Submitted on August 8, 2018 to the Notre Dame Journal of Formal Logic
Volume ??, Number ??,

On the degree structure of equivalence relations under computable reducibility

Keng Meng Ng and Hongyuan Yu

Abstract We study the degree structure of the ω-c.e., n-c.e. and Π^0_1 equivalence relations under the computable many-one reducibility. In particular we investigate for each of these classes of degrees the most basic questions about the structure of the partial order. We prove the existence of the greatest element for the ω-c.e. and n-c.e. equivalence relations. We provide computable enumerations of the degrees of ω-c.e., n-c.e. and Π^0_1 equivalence relations. We prove that for all the degree classes considered, upward density holds and downward density fails.

1 Introduction

An important theme in the study of mathematics is the classification of mathematical structures according to various different criteria. If we can identify when two different structures are "equivalent", then we could reduce the study of all structures in a certain class to just examining the essential properties in each equivalence class of structures. The notion of dimension in algebra is an example of such a classification. A common tool in this endeavour is to define a *reduction*; this is a map which reduces a (possibly) more complicated problem to a simpler question.

In the context of Borel theory, a large body of work exists on classifying problems via the Borel reducibility. These reducibilities focus on classifying equivalence relations on an uncountable domain, for example, Friedman and Stanley [7] applied this to study the isomorphism problem on the countable models of a theory.

A more recent direction of investigation has been directed towards studying equivalence relations on the domain ω. By restricting the domain down to a countable set, the usual tools of computability theory can be applied to supply meaningful definitions, concepts and proof techniques. The main

Keywords: Degrees, equivalence relations, density, effective enumerations, universality

reducibility used to calibrate the relative complexity of equivalence relations on ω is *computable reducibility*:

Definition 1.1 *Let E and F be equivalence relations on ω. We say that E is computably reducible to F, denoted as $E \leq_c F$, if there exists a computable function $f : \omega \to \omega$ such that*

$$\forall x, y \in \omega, \quad (x, y) \in E \Leftrightarrow (f(x), f(y)) \in F.$$

This definition is obviously closely related to the Borel reducibility; in the Borel case one studies equivalence relations on 2^ω and we require the reductions to be Borel, or sometimes merely continuous. Indeed, Coskey, Hamkins and Miller [4] and later Miller and Ng [11] studied the countable analogues of several standard equivalence relations arising in Borel theory. This reducibility has gone by many different names in the literature, having been called m-reducibility in [1, 8, 2] and FF-reducibility in [6], in addition to a version on first-order theories which was called Turing-computable reducibility [13, 3]. Computable reducibility was also used in [9, 11]; we prefer this name since we wish to avoid confusion with the usual m-reducibility and Turing-reducibility between sets of numbers.

The computable reducibility was first introduced by Ershov in [5]. However his approach was much more indirect in that he phrased the notions inside the language of categories. Indeed his motivation was to study the theory of numberings, which is of fundamental interest in computable mathematics. In recent years there has been a sudden increase in interest in the computable reducibility between equivalence relations on ω.

The objective of this paper is to initiate a systematic study of the degrees induced by the computable reducibility: The preordering $\leq_c$ induces an equivalence relation on the class of all equivalence relations. The class of all R that are c-equivalent to S is called the c-degree of S, denoted as $deg_c(S)$. In this paper we shall be mainly interested in the α-c.e. equivalence relations for $\alpha \leq \omega$ and the Π^0_1 equivalence relations; the descriptive complexity of an equivalence relation is obviously evaluated when it is viewed as a subset of $\mathbb{N} \times \mathbb{N}$. A c-degree is α-c.e. if it contains an α-c.e. member.

The concept of *degrees* has been a central notion since the dawn of computability theory, and much effort has been directed towards the understanding of the algebraic structure of the degrees of different classes of sets. For instance, the c.e. m-degrees, 1-degrees, (weak) truth table degrees and Turing degrees have all been well-studied. The original view that many people had of these structures were that they would be very simple and easy to classify. However, it turns out that these structures, particularly the c.e. Turing degrees, are much more complicated than originally thought. The study of the Turing degrees of larger classes of sets have also been extensively carried out, for instance, the d-c.e. Turing degrees and the Δ^0_2 Turing degrees have both turned out to be of interest, and shown to be structurally very different from the c.e. Turing degrees. Our aim here is not to dwell upon these results as these are all readily available in the literature. Rather, we wish to point out that this earlier work serves as the primary motivation for the questions we wish to investigate in this paper.

One of the most basic questions to ask of a partial order is the existence of a greatest element. Given a class $\mathcal{C}$ of c-degrees, we call $\boldsymbol{u} \in \mathcal{C}$ *universal* if $\boldsymbol{u} \geq \boldsymbol{a}$ for every c-degree $\boldsymbol{a} \in \mathcal{C}$. We note here that the question of a *least* c-degree is uninteresting, for the computable equivalence relations generate an initial segment of c-degrees of length $\omega + 1$.

The question of universality has already been studied for some classes of c-degrees. For instance, Lachlan [10] studied certain kinds of universal c.e. equivalence relations (ceers), called precomplete ceers. We remark that ceers are also known as positive equivalence relations, since they are closely related to positive numberings. Gao and Gerdes [8] and Andrews et al. [1] studied the c.e. c-degrees extensively, particularly in [1], where different types of universal c.e. c-degrees were identified and compared. The question of universality was then studied further in [9], where the authors showed the existence of a universal Π^0_1 c-degree, and that no universal degree exists for the class of Δ^0_2 c-degrees.

The notion of a universal element of a class is closely related to the existence of a computable enumeration of the members of that class. If one can effectively list out all elements of the class, then taking the disjoint union of the members of the effective list will usually produce a universal element of the class. For instance, the Halting problem, which has universal m-degree amongst the c.e. sets, can be viewed as being the disjoint union of all c.e. sets (we can even effectively list all c.e. sets without repetition, a famous result of Friedberg). Indeed, the existence of a universal c.e. c-degree can be easily proved in the same way; first, notice that the transitive closure of any c.e. binary relation is again c.e., and hence we can easily produce an effective list containing all c.e. equivalence relations. Taking the disjoint union of this list (with no relations between different members of the list) produces a universal c.e. equivalence relation.

The same trick works to produce a universal Σ^0_n c-degree for each $n > 0$. Unfortunately, an effective listing of all Π^0_1 equivalence relations is less obvious. Indeed, the authors of [9] proved the existence of a universal Π^0_1 c-degree in an indirect way. For α-c.e. equivalence relations, where $1 < \alpha \leq \omega$, the question of universality was not known before this article. In Section 2, we answer the question of universality in these classes and also investigate what kinds of effective enumerations are possible for these classes. The results are summarized in the table on page 4.

Another fundamental question one can ask about a partial order is whether the structure is dense. Interest in this particular property is partially motivated by results in the c.e. and Δ^0_2 Turing degrees, where similar questions about density have led to a rich collection of knowledge and to new techniques being invented. Related questions such as those on lattice embeddings have also enriched the field. We will investigate the question of density in the classes of c-degrees we are interested in. We prove that in each of these classes (α-c.e. for $1 \leq \alpha \leq \omega$, Π^0_1 and Δ^0_2 c-degrees), downward density fails but upward density holds. We believe that this is due to the fact that it is much easier to *code* given information into a relation we build, compared to *permitting* below a given equivalence relation.

In Section 3 we consider the problem of downward density in the structure of c.e. c-degrees. It is not hard to see that the map $A \mapsto \{(x,y) \mid x = y$ or $x, y \in A\}$ is an embedding (of partial orders) of the c.e. 1-degrees into the c.e. c-degrees. We call a c-degree *set-induced* if it contains a relation in the range of this embedding. An old result of Lachlan implies the existence of a set-induced c-degree for which downward density fails. These results and terminologies will be further elaborated upon in Section 3. We construct two different counter-examples to the downward density of the c.e. c-degrees, one which is above $deg_c(\texttt{id})$, and another which is not above $deg_c(\texttt{id})$. Both our counter-examples are, of course, not set-induced.

In Section 4 we show that downward density fails for the Π^0_1 c-degrees, and generalize this theorem by constructing an infinite sequence of Π^0_1 c-degrees $deg_c(\texttt{id}) < deg_c(A_2) < deg_c(A_3) < \cdots$, each of which is a strong minimal cover over the previous. All our counter-examples are not set-induced.

