_s \right\},
\]
which is not a stopping time, will satisfy
\[
\mathbb{E}[M_\tau] > \mathbb{E}[M_T],
\]
although \(\tau \leq T\) almost surely. Similarly,
\[
\tau := \inf \left\{ t \in [0, T] : M_t = \inf_{s \in [0, T]} M_s \right\},
\]
is not a stopping time and satisfies
\[
\mathbb{E}[M_\tau] < \mathbb{E}[M_T].
\]
When \((M_t)_{t \in [0, T]}\) is a martingale, e.g. a centered random walk with independent increments, the message of the Stopping Time Theorem 14.7 is that the expected gain of the exit strategy \(\tau_{x,y}\) of (14.4) remains zero on average since
\[
\mathbb{E}[M_{\tau_{x,y}}] = \mathbb{E}[M_0] = 0,
\]
if \(M_0 = 0\). This shows that, on average, this exit strategy does not increase the average gain of the player. More precisely we have
\[
0 = M_0 = \mathbb{E}[M_{\tau_{x,y}}] = xP(M_{\tau_{x,y}} = x) + yP(M_{\tau_{x,y}} = y)
\]
\[
= -10 \times P(M_{\tau_{x,y}} = -10) + 100 \times P(M_{\tau_{x,y}} = 100),
\]
which shows that
\[
P(M_{\tau_{x,y}} = -10) = \frac{10}{11} \quad \text{and} \quad P(M_{\tau_{x,y}} = 100) = \frac{1}{11},
\]
provided that the relation \(P(M_{\tau_{x,y}} = x) + P(M_{\tau_{x,y}} = y) = 1\) is satisfied, see below for further applications to Brownian motion.

In the next Table 14.1 we summarize some of the results of this section for bounded stopping times.

* \(\mathbb{E}[\lim_{n \to \infty} F_n] \leq \lim_{n \to \infty} \mathbb{E}[F_n]\) for any sequence \((F_n)_{n \in \mathbb{N}}\) of nonnegative random variables, provided that the limits exist.
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<table>
<thead>
<tr>
<th>bounded stopping times $\tau, \nu$</th>
</tr>
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<tbody>
<tr>
<td>$(M_t)_{t \in \mathbb{R}^+}$</td>
</tr>
<tr>
<td>supermartingale</td>
</tr>
<tr>
<td>$\mathbb{E}[M_\tau] \geq \mathbb{E}[M_\nu]$ if $\tau \leq \nu$.</td>
</tr>
<tr>
<td>martingale</td>
</tr>
<tr>
<td>$\mathbb{E}[M_\tau] = \mathbb{E}[M_\nu]$.</td>
</tr>
<tr>
<td>submartingale</td>
</tr>
<tr>
<td>$\mathbb{E}[M_\tau] \leq \mathbb{E}[M_\nu]$ if $\tau \leq \nu$.</td>
</tr>
</tbody>
</table>

Table 14.1: Martingales and stopping times.

Examples of application

In this section we note that, as an application of the Stopping Time Theorem 14.7, a number of expectations can be computed in a simple and elegant way.

**Brownian motion hitting a barrier**

Given $a, b \in \mathbb{R}$, $a < b$, let the hitting* time $\tau_{a,b} : \Omega \rightarrow \mathbb{R}^+$ be defined by

$$\tau_{a,b} = \inf\{t \geq 0 : B_t = a \text{ or } B_t = b\},$$

which is the hitting time of the boundary $\{a, b\}$ of Brownian motion $(B_t)_{t \in \mathbb{R}^+}$, $a < b \in \mathbb{R}$.

Recall that Brownian motion $(B_t)_{t \in \mathbb{R}^+}$ is a martingale since it has independent increments, and those increments are centered:

$$\mathbb{E}[B_t - B_s] = 0, \quad 0 \leq s \leq t.$$ 

Consequently, $(B_{\tau_{a,b}\wedge t})_{t \in \mathbb{R}^+}$ is still a martingale and by (14.9) we have

$$\mathbb{E}[B_{\tau_{a,b}} \mid B_0 = x] = \mathbb{E}[B_0 \mid B_0 = x] = x,$$

as the exchange between limit and expectation in (14.9) can be justified since

$$|B_{t\wedge \tau_{a,b}}| \leq \text{Max}(|a|, |b|), \quad t \in \mathbb{R}^+.$$ 

Hence we have

* A hitting time is a stopping time.
\[
\begin{align*}
    x &= \mathbb{E}[B_{\tau_{a,b}} | B_0 = x] = a \times \mathbb{P}(B_{\tau_{a,b}} = a | B_0 = x) + b \times \mathbb{P}(B_{\tau_{a,b}} = b | B_0 = x), \\
    \mathbb{P} \left( X_{\tau_{a,b}} = a | X_0 = x \right) + \mathbb{P}(X_{\tau_{a,b}} = b | X_0 = x) &= 1,
\end{align*}
\]
which yields
\[
\mathbb{P}(B_{\tau_{a,b}} = b | B_0 = x) = \frac{x-a}{b-a}, \quad a \leq x \leq b,
\]
which also shows that
\[
\mathbb{P}(B_{\tau_{a,b}} = a | B_0 = x) = \frac{b-x}{b-a}, \quad a \leq x \leq b.
\]
Note that the above result and its proof actually apply to any continuous martingale, and not only to Brownian motion.

**Drifted Brownian motion hitting a barrier**

Next, let us turn to the case of drifted Brownian motion
\[
X_t = x + B_t + \mu t, \quad t \in \mathbb{R}_+.
\]
In this case, the process \((X_t)_{t \in \mathbb{R}_+}\) is no longer a martingale and in order to use Theorem 14.7 we need to construct a martingale of a different type. Here we note that the process
\[
M_t := e^{\sigma B_t - \sigma^2 t/2}, \quad t \in \mathbb{R}_+,
\]
is a martingale with respect to \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\). Indeed, we have
\[
\mathbb{E}[M_t | \mathcal{F}_s] = \mathbb{E} \left[ e^{\sigma B_t - \sigma^2 t/2} | \mathcal{F}_s \right] = e^{\sigma B_s - \sigma^2 s/2}, \quad 0 \leq s \leq t,
\]
cf. e.g. Example 1 page 234.

By Theorem 14.7 we know that the stopped process \((M_{\tau_{a,b}})_{t \in \mathbb{R}_+}\) is a martingale, hence its expectation is constant by Proposition 14.2, and (14.9) gives
\[
1 = \mathbb{E}[M_0] = \mathbb{E}[M_{\tau_{a,b}}],
\]
as the exchange between limit and expectation in (14.9) can be justified since
\[
|M_{t \wedge \tau_{a,b}}| \leq \text{Max} \left( e^{\sigma |a|}, e^{\sigma |b|} \right), \quad t \in \mathbb{R}_+.
\]
Next, we note that letting \(\mu = -\sigma/2\) we have
\[
e^{\sigma X_t} = e^{\sigma x + \sigma B_t + \sigma t} = e^{\sigma x + \sigma B_t - \sigma^2 t/2} = e^{\sigma x} M_t,
\]

or $M_t = e^{-\sigma x} e^{\sigma X_t}$, hence

$$1 = \mathbb{E}[M_{\tau_{a,b}}] = e^{-\sigma x} \mathbb{E}[e^{\sigma X_{\tau_{a,b}}}]$$

$$= e^{(a-x)\sigma} \mathbb{P}(X_{\tau_{a,b}} = a \mid X_0 = x) + e^{(b-x)\sigma} \mathbb{P}(X_{\tau_{a,b}} = b \mid X_0 = x),$$

under the additional condition

$$\mathbb{P}(X_{\tau_{a,b}} = a \mid X_0 = x) + \mathbb{P}(X_{\tau_{a,b}} = b \mid X_0 = x) = 1.$$

Finally this gives

$$\begin{align*}
\mathbb{P}(X_{\tau_{a,b}} = a \mid X_0 = x) &= \frac{e^{\sigma x} - e^{\sigma b}}{e^{\sigma a} - e^{\sigma b}} = \frac{e^{-2\mu x} - e^{-2\mu b}}{e^{-2\mu a} - e^{-2\mu b}}, \\
\mathbb{P}(X_{\tau_{a,b}} = b \mid X_0 = x) &= \frac{e^{-2\mu a} - e^{-2\mu x}}{e^{-2\mu a} - e^{-2\mu b}},
\end{align*}$$

(14.13a)

(14.13b)

for $a \leq x \leq b$. Letting $b$ tend to infinity in the above equalities shows that the probability $\mathbb{P}(\tau_a = +\infty)$ of escape to infinity of Brownian motion started from $x \in [a, \infty)$ is equal to

$$\mathbb{P}(\tau_a = +\infty) = \begin{cases} 1 - \mathbb{P}(X_{\tau_{a,\infty}} = a \mid X_0 = x) = 1 - e^{-2\mu(x-a)}, & \mu \geq 0, \\
0, & \mu \leq 0, \end{cases}$$

(14.14)

i.e.

$$\mathbb{P}(\tau_a < +\infty) = \begin{cases} \mathbb{P}(X_{\tau_{a,\infty}} = a \mid X_0 = x) = e^{-2\mu(x-a)}, & \mu \geq 0, \\
1, & \mu \leq 0. \end{cases}$$

(14.15)

Similarly for $x \in (-\infty, b]$, letting $a$ tend to infinity we have

$$\mathbb{P}(\tau_b = +\infty) = \begin{cases} 1 - \mathbb{P}(X_{\tau_{-\infty,b}} = b \mid X_0 = x) = 1 - e^{-2\mu(x-b)}, & \mu \leq 0, \\
0, & \mu \geq 0, \end{cases}$$

(14.16)

i.e.
\[ \Pr(\tau_b < +\infty) = \begin{cases} 0 & \Pr(X_{\tau_{-\infty,b}} = b \mid X_0 = x) = e^{-2\mu(x-b)}, \quad \mu \leq 0, \\ 1 & \mu > 0. \end{cases} \] (14.17)

**Mean hitting time for Brownian motion**

The martingale method also allows us to compute the expectation \( \mathbb{E}[B_{\tau_{a,b}}] \), after rechecking that

\[ B_t^2 - t = 2 \int_0^t B_s dB_s, \quad t \in \mathbb{R}_+, \]

is also a martingale. Indeed we have

\[
\begin{align*}
\mathbb{E}[B_t^2 - t \mid \mathcal{F}_s] &= \mathbb{E}[(B_s + (B_t - B_s))^2 - t \mid \mathcal{F}_s] \\
&= \mathbb{E}[B_s^2 + (B_t - B_s)^2 + 2B_s(B_t - B_s) - t \mid \mathcal{F}_s] \\
&= \mathbb{E}[B_s^2 - s \mid \mathcal{F}_s] - (t-s) + \mathbb{E}[(B_t - B_s)^2 \mid \mathcal{F}_s] + 2 \mathbb{E}[B_s(B_t - B_s) \mid \mathcal{F}_s] \\
&= B_s^2 - s - (t-s) + \mathbb{E}[(B_t - B_s)^2 \mid \mathcal{F}_s] + 2B_s \mathbb{E}[B_t - B_s] \\
&= B_s^2 - s, \quad 0 \leq s \leq t.
\end{align*}
\]