In Section 5 we investigate upward density in the c-degrees. We show that in contrast to downward density, upward density holds in all the classes of interest. We do this by constructing a pair of incomparable Π^0_1 c-degrees above any given non-universal Π^0_1 c-degree.

Degree class	Existence of a universal element	Existence of a computable enumeration	Downward density	Upward density
Σ^0_n, $n \geq 1$	✓ (Folklore)	✓ (Folklore)	× (Cor 3.10)	✓ (Cor 5.5)
n-c.e., $n > 1$	✓ (Cor 2.2)	✓ (Thm 2.1)	× (Cor 3.10)	✓ (Cor 5.5)
ω-c.e.	✓ (Cor 2.6)	– of all relations × (Folklore) – of all degrees ✓(Thm 2.5)	× (Cor 3.10)	✓ (Cor 5.5)
Π^0_1	✓ (Thm 3.3 of [9])	– of all relations × (Prop 2.10) – of all degree upper cones ✓(Thm 2.8)	× (Thm 4.1)	✓ (Thm 5.1)
Δ^0_2	× (Thm 3.7 of [9])	× (no universal element)	× (Cor 3.10)	✓ (Thm 3.7 of [9])

1.1 Preliminaries Our notation is standard, and for these and other computability theory concepts, we refer the reader to [12]. For the rest of this paper, an (equivalence) relation S always refers to a subset of ordered pairs, i.e. $S \subseteq \mathbb{N}^2$, and the domain of each relation is always $\mathbb{N}$.

If S and T are equivalence relations, denote $S \sqcup T$ by the equivalence relation $\{(2x, 2y) \mid (x, y) \in S\} \cup \{(2x+1, 2y+1) \mid (x, y) \in T\}$. Notice that $deg_c(S \sqcup T)$ is not obviously the least upper bound of the pair $deg_c(S)$ and $deg_c(T)$, since if f witnesses $S \leq_c X$ and g witnesses $T \leq_c X$ then $rng(f)$ and $rng(g)$ might contain elements that are X-related; we will not know how to identify these elements inside $S \sqcup T$.

Given a binary relation S and an effective enumeration $\{S_s\}_{s\in\omega}$ of S, define $S \upharpoonright t := \{(x, y) \in S \mid x, y < t\}$ and $S_s \upharpoonright t := \{(x, y) \in S_s \mid x, y < t\}$. We sometimes append $[s]$ to an expression to denote the value of the expression evaluated at step or stage s. A number x is said to be *isolated* with respect to an equivalence relation T if $T(x, y) = 0$ for every $y \neq x$.

We fix notations for several standard equivalence relations. Let $\equiv_n$ be the computable equivalence relation with exactly n many distinct equivalence classes. (Note that not every equivalence relation with finitely many classes is computable, for example, take the equivalence relation with two classes $\{A, \omega - A\}$ for any non-computable set A). Also fix `id` to be the computable equivalence relation with infinitely many classes. Of course we have

$$\equiv_1 \quad <_c \quad \equiv_2 \quad <_c \quad \cdots \quad <_c \quad \mathtt{id},$$

and every computable equivalence relation is of the same degree as one of these. Furthermore, this chain of degrees form an initial segment of the global c-degrees - since if R is reducible to any degree in the chain then R is computable.

In order for us to work with the various classes of equivalence relations (ceers, Π^0_1 and n-c.e.), we wish to fix an effective enumeration of all members of the class. This is unfortunately not always possible, so we fix a computable enumeration $\{S_e\}_{e\in\omega}$ of all reflexive and symmetric (c.e., Π^0_1 or n-c.e., depending on the context) binary relations. In fact, we will fix the approximation to $\{S_e\}_{e\in\omega}$ such that for all e, x, y, t, $S_e(x, x)[t] = 1$ and $S_e(x, y)[t] = S_e(y, x)[t]$.

Notice that the reflexive and symmetric closure of a binary relation has the same descriptive complexity as the relation, while the transitive closure of a binary relation S is Σ^0_1 relative to S. Thus, the issue of whether $\{S_e\}_{e\in\omega}$ can be turned into a computable enumeration of all members of the class comes down to the difficulty of recognizing transitivity in the class; for ceers this is trivial (the transitive closure of a c.e. binary relation is still c.e.), while for Π^0_1 and n-c.e. equivalence relations this question will be addressed in the respective section. The existence of an enumeration $\{S_e\}_{e\in\omega}$ for ω-c.e. equivalence relations has to be defined in the right way, and we will discuss this in Section 2.2.

1.2 Organization of the paper

The paper is organized as follows. We shall be primarily interested in the class of Π^0_1 equivalence relations, the class of ω-c.e. equivalence relations and the class of n-c.e. equivalence relations for each $n \geq 1$.

In Section 2 we consider the problem of producing a computable enumeration of each class as well as the existence of a universal equivalence relation of the class. In Theorem 2.1 we provide a computable enumeration of all n-c.e. equivalence relations and deduce the existence of a universal n-c.e. equivalence relation for each $n \geq 1$. In Theorem 2.5 we provide a computable enumeration of ω-c.e. equivalence relations in which every degree is represented, and deduce the existence of a universal ω-c.e. equivalence relation. Then in Theorem 2.8 we provide a computable enumeration of Π^0_1 equivalence relations in which the upper cone of every degree is represented, and deduce the existence of a universal Π^0_1 equivalence relation. Note that the existence of a universal Π^0_1 equivalence relation is proved in [9], while a universal α-c.e. equivalence relation (for $1 < \alpha \leq \omega$) was not known before. We also show in Proposition 2.10 that a computable enumeration of all Π^0_1 equivalence relations is impossible.

In Section 3 we consider the problem of downward density in the structure of the c-degrees of c.e. equivalence relations. In Theorem 3.7 we construct a c.e. c-degree which is not set-induced, minimal modulo the computable c-degrees and not above $deg_c(\texttt{id})$. In Theorem 3.9 we construct another c.e. c-degree which is not set-induced and minimal, and this time above $deg_c(\texttt{id})$.

In Section 4 we show that downward density fails for the Π^0_1 c-degrees. Next we generalize this theorem by constructing an infinite sequence of Π^0_1 c-degrees $deg_c(\texttt{id}) < deg_c(A_2) < deg_c(A_3) < \cdots$, each of which is a strong minimal cover over the previous. All our counter-examples are not set-induced.

Finally in Section 5 we prove that in contrast to downward density, upward density holds in all the classes of interest.

2 Enumerations of a class and universality

2.1 n-c.e. equivalence relations In this section we address the question of the existence of a computable enumeration of all n-c.e. equivalence relations, and the existence of a universal n-c.e. equivalence relation. We fix some $n \in \omega$, $n \geq 1$.

Theorem 2.1 *There is a computable enumeration of all n-c.e. equivalence relations. More specifically, there exists a computable function $f(e, x, y, s)$ such that:*

- *For all e, x, y, $\lim_{s\to\infty} f(e, x, y, s) = E_e(x, y)$.*
- *For all e, E_e is a n-c.e. equivalence relation.*
- *For each n-c.e. equivalence relation S there exists some e such that $S = E_e$[1].*
- *For all e, x, y, $f(e, x, y, 0) = 0$ and $\#\{s \mid f(e, x, y, s) \neq f(e, x, y, s+1)\} \leq n$.*

Proof As mentioned in Section 1.1 we fix $\{S_e\}_{e\in\omega}$ to be a computable enumeration of all n-c.e. binary relations which are reflexive and symmetric. For each e, we construct (an approximation to) E_e uniformly in e.

At stage $v = 0$, define $u_0 := 0$ and $E_{e,0} := \{(x, x) : x \in \omega\}$, and do nothing else. At stage $v > 0$, let t be largest such that $u_t \downarrow$, and assume that $E_{e,v-1}$ has also been defined (note that t always exists). Check if $S_{e,v} \upharpoonright t + 1$ is transitive. If it is, define $u_{t+1} = v$ and $E_{e,v} = S_{e,v} \upharpoonright t + 1$. If not, define $E_{e,v} := E_{e,v-1}$ and leave $u_{t+1} \uparrow$.

We now verify that E_e has the properties we want.

E_e is n-c.e.: For each x, y, t, we have that $E_{e,v-1}(x,y) \neq E_{e,v}(x,y)$ iff $v = u_{t+1}$ and $(S_{e,u_t} \upharpoonright t)(x,y) \neq \left(S_{e,u_{t+1}} \upharpoonright t+1\right)(x,y)$. In particular, if v_0 is least (if it exists) such that $E_{e,v_0-1}(x,y) \neq E_{e,v_0}(x,y)$, then $E_{e,v_0}(x,y) = \left(S_{e,u_{t+1}} \upharpoonright t+1\right)(x,y) = 1$ where $v_0 = u_{t+1}$, and hence $x, y < t+1$. This means that after stage v_0, all further changes in $E_{e,v}(x,y)$ must correspond to a change in $S_e(x,y)$. Therefore, $E_e(x,y)$ changes no more often than $S_e(x,y)$.

E_e is an equivalence relation: E_e is obviously reflexive since $E_{e,0}$ is reflexive. Now note that for every x, y, if $E_e(x,y) = 1$ then $S_e(x,y) = 1$ and $u_t \downarrow$ for some $t > x, y$. Therefore E_e is symmetric. Clearly $E_{e,v}$ is transitive for every v, hence E_e is transitive.

If S_e is transitive then $S_e = E_e$: If S_e is transitive then $u_t \downarrow$ for every t, and so E_e is set to copy $S_e \upharpoonright t$ for larger and larger t. □

The existence of a computable enumeration of all n-c.e. equivalence relation immediately yields:

Corollary 2.2 *There exists a universal n-c.e. equivalence relation for any fixed $n \geq 1$.*

However, if one considers varying n over all of ω, then there will not be a universal element of the class:

Proposition 2.3 *The collection*

$$\{S : \ S \textit{ is an } n\textit{-c.e. equivalence relation for some } n \in \omega\}$$

has no universal element.

Proof Given an n-c.e. equivalence relation S, the obvious thing to do is to consider the following equivalence relation (generated by) E:

$$E := \{\langle\langle e,0\rangle, \langle e,1\rangle\rangle \mid \varphi_e(\langle e,0\rangle)\downarrow = x \ \wedge \ \varphi_e(\langle e,1\rangle)\downarrow = y \ \wedge \ S(\langle x,y\rangle) = 0\}.$$

By the definition of E, we have for each e such that φ_e is total, $E(\langle e,0\rangle, \langle e,1\rangle) \neq S(\varphi_e(\langle e,0\rangle), \varphi_e(\langle e,1\rangle))$, which implies $E \not\leq_c S$ via φ_e.

Now for each e, $E(\langle e,0\rangle, \langle e,1\rangle)$ initially starts off taking value 0, before φ_e converges on the two inputs. Once $\varphi_e(\langle e,0\rangle)$ and $\varphi_e(\langle e,1\rangle)$ both converge, we might see $S(x,y) = 0$ where we now have to let $E(\langle e,0\rangle, \langle e,1\rangle) = 1$, and subsequently we follow the changes in $S(x,y)$. Hence E changes at most one more time than S, and so E is $n+1$-c.e. □

2.2 ω-c.e. equivalence relations Now we turn to ω-c.e. equivalence relations. In particular, we wish to investigate the question of whether there exists a universal ω-c.e. equivalence relation, and what kind of enumerations are possible of this class. The best kind of enumeration one could ask for is an effective listing $\{\langle S_e, f_e\rangle\}_{e\in\omega}$ of all ω-c.e. equivalence relations S_e along with a computable bound f_e for the number of changes in S_e. This is of course impossible, due to general computability theoretic reasons. Therefore, we will fix an enumeration $\{\langle S_e, h_e\rangle\}_{e\in\omega}$ in the following sense:

Lemma 2.4 *There exists an effective listing $\{\langle S_e, h_e\rangle\}_{e\in\omega}$ of all ω-c.e. equivalence relations S_e such that h_e is partial computable. Here every S_e is an ω-c.e. equivalence relation and every ω-c.e. equivalence relation S is equal to*

S_e for some e such that h_e is total and bounds the number of mind changes of S_e.

Proof S_e is taken to be a slowed down version of the e^{th} possible ω-c.e. binary relation T_e. We follow the proof of Theorem 2.1. We start with $S_e(x,y)[0] = 0$ for all $x \neq y$. For some t, if we discover that $\varphi_e(x,y)$ has converged for all $x,y < t$, we proceed to wait for convergence of all $x,y < t+1$. Before that happens, we keep $S_e(x,y) = 0$ for all (x,y) such that $x > t$ or $y > t$, and update S_e to copy changes in $T_e \upharpoonright t$ only when T looks transitive. When $\varphi(x,y)$ converges for all $x,y < t+1$, if ever, and when $T_e \upharpoonright t+1$ again looks transitive, we extend the domain of h_e and then set S_e to now copy $T_e \upharpoonright t+1$, again updating only if $T_e \upharpoonright t+1$ looks transitive. This ensures that S_e is always an equivalence relation, and the domain of f_e is $\{(x,y) \mid x,y < t\}$ for some $t \leq \omega$, and where $S_e = T_e$ and $h(e)$ is total in the event that T_e turns out to be ω-c.e. and transitive.

Note that if h_e is partial, then $S_e - \{(x,x) \mid x \in \omega\}$ is finite and certainly ω-c.e. □

Unlike Theorem 2.1 for n-c.e. equivalence relations, Lemma 2.4 is not sufficient to deduce the existence of a universal ω-c.e. equivalence relation, because the effective disjoint union of all S_e is not ω-c.e. However, recall that every ω-c.e. m-degree (of subsets of ω) contains an ω-c.e. set with an approximation that changes at most identity bounded many times. If we could prove, for example, the analogous statement that there exists a computable enumeration $\{E_e\}_{e\in\omega}$ of equivalence relations with identity bounded changes, and which contains an equivalence relation of every ω-c.e. degree, then the disjoint union of E_e will provide a universal element of the class. This is provided by the following theorem:

Theorem 2.5 *For each e, there exists (uniformly in e) an approximation to an equivalence relation E_e such that:*

- *For every x, y, the number of changes in the approximation of $E_e(x,y)[s]$ is bounded by $\max\{x,y\}$.*
- *If h_e is total then $S_e \equiv_c E_e$.*

Proof Fix e. Let $\hat{h}(y) = \max_{x,x' \leq y} h_e(x,x')$; then $\hat{h}$ is a partial computable function whose domain is an initial segment of ω. Without loss of generality, we assume that $\hat{h}$ is strictly increasing on its domain. Define $h^*(x) = \max\{z-1, 0\}$ where z is least such that $\hat{h}(z) \geq x$. Now define

$$E_e(x,y) = \begin{cases} S_e(h^*(x), h^*(y)), & \text{if } h^*(x), h^*(y) \text{ both exist,} \\ 1, & \text{if } x = y, \\ 0, & \text{otherwise.} \end{cases}$$

Now $E_e(x,y)$ has the following obvious approximation for $x \neq y$: Begin with $E_e(x,y) = 0$ and wait for both $h^*(x)$ and $h^*(y)$ to become defined. When this happens, copy $S_e(h^*(x), h^*(y))$. Thus, the number of changes in the approximation of $E_e(x,y)$ is at most that of $S_e(h^*(x), h^*(y))$, which is in turn bounded by $h_e(h^*(x), h^*(y))$. Suppose $x < y$ (the other case $y < x$ follows symmetrically). Therefore $h^*(x) \leq h^*(y)$ by the definition of h^*. Let z_y be least such that $\hat{h}(z_y) \geq y$ (this is defined as $h^*(y)\downarrow$). If $h^*(y) = 0$ then $h^*(x) = 0$ and so $E_e(x,y)$ changes at most once. So suppose that $h^*(y) > 0$, and in particular, $h^*(y) = z_y - 1$. Since $h^*(x), h^*(y) \leq z_y - 1$, this means that $h_e(h^*(x), h^*(y)) \leq \hat{h}(z_y - 1)$. However $\hat{h}(z_y - 1) < y$ by the minimality of z_y, and so $S_e(h^*(x), h^*(y))$ changes no more than $\max\{x,y\}$ many times. This verifies the first property claimed in the theorem.