Consequently the stopped process \((B_{\tau_{a,b} \wedge t}^2 - \tau_{a,b} \wedge t)_{t \in \mathbb{R}_+}\) is still a martingale by Theorem 14.7 hence the expectation \( \mathbb{E}[B_{\tau_{a,b} \wedge t}^2 - \tau_{a,b} \wedge t] \) is constant in \( t \in \mathbb{R}_+ \), hence by (14.9) we get*

\[
\begin{align*}
x^2 &= \mathbb{E}[B_{\tau_{a,b}}^2 - B_0^2 \mid B_0 = x] \\
&= \mathbb{E}[B_{\tau_{a,b}}^2 - \tau_{a,b} \mid B_0 = x] \\
&= \mathbb{E}[B_{\tau_{a,b}}^2 \mid B_0 = x] - \mathbb{E}[\tau_{a,b} \mid B_0 = x] \\
&= b^2 \Pr(B_{\tau_{a,b}} = b \mid B_0 = x) + a^2 \Pr(B_{\tau_{a,b}} = a \mid B_0 = x) - \mathbb{E}[\tau_{a,b} \mid B_0 = x],
\end{align*}
\]

i.e.

\[
\begin{align*}
\mathbb{E}[\tau_{a,b} \mid B_0 = x] &= b^2 \Pr(B_{\tau_{a,b}} = b \mid B_0 = x) + a^2 \Pr(B_{\tau_{a,b}} = a \mid B_0 = x) - x^2 \\
&= \frac{b^2 x - a}{b - a} + \frac{a^2 b - x}{b - a} - x^2 \\
&= (x - a)(b - x), \quad a \leq x \leq b.
\end{align*}
\]

* Here we note that it can be showed that \( \mathbb{E}[\tau_{a,b}] < \infty \) in order to apply (14.9).
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Mean hitting time for drifted Brownian motion

Finally we show how to recover the value of the mean hitting time $\mathbb{E}[\tau_{a,b} \mid X_0 = x]$ of drifted Brownian motion $X_t = x + B_t + \mu t$. As above, the process $X_t - \mu t$ is a martingale the stopped process $(X_{\tau_{a,b} \wedge t} - \mu(\tau_{a,b} \wedge t))_{t \in \mathbb{R}_+}$ is still a martingale by Theorem 14.7. Hence the expectation $\mathbb{E}[X_{\tau_{a,b} \wedge t} - \mu(\tau_{a,b} \wedge t)]$ is constant in $t \in \mathbb{R}_+$.

Since the stopped process $(X_{\tau_{a,b} \wedge t} - \mu(\tau_{a,b} \wedge t))_{t \in \mathbb{R}_+}$ is a martingale, we have

$$x = \mathbb{E}[X_{\tau_{a,b}} - \mu \tau_{a,b} \mid X_0 = x],$$

which gives

$$x = \mathbb{E}[X_{\tau_{a,b}} - \mu \tau_{a,b} \mid X_0 = x] = \mathbb{E}[X_{\tau_{a,b}} \mid X_0 = x] - \mu \mathbb{E}[\tau_{a,b} \mid X_0 = x] = bP(X_{\tau_{a,b}} = b \mid X_0 = x) + aP(X_{\tau_{a,b}} = a \mid X_0 = x) - \mu \mathbb{E}[\tau_{a,b} \mid X_0 = x],$$

i.e. by (14.13a),

$$\mu \mathbb{E}[\tau_{a,b} \mid X_0 = x] = bP(X_{\tau_{a,b}} = b \mid X_0 = x) + aP(X_{\tau_{a,b}} = a \mid X_0 = x) - x$$

$$= b \frac{e^{-2\mu a} - e^{-2\mu x}}{e^{-2\mu a} - e^{-2\mu b}} + a \frac{e^{-2\mu x} - e^{-2\mu b}}{e^{-2\mu a} - e^{-2\mu b}} - x$$

$$= b(e^{-2\mu a} - e^{-2\mu x}) + a(e^{-2\mu x} - e^{-2\mu b}) - x(e^{-2\mu a} - e^{-2\mu b}),$$

hence

$$\mathbb{E}[\tau_{a,b} \mid X_0 = x] = \frac{b(e^{-2\mu a} - e^{-2\mu x}) + a(e^{-2\mu x} - e^{-2\mu b}) - x(e^{-2\mu a} - e^{-2\mu b})}{\mu(e^{-2\mu a} - e^{-2\mu b})},$$

$$a \leq x \leq b.$$

14.4 Perpetual American Options

The price of an American put option with finite expiration time $T > 0$ and strike price $K$ can be expressed as the expected value of its discounted payoff:

$$f(t, S_t) = \sup_{t \leq \tau \leq T} \mathbb{E}^* \left[ e^{-(\tau-t)r} (K - S_\tau)^+ \mid S_t \right],$$

under the risk-neutral probability measure $\mathbb{P}^*$, where the supremum is taken over stopping times between $t$ and a fixed maturity $T$. Similarly, the price of a finite expiration American call option with strike price $K$ is expressed as

\[ \]
$f(t, S_t) = \sup_{t \leq \tau \leq T} \mathbb{E}^* \left[ e^{-(\tau-t)r} (S_{\tau} - K)^+ \mid S_t \right].$

In this section we take $T = +\infty$, in which case we refer to these options as *perpetual* options, and the corresponding put and call options are respectively priced as

$$f(t, S_t) = \sup_{\tau \geq t} \mathbb{E}^* \left[ e^{-(\tau-t)r} (K - S_{\tau})^+ \mid S_t \right],$$

and

$$f(t, S_t) = \sup_{\tau \geq t} \mathbb{E}^* \left[ e^{-(\tau-t)r} (S_{\tau} - K)^+ \mid S_t \right].$$

**Two-choice optimal stopping at a fixed price level for perpetual put options**

In this section we consider the pricing of perpetual put options. Given $L \in (0, K)$ a fixed price, consider the following choices for the exercise of a put option with strike price $K$:

1. If $S_t \leq L$, then exercise at time $t$.

2. Otherwise if $S_t > L$, wait until the first hitting time

$$\tau_L := \inf\{u \geq t : S_u \leq L\}$$

(14.18)

of the level $L > 0$, and exercise the option at time $\tau_L$ if $\tau_L < \infty$.

Note that by definition of (14.18) we have $\tau_L = t$ if $S_t \leq L$.

In case $S_t \leq L$, the payoff will be

$$(K - S_t)^+ = K - S_t$$

since $K > L \geq S_t$, however in this case one would buy the option at price $K - S_t$ only to exercise it immediately for the same amount.

In case $S_t > L$, the price of the option will be

$$f_L(t, S_t) = \mathbb{E}^* \left[ e^{-(\tau_L-t)r} (K - S_{\tau_L})^+ \mid S_t \right]$$

$$= \mathbb{E}^* \left[ e^{-(\tau_L-t)r} (K - L)^+ \mid S_t \right]$$

$$= (K - L) \mathbb{E}^* \left[ e^{-(\tau_L-t)r} \mid S_t \right].$$

(14.19)

We note that the starting date $t$ does not matter when pricing perpetual options which have an infinite time horizon. Hence, $f_L(t, x)$ does not depend
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on \( t \in \mathbb{R}_+ \), and the pricing of the perpetual put option can be performed by taking \( t = 0 \) and in the sequel we will work under

\[
\mathcal{F}_t = \mathcal{F}_0 e^{r \mathcal{B}_t - \sigma^2 t / 2}, \quad t \in \mathbb{R}_+, \tag{14.20}
\]

where \( (\mathcal{B}_t)_{t \in \mathbb{R}_+} \) is a standard Brownian motion under the risk-neutral probability measure \( \mathcal{P}^* \), \( r \) is the risk-free interest rate, and \( \sigma > 0 \) is the volatility coefficient.

**Lemma 14.8.** Assume that \( r > 0 \). We have

\[
\mathbb{E}^* \left[ e^{-r \tau} \mid S_t = x \right] = \left( \frac{x}{L} \right)^{-2r / \sigma^2}, \quad x \geq L.
\]

**Proof.** We take \( t = 0 \) without loss of generality. We note that from (14.20), for all \( \lambda \in \mathbb{R} \) the process \( (Z_{t}^{(\lambda)})_{t \in \mathbb{R}_+} \) defined as

\[
Z_{t}^{(\lambda)} := S_0^\lambda e^{\lambda \mathcal{B}_t - \lambda^2 \sigma^2 t / 2} = S_t^\lambda e^{-(r\lambda - \lambda(1-\lambda)\sigma^2 / 2)t}, \quad t \in \mathbb{R}_+, \tag{14.22}
\]

is a martingale under the risk-neutral probability measure \( \mathcal{P}^* \). Choosing \( \lambda \in \mathbb{R} \) such that

\[
r = r\lambda - \lambda(1-\lambda)\sigma^2 / 2, \tag{14.23}
\]

we have

\[
Z_{t}^{(\lambda)} = S_t^\lambda e^{-rt}, \quad t \in \mathbb{R}_+.
\]

The equation (14.23) rewrites as

\[
0 = \lambda^2 \sigma^2 / 2 + \lambda \left( r - \sigma^2 / 2 \right) - r = \sigma^2 \left( \lambda + \frac{2r}{\sigma^2} \right) (\lambda - 1), \tag{14.24}
\]

with solutions

\[
\lambda_+ = 1 \quad \text{and} \quad \lambda_- = -\frac{2r}{\sigma^2}.
\]

Choosing the negative solution\(^*\) \( \lambda_- = -\frac{2r}{\sigma^2} < 0 \), we have

\[
0 \leq Z_{t}^{(\lambda_-)} = e^{-rt} (S_t)^{\lambda_-} \leq e^{-rt} L^{\lambda_-} \leq L^{\lambda_-}, \quad 0 \leq t < \tau_L, \tag{14.25}
\]

\(^*\) Note that \( \mathbb{P}(\tau_L = \infty) > 0 \) since \( (S_t)_{t \in \mathbb{R}_+} \) is a submartingale, cf. (14.14), and the bound (14.25) does not hold for the positive solution \( \lambda_+ = 1 \).
since \( r > 0 \), hence \( \lim_{t \to \infty} Z_t^{(\lambda^-)} = 0 \) and \( \lim_{t \to \infty} Z_{\tau_L \wedge t}^{(\lambda^-)} = 0 \) on \( \{ \tau_L < \infty \} \). Therefore, since \( r > 0 \) we have

\[
L^{\lambda^-} \mathbb{E}^* \left[ e^{-r \tau_L} \right] = L^{\lambda^-} \mathbb{E}^* \left[ e^{-r \tau_L} \mathbb{1}_{\{\tau_L < \infty\}} \right] \\
= \mathbb{E}^* \left[ e^{-r \tau_L} (S_{\tau_L})^{\lambda^-} \mathbb{1}_{\{\tau_L < \infty\}} \right] \\
= \mathbb{E}^* \left[ Z_{\tau_L}^{(\lambda^-)} \mathbb{1}_{\{\tau_L < \infty\}} \right] \\
= \mathbb{E}^* \left[ \mathbb{1}_{\{\tau_L < \infty\}} \lim_{t \to \infty} Z_{\tau_L \wedge t}^{(\lambda^-)} \right] \\
= \mathbb{E}^* \left[ \lim_{t \to \infty} Z_{\tau_L \wedge t}^{(\lambda^-)} \right] \\
= \lim_{t \to \infty} \mathbb{E}^* \left[ Z_{\tau_L \wedge t}^{(\lambda^-)} \right] \\
= \lim_{t \to \infty} \mathbb{E}^* \left[ Z_{0}^{(\lambda^-)} \right] \\
= (S_0)^{\lambda^-},
\]

where by (14.25) we used the dominated convergence theorem from (14.26) to (14.27), hence we find

\[
\mathbb{E}^* \left[ e^{-r \tau_L} \middle| S_0 = x \right] = \left( \frac{x}{L} \right)^{-2r/\sigma^2}, \quad x \geq L.
\]

Next, we apply Lemma 14.8 in order to price the perpetual American put option.

**Proposition 14.9.** Assume that \( r > 0 \). We have

\[
f_L(x) = \mathbb{E}^* \left[ e^{-(\tau_L - t)r} (K - S_{\tau_L})^+ \middle| S_t = x \right] \\
= \begin{cases} 
K - x, & 0 < x \leq L, \\
(K - L) \left( \frac{x}{L} \right)^{-2r/\sigma^2}, & x \geq L.
\end{cases}
\]

**Proof.** We take \( t = 0 \) without loss of generality.

i) The result is obvious for \( S_0 = x \leq L \) since in this case we have \( \tau_L = t = 0 \) and \( S_{\tau_L} = S_0 = x \), so that we only focus on the case \( x \geq L \).

ii) Next, we consider the case \( S_0 = x > L \). We have

\[
\mathbb{E}^* \left[ e^{-r \tau_L} (K - S_{\tau_L})^+ \middle| S_0 = x \right] = \mathbb{E}^* \left[ \mathbb{1}_{\{\tau_L < \infty\}} e^{-r \tau_L} (K - S_{\tau_L})^+ \middle| S_0 = x \right] \\
= \mathbb{E}^* \left[ \mathbb{1}_{\{\tau_L < \infty\}} e^{-r \tau_L} (K - S_{\tau_L}) \middle| S_0 = x \right] \\
= (K - L) \mathbb{E}^* \left[ e^{-r \tau_L} \middle| S_0 = x \right], \quad (14.28)
\]
and we conclude by the expression of $\mathbb{E}^* \left[ e^{-r\tau_L} \mid S_0 = x \right]$ given in Lemma 14.8.

We note that taking $L = K$ would yield a payoff always equal to 0 for the option holder, hence the value of $L$ should be strictly lower than $K$. On the other hand, if $L = 0$ the value of $\tau_L$ will be infinite almost surely, hence the option price will be 0 when $r \geq 0$ from (14.19). Therefore there should be an optimal value $L^*$, which should be strictly comprised between 0 and $K$.

Figure 14.4 shows for $K = 100$ that there exists an optimal value $L^* = 85.71$ which maximizes the option price for all values of the underlying asset price.

![Figure 14.4: Graphs of American put prices by exercise at $\tau_L$ for several values of $L$.](https://www.ntu.edu.sg/home/nprivault/indext.html)

In order to compute $L^*$ we observe that, geometrically, the slope of $f_L(x)$ at $x = L^*$ is equal to $-1$, \( i.e. \)

$$f'_L(L^*) = -\frac{2r}{\sigma^2}(K - L^*) \left(\frac{L^*}{L^*}\right) - \frac{2r}{\sigma^2} - 1 = -1,$$

\( i.e. \)

$$\frac{2r}{\sigma^2}(K - L^*) = L^*,$$

or

$$L^* = \frac{2r}{2r + \sigma^2} K < K.$$

The same conclusion can be reached by the vanishing of the derivative of $L \mapsto f_L(x)$:

$$\frac{\partial f_L(x)}{\partial L} = -\left(\frac{x}{L}\right)^{-2r/\sigma^2} + \frac{2r}{\sigma^2} \frac{K - L}{L} \left(\frac{x}{L}\right)^{-2r/\sigma^2} = 0,$$
cf. page 351 of Shreve (2004). The next Figure 14.5 is a 2-dimensional animation that also shows the optimal value $L^*$ of $L$.

Fig. 14.5: Animated graph of American put prices depending on $L$.

The next Figure 14.6 gives another view of the put option prices according to different values of $L$, with the optimal value $L^* = 85.71$.

Fig. 14.6: Option price as a function of $L$ and of the underlying asset price.

In Figure 14.7, which is based on the stock price of HSBC Holdings (0005.HK) over year 2009 as in Figures 6.7-6.13, the optimal exercise strategy for an American put option with strike price $K=\$77.67$ would have been to exercise whenever the underlying asset price goes above $L^* = \$62$, i.e. at approximately 54 days, for a payoff of $\$25.67$. Exercising after a longer time, e.g. 85 days, could yield an even higher payoff of over $\$65$, however this choice is not made because decisions are taken based on existing (past) information, and optimization is in expected value (or average) over all possible future paths.

* The animation works in Acrobat Reader on the entire pdf file.
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Fig. 14.7: Path of the American put option price on the HSBC stock.

PDE approach

We can check by hand calculations that the function

\[ f_{L^*}(x) := \begin{cases} 
K - x, & 0 < x \leq L^* = \frac{2r}{2r + \sigma^2} K, \\
\frac{K\sigma^2}{2r + \sigma^2} \left( \frac{2r + \sigma^2}{2r} \frac{x}{K} \right)^{-2r/\sigma^2}, & x \geq L^* = \frac{2r}{2r + \sigma^2} K,
\end{cases} \]

satisfies the PDE

\[
-r f_{L^*}(x) + r x f'_{L^*}(x) + \frac{1}{2} \sigma^2 x^2 f''_{L^*}(x) = \begin{cases} 
-rK < 0, & 0 < x \leq L^* < K, \\
0, & x > L^*.
\end{cases}
\]

in addition to the conditions

\[
\begin{cases} 
f_{L^*}(x) = K - x, & 0 < x \leq L^* < K, \\
f_{L^*}(x) > (K - x)^+, & x > L^*,
\end{cases}
\]

which can be checked from (14.29). The above statements can be summarized in the following set of differential inequalities, or variational differential equation:
\[
\begin{aligned}
f_{L^*}(x) \geq (K - x)^+, \quad (14.31a) \\
r_x f_{L^*}'(x) + \frac{\sigma^2}{2} x^2 f_{L^*}''(x) \leq r f_{L^*}(x), \quad (14.31b) \\
\left(r f_{L^*}(x) - r x f_{L^*}'(x) - \frac{\sigma^2}{2} x^2 f_{L^*}''(x)\right) \left(f_{L^*}(x) - (K - x)^+\right) = 0, \quad (14.31c)
\end{aligned}
\]

which admits an interpretation in terms of absence of arbitrage, as shown below. By (14.30) and Itô’s formula the discounted portfolio price
\[
\tilde{f}_{L^*}(S_t) = e^{-rt} f_{L^*}(S_t), \quad t \in \mathbb{R}_+,
\]
satisfies
\[
d(\tilde{f}_{L^*}(S_t))
\]
\[
= \left(-r f_{L^*}(S_t) + r S_t f_{L^*}'(S_t) + \frac{1}{2} \sigma^2 S_t^2 f_{L^*}''(S_t)\right) e^{-rt} dt + e^{-rt} \sigma S_t f_{L^*}'(S_t) d\tilde{B}_t
\]
\[
= -\mathbb{1}\{S_t \leq L^*\} r K e^{-rt} dt + e^{-rt} \sigma S_t f_{L^*}'(S_t) d\tilde{B}_t
\]
\[
= -\mathbb{1}\{f_{L^*}(S_t) \leq (K - S_t)^+\} r K e^{-rt} dt + e^{-rt} \sigma S_t f_{L^*}'(S_t) d\tilde{B}_t,
\]
(14.32)
hence we have the relation
\[
\tilde{f}_{L^*}(S_T) - \tilde{f}_{L^*}(S_t)
\]
\[
= -r K \int_t^T \mathbb{1}\{f_{L^*}(S_u) \leq (K - S_u)^+\} e^{-ru} du + \int_t^T e^{-ru} \sigma S_u f_{L^*}'(S_u) d\tilde{B}_u,
\]
which implies
\[
\tilde{f}_{L^*}(S_t)
\]
\[
= \mathbb{E}^* \left[\tilde{f}_{L^*}(S_T) \mid \mathcal{F}_t\right] - r K \mathbb{E}^* \left[\int_t^T \mathbb{1}\{f_{L^*}(S_u) \leq (K - S_u)^+\} e^{-ru} du \mid \mathcal{F}_t\right]
\]
\[
= e^{-rT} \mathbb{E}^* \left[(K - S_T)^+ \mid \mathcal{F}_t\right] + r K \mathbb{E}^* \left[\int_t^T \mathbb{1}\{S_u \leq L^*\} e^{-ru} du \mid \mathcal{F}_t\right],
\]
\[0 \leq t \leq T, \text{ see also Theorem 8.4.1 in § 8.4 of Elliott and Kopp (2005) on early exercise premium. From (14.32) we also make the following observations.}
\]
a) From Equation (14.31c), \( \tilde{f}_{L^*}(S_t) \) is a martingale when
\[
f_{L^*}(S_t) > (K - S_t)^+, \quad \text{i.e.} \quad S_t > L^*,
\]
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and in this case the expected rate of return of the hedging portfolio price $f_{L^*}(S_t)$ equals the rate $r$ of the riskless asset as

$$d(\tilde{f}_{L^*}(S_t)) = e^{-rt} \sigma S_t f_{L^*}'(S_t) d\tilde{B}_t,$$

or

$$d(f_{L^*}(S_t)) = d(e^{rt} \tilde{f}_{L^*}(S_t)) = r f_{L^*}(S_t) dt + \sigma S_t f_{L^*}'(S_t) d\tilde{B}_t,$$

and the investor prefers to wait.

b) On the other hand, if

$$f_{L^*}(S_t) = (K - S_t)^+, \quad \text{i.e.} \quad 0 < S_t < L^*,$$

the return of the hedging portfolio becomes lower than $r$ as $d(\tilde{f}_{L^*}(S_t)) = -r K e^{-rt} dt + e^{-rt} \sigma S_t f_{L^*}'(S_t) d\tilde{B}_t$ and

$$d(f_{L^*}(S_t)) = d(e^{rt} \tilde{f}_{L^*}(S_t))$$

$$= r f_{L^*}(S_t) dt - r K dt + e^{-rt} \sigma S_t f_{L^*}'(S_t) d\tilde{B}_t.$$

In this case it is not worth waiting as (14.31b)-(14.31c) show that the return of the hedging portfolio is lower than that of the riskless asset, i.e.:

$$-r f_{L^*}(S_t) + r S_t f_{L^*}'(S_t) + \frac{1}{2} \sigma^2 S_t^2 f_{L^*}''(S_t) = -r K < 0,$$

exercise becomes immediate since the process $\tilde{f}_{L^*}(S_t)$ becomes a (strict) supermartingale, and (14.31c) implies $f_{L^*}(x) = (K - x)^+.$

In view of the above derivation, it should make sense to assert that $f_{L^*}(S_t)$ is the price at time $t$ of the perpetual American put option. The next proposition confirms that this is indeed the case, and that the optimal exercise time is $\tau^* = \tau_{L^*}.$

**Proposition 14.10.** The price of the perpetual American put option is given for all $t \geq 0$ by
\[
\begin{align*}
\mathbf{f}_{L^*}(S_t) &= \sup_{\tau \geq t} \mathbb{E}^* \left[ e^{-(\tau-t)r} (K - S_\tau)^+ \mid S_t \right] \\
&= \mathbb{E}^* \left[ e^{-(\tau L^* - t)r} (K - S_{\tau L^*})^+ \mid S_t \right] \\
&= \begin{cases}
K - S_t, & 0 < S_t \leq L^*, \\
\frac{K \sigma^2}{2r + \sigma^2} \left( \frac{2r + \sigma^2 S_t}{2r K} \right)^{-2r/\sigma^2}, & S_t \geq L^*.
\end{cases}
\end{align*}
\]