It is also clear that E_e is an equivalence relation, because S_e is. Finally, suppose that h_e is total. We wish to see that $S_e \equiv_c E_e$. Now $E_e \leq_c S_e$ is given by h^*, which must be total since h_e is total and $\hat{h}$ is strictly increasing. For the direction $S_e \leq_c E_e$, we simply take the map $z \mapsto \hat{h}(z) + 1$. Notice that $h^*(\hat{h}(z) + 1) = z$. □

Corollary 2.6 *There exists a universal ω-c.e. equivalence relation.*

Proof The effective disjoint union $\sqcup_{e\in\omega} E_e$ from Theorem 2.5 is clearly an ω-c.e. equivalence relation. Every ω-c.e. equivalence relation is equal to S_e for some e where h_e is total, which has the same degree as E_e. □

2.3 $\mathbf{\Pi^0_1}$ equivalence relations We recall Theorem 3.3 of [9], where it is shown that there is a universal Π^0_1 equivalence relation:

Theorem 2.7 ([9, Theorem 3.3]) *There exists a universal Π^0_1 equivalence relation.*

The proof in [9] does not mention enumerations of the class, nor does it explicitly show how to uniformly transform a Π^0_1 binary relation S_e into a transitive equivalence relation. For the sake of completeness and to fit Π^0_1 into the context of our paper, we will give a slightly different presentation in this section.

First, we fix $\{S_e\}_{e\in\omega}$ as in Section 1.1. We first show that it is possible to construct an effective enumeration $\{E_e\}_{e\in\omega}$ of Π^0_1 equivalence relations in which the upper cone of every Π^0_1 degree is represented:

Theorem 2.8 *There exists an effective Π^0_1 enumeration of a sequence $\{E_e\}_{e\in\omega}$ such that:*

- *For every e, E_e is a Π^0_1 equivalence relation.*
- *For any Π^0_1 equivalence relation S, there exists some e such that $S \leq_c E_e$;*

Proof Fix e, and we shall build a Π^0_1 enumeration of E_e uniformly in e such that if S_e is transitive, then $S_e \leq_c E_e$. The issue that we *must* keep E_e transitive, while S_e might not be, and we have to do it in a way where we can extend our reduction f_e witnessing $S_e \leq_c E_e$ whenever S_e looks transitive on a larger fragment.

Define $f_e(x)$ inductively by setting $f_e(0) = 0$, and $f_e(x+1)$ to be the least stage larger than $f_e(x)$ such that $S_e[f_e(x+1)] \upharpoonright x+2$ looks like an equivalence relation. Then f_e is partial computable, and is total if and only if S_e is an equivalence relation. Now define $E_e(x,y)$ (for $x \neq y$) by the following. If $x \notin rng(f_e)$ or $y \notin rng(f_e)$ then $E_e(x,y) = 0$, otherwise let a, b be such that $f_e(a) = x$ and $f_e(b) = y$, and define $E_e(x,y) = 0$ if and only if $S_e(a,b)[f_e(z)] = 0$ for some z in $dom(f_e)$. Obviously if $x = y$ we set $E_e(x,y) = 1$.

There is an obvious Π^0_1 enumeration of E_e: First, note that $rng(f_e)$ is computable (uniformly in e), since $s \in rng(f_e)$ if and only if this has happened before querying $S_e[s+1]$ (this is true even if $rng(f_e)$ is finite). Furthermore the condition "$S_e(a,b)[f_e(z)] = 0$ for some z in $dom(f_e)$" is c.e. Hence E_e is Π^0_1, with an enumeration effectively found in e.

It is obvious that E_e is reflexive and symmetric (since S_e is at every stage). For transitivity, suppose $E(x,y) = E(y,w) = 1$ and $E(x,w) = 0$ for distinct x, y, w. Then clearly there must exist a, b, c such that $f_e(a) = x$, $f_e(b) = y$ and $f_e(c) = w$. Also fix $z \in dom(f_e)$ such that $S_e(a,c)[f_e(z)] = 0$ triggering us to define $E_e(x,w) = 0$. Now observe that $S_e\left[f_e\left(\max\{a,b,c,z\}\right)\right] \upharpoonright (\max\{a,b,c,z\}+1)$ cannot be transitive, contradicting the properties of f_e.

Finally suppose that S_e is an equivalence relation, hence f_e is total. Then f_e witnesses that $S_e \leq_c E_e$. □

We deduce immediately [9, Theorem 3.3]:

Corollary 2.9 *There exists a universal* Π^0_1 *equivalence relation.*

We now turn to the interesting question of whether there exists an effective enumeration of all Π^0_1 equivalence relations, or even an enumeration representing all Π^0_1 degrees. First of all, it is easy to see that an enumeration of all Π^0_1 equivalence relations is impossible:

Proposition 2.10 *There does not exist an effective sequence* $\{E_e\}_{e\in\omega}$ *of* Π^0_1 *equivalence relations such that for every* Π^0_1 *equivalence relation* S*, there is some* e *such that* $S = E_e$*. (We can even replace "*$S = E_e$*" by "*$S \leq_c E_e$ *via a primitive recursive function").*

Note that Proposition 2.10 does not contradict Theorem 2.8. In fact, it shows that the proof of Theorem 2.8 is sharp in the sense that the reductions f_e constructed there cannot all be total.

Proof of Proposition 2.10 Fix the sequence $\{E_e\}_{e\in\omega}$ as above, and we shall construct a Π^0_1 equivalence relation S such that $S \neq E_e$ for every e. In fact, we shall ensure that for every e, p_e does not witness $S \leq_c E_e$, where p_e is the e^{th} primitive recursive function.

At the beginning we define $S(\langle e,i\rangle, \langle e',j\rangle)[0] = 0$ for all e, e', i, j such that $e \neq e'$, and $S(\langle e,i\rangle, \langle e,j\rangle)[0] = 1$ for all e, i, j. For each e, we do the following uniformly. We begin our strategy for e by first setting $S(\langle e,0\rangle, \langle e,1\rangle) = 0$, and wait for a stage s such that $E_e\left(p_e(\langle e,0\rangle), p_e(\langle e,1\rangle)\right) = 0$. For every stage t where $E_e\left(p_e(\langle e,0\rangle), p_e(\langle e,1\rangle)\right)[t]$ is still $= 1$, we set $S\left(\langle e,t+2\rangle, \langle e,i\rangle\right) = 0$ for every $i \neq t+2$ (that is, declare $\langle e,t+2\rangle$ to be an isolated point).

Let s be the first stage such that $E_e\left(p_e(\langle e,0\rangle), p_e(\langle e,1\rangle)\right)[s] = 0$ (note that s may not exist). At this point, we have made $\langle e,i\rangle$ an isolated point for all $2 \leq i < s+2$. However $S(\langle e,s+2\rangle, \langle e,i\rangle) = 1$ for all i. We wait for either $E_e\left(p_e(\langle e,0\rangle), p_e(\langle e,s+2\rangle)\right) = 0$ or $E_e\left(p_e(\langle e,1\rangle), p_e(\langle e,s+2\rangle)\right) = 0$; at least one of these must hold because E_e is transitive. Suppose $E_e\left(p_e(\langle e,0\rangle), p_e(\langle e,s+2\rangle)\right) = 0$. Then we declare $\langle e,1\rangle$ to be isolated and do nothing else for e; otherwise declare $\langle e,0\rangle$ to be isolated.

We now verify that S has the properties we want. Firstly, it is clear that the above describes a Π^0_1 approximation to S, as E_e is transitive and p_e is total. Now if s is never found by the strategy for e then the e^{th} column of S will consist only of isolated points. If s is found by the strategy then the e^{th} column $S^{[e]} = \{\langle e, i\rangle \mid i \in \omega\}$ will consist of $s+1$ many isolated points, while the rest of the points belong to a single equivalence class. In either case, $S^{[e]}$ is an equivalence relation. Hence S is an equivalence relation. Finally, it is straightforward to check that for every e, p_e does not witness $S \leq_c E_e$. □

Given Proposition 2.10 we can ask if there is an enumeration of Π^0_1 equivalence relations in which every degree is represented, like in the case of ω-c.e. equivalence relations. Part of the difficulty of obtaining such an enumeration is that the reductions from S_e to E_e must be allowed to be partial. We leave open the question:

Question 2.11 *Is there an effective enumeration of Π^0_1 equivalence relations $\{E_e\}_{e\in\omega}$ such that if S is a Π^0_1 equivalence relation, then there is some e such that $S \equiv_c E_e$?*

3 Downward density and c.e. equivalence relations

We now investigate the structure of the c.e. c-degrees, in particular, the problem of downward density. Unlike the Π^0_1 c-degrees (see Section 4), it is not the case that every c.e. c-degree is comparable with $deg_c(\texttt{id})$. Given any c.e. set $A \subseteq \omega$, define the ceer $R_A = \{(x,y) \mid x = y \text{ or } x,y \in A\}$. Then observe that $\texttt{id} \leq_c R_A$ if and only if A is not simple. Thus, even though the computable c-degrees $deg_c(\equiv_1) < deg_c(\equiv_2) < \cdots < deg_c(\texttt{id})$ is an initial segment of the c.e. c-degrees, we now get non-trivial c.e. c-degrees which are incomparable with this initial segment.