Proof. i) Since the drift
\[
-r \mathbf{f}_{L^*}(S_t) + r S_t \mathbf{f}'_{L^*}(S_t) + \frac{1}{2} \sigma^2 S_t^2 \mathbf{f}''_{L^*}(S_t)
\]
in Itô’s formula (14.32) is nonpositive by the inequality (14.31b), the discounted portfolio price
\[
u \mapsto e^{-r u} \mathbf{f}_{L^*}(S_u), \quad u \in [t, \infty),
\]
is a supermartingale. As a consequence, for all (a.s. finite) stopping times \( \tau \in [t, \infty) \) we have, by (14.12),
\[
e^{-r \tau} \mathbf{f}_{L^*}(S_t) \geq \mathbb{E}^* \left[ e^{-r \tau} \mathbf{f}_{L^*}(S_\tau) \mid S_t \right] \geq \mathbb{E}^* \left[ e^{-r \tau} (K - S_\tau)^+ \mid S_t \right],
\]
from (14.31a), which implies
\[
e^{-r \tau} \mathbf{f}_{L^*}(S_t) \geq \sup_{\tau \geq t} \mathbb{E}^* \left[ e^{-r \tau} (K - S_\tau)^+ \mid S_t \right]. \quad \text{(14.33)}
\]

ii) The converse inequality is obvious by Proposition 14.9, as
\[
\mathbf{f}_{L^*}(S_t) = \mathbb{E}^* \left[ e^{-(\tau L^* - t)r} (K - S_{\tau L^*})^+ \mid S_t \right] \\
\leq \sup_{\tau \geq t} \mathbb{E}^* \left[ e^{-(\tau-t)r} (K - S_\tau)^+ \mid S_t \right], \quad \text{(14.34)}
\]
since \( \tau_{L^*} \) is a stopping time larger than \( t \in \mathbb{R}_+ \). The inequalities (14.33) and (14.34) allow us to conclude to the equality
\[
\mathbf{f}_{L^*}(S_t) = \sup_{\tau \geq t} \mathbb{E}^* \left[ e^{-(\tau-t)r} (K - S_\tau)^+ \mid S_t \right].
\]
\[\square\]

Remark. Note that the converse inequality (14.34) can also be obtained from the variational PDE (14.31a)-(14.31c) itself, without relying on Proposition 14.9.
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Section 14.9. For this, taking $\tau = \tau_{L^*}$ we note that the process

$$u \mapsto e^{-rt} f_{L^*}(S_u \wedge \tau_{L^*}), \quad u \geq t,$$

is not only a supermartingale, it is also a martingale until exercise at time $\tau_{L^*}$ by (14.30) since $S_u \wedge \tau_{L^*} \geq L^*$, hence we have

$$e^{-rt} f_{L^*}(S_t) = \mathbb{E}^*[e^{-(u \wedge \tau_{L^*})r} f_{L^*}(S_{u \wedge \tau_{L^*}}) \mid S_t], \quad u \geq t,$$

hence after letting $u$ tend to infinity we obtain

$$e^{-rt} f_{L^*}(S_t) = \mathbb{E}^*[e^{-r \tau_{L^*}} f_{L^*}(S_{\tau_{L^*}}) \mid S_t]$$

$$= \mathbb{E}^*[e^{-r \tau_{L^*}} f_{L^*}(L^*) \mid S_t]$$

$$= \mathbb{E}^*[e^{-r \tau_{L^*}} (K - S_{\tau_{L^*}})^+ \mid S_t]$$

$$\leq \sup_{\tau \geq t} \mathbb{E}^*[e^{-r \tau_{L^*}} (K - S_{\tau_{L^*}})^+ \mid S_t],$$

which shows that

$$e^{-rt} f_{L^*}(S_t) \leq \sup_{\tau \geq t} \mathbb{E}^*[e^{-r \tau} (K - S_\tau)^+ \mid S_t], \quad t \in \mathbb{R}_+.$$

Two-choice optimal stopping at a fixed price level for perpetual call options

In this section we consider the pricing of perpetual call options. Given $L > K$ a fixed price, consider the following choices for the exercise of a call option with strike price $K$:

1. If $S_t \geq L$, then exercise at time $t$.
2. Otherwise, wait until the first hitting time

$$\tau_L = \inf\{u \geq t : S_u = L\}$$

and exercise the option and time $\tau_L$.

In case $S_t \geq L$, the immediate exercise (or intrinsic) payoff will be

$$(S_t - K)^+ = S_t - K,$$

since $K < L \leq S_t$.

In case $S_t < L$, as $r > 0$ the price of the option will be
Proposition 14.11. In case $S_t < L$, as $r > 0$ the price of the option is given by $f_L(S_t)$, where

$$f_L(x) = \begin{cases} 
  x - K, & x \geq L > K, \\
  (L - K) \frac{x}{L}, & 0 < x \leq L.
\end{cases} \quad (14.35)$$

**Proof.** We only need to consider the case $S_0 = x < L$. Note that for all $\lambda \in \mathbb{R}$ the process

$$Z_t^{(\lambda)} := S_t^\lambda e^{-r\lambda t + \lambda \sigma^2 t^2/2 - \lambda^2 \sigma^2 t/2} = S_0^\lambda e^{\lambda \sigma \tilde{B}_t - \lambda^2 \sigma^2 t/2}, \quad t \in \mathbb{R}_+,$$

defined in (14.22) is a martingale under the risk-neutral probability measure $\tilde{P}$. Hence the stopped process $(Z_{t \wedge \tau_L}^{(\lambda)})_{t \in \mathbb{R}_+}$ is a martingale and it has constant expectation, i.e. we have

$$\mathbb{E}^* \left[ Z_{t \wedge \tau_L}^{(\lambda)} \right] = \mathbb{E}^* \left[ Z_0^{(\lambda)} \right] = S_0^\lambda, \quad t \in \mathbb{R}_+. \quad (14.36)$$

Choosing $\lambda$ such that

$$r = r \lambda - \lambda \sigma^2/2 + \lambda^2 \sigma^2/2,$$

i.e.

$$0 = \lambda^2 \sigma^2/2 + \lambda(r - \sigma^2/2) - r = \frac{\sigma^2}{2} (\lambda + 2r/\sigma^2)(\lambda - 1),$$

Relation (14.36) rewrites as

$$\mathbb{E}^* \left[ (S_{t \wedge \tau_L})^\lambda e^{-(t \wedge \tau_L)r} \right] = S_0^\lambda, \quad t \in \mathbb{R}_+. \quad (14.37)$$

Choosing the positive solution $\lambda_+ = 1$ yields the bound

$$0 \leq Z_t^{(\lambda_+)} = e^{-rt} S_t \leq S_t \leq L, \quad 0 \leq t < \tau_L; \quad (14.38)$$

* We actually have $P(\tau_L = \infty) = 0$ since $(S_t)_{t \in \mathbb{R}_+}$ is a submartingale, cf. (14.16), and the bound (14.38) does not hold for the negative solution $\lambda_- = -2r/\sigma^2$. 

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since $S_0 = x < L$. Hence, noting that $\lim_{t \to \infty} Z_t^{(\lambda_+)} = 0$ on $\{ \tau_L = +\infty \}$ by letting $t$ go to infinity in (14.37), by (14.38) and the dominated convergence theorem we get, since $r > 0$,

$$
L \mathbb{E}^* \left[ e^{-r\tau_L} \right] = \mathbb{E}^* \left[ e^{-r\tau_L} S_{\tau_L} \mathbb{1}_{\{ \tau_L < \infty \}} \right]
= \mathbb{E}^* \left[ \lim_{t \to \infty} e^{-(\tau_L \wedge t)r} S_{\tau_L \wedge t} \right]
= \mathbb{E}^* \left[ \lim_{t \to \infty} Z_{\tau_L \wedge t}^{(\lambda_+)} \right]
= \lim_{t \to \infty} \mathbb{E}^* \left[ Z_{\tau_L \wedge t}^{(\lambda_+)} \right]
= \lim_{t \to \infty} \mathbb{E}^* \left[ Z_{0}^{(\lambda_+)} \right]
= S_0,
$$

which yields

$$
\mathbb{E}^* \left[ e^{-r\tau_L} \right] = \frac{S_0}{L}. \tag{14.39}
$$

One can check from Figures 14.8 and 14.9 that the situation completely differs from the perpetual put option case, as there does not exist an optimal value $L^*$ that would maximize the option price for all values of the underlying asset price.

Fig. 14.8: Graphs of American call prices by exercising at $\tau_L$ for several values of $L$.  

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Fig. 14.9: Animated graph of American call prices depending on $L$. *

The intuition behind this picture is that there is no upper limit above which one should exercise the option, and in order to price the American perpetual call option we have to let $L$ go to infinity, i.e. the “optimal” exercise strategy is to wait indefinitely.

Fig. 14.10: Graphs of American call option prices for different values of $L$.

We check from (14.35) that

$$\lim_{L \to \infty} f_L(x) = x - \lim_{L \to \infty} \frac{Kx}{L} = x, \quad x > 0. \quad (14.40)$$

As a consequence we have the following proposition.

**Proposition 14.12.** Assume that $r \geq 0$. The price of the perpetual American call option is given by

* The animation works in Acrobat Reader on the entire pdf file.
American Options

\[ \sup_{\tau \geq t} \mathbb{E}^\ast \left[ e^{-(\tau-t)r} (S_\tau - K)^+ \mid S_t \right] = S_t, \quad t \in \mathbb{R}_+. \quad (14.41) \]

**Proof.** For all \( L > K \) we have

\[
\begin{align*}
f_L(S_t) &= \mathbb{E}^\ast \left[ e^{-(\tau_L-t)r} (S_{\tau_L} - K)^+ \mid S_t \right] \\
&\leq \sup_{\tau \geq t} \mathbb{E}^\ast \left[ e^{-(\tau-t)r} (S_\tau - K)^+ \mid S_t \right], \quad t \in \mathbb{R}_+,
\end{align*}
\]

hence from (14.40), taking the limit as \( L \to \infty \) yields

\[
S_t \leq \sup_{\tau \geq t} \mathbb{E}^\ast \left[ e^{-(\tau-t)r} (S_\tau - K)^+ \mid S_t \right]. \quad (14.42)
\]

On the other hand, since \( u \mapsto e^{-(u-t)r} S_u \) is a martingale, by (14.12) we have, for all stopping times \( \tau \in [t, \infty) \),

\[
\mathbb{E}^\ast \left[ e^{-(\tau-t)r} (S_\tau - K)^+ \mid S_t \right] \leq \mathbb{E}^\ast \left[ e^{-(\tau-t)r} S_\tau \mid S_t \right] \leq S_t, \quad t \in \mathbb{R}_+,
\]

hence

\[
\sup_{\tau \geq t} \mathbb{E}^\ast \left[ e^{-(\tau-t)r} (S_\tau - K)^+ \mid S_t \right] \leq S_t, \quad t \in \mathbb{R}_+,
\]

which shows (14.41) by (14.42). \( \square \)

We may also check that since \( (e^{-rt} S_t)_{t \in \mathbb{R}_+} \) is a martingale, the process \( t \mapsto (e^{-rt} S_t - K)^+ \) is a submartingale since the function \( x \mapsto (x - K)^+ \) is convex, hence for all bounded stopping times \( \tau \) such that \( t \leq \tau \) we have

\[
(S_t - K)^+ \leq \mathbb{E}^\ast \left[ (e^{-(\tau-t)r} S_\tau - K)^+ \mid S_t \right] \leq \mathbb{E}^\ast \left[ e^{-(\tau-t)r} (S_\tau - K)^+ \mid S_t \right],
\]

\( t \in \mathbb{R}_+ \), showing that it is always better to wait than to exercise at time \( t \), and the optimal exercise time is \( \tau^* = +\infty \). This argument does not apply to American put options.

### 14.5 Finite Expiration American Options

In this section we consider finite expirations American put and call options with strike price \( K \), whose prices can be expressed as

\[
f(t, S_t) = \sup_{t \leq \tau \leq T} \mathbb{E}^\ast \left[ e^{-(\tau-t)r} (K - S_\tau)^+ \mid S_t \right],
\]
Two-choice optimal stopping at fixed times with finite expiration

We start by considering the optimal stopping problem in a simplified setting where \( \tau \in \{ t, T \} \) is allowed at time \( t \) to take only two values \( t \) (which corresponds to immediate exercise) and \( T \) (wait until maturity).

**Proposition 14.13.** For any stopping time \( \tau \geq t \) we have

\[
(x - K)^+ \leq \mathbb{E}^*[e^{-(\tau-t)r}(S_{\tau} - K)^+ | S_t = x], \quad x, t > 0. \tag{14.43}
\]

**Proof.** Since the function \( x \mapsto x^+ = \text{Max}(x, 0) \) is convex non-decreasing and the process \( (e^{-rt}S_t - e^{-rt}K)_{t \in \mathbb{R}_+} \) is a submartingale under \( \mathbb{P}^* \), we know that \( t \mapsto (e^{-rt}S_t - e^{-rt}K)^+ \) is a submartingale by the Jensen inequality (14.2), hence by (14.12), for any stopping time \( \tau \geq t \) we have

\[
(S_t - K)^+ = e^{rt}(e^{-rt}S_t - e^{-rt}K)^+
\leq e^{rt} \mathbb{E}^*[\left( e^{-rt}S_\tau - e^{-rt}K \right)^+ | \mathcal{F}_t]
= \mathbb{E}^*[e^{-(\tau-t)r}(S_\tau - K)^+ | \mathcal{F}_t],
\]

which yields (14.43). \qed

In particular, for the deterministic time \( \tau := T \geq t \) we get

\[
(x - K)^+ \leq e^{-(T-t)r} \mathbb{E}^*[\left( S_T - K \right)^+ | S_t = x], \quad x, t > 0.
\]

as illustrated in Figure 14.11 using the Black-Scholes formula for European call options. In other words, taking \( x = S_t \), the payoff \( (S_t - K)^+ \) of immediate exercise at time \( t \) is always lower than the expected payoff \( e^{-(T-t)r} \mathbb{E}^*[\left( S_T - K \right)^+ | S_t = x] \) given by exercise at maturity \( T \). As a consequence, the optimal strategy for the investor is to wait until time \( T \) to exercise an American call option, rather than exercising earlier at time \( t \).
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More generally, it can be shown that the price of the American call option equals the price of the corresponding European call option with maturity $T$, i.e.

$$f(t, S_t) = e^{-(T-t)r} \mathbb{E}^*[(S_T - K)^+ | S_t],$$

i.e. $T$ is the optimal exercise date, see Proposition 14.14 below or §14.4 of Steele (2001) for a proof.

**Put options**

For put options the situation is entirely different. The Black-Scholes formula for European put options shows that the inequality

$$(K - x)^+ \leq e^{-(T-t)r} \mathbb{E}^*[(K - S_T)^+ | S_t = x],$$

do not always hold, as illustrated in Figure 14.12.

As a consequence, the optimal exercise decision for a put option depends on whether $(K - S_t)^+ \leq e^{-(T-t)r} \mathbb{E}^*[(K - S_T)^+ | S_t]$ (in which case one
chooses to exercise at time \( T \) or \( (K - S_t)^+ > e^{-(T-t)r} \mathbb{E}^+[(K - S_T)^+ | S_t] \) (in which case one chooses to exercise at time \( t \)).

A view from above of the graph of Figure 14.12 shows the existence of an optimal frontier depending on time to maturity and on the price of the underlying asset instead of being given by a constant level \( L^* \) as in Section 14.4, cf. Figure 14.13:

Fig. 14.13: Optimal frontier for the exercise of a put option.

At a given time \( t \), one will choose to exercise immediately if \((S_t, T - t)\) belongs to the blue area on the right, and to wait until maturity if \((S_t, T - t)\) belongs to the red area on the left.

**PDE characterization of the finite expiration American put option price**

Let us describe the PDE associated to American put options. After discretization \( \{0 = t_0 < t_1 < \ldots < t_N = T\} \) of the time interval \([0, T]\), the optimal exercise strategy for the American put option can be described as follow at each time step:

- If \( f(t, S_t) > (K - S_t)^+ \), wait.
- If \( f(t, S_t) = (K - S_t)^+ \), exercise the option at time \( t \).

Note that we cannot have \( f(t, S_t) < (K - S_t)^+ \).

If \( f(t, S_t) > (K - S_t)^+ \) the expected return of the hedging portfolio equals that of the riskless asset. This means that \( f(t, S_t) \) follows the Black-Scholes PDE

\[
rf(t, S_t) = \frac{\partial f}{\partial t}(t, S_t) + rS_t \frac{\partial f}{\partial x}(t, S_t) + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 f}{\partial x^2}(t, S_t),
\]

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whereas if \( f(t, S_t) = (K - S_t)^+ \) it is not worth waiting as the return of the hedging portfolio is lower than that of the riskless asset:

\[
rf(t, S_t) \geq \frac{\partial f}{\partial t}(t, S_t) + rS_t \frac{\partial f}{\partial x}(t, S_t) + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 f}{\partial x^2}(t, S_t).
\]

As a consequence, \( f(t, x) \) should solve the following variational PDE:

\[
\begin{cases}
f(t, x) \geq (K - x)^+, \\ 
\frac{\partial f}{\partial t}(t, x) + rx \frac{\partial f}{\partial x}(t, x) + \frac{\sigma^2}{2} x^2 \frac{\partial^2 f}{\partial x^2}(t, x) \leq rf(t, x), \\ 
\left( \frac{\partial f}{\partial t}(t, x) + rx \frac{\partial f}{\partial x}(t, x) + \frac{\sigma^2}{2} x^2 \frac{\partial^2 f}{\partial x^2}(t, x) - rf(t, x) \right) \times (f(t, x) - (K - x)^+) = 0,
\end{cases}
\]

\( x > 0, 0 \leq t \leq T, \) subject to the terminal condition \( f(T, x) = (K - x)^+ \).

In other words, equality holds either in (14.44a) or in (14.44b) due to the presence of the term \( (f(t, x) - (K - x)^+) \) in (14.44c).

The optimal exercise strategy consists in exercising the put option as soon as the equality \( f(u, S_u) = (K - S_u)^+ \) holds, \( i.e. \) at the time

\[
\tau^* = \inf \{ u \geq t : f(u, S_u) = (K - S_u)^+ \},
\]

after which the process \( \tilde{f}_{L^*}(S_t) \) ceases to be a martingale and becomes a (strict) supermartingale.

A simple procedure to compute numerically the price of an American put option is to use a finite difference scheme while simply enforcing the condition \( f(t, x) \geq (K - x)^+ \) at every iteration by adding the condition

\[
f(t_i, x_j) := \text{Max}(f(t_i, x_j), (K - x_j)^+)
\]

right after the computation of \( f(t_i, x_j) \).

The next figure shows a numerical resolution of the variational PDE (14.44a)-(14.44c) using the above simplified (implicit) finite difference scheme, see also Jacka (1991) for properties of the optimal boundary function. In comparison with Figure 14.7, one can check that the PDE solution becomes time-dependent in the finite expiration case.
In general, one will choose to exercise the put option when

\[ f(t, S_t) = (K - S_t)^+ , \]

\textit{i.e.} within the blue area in Figure (14.14). We check that the optimal threshold \( L^* = 90.64 \) of the corresponding perpetual put option is within the exercise region, which is consistent since the perpetual optimal strategy should allow one to wait longer than in the finite expiration case.

The numerical computation of the American put option price

\[ f(t, S_t) = \sup_{t \leq \tau \leq T} \mathbb{E}^* \left[ e^{-(\tau-t)r} (K - S_\tau)^+ \mid S_t \right] \]

can also be done by dynamic programming and backward optimization using the Longstaff-Schwartz (or Least Square Monte Carlo, LSM) algorithm \cite{Longstaff2001}, as in Figure 14.15.
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In Figure 14.15 above and Figure 14.16 below the optimal threshold of the corresponding perpetual put option is again $L^* = 90.64$ and falls within the exercise region. Also, the optimal threshold is closer to $L^*$ for large time to maturities, which shows that the perpetual option approximates the finite expiration option in that situation. In the next Figure 14.16 we compare the numerical computation of the American put option price by the finite difference and Longstaff-Schwartz methods.

![Comparison between Longstaff-Schwartz and finite differences](image)

**Fig. 14.16:** Comparison between Longstaff-Schwartz and finite differences.

It turns out that, although both results are very close, the Longstaff-Schwartz method performs better in the critical area close to exercise at it yields the expected continuously differentiable solution, and the simple numerical PDE solution tends to underestimate the optimal threshold. Also, a small error in the values of the solution translates into a large error on the value of the optimal exercise threshold.

**The finite expiration American call option**

In the next proposition we compute the price of a finite expiration American call option with an arbitrary convex payoff function $\phi$.

**Proposition 14.14.** Let $\phi : \mathbb{R} \to \mathbb{R}$ be a nonnegative convex function such that $\phi(0) = 0$. The price of the finite expiration American call option with payoff function $\phi$ on the underlying asset price $(S_t)_{t \in \mathbb{R}_+}$ is given by

$$ f(t, S_t) = \sup_{t \leq \tau \leq T} \mathbb{E}^* \left[ e^{-(\tau-t)r} \phi(S_\tau) \mid S_t \right] = e^{-(T-t)r} \mathbb{E}^* \left[ \phi(S_T) \mid S_t \right], $$

i.e. the optimal strategy is to wait until the maturity time $T$ to exercise the option, and $\tau^* = T$.

**Proof.** Since the function $\phi$ is convex and $\phi(0) = 0$ we have

$$ \phi(px) = \phi((1-p) \times 0 + px) \leq (1-p) \times \phi(0) + p\phi(x) = p\phi(x), \quad (14.45) $$
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for all \( p \in [0,1] \) and \( x \geq 0 \). Hence the process \( s \mapsto e^{-rs}\phi(S_{t+s}) \) is a submartingale since taking \( p = e^{-rs} \) in (14.45) we have

\[
e^{-rs} \mathbb{E}^* \left[ \phi(S_{t+s}) \mid \mathcal{F}_t \right] \geq e^{-rs} \phi \left( \mathbb{E}^* \left[ S_{t+s} \mid \mathcal{F}_t \right] \right)
\]

\[
\geq \phi \left( e^{-rs} \mathbb{E}^* \left[ S_{t+s} \mid \mathcal{F}_t \right] \right) = \phi(S_t),
\]

where we used Jensen’s inequality (14.2) applied to the convex function \( \phi \).

Hence by the optional stopping theorem for submartingales, cf (14.8), for all (bounded) stopping times \( \tau \) comprised between \( t \) and \( T \) we have,

\[
\mathbb{E}^* \left[ e^{-(\tau-t)r} \phi(S_{\tau}) \mid \mathcal{F}_t \right] \leq e^{-(T-t)r} \mathbb{E}^* \left[ \phi(S_T) \mid S_t \right],
\]

i.e. it is always better to wait until time \( T \) than to exercise at time \( \tau \in [t, T] \), and this yields

\[
\sup_{\tau \text{ stopping time}} \mathbb{E}^* \left[ e^{-(\tau-t)r} \phi(S_{\tau}) \mid S_t \right] \leq e^{-(T-t)r} \mathbb{E}^* \left[ \phi(S_T) \mid S_t \right].
\]

The converse inequality

\[
e^{-(T-t)r} \mathbb{E}^* \left[ \phi(S_T) \mid S_t \right] \leq \sup_{\tau \text{ stopping time}} \mathbb{E}^* \left[ e^{-(\tau-t)r} \phi(S_{\tau}) \mid S_t \right],
\]

being obvious because \( T \) is a stopping time.

As a consequence of Proposition 14.14 applied to the convex function \( \phi(x) = (x-K)^+ \), the price of the finite expiration American call option is given by

\[
f(t, S_t) = \sup_{\tau \text{ stopping time}} \mathbb{E}^* \left[ e^{-(\tau-t)r} (S_{\tau} - K)^+ \mid S_t \right]
\]

\[
= e^{-(T-t)r} \mathbb{E}^* \left[ (S_T - K)^+ \mid S_t \right],
\]

i.e. the optimal strategy is to wait until the maturity time \( T \) to exercise the option. In the following Table 14.2 we summarize the optimal exercise strategies for the pricing of American options.