This situation is very similar to that of the c.e. 1-degrees, where it had been observed that the "bottom" degree, $\mathbf{0_1} = deg_1(\emptyset \oplus \omega)$, isn't really the least element of the c.e. 1-degrees: The simple c.e. sets all have 1-degree incomparable with $\mathbf{0_1}$. Nevertheless, much of the earlier studies of the c.e. 1-degrees have been focused on the interval $[\mathbf{0_1}, deg_1(K)]$. For example, Lachlan [10] showed that every finite initial segment in this interval is a distributive lattice, and hence the first-order theory of the c.e. 1-degrees is undecidable. Young [14] showed that this interval is neither an upper semilattice nor a lower semillatice. Part of the reason for this is due to the fact that results about m-degrees of c.e. sets can often be transferred to the interval $[\mathbf{0_1}, deg_1(K)]$, as the non-simplicity of the c.e. sets in this interval often allow m-reductions to be turned into 1-reductions.

It was observed in [1] that the interval $[\mathbf{0_1}, deg_1(K)]$ embeds into the c.e. c-degrees; in fact, the interval $[\mathbf{0_1}, deg_1(K)]$ of c.e. 1-degrees was shown to be order-isomorphic with the interval $[deg_c(\texttt{id}), deg_c(R_K)]$ of c-degrees. Note that $deg_c(R_K)$ is *not* the top c.e. c-degree; in fact, R_K is far from being universal amongst the c.e. c-degrees.

Definition 3.1 *We call a c-degree $\boldsymbol{a}$* set-induced *if there exists an infinite and co-infinite set $A \subseteq \omega$ such that $R_A \in \boldsymbol{a}$.*

It was shown in [1] that every c-degree in the interval $[deg_c(\mathtt{id}), deg_c(R_K)]$ is set-induced. In fact, the same proof shows a little more; for completeness, we reproduce the argument here:

Lemma 3.1 *Suppose R is a non-computable equivalence relation and $A \subseteq \omega$ is any set such that $R \leq_c R_A$. Then there exists a set $B \subseteq \omega$ such that $R \equiv_c R_B$.*

Proof Fix a computable function f witnessing $R \leq_c R_A$. Since R is not computable, $rng(f)$ is an infinite c.e. set. We let $\{d_n\}_{n\in\omega}$ be a one-one computable enumeration of $rng(f)$. Let $B = \{n \in \omega \mid d_n \in A\}$. Notice that this gives $B \leq_1 A$, as expected.

To see that $R \leq_c R_B$, use the reduction $(d_n)^{-1} \circ f$. For the other direction $R_B \leq_c R$, we map $n \mapsto$ the least x such that $f(x) = d_n$. □

Lemma 3.1 shows not only that every c-degree in the interval $[deg_c(\mathtt{id}), deg_c(R_K)]$ is set-induced, but also that:

Corollary 3.2 *A non-computable c.e. c-degree $\boldsymbol{a}$ is set-induced if and only if $\boldsymbol{a} \leq deg_c(R_K)$.*

Before examining the question of downward density for c-degrees, we mention one related result about c.e. 1-degrees. Recall the following definition:

Definition 3.2 *If $\boldsymbol{a} < \boldsymbol{b}$ are two elements of a partial order, then $\boldsymbol{b}$ is said to be a* strong minimal cover *of $\boldsymbol{a}$ if the following is true of the partial order: $\forall \boldsymbol{c} \leq \boldsymbol{b}\,(\boldsymbol{c} = \boldsymbol{b} \ \vee\ \boldsymbol{c} \leq \boldsymbol{a})$.*

Lachlan [10] proved the existence of a minimal element of the interval $[\mathbf{0_1}, deg_1(K)]$. In fact, a closer examination of the proof reveals that there exists a c.e. 1-degree which is a strong minimal cover of $\mathbf{0_1}$ within the structure of all c.e. 1-degrees:

Theorem 3.3 ([10, Lachlan]) *There exists a non-simple and non-computable c.e. set A such that for every set $B \leq_1 A$, we have $A \equiv_1 B$ or B is computable.*

We now turn to the matter at hand in this section. The statement of downward density for c-degrees says that given any non-computable c.e. c-degree $\boldsymbol{a}$, there exists a non-computable (c.e.) c-degree $\boldsymbol{b} < \boldsymbol{a}$. We show that this fails by constructing two counter-examples: Theorem 3.7 provides a "minimal" c.e. c-degree $\boldsymbol{a} \not\geq deg_c(\mathtt{id})$, while Theorem 3.9 provides a strong minimal cover $\boldsymbol{a}$ of $deg_c(\mathtt{id})$. However the existence of a strong minimal cover of $deg_c(\mathtt{id})$ in the c.e. c-degrees can already be deduced from Theorem 3.3:

Proposition 3.4 *There is a c.e. set-induced c-degree $\boldsymbol{a} > deg_c(\mathtt{id})$ such that $\boldsymbol{a}$ is a strong minimal cover of $deg_c(\mathtt{id})$.*

Proof Consider $R_A >_c \mathtt{id}$ where A is from Theorem 3.3. Now if $R \leq_c R_A$ then by Lemma 3.1 either R is computable, or else $R \equiv_c R_B$ for some $B \leq_1 A$. As B is not computable we have $B \geq_1 A$, which means that $R \geq_c R_A$. □

Thus our counter-examples to downward density in Theorems 3.7 and 3.9 will be c.e. c-degrees that are not set-induced. It shall be convenient to isolate properties that imply that a certain c-degree is not set-induced. We identify two conditions sufficient for a degree to be not set-induced.

The first and probably the easiest way to produce an equivalence relation R that does not have set-induced c-degree is to make every R-class finite:

Lemma 3.5 *If R is any non-computable equivalence relation where every class is finite, then $deg_c(R)$ is incomparable with every non-computable set-induced c-degree.*

Proof If $R_A \leq_c R$ via a computable function f, then $f(A)$ is a finite set and hence A is computable. On the other hand if $R \leq_c R_A$ and R is non-computable, then by Lemma 3.1 there is a B such that $R \equiv_c R_B$. Since $R_B \leq_c R$, B is computable, a contradiction. □

Unfortunately we will not use Lemma 3.5 to ensure that the ceers constructed in Theorems 3.7 and 3.9 are not of set-induced degree. This is because it is generally a messy task to construct non-trivial c.e. equivalence relations where every class is finite. However, we will be able to easily do this for Π^0_1 equivalence relations. Therefore Lemma 3.5 will be used in the proof of Theorems 4.1, 4.2 and 4.8 in the next section.

To ensure that the c.e. equivalence relations constructed in Theorems 3.7 and 3.9 are not of set-induced degree, we will introduce a second sufficient condition:

Lemma 3.6 *Let R be any non-computable c.e. equivalence relation with the property that given any c.e. set W, if infinitely many R-classes contain an element of W, then every R-class contains an element of W. Then R is a counter-example to downward density. If additionally each R-class contains at least two elements of each W which intersects infinitely many R-classes, then there is no set-induced degree below $deg_c(R)$.*

Proof Fix R as above. Let $S \leq_c R$ via a computable function f. If $rng(f)$ intersects only finitely many R-classes then as R is c.e., S is computable. Otherwise each R-class must contain an element of $rng(f)$. Since R is c.e., this obviously allows us to show that $R \leq_c S$.

Now suppose $R_A \leq_c R$ via f for some co-infinite set A. Then $rng(f)$ must intersect infinitely many different R-classes. Hence each R-class will contain at least two elements of $rng(f)$. Since R is not computable, there are at least two R-classes. This means that $f(x)\ R\ f(y)$ for some pair $x, y \notin A$ and $x \neq y$, a contradiction. □

In Theorem 3.7 we will use Lemma 3.6 to ensure that the degree of the constructed ceer is not above a set-induced degree.

Theorem 3.7 *There is a non-computable c.e. c-degree $\boldsymbol{a} \not\geq deg_c(\mathtt{id})$ such that $\boldsymbol{a}$ is not set-induced and for every $\boldsymbol{b} \leq \boldsymbol{a}$, either $\boldsymbol{b}$ is computable or $\boldsymbol{b} \geq \boldsymbol{a}$.*

Proof We are going to build a c.e. equivalence relation A and ensure the following requirements:

$\boldsymbol{R_{2e}}$: $A \neq \varphi_e$.