**Exercises**

**Exercise 14.1** Let \((B_t)_{t \in \mathbb{R}^+}\) be a standard Brownian motion started at 0, i.e. \(B_0 = 0\).

a) Is the process \( t \mapsto (2 - B_t)^+ \) a submartingale, a martingale or a supermartingale?
Table 14.2: Optimal exercise strategies.

<table>
<thead>
<tr>
<th>Option type</th>
<th>Perpetual</th>
<th>Finite expiration</th>
</tr>
</thead>
</table>
| Put option        | \[
\begin{align*}
    \tau^* &= \tau_L^*, \\
    (K - S_t) \left( \frac{S_t}{L^*} \right)^{-2r/\sigma^2}, & S_t \leq L^*, \\
    (K - L^*) \left( \frac{S_t}{L^*} \right)^{-2r/\sigma^2}, & S_t > L^*.
\end{align*}
\]
| Solve the PDE (14.44a)-(14.44c) for \( f(t, x) \) or use Longstaff-Schwartz Longstaff and Schwartz (2001). |
| Call option       | \( S_t \), \( \tau^* = +\infty \).          | \( e^{-(T-t)r} \mathbb{E}^*[(S_T - K)^+ | S_t], \quad \tau^* = T. \) |

b) Is the process \( (e^{B_t})_{t \in \mathbb{R}^+} \) a submartingale, a martingale, or a supermartingale?

c) Consider the random time \( \nu \) defined by
\[
\nu := \inf \{ t \in \mathbb{R}^+ : B_t = B_{2t} \},
\]
which represents the first time the curves \( (B_t)_{t \in \mathbb{R}^+} \) and \( (B_{2t})_{t \in \mathbb{R}^+} \) intersect.

Is \( \nu \) a stopping time?

d) Consider the random time \( \tau \) defined by
\[
\tau := \inf \{ t \in \mathbb{R}^+ : e^{B_t} = \left( \alpha + \beta t \right) e^{t/2} \},
\]
which represents the first time geometric Brownian motion \( B_t \) crosses the straight line \( t \mapsto \alpha + \beta t \). Is \( \tau \) a stopping time?

e) If \( \tau \) is a stopping time, compute \( \mathbb{E}[\tau] \) by the Doob Stopping Time Theorem 14.7 in each of the following two cases:

i) \( \alpha > 1 \) and \( \beta < 0 \),

ii) \( \alpha < 1 \) and \( \beta > 0 \).

Exercise 14.2 Stopping times. Let \( (B_t)_{t \in \mathbb{R}^+} \) be a standard Brownian motion started at 0.

a) Consider the random time \( \nu \) defined by
\[
\nu := \inf \{ t \in \mathbb{R}^+ : B_t = B_1 \},
\]
which represents the first time Brownian motion \( B_t \) hits the level \( B_1 \). Is \( \nu \) a stopping time?

b) Consider the random time \( \tau \) defined by
\[ \tau := \inf \{ t \in \mathbb{R}_+ : e^{B_t} = \alpha e^{-t/2} \}, \]

which represents the first time the exponential of Brownian motion \( B_t \) crosses the path of \( t \mapsto \alpha e^{-t/2} \), where \( \alpha > 1 \).

Is \( \tau \) a stopping time? If \( \tau \) is a stopping time, compute \( \mathbb{E}[e^{-\tau}] \) by applying the Stopping Time Theorem 14.7.

c) Consider the random time \( \tau \) defined by

\[ \tau := \inf \{ t \in \mathbb{R}_+ : B_t^2 = 1 + \alpha t \}, \]

which represents the first time the process \( (B_t^2)_{t \in \mathbb{R}_+} \) crosses the straight line \( t \mapsto 1 + \alpha t \), with \( \alpha < 1 \).

Is \( \tau \) a stopping time? If \( \tau \) is a stopping time, compute \( \mathbb{E}[\tau] \) by the Doob Stopping Time Theorem 14.7.

Exercise 14.3 Consider a standard Brownian motion \( (B_t)_{t \in \mathbb{R}_+} \) started at \( B_0 = 0 \), and let \( \tau_L = \inf \{ t \in \mathbb{R}_+ : B_t = L \} \) denote the first hitting time of level \( L > 0 \).

a) Compute the Laplace transform \( \mathbb{E}[e^{-r\tau_L}] \) of \( \tau_L \) for all \( r \geq 0 \).

*Hint:* Use the Stopping Time Theorem 14.7 and the fact that \( (e^{\sqrt{2}B_t - rt})_{t \in \mathbb{R}_+} \) is a martingale when \( r > 0 \).

b) Find the optimal level stopping strategy depending on the value of \( r > 0 \) for the maximization problem

\[ \sup_{L > 0} \mathbb{E} \left[ e^{-r\tau_L} B_{\tau_L} \right]. \]

Exercise 14.4 Consider \( (B_t)_{t \in \mathbb{R}_+} \) a Brownian motion started at \( B_0 = x \in [a, b] \) with \( a < b \), and let

\[ \tau := \inf \{ t \in \mathbb{R}_+ : B_t \in \{a, b\} \} \]

denote the first exit time of the interval \( [a, b] \). Show that the solution \( f(x) \) of the differential equation \( f''(x) = -2 \) with \( f(a) = f(b) = 0 \) satisfies \( f(x) = \mathbb{E}[\tau \mid B_0 = x] \).

*Hint:* Consider the process \( X_t := f(B_t) - \frac{1}{2} \int_0^t f''(B_s) \, ds \), and apply the Doob Stopping Time Theorem 14.7.
Exercise 14.5  (Doob-Meyer decomposition in discrete time). Let \((M_n)_{n \in \mathbb{N}}\) be a discrete-time submartingale with respect to a filtration \((\mathcal{F}_n)_{n \in \mathbb{N}}\), with \(\mathcal{F}_{-1} = \{\emptyset, \Omega\}\).

a) Show that there exists two processes \((N_n)_{n \in \mathbb{N}}\) and \((A_n)_{n \in \mathbb{N}}\) such that

i) \((N_n)_{n \in \mathbb{N}}\) is a martingale with respect to \((\mathcal{F}_n)_{n \in \mathbb{N}}\),

ii) \((A_n)_{n \in \mathbb{N}}\) is non-decreasing, i.e. \(A_n \leq A_{n+1}\) a.s., \(n \in \mathbb{N}\),

iii) \((A_n)_{n \in \mathbb{N}}\) is predictable in the sense that \(A_n\) is \(\mathcal{F}_{n-1}\)-measurable, \(n \in \mathbb{N}\), and

iv) \(M_n = N_n + A_n, n \in \mathbb{N}\).

Hint: Let \(A_0 := 0,\)

\[ A_{n+1} := A_n + \mathbb{E}[M_{n+1} - M_n | \mathcal{F}_n], \quad n \geq 0, \]

and define \((N_n)_{n \in \mathbb{N}}\) in such a way that it satisfies the four required properties.

b) Show that for all bounded stopping times \(\sigma\) and \(\tau\) such that \(\sigma \leq \tau\) a.s., we have

\[ \mathbb{E}[M_\sigma] \leq \mathbb{E}[M_\tau]. \]

Hint: Use the Stopping Time Theorem 14.7 for martingales and (14.6).

Exercise 14.6  Consider a two-step binomial model \((S_k)_{k=0,1,2}\) with interest rate \(r = 0\%\) and risk-neutral probability measure \((p^*, q^*)\):

\[
\begin{align*}
S_0 &= 1 & p^* &= 2/3 & S_1 &= 1.2 & p^* &= 2/3 & S_2 &= 1.44 \\
& & q^* &= 1/3 & S_1 &= 0.9 & q^* &= 1/3 & S_2 &= 0.81
\end{align*}
\]

a) At time \(t = 1\), would you exercise the American put option with strike price \(K = 1.25\) if \(S_1 = 1.2\)? If \(S_1 = 0.9\)?

b) What would be your strategy at time \(t = 0\)?

* Download the corresponding discrete-time Python notebook that can be run here.
Exercise 14.7 Let $r > 0$ and $\sigma > 0$.

a) Show that for every $C > 0$, the function $f(x) := Cx^{-2r/\sigma^2}$ solves the differential equation

$$rf(x) = rxf'(x) + \frac{1}{2}\sigma^2 x^2 f''(x),$$

$$\lim_{x \to \infty} f(x) = 0.$$ 

b) Show that for every $K > 0$ there exists a unique level $L^* \in (0, K)$ and constant $C > 0$ such that $f(x)$ also solves the smooth fit conditions $f(L^*) = K - L^*$ and $f'(L^*) = -1$.

Exercise 14.8 (Barone-Adesi and Whaley (1987)) We approximate the finite expiration American put option price with strike price $K$ as

$$f(x, T) \simeq \begin{cases} 
\text{BS}_p(x, T) + \alpha(x/S^*)^{-2r/\sigma^2}, & x > S^*, \\
K - x, & x \leq S^*, 
\end{cases}$$

(14.46) (14.47)

where $S^*$ is called the critical price and $\text{BS}_p(x, T) = e^{-rT}K\Phi(-d_-(x, T)) - x\Phi(-d_+(x, T))$ is the Black-Scholes put function.

a) Find the value of $\alpha$ by assuming a smooth fit (equality of derivatives in $x$) between (14.46) and (14.47) at $x = S^*$.

b) Derive the equation satisfied by the critical price $S^*$.

Exercise 14.9 Consider the process $(X_t)_{t \in \mathbb{R}_+}$ given by $X_t := t Z$, $t \in \mathbb{R}_+$, where $Z \in \{0, 1\}$ is a Bernoulli random variable with $\mathbb{P}(Z = 1) = \mathbb{P}(Z = 0) = 1/2$. Given $\epsilon \geq 0$, let the random time $\tau_\epsilon$ be defined as

$$\tau_\epsilon := \inf\{t > 0 : X_t > \epsilon\},$$

with $\inf \emptyset = +\infty$, and let $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ denote the filtration generated by $(X_t)_{t \in \mathbb{R}_+}$.

a) Give the possible values of $\tau_\epsilon$ in $[0, \infty]$ depending on the value of $Z$.

b) Take $\epsilon = 0$. Is $\tau_0 := \inf\{t > 0 : X_t > 0\}$ an $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$-stopping time? 

*Hint:* Consider the event $\{\tau_0 > 0\}$.

c) Take $\epsilon > 0$. Is $\tau_\epsilon := \inf\{t > 0 : X_t > \epsilon\}$ an $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$-stopping time? 

*Hint:* Consider the event $\{\tau_\epsilon > t\}$ for $t \geq 0$.

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\[ S_t = S_0 e^{(r-\delta)t + \sigma \bar{B}_t - \sigma^2 t/2}, \quad t \in \mathbb{R}_+, \]

where \( r > 0 \) is the risk-free interest rate, \( \delta \geq 0 \) is a continuous dividend rate, \((\bar{B}_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion under the risk-neutral probability measure \( \mathbb{P}^* \), and \( \sigma > 0 \) is the volatility coefficient. Consider the american put option with payoff

\[
(k - S_\tau)^+ = \begin{cases}
    k - S_\tau & \text{if } S_\tau \leq k, \\
    0 & \text{if } S_\tau > k,
\end{cases}
\]

when exercised at the stopping time \( \tau > 0 \). Given \( L \in (0, \kappa) \) a fixed level, consider the following exercise strategy for the above option:

- If \( S_t \leq L \), then exercise at time \( t \).
- If \( S_t > L \), wait until the hitting time \( \tau_L := \inf\{u \geq t : S_u = L\} \), and exercise the option at time \( \tau_L \).

a) Give the intrinsic option value at time \( t = 0 \) in case \( S_0 \leq L \).

In the sequel we work with \( S_0 = x > L \).

b) Show that for all \( \lambda \in \mathbb{R} \) the process \( (Z_t^{(\lambda)})_{t \in \mathbb{R}_+} \) defined as

\[
Z_t^{(\lambda)} := \left( \frac{S_t}{S_0} \right)^\lambda e^{-(r-\delta)\lambda + \lambda(1-\lambda)\sigma^2/2} t
\]

is a martingale under the risk-neutral probability measure \( \mathbb{P}^* \).

c) Show that \( (Z_t^{(\lambda)})_{t \in \mathbb{R}_+} \) can be rewritten as

\[
Z_t^{(\lambda)} = \left( \frac{S_t}{S_0} \right)^\lambda e^{-rt}, \quad t \in \mathbb{R}_+,
\]

for two values \( \lambda_- \leq 0 \leq \lambda_+ \) of \( \lambda \) that can be computed explicitly.

d) Choosing the negative solution \( \lambda_- \), show that

\[
0 \leq Z_t^{(\lambda_-)} = \left( \frac{S_t}{S_0} \right)^{\lambda_-} e^{-rt} \leq \left( \frac{L}{S_0} \right)^{\lambda_-}, \quad 0 \leq t < \tau_L.
\]

e) Let \( \tau_L \) denote the hitting time

\[
\tau_L = \inf\{u \in \mathbb{R}_+ : S_u \leq L\}.
\]

By application of the Stopping Time Theorem 14.7 to the martingale \( (Z_t)_{t \in \mathbb{R}_+} \), show that
\[ \mathbb{E}^* \left[ e^{-r \tau_L} \right] = \left( \frac{S_0}{L} \right)^{\lambda_-}, \quad (14.48) \]

with
\[ \lambda_- := \frac{-(r - \delta - \sigma^2/2) - \sqrt{(r - \delta - \sigma^2/2)^2 + 4r\sigma^2/2}}{\sigma^2}. \quad (14.49) \]

f) Show that for all \( L \in (0, K) \) we have
\[
\mathbb{E}^* \left[ e^{-r \tau_L} (K - S_{\tau_L})^+ \mid S_0 = x \right]
= \begin{cases} 
K - x, & 0 < x \leq L, \\
(K - L) \left( \frac{x}{L} \right)^{-(r - \delta - \sigma^2/2) - \sqrt{(r - \delta - \sigma^2/2)^2 + 4r\sigma^2/2}} / \sigma^2, & x \geq L.
\end{cases}
\]

g) Show that the value \( L^* \) of \( L \) that maximizes
\[ f_L(x) := \mathbb{E}^* \left[ e^{-r \tau_L} (K - S_{\tau_L})^+ \mid S_0 = x \right] \]
for every \( x > 0 \) is given by
\[ L^* = \frac{\lambda_-}{\lambda_- - 1} K. \]

h) Show that
\[
f_{L^*}(x) = \begin{cases} 
K - x, & 0 < x \leq L^* = \frac{\lambda_-}{\lambda_- - 1} K, \\
\left( \frac{1 - \lambda_-}{K} \right)^{\lambda_- - 1} \left( \frac{x}{-\lambda_-} \right)^{\lambda_-}, & x \geq L^* = \frac{\lambda_-}{\lambda_- - 1} K,
\end{cases}
\]

i) Show by hand computation that \( f_{L^*}(x) \) satisfies the variational differential equation
\[
\begin{cases} 
f_{L^*}(x) \geq (K - x)^+, \quad (14.50a) \\
(r - \delta) x f'_{L^*}(x) + \frac{1}{2} \sigma^2 x^2 f''_{L^*}(x) \leq r f_{L^*}(x), \quad (14.50b) \\
\left( r f_{L^*}(x) - (r - \delta) x f'_{L^*}(x) - \frac{1}{2} \sigma^2 x^2 f''_{L^*}(x) \right) \times (f_{L^*}(x) - (K - x)^+) = 0. \quad (14.50c)
\end{cases}
\]
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j) Using Itô’s formula, check that the discounted portfolio price

\[ t \mapsto e^{-rt} f_{L^*}(S_t) \]

is a supermartingale.

k) Show that we have

\[ f_{L^*}(S_0) \geq \sup_{\tau \text{ stopping time}} \mathbb{E}^*[e^{-r\tau} (K - S_\tau)^+] | S_0]. \]

l) Show that the stopped process

\[ s \mapsto e^{-(s \wedge \tau_{L^*})r} f_{L^*}(S_{s \wedge \tau_{L^*}}), \quad s \in \mathbb{R}_+, \]

is a martingale, and that

\[ f_{L^*}(S_0) \leq \sup_{\tau \text{ stopping time}} \mathbb{E}^*[e^{-r\tau} (K - S_\tau)^+] \]

m) Fix \( t \in \mathbb{R}_+ \) and let \( \tau_{L^*} \) denote the hitting time

\[ \tau_{L^*} = \inf\{u \geq t : S_u = L^*\}. \]

Conclude that the price of the perpetual American put option with dividend is given for all \( t \in \mathbb{R}_+ \) by

\[
f_{L^*}(S_t) = \mathbb{E}^*[e^{-(\tau_{L^*} - t)r} (K - S_{\tau_{L^*}})^+] | S_t]
\]

\[ = \begin{cases} 
K - S_t, & 0 < S_t \leq \frac{\lambda_-}{\lambda_- - 1} K, \\
\left(1 - \frac{\lambda_-}{K}\right) \frac{\lambda_-}{\lambda_- - 1} \left(\frac{S_t}{-\lambda_-}\right)^{\lambda_-}, & S_t \geq \frac{\lambda_-}{\lambda_- - 1} K,
\end{cases} \]

where \( \lambda_- < 0 \) is given by (14.49), and

\[ \tau_{L^*} = \inf\{u \geq t : S_u \leq L\}. \]

Exercise 14.11 American call options with dividends, see § 9.3 of Wilmott (2006). Consider a dividend-paying asset priced as \( S_t = S_0 e^{(r - \delta)t + \sigma \tilde{B}_t - \frac{\sigma^2}{2} t}, \) \( t \in \mathbb{R}_+ \), where \( r > 0 \) is the risk-free interest rate, \( \delta \geq 0 \) is a continuous dividend rate, and \( \sigma > 0 \).

a) Show that for all \( \lambda \in \mathbb{R} \) the process \( Z_t^{(\lambda)} := (S_t)^\lambda e^{-(r - \delta)\lambda - \lambda(1 - \lambda)\sigma^2/2) t} \) is a martingale under \( \mathbb{P}^* \).

b) Show that we have \( Z_t^{(\lambda)} = (S_t)^\lambda e^{-r t} \) for two values \( \lambda_- \leq 0, 1 \leq \lambda_+ \) of \( \lambda \) satisfying a certain equation.
c) Show that $0 \leq Z_t^{(\lambda_+)} \leq L^{\lambda_+}$ for $0 \leq t < \tau_L := \inf \{ u \geq t : S_u = L \}$, and compute $\mathbb{E}^*\left[ e^{-r\tau_L} (S_{\tau_L} - K)^+ \mid S_0 = x \right]$ when $S_0 = x < L$ and $K < L$.

Exercise 14.12 Optimal stopping for exchange options (Gerber and Shiu (1996)). We consider two risky assets $S_1$ and $S_2$ modeled by

$$S_1(t) = S_1(0) e^{\sigma_1 W_t + rt - \sigma_2^2 t/2} \quad \text{and} \quad S_2(t) = S_2(0) e^{\sigma_2 W_t + rt - \sigma_2^2 t/2},$$

$t \in \mathbb{R}_+$, with $\sigma_2 > \sigma_1 \geq 0$, and the perpetual optimal stopping problem

$$\sup_{\tau \text{ stopping time}} \mathbb{E}[e^{-r\tau} (S_1(\tau) - S_2(\tau))^+] ,$$

where $(W_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}$.

a) Find $\alpha > 1$ such that the process

$$Z_t := e^{-rt} S_1(t)^\alpha S_2(t)^{1-\alpha}, \quad t \in \mathbb{R}_+,$$

(14.52)

is a martingale.

b) For some fixed $L \geq 1$, consider the hitting time

$$\tau_L = \inf \{ t \in \mathbb{R}_+ : S_1(t) \geq LS_2(t) \},$$

and show that

$$\mathbb{E}[e^{-r\tau_L} (S_1(\tau_L) - S_2(\tau_L))^+] = (L - 1) \mathbb{E}[e^{-r\tau_L} S_2(\tau_L)].$$

c) By an application of the Stopping Time Theorem 14.7 to the martingale (14.52), show that we have

$$\mathbb{E}[e^{-r\tau_L} (S_1(\tau_L) - S_2(\tau_L))^+] = \frac{L - 1}{L^\alpha} S_1(0)^\alpha S_2(0)^{1-\alpha}.$$

d) Show that the price of the perpetual exchange option is given by

$$\sup_{\tau \text{ stopping time}} \mathbb{E}[e^{-r\tau} (S_1(\tau) - S_2(\tau))^+] = \frac{L^* - 1}{(L^*)^\alpha} S_1(0)^\alpha S_2(0)^{1-\alpha},$$

where

$$L^* = \frac{\alpha}{\alpha - 1}.$$

e) As an application of Question (d), compute the perpetual American put option price

$$\sup_{\tau \text{ stopping time}} \mathbb{E}[e^{-r\tau} (\kappa - S_2(\tau))^+]$$
when \( r = \sigma^2 / 2 \).

Exercise 14.13 Consider an underlying asset whose price is written as

\[
S_t = S_0 e^{r t + \sigma B_t - \sigma^2 t / 2}, \quad t \in \mathbb{R}_+,
\]

where \((B_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion under the risk-neutral probability measure \(\mathbb{P}^*\), \(\sigma > 0\) denotes the volatility coefficient, and \(r \in \mathbb{R}\) is the risk-free interest rate. For any \(\lambda \in \mathbb{R}\) we consider the process \((Z_t(\lambda))_{t \in \mathbb{R}_+}\) defined by

\[
Z_t(\lambda) := e^{-rt}(S_t)^\lambda = (S_0)^\lambda e^{\lambda \sigma B_t - \lambda^2 \sigma^2 t / 2 + (\lambda - 1)(\lambda + 2r / \sigma^2) \sigma^2 t / 2}, \quad t \in \mathbb{R}_+. \tag{14.53}
\]

a) Assume that \(r \geq -\sigma^2 / 2\). Show that, under \(\mathbb{P}^*\), the process \((Z_t(\lambda))_{t \in \mathbb{R}_+}\) is a supermartingale when \(-2r / \sigma^2 \leq \lambda \leq 1\), and that it is a submartingale when \(\lambda \in (-\infty, -2r / \sigma^2] \cup [1, \infty)\).

b) Assume that \(r \leq -\sigma^2 / 2\). Show that, under \(\mathbb{P}^*\), the process \((Z_t(\lambda))_{t \in \mathbb{R}_+}\) is a supermartingale when \(1 \leq \lambda \leq -2r / \sigma^2\), and that it is a submartingale when \(\lambda \in (-\infty, 1] \cup [-2r / \sigma^2, \infty)\).

c) From now on we assume that \(r < 0\). Given \(L > 0\), let \(\tau_L\) denote the hitting time

\[
\tau_L = \inf\{u \in \mathbb{R}_+ : S_u = L\}.
\]

By application of the Stopping Time Theorem 14.7 to \((Z_t(\lambda))_{t \in \mathbb{R}_+}\) to suitable values of \(\lambda\), show that

\[
\mathbb{E}^*\left[ e^{-r \tau_L} \mathbb{1}_{\{\tau_L < \infty\}} \mid S_0 = x \right] \leq \begin{cases} \frac{x}{L} \max(1, -2r / \sigma^2), & x \geq L, \\ \frac{x}{L} \min(1, -2r / \sigma^2), & 0 < x \leq L. \end{cases}
\]

d) Deduce an upper bound on the price

\[
\mathbb{E}^*\left[ e^{-r \tau_L} (K - S_{\tau_L})^+ \mid S_0 = x \right]
\]

of a European put option exercised in finite time under the stopping strategy \(\tau_L\) when \(L \in (0, K)\) and \(x \geq L\).

e) Show that when \(r \leq -\sigma^2 / 2\), the upper bound of Question (d) increases and tends to \(+\infty\) when \(L\) decreases to 0.

f) Find an upper bound on the price
of a European call option exercised in finite time under the stopping strat-
egy \tau_L when \( L \geq K \) and \( x \leq L \).
g) Show that when \(-\sigma^2/2 \leq r < 0\), the upper bound of Question (f) increases in \( L \geq K \) and tends to \( S_0 \) as \( L \) increases to \(+\infty\).


a) Compute the price

\[
C_{b Am}^\tau (t, S_t) = \sup_{\tau \geq t} \mathbb{E}^* \left[ e^{-(\tau-t)r} 1_{\{S_{\tau} \geq K\}} \mid S_t \right]
\]

of the perpetual American binary call option.
b) Compute the price

\[
P_{b Am}^\tau (t, S_t) = \sup_{\tau \geq t} \mathbb{E}^* \left[ e^{-(\tau-t)r} 1_{\{S_{\tau} \leq K\}} \mid S_t \right]
\]

of the perpetual American binary put option.