$\boldsymbol{R_{2e+1}}$: W_e intersects infinitely many A-classes $\Rightarrow$ every A-class contains at least two elements of W_e.

It should be clear that these requirements ensure the properties needed for A. The R_{2e} requirements ensure that A is not computable. Lemma 3.6 and the R_{2e+1} requirements together ensure that $deg_c(A)$ is a counterexample to downward density and does not bound a set-induced degree. Since $\mathtt{id} \equiv_c R_{\omega\oplus\emptyset}$, it follows that $deg_c(A) \not\geq deg_c(\mathtt{id})$.

During the construction, the classes of A are ordered according to the magnitude of the least element of the class. The n-th class of A_s is denoted by $C_{n,s}$. Since A is c.e., the basic operation we will perform at each step s is to *collapse* C_n *and* C_m for certain classes C_n and C_m. This has the obvious meaning, i.e. define $C_{n,s+1} = C_{n,s} \cup C_{m,s}$ if C_n is the smaller class, and rearrange the names of all affected classes. To say that a class C_n *is larger than* m *(or another class* C_m*)* means that $n > m$.

The reader might expect the construction of A to be carried out on a priority tree using infinite injury, since one might want to guess if a certain c.e. set W intersects infinitely many different A-classes or not. However, the actions of the different requirements are sufficiently independent from each other that we will construct A in an "injury-free" way.

We will always prevent R_e from collapsing two classes smaller than e. Of course, R_e is still allowed to collapse two or more classes if at most one class being collapsed is smaller than e.

We now describe the basic strategy for R_{2e+1}. We act for R_{2e+1} at a stage $s > 2e+1$ if at least $2\bar{s}$ many different A_s-classes contain an element of $W_{e,s}$, where $\bar{s}$ is the largest stage $< s$ where we had acted for R_{2e+1}. (Obviously, if $\bar{s}$ does not exist, set it to be equal to $2e+1$). At such a stage s, we act for R_{2e+1} by doing the following. Let N be the least such that $C_{N,s}$ does not yet contain at least two elements of W_e. If $N > \bar{s}$ do nothing else. Otherwise find the least pair $M > M' > \max\{N, 2e+1\}$ such that $C_{M,s}$ and $C_{M',s}$ each contains an element of W_e; such a pair $M' < M$ exists by assumption (note that $\max\{N, 2e+1\} \leq \bar{s}$). Collapse the classes $C_N, C_{M'}$ and C_M.

We now describe the basic strategy for R_{2e}. At a stage $s > 2e$, we check to see if:

- For every $\langle i', j'\rangle < s$, $\varphi(i', j') \downarrow= 0$ implies that i' and j' are in different classes, and
- There exists some pair of numbers $\langle i, j\rangle < s$ such that $\varphi_e(i, j) \downarrow= 0$ and i and j currently belong to different A-classes which are both larger than $2e$.

If this is true, collapse the classes containing i and j. Otherwise do nothing else at this stage.

Construction of A: At the beginning, let $A_0 = \mathtt{id}$, i.e. $C_{n,0} = \{n\}$ for all n. At stage $s > 0$, let all requirements R_e for $e < s$ act (if necessary) according to the basic strategy above.

We now verify that all the requirements are met. First of all, note that each class C_n can be collapsed with another class C_m infinitely often, and so each class can be infinite. However, each pair of classes C_n and C_m are collapsed with each other only finitely often. We now verify this fact. More specifically, suppose there is a pair $n < m$ such that there are infinitely many steps s in the construction in which $C_{n,s}$ and $C_{m,s}$ are collapsed with each other. Fix a pair such that $\langle n, m\rangle$ is the least; recall that the standard pairing function $\langle \cdot, \cdot \rangle$ has the property that $\langle x, y\rangle < \langle x+1, y\rangle$ and $\langle x, y\rangle < \langle x, y+1\rangle$ for all x, y.

Now by examining the construction, only finitely many requirements (specifically, only those R_e for $e < m$) can cause the current C_n and the current C_m to be collapsed. Therefore, we can fix an e where R_e does this infinitely often. This means that e cannot be even, because if R_e ever collapses any two classes, it must be that $\varphi(i, j) = 0$ and that i, j are put the same A-class, which will block R_e from doing anything else. Therefore, e has to be odd.

By the minimality of the pair $\langle n, m\rangle$, there is a time after which every class C_k for $k \le n$ is eventually growing monotonically (as eventually no two classes smaller or equal to n is collapsed). Formally, this means that there exists a stage t_0 such that for all $s \ge t_0$, and $k \le n$, we have $C_{k,s} \subseteq C_{k,s+1}$. If R_e collapses C_n and C_m after stage t_0, then some class less than or equal to C_n will now contain at least two elements of W. By monotonicity, eventually every class less than or equal to C_n will contain at least two elements of W, and so makes it impossible for R_e to collapse C_n and C_m (in fact, any class larger than C_n) again.

This means that *every* A-class C_n eventually grows monotonically. This implies that A will (at the end) consist of infinitely many different classes. We now apply this to show that every requirement is satisfied. To show that R_{2e} is met, suppose that φ_e is total and correctly computes A. Fix any pair of elements i, j belonging to the $2e+1$-th and the $2e+2$-th classes respectively. Then $\varphi_e(i, j)$ must be equal to 0. But this means that at all large enough stages, R_{2e} would want to act to collapse these two classes. Since this is impossible, it must be because at all large enough stages, the first clause in the basic strategy for R_{2e} fails. This is also impossible if $\varphi_e = A$.

Now we argue that R_{2e+1} is met. Suppose that W_e intersects infinitely many A-classes, and let C_N be the smallest A-class which contains at most one element of W_e. Eventually from some point on, the classes $C_0, C_1, \cdots, C_N$ will be growing monotonically. Since W_e intersects infinitely many A-classes, there must be infinitely many stages where we will act for R_{2e+1}. Eventually $\bar{s}$ will be larger than N, and we will act to put two elements of W_e into C_N. □

We now turn to our second counter-example to downward density, this time constructing a c.e. c-degree which is above $deg_c(\mathtt{id})$. Unlike in the proof of Theorem 3.7, we cannot directly use Lemma 3.6 to show that the constructed c-degree is not set-induced, because we have to now make the constructed degree bound $deg_c(\mathtt{id})$. A modification is needed:

Lemma 3.8 (Modified Lemma 3.6) *Let R be any non-computable c.e. equivalence relation with the property that $(2n, 2m) \notin R$ for every $n \neq m$. We call an R-class even if it contains an even number, otherwise we call the R-class odd.*

Suppose R has the property that given any c.e. set W, if infinitely many odd R-classes contain an element of W, then every R-class contains at least *two elements of W. Then R is a counter-example to downward density and the only set-induced degree below $deg_c(R)$ is* $\deg_c(\texttt{id})$.

Proof Fix R as above. Let $S \leq_c R$ via a computable function f. If $rng(f)$ is lies in only finitely many odd R-classes then as R is c.e., S is computable. Otherwise each R-class must contain an element of $rng(f)$. Since R is c.e., this allows us as before to show that $R \leq_c S$.

Now suppose $R_A \leq_c R$ via f for some co-infinite set A. Then $rng(f)$ must intersect infinitely many different odd R-classes (otherwise $R_A \equiv_c \texttt{id}$). Hence each R-class will contain at least two elements of $rng(f)$. Since R is not computable, there are at least two R-classes. This means that $f(x)\ R\ f(y)$ for some pair $x, y \notin A$ and $x \neq y$, a contradiction. □

In Theorem 3.9 we will use Lemma 3.8 to ensure that the degree of the constructed ceer is not above a set-induced degree. Notice that the equivalence relation with only finitely many distinct classes is not of set-induced degree.

Theorem 3.9 *There is a c.e. c-degree $\boldsymbol{a} > deg_c(\texttt{id})$ such that $\boldsymbol{a}$ is not set-induced and for every $\boldsymbol{b} \leq \boldsymbol{a}$, either $\boldsymbol{b}$ is computable or $\boldsymbol{b} \geq \boldsymbol{a}$. In other words, $\boldsymbol{a}$ is a strong minimal cover of $deg_c(\texttt{id})$.*

Proof We will modify the proof of Theorem 3.7. We are going to build a c.e. equivalence relation A and ensure that $(2n, 2m) \notin A$ for every $n \neq m$. We will ensure the following requirements:

- $\boldsymbol{R_{2e}}$: $A \neq \varphi_e$.
- $\boldsymbol{R_{2e+1}}$: W_e intersects infinitely many odd A-classes $\Rightarrow$ every A-class contains at least two elements of W_e.