Exercise 14.15 Finite expiration American binary options. An American binary (or digital) call (resp. put) option with maturity \( T > 0 \) can be exer-
cised at any time \( t \in [0, T] \), at the choice of the option holder.

The call (resp. put) option exercised at time \( t \) yields the payoff \( 1_{[K,\infty)}(S_t) \) (resp. \( 1_{[0,K]}(S_t) \)), and the option holder wants to find an exercise strategy that will maximize his payoff.

a) Consider the following possible situations at time \( t \):

i) \( S_t \geq K \),
ii) \( S_t < K \).

In each case (i) and (ii), tell whether you would choose to exercise the call option immediately, or to wait.
b) Consider the following possible situations at time \( t \):

i) \( S_t > K \),
ii) \( S_t \leq K \).

In each case (i) and (ii), tell whether you would choose to exercise the put option immediately, or to wait.
c) The price \( C_d Am(T, S_t) \) of an American binary call option is known to satisfy the Black-Scholes PDE

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\[ rC_d^{Am}(t, T, x) = \frac{\partial C_d^{Am}}{\partial t}(t, T, x) + rx \frac{\partial C_d^{Am}}{\partial x}(t, T, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 C_d^{Am}}{\partial x^2}(t, T, x). \]

Based on your answers to Question (a), how would you set the boundary conditions \( C_d^{Am}(t, T, K), 0 \leq t < T, \) and \( C_d^{Am}(T, T, x), 0 \leq x < K? \)

d) The price \( P_d^{Am}(t, T, S_t) \) of an American binary put option is known to satisfy the same Black-Scholes PDE

\[ rP_d^{Am}(t, T, x) = \frac{\partial P_d^{Am}}{\partial t}(t, T, x) + rx \frac{\partial P_d^{Am}}{\partial x}(t, T, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 P_d^{Am}}{\partial x^2}(t, T, x). \]

Based on your answers to Question (b), how would you set the boundary conditions \( P_d^{Am}(t, T, K), 0 \leq t < T, \) and \( P_d^{Am}(T, T, x), x > K? \)

e) Show that the optimal exercise strategy for the American binary call option with strike price \( K \) is to exercise as soon as the price of the underlying asset reaches the level \( K, \) at the time

\[ \tau_K = \inf \{ u \geq t : S_u = K \}, \]

starting from any level \( S_t \leq K, \) and that the price \( C_d^{Am}(t, T, S_t) \) of the American binary call option is given by

\[ C_d^{Am}(t, x) = \mathbb{E}[e^{-(\tau_K-t)r} \mathbb{1}_{\{\tau_K<T\}} | S_t = x]. \]

f) Show that the price \( C_d^{Am}(t, T, S_t) \) of the American binary call option is equal to

\[ C_d^{Am}(t, T, x) = \frac{x}{K} \Phi \left( \frac{(r + \sigma^2/2)(T - t) + \log(x/K)}{\sigma \sqrt{T - t}} \right) + \left( \frac{x}{K} \right)^{-2r/\sigma^2} \Phi \left( \frac{-(r + \sigma^2/2)(T - t) + \log(x/K)}{\sigma \sqrt{T - t}} \right), \quad 0 \leq x \leq K. \]

Show that this formula is consistent with the answer to Question (c), and that it recovers the answer to Question (a) of Exercise 14.14 as \( T \) tends to infinity.

g) Show that the optimal exercise strategy for the American binary put option with strike price \( K \) is to exercise as soon as the price of the underlying asset reaches the level \( K, \) at the time

\[ \tau_K = \inf \{ u \geq t : S_u = K \}, \]

starting from any level \( S_t \geq K, \) and that the price \( P_d^{Am}(t, T, S_t) \) of the American binary put option is

\[ P_d^{Am}(t, T, x) = \mathbb{E}[e^{-(\tau_K-t)r} \mathbb{1}_{\{\tau_K<T\}} | S_t = x], \quad x \geq K. \]
h) Show that the price $P^\text{Am}(t,T,S_t)$ of the American binary put option is equal to

$$
P^\text{Am}(t,T,x) = \frac{x}{K} \Phi\left( \frac{-(r + \sigma^2/2)\tau - \log(x/K)}{\sigma\sqrt{\tau}} \right) + \left( \frac{x}{K} \right)^{-2r/\sigma^2} \Phi\left( \frac{(r + \sigma^2/2)\tau - \log(x/K)}{\sigma\sqrt{\tau}} \right), \quad x \geq K,
$$

and that this formula is consistent with the answer to Question (d), and that it recovers the answer to Question (b) of Exercise 14.14 as $T$ tends to infinity.

i) Does the usual call-put parity relation hold for American binary options?

Exercise 14.16 American forward contracts. Consider $(S_t)_{t \in \mathbb{R}_+}$ an asset price process given by

$$
\frac{dS_t}{S_t} = rdt + \sigma dB_t,
$$

where $r > 0$ and $(B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion.

a) Compute the price

$$
f(t,S_t) = \sup_{t \leq \tau \leq T} \mathbb{E}^* \left[ e^{-(\tau-t)r} (K - S_\tau) \mid S_t \right],
$$

and optimal exercise strategy of a finite expiration American-type short forward contract with strike price $K$ on the underlying asset priced $(S_t)_{t \in \mathbb{R}_+}$, with payoff $K - S_T$.

b) Compute the price

$$
f(t,S_t) = \sup_{t \leq \tau \leq T} \mathbb{E}^* \left[ e^{-(\tau-t)r} (S_\tau - K) \mid S_t \right],
$$

and optimal exercise strategy of a finite expiration American-type long forward contract with strike price $K$ on the underlying asset priced $(S_t)_{t \in \mathbb{R}_+}$, with payoff $S_T - K$.

c) How are the answers to Questions (a) and (b) modified in the case of perpetual options?

Exercise 14.17 Consider an underlying asset price process written as

$$
S_t = S_0 e^{r t + \sigma \hat{B}_t - \sigma^2 t / 2}, \quad t \in \mathbb{R}_+,
$$

where $(\hat{B}_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$, with $\sigma, r > 0$. 
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a) Show that the processes \((Y_t)_{t \in \mathbb{R}_+}\) and \((Z_t)_{t \in \mathbb{R}_+}\) defined as

\[ Y_t := e^{-rt} S_t^{-2r/\sigma^2} \quad \text{and} \quad Z_t := e^{-rt} S_t, \quad t \in \mathbb{R}_+, \]

are both martingales under \(\mathbb{P}^*\).

b) Let \(\tau_L\) denote the hitting time

\[ \tau_L = \inf\{u \in \mathbb{R}_+ : S_u = L\}. \]

By application of the Stopping Time Theorem 14.7 to the martingales \((Y_t)_{t \in \mathbb{R}_+}\) and \((Z_t)_{t \in \mathbb{R}_+}\), show that

\[ \mathbb{E}^{*}[e^{-r\tau_L} | S_0 = x] = \begin{cases} x/L, & 0 < x \leq L, \\ (x/L)^{-2r/\sigma^2}, & x \geq L. \end{cases} \]

c) Compute the price \(\mathbb{E}^{*}[e^{-r\tau_L} (K - S_{\tau_L})]\) of a short forward contract under the exercise strategy \(\tau_L\).

d) Show that for every value of \(S_0 = x\) there is an optimal value \(L^*_x\) of \(L\) that maximizes \(L \mapsto \mathbb{E}[e^{-r\tau_L} (K - S_{\tau_L})]\).

e) Would you use the stopping strategy

\[ \tau_{L^*_x} = \inf\{u \in \mathbb{R}_+ : S_u = L^*_x\} \]

as an optimal exercise strategy for the short forward contract with payoff \(K - S_\tau\)?

Exercise 14.18 Let \(p \geq 1\) and consider a power put option with payoff

\[ (\left(\kappa - S_\tau\right)^+)^p = \begin{cases} \left(\kappa - S_\tau\right)^p & \text{if } S_\tau \leq \kappa, \\ 0 & \text{if } S_\tau > \kappa, \end{cases} \]

exercised at time \(\tau\), on an underlying asset whose price \(S_t\) is written as

\[ S_t = S_0 e^{rt + \sigma B_t - \sigma^2 t/2}, \quad t \in \mathbb{R}_+, \]

where \((B_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion under the risk-neutral probability measure \(\mathbb{P}^*\), \(r \geq 0\) is the risk-free interest rate, and \(\sigma > 0\) is the volatility coefficient.

Given \(L \in (0, \kappa)\) a fixed price, consider the following choices for the exercise of a put option with strike price \(\kappa\):

i) If \(S_t \leq L\), then exercise at time \(t\).
ii) Otherwise, wait until the first hitting time \( \tau_L := \inf\{u \geq t : S_u = L\} \), and exercise the option at time \( \tau_L \).

a) Under the above strategy, what is the option payoff equal to if \( S_t \leq L \)?

b) Show that in case \( S_t > L \), the price of the option is equal to

\[
\begin{align*}
f_L(S_t) &= (\kappa - L)^p \mathbb{E}^* \left[ e^{-(\tau_L - t)r} \mid S_t \right].
\end{align*}
\]

c) Compute the price \( f_L(S_t) \) of the option at time \( t \).

\textbf{Hint.} Recall that by (14.21) we have \( \mathbb{E}^* \left[ e^{-(\tau_L - t)r} \mid S_t = x \right] = \left( x/L \right)^{-2r/\sigma^2} \), \( x \geq L \).

d) Compute the optimal value \( L^* \) that maximizes \( L \mapsto f_L(x) \) for all fixed \( x > 0 \).

\textbf{Hint.} Observe that, geometrically, the slope of \( x \mapsto f_L(x) \) at \( x = L^* \) is equal to \( -p(\kappa - L^*)^{p-1} \).

e) How would you compute the American put option price

\[
\begin{align*}
f(t, S_t) &= \sup_{\tau \geq t} \mathbb{E}^* \left[ e^{-(\tau - t)r} ((\kappa - S_\tau)^+)^p \mid S_t \right].
\end{align*}
\]

Exercise 14.19 Same questions as in Exercise 14.18, this time for the option with payoff \( \kappa - (S_\tau)^p \) exercised at time \( \tau \), with \( p > 0 \).