It should be clear that these requirements ensure the properties needed for A. The R_{2e} requirements ensure that A is not computable. Lemma 3.8 and the R_{2e+1} requirements together ensure that $deg_c(A)$ is a counter-example to downward density and is not of set-induced degree. Clearly $\texttt{id} \leq_c A$ by sending $x \mapsto 2x$.

We remark that we can satisfy R_{2e+1} (trivially) by ensuring that A only has finitely many odd classes at the end; however it is for the sake of R_{2e} that we will deliberately ensure that we end up with infinitely many odd classes at the end.

We follow closely the proof of Theorem 3.7. Let $o(n, s)$ be the n-th odd class at stage s, i.e. $C_{o(n,s),s}$ is an odd class and there are exactly n many odd classes amongst $C_{0,s}, C_{1,s}, \cdots, C_{o(n,s),s}$.

We now describe the basic strategy for R_{2e+1}. We act for R_{2e+1} at a stage $s > 2e+1$ if at least $2\bar{s}$ many different odd A_s-classes contain an element of $W_{e,s}$, where $\bar{s}$ is the largest stage $< s$ where we had acted for R_{2e+1}. (Obviously, if $\bar{s}$ does not exist, set it to be equal to $2e+1$). At such a stage s, we act for R_{2e+1} by doing the following. Let N be the least such that $C_{N,s}$ does not yet contain at least two elements of W_e (note that $C_{N,s}$ might be an even class). If $N > \bar{s}$ do nothing else. Otherwise find the least pair $M > M' > \max\{o(N, s), o(2e+1, s)\}$ such that $C_{M,s}$ and $C_{M',s}$ each contains an element of W_e and are both odd classes; such a pair $M' < M$ exists by assumption (note that $\max\{N, 2e+1\} \leq \bar{s}$). Collapse the classes $C_N, C_{M'}$ and C_M.

We now describe the basic strategy for R_{2e}. At a stage $s > 2e$, we check to see if:

- For every $\langle i', j' \rangle < s$, $\varphi(i', j') \downarrow= 0$ implies that i' and j' are in different classes, and
- There exists some pair of numbers $\langle i, j \rangle < s$ such that $\varphi_e(i, j) \downarrow= 0$ and i and j currently belong to different A-classes which are both larger than $o(2e, s)$ and are both odd classes.

If this is true, collapse the classes containing i and j. Otherwise do nothing else at this stage.

Construction of A: At the beginning, let $A_0 = \texttt{id}$, i.e. $C_{n,0} = \{n\}$ for all n. At stage $s > 0$, let all requirements R_e for $e < s$ act (if necessary) according to the basic strategy above.

We now verify that all the requirements are met. First of all, note that no two even classes are ever collapsed together, and thus $(2n, 2m) \notin A$ for every $n \neq m$.

Next, we observe the following facts:

- Every A-class C_n eventually grows monotonically. This is because $o(n, s) \geq n$ for every n and s, and so the same argument from Theorem 3.7 applies.
- For each n, there are only finitely many steps in the construction where we collapse $C_{m,s}$ and $C_{o(n,s),s}$ for some $m < o(n, s)$. To see this, note that this can only be done by some requirement R_e for $e < n$. Fix some odd R_e which does this infinitely often. Since each class eventually grows monotonically, this means that eventually R_e will be collapsing the three classes $N < M' < M$ where $N > n$. This of course means that M' and M are both larger than $o(n, s)$ which is a contradiction.
- At the end, A must contain infinitely many odd classes. This follows because of the preceding fact and also that if $C_{n,s}$ is collapsed with $C_{m,s}$ and $n < m$, then $C_{m,s}$ must be an odd class.

Since A contains infinitely many odd classes, the same argument from before works to show that R_e is met for every e. □

Notice that if $R \leq_c S$ and S is a ceer, then so is R. Therefore we can immediately conclude that any class of equivalence relations containing the c.e. equivalence relations is not downwards dense:

Corollary 3.10 *The class of Σ^0_n equivalence relations, the n-c.e. equivalence relations and the ω-c.e. equivalence relations are not downwards dense for $n \geq 1$.*

4 Downward density and $\mathbf{\Pi^0_1}$ equivalence relations

We now turn to the structure of the Π^0_1 c-degrees. In particular, we will show in this section that downward density fails in the structure of Π^0_1 c-degrees. Before making our statements precise, we first make a few trivial observations about the Π^0_1 c-degrees. Recall that the computable c-degrees form an initial segment of the Π^0_1 c-degrees. In fact, each Π^0_1 equivalence relation has comparable c-degree with $\texttt{id}$: If R is Π^0_1 and has infinitely many classes, then we can compute an infinite set of elements which are pairwise R-inequivalent, allowing us to reduce `id` to R. On the other hand, if R is Π^0_1 with finitely many distinct classes then every element is eventually R-inequivalent with all but one class, so it is computable.

Since every Π^0_1 c-degree is comparable with $deg_c(\mathtt{id})$, we will construct a strong minimal cover of $deg_c(\mathtt{id})$ which is not of set-induced degree.

Theorem 4.1 *There exists a Π^0_1 c-degree $\boldsymbol{a_2} > deg_c(\mathtt{id})$ such that $\boldsymbol{a_2}$ is a strong minimal cover of $deg_c(\mathtt{id})$, and where $\boldsymbol{a_2}$ is not of set-induced degree.*

As we will later formally prove Theorem 4.2, a stronger version of this theorem, we not prove Theorem 4.1. Instead, we give some relevant definitions. For any integer $x \in \omega$ and equivalence relation E, $[x]_E$ is defined to be the equivalence class of E containing x; where the context is clear we simply write $[x]$. (Note that $[s]$ also used as a stage notation earlier, but there should be no confusion). An equivalence relation E is said to be *size-n* if for every x, y such that $E(x, y) = 1$, we have $\lfloor \frac{x}{n} \rfloor = \lfloor \frac{y}{n} \rfloor$. In other words, each size-n equivalence relation has $\#[x] \leq n$ as well as an upper bound on the set $[x]$ for every x. Notice that `id` is a size-1 equivalence relation.

It is easy to see that a size-n Π^0_1 equivalence relation E is computable iff the function $F_E : x \mapsto \#[x]_E$ is computable.

Our goal is to extend Theorem 4.1 to an infinite sequence $deg_c(\mathtt{id}) = \boldsymbol{a_1} <_c \boldsymbol{a_2} <_c \boldsymbol{a_3} <_c \cdots$ such that $\boldsymbol{a_{j+2}}$ is a strong minimal cover of $\boldsymbol{a_{j+1}}$ for every $j \in \omega$. In the next theorem, Theorem 4.2, we give the formal proof for the existence of $\boldsymbol{a_2}$ and $\boldsymbol{a_3}$. The infinite sequence $deg_c(\mathtt{id}) = \boldsymbol{a_1} <_c \boldsymbol{a_2} <_c \boldsymbol{a_3} <_c \cdots$ is a straightforward generalization of Theorem 4.2 and there are no new mathematical ideas needed apart from increased notational complexity.

Theorem 4.2 *There exist Π^0_1 c-degrees $\boldsymbol{a_2}$ and $\boldsymbol{a_3}$ which are not set-induced such that $deg_c(\mathtt{id}) < \boldsymbol{a_2} < \boldsymbol{a_3}$, $\boldsymbol{a_2}$ is a strong minimal cover of $deg_c(\mathtt{id})$, and $\boldsymbol{a_3}$ is a strong minimal cover of $\boldsymbol{a_2}$.*

Proof We shall construct a size-2 Π^0_1 equivalence relation A_2 and a size-3 Π^0_1 equivalence relation A_3 satisfying the following requirements.

- $\boldsymbol{Q}$ **:** $A_2 \leq_c A_3$.
- $\boldsymbol{R^1_{\langle m,n\rangle}}$ **:** $E_m \leq_c A_2$ via $f_n \Rightarrow \exists$ computable function $g_{\langle n,m\rangle}$ s.t. $A_2 \leq_c E_m$ via $g_{\langle n,m\rangle}$ or E_m is computable.
- $\boldsymbol{R^2_{\langle i,j\rangle}}$ **:** $E_i \leq_c A_3$ via $f_j \Rightarrow \exists$ computable function $h_{\langle i,j\rangle}$ s.t. $A_3 \leq_c E_i$ via $h_{\langle i,j\rangle}$ or $E_i \leq_c A_2$.
- $\boldsymbol{P^1_e}$ **:** $F_{A_2} \neq \varphi_e$.
- $\boldsymbol{P^2_e}$ **:** $A_3 \not\leq_c A_2$ via φ_e.

Here, we think of E_m and E_i as the "pull-back" of A_2 and A_3 under f_n and f_j respectively, and enumerate E_n and E_m in the obvious way.

By Lemma 3.5, $deg_c(A_2)$ and $deg_c(A_3)$ are not set-induced. We fix the starting values $A_{2,0} = \{(i,i), (2i, 2i+1), (2i+1, 2i) \mid i \in \omega\}$ and $A_{3,0} = \{(i,i), (3i, 3i{+}1), (3i{+}1, 3i{+}2), (3i{+}2, 3$
Q is a global requirement and will be satisfied by ensuring that for every z, $[2z]_{A_2}$ is isolated if and only if $[3z]_{A_3}$ is isolated for every z.

Tree of strategies Effectively order the requirements (of order type ω). We will carry out the construction on a finitely branching priority tree, where every node at the same (next) level is assigned the same (next) requirement. The outcomes of each node will be given later.

Notation For any node α, define α^- to be β of the longest length such that $\beta * \infty \subseteq \alpha$. Of course, α^- need not exist. During the construction, we define δ_s to be the stage s approximation to the true path.

The length function for $R^1_{\langle m,n\rangle}$ is defined as $l(\langle m,n\rangle, s) :=$ the largest k such that for every $\langle x,y\rangle < k$, we have $f_{n,s}(x)\downarrow$, $f_{n,s}(y)\downarrow$ and $E_{m,s}(x,y) = A_{2,s}(f_{n,s}(x), f_{n,s}(y))$.

The length function for $R^2_{\langle i,j\rangle}$ is defined similarly as $L(\langle i,j\rangle, s) :=$ the largest k such that for every $\langle x,y\rangle < k$, we have $f_{j,s}(x)\downarrow$, $f_{j,s}(y)\downarrow$, and $E_{i,s}(x,y) = A_{3,s}(f_{j,s}(x), f_{j,s}(y))$.

Definition 4.1 (R^1-expansionary stage) *If α is assigned an R^1 strategy, then we call a stage $s > 0$ an α-expansionary stage if the following hold.*

- $\alpha \subset \delta_s$.
- $l(\alpha, s) > l(\alpha, t)$, *where t is the largest α-expansionary stage less than s (if it exists).*
- $\#(rng(f_{\alpha,s}) - \{0,1,\ldots,t\}) > 2t$.
- *There exists an even number y such that:*
  - *If α^- exists and α^- is assigned an R^1 requirement then $y \in S_{\alpha^-}$ (this parameter S_{α^-} will be defined later). If α^- is assigned an R^2 requirement then $\frac{3y}{2} \in S_{\alpha^-}$.*
  - *$y > \max S_\beta$ for all β assigned an R^1 requirement such that $\beta <_L \alpha$ or $\beta * fin \subseteq \alpha$.*
  - *$\frac{3y}{2} > \max S_\beta$ for all β assigned some R^2 requirement such that $\beta <_L \alpha$ or $\beta * fin \subseteq \alpha$.*
  - *$y > x_\beta$ (the follower assigned to the positive strategy β) for each β of higher priority than α assigned some P^1 requirement.*
  - *$y > \frac{2x_\beta}{3}$ for each higher priority β assigned some P^2 requirement.*
  - $[y]_{A_{2,s}} \subseteq rng(f_{\alpha,s})$.
  - $\#[y]_{A_{2,s}} = 2$.
- *$f_{\alpha,s}$ looks like a reduction on all the numbers above.*

Definition 4.2 (R^2-expansionary stage) *If α is assigned an R^2 strategy, then we call a stage $s > 0$ an α-expansionary stage if the following hold.*

- $\alpha \subset \delta_s$.
- $L(\alpha, s) > L(\alpha, t)$, *where t is the largest α-expansionary stage less than s (if it exists).*
- $\#(rng(f_{\alpha,s}) - \{0,1,\ldots,t\}) > 3t$.
- *There exists a number y divisible by 3 such that:*
  - *If α^- exists and α^- is assigned an R^1 requirement then $\frac{2y}{3} \in S_{\alpha^-}$ (this parameter S_α will be defined later). If α^- is assigned an R^2 requirement then $y \in S_{\alpha^-}$.*
  - *$\frac{2y}{3} > \max S_\beta$ for all β assigned an R^1 requirement such that $\beta <_L \alpha$ or $\beta * fin \subseteq \alpha$.*
  - *$y > \max S_\beta$ for all β assigned some R^2 requirement such that $\beta <_L \alpha$ or $\beta * fin \subseteq \alpha$.*

  - $\frac{2y}{3} > x_\beta$ *(the follower assigned to the positive strategy* β*) for each* β *of higher priority than* α *assigned some* P^1 *requirement.*
  - $y > x_\beta$ *for each higher priority* β *assigned some* P^2 *requirement.*
  - $[y]_{A_{3,s}} \subseteq rng(f_{\alpha,s})$.
  - $\#[y]_{A_{3,s}} = 3$.
- $f_{\alpha,s}$ *looks like a reduction on all the numbers above.*

Strategies of individual nodes We now describe the formal strategy for each requirement. Clearly, all the parameters and values described below are to be interpreted relative to the last stage at which a node is initialized.

Strategy for an R^1 *node* α. Suppose the current stage $s > 0$ is an α-expansionary stage. Let y_α be the least y satisfying the conditions of Definition 4.1. We fix $\{c_\alpha, c'_\alpha\} \subseteq dom(f_\alpha)$ such that $f_\alpha(c_\alpha) = y_\alpha$ and $f_\alpha(c'_\alpha) = y_\alpha + 1$.

Our action for α is to enumerate y_α into S_α, which is a collection of all such ys chosen by α in this way. Definition 4.1 ensures that at s, there are at least $t+1$ many distinct equivalence classes in $rng(f_\alpha)$ which are larger than t. Here we let $t' < t$ be the previous two α-expansionary stages before s (if s is the first or second expansionary stage, we give a separate description below). We know there are at most $y_{\alpha,t} < t$ many elements strictly between the class $[y_{\alpha,t'}]$ and $[y_{\alpha,t}]$, where $y_{\alpha,t'}, y_{\alpha,t}$ are the elements enumerated into S_α at stage t' and t.

We isolate every z strictly between $[y_{\alpha,t'}]$ and $[y_{\alpha,t}]$. That is, for every z with $\max[y_{\alpha,t'}] < z < \min[y_{\alpha,t}]$, we make z isolated. Also isolate $\left[3\lfloor \frac{z}{2} \rfloor\right]_{A_3}$ for each such z. We now have $k_\alpha < y_{\alpha,t} < t$ many isolated points there. Pick k_α many elements $b_{\alpha,1}, b_{\alpha,2}, \cdots, b_{\alpha,k_\alpha}$ from $dom(f_\alpha)$ all of which are in distinct E-classes, and which are distinct from $\{c_\alpha, c'_\alpha\}$, and all previous values in $rng(g_\alpha)$. This exists because f_α looks like a reduction, and $k_\alpha < t$. Note that if b' is a previous value in $rng(g_\alpha)$ then $f_\alpha(b')$ converges at stage t and hence $f_\alpha(b') < t$ (by the usual convention).

Now define $g_\alpha(\max[y_{\alpha,t'}] + i) = b_{\alpha,i}, i = 1, 2, \ldots, k_\alpha$, $g_\alpha(y_\alpha) := c_\alpha$ and $g_\alpha(y_\alpha + 1) := c'_\alpha$.

For clarity, we now describe our actions at the first and second α-expansionary stages. If the current stage s is the first α-expansionary stage after an initialization, define $y_{\alpha,s}$ as in Definition 4.1, and $g_\alpha(y_\alpha)$ and $g_\alpha(y_\alpha + 1)$, and do nothing else. Note that we should not allow α to isolate any number less than $y_{\alpha,s}$, as these classes might be restrained by a higher priority requirement. If s is the second α-expansionary stage after an initialization, define $y_{\alpha,s}$, $g_\alpha(y_\alpha)$ and $g_\alpha(y_\alpha + 1)$ as usual. *Do not* isolate any number less than $[y_{\alpha,t}]$, and define $g_\alpha(i)$ as above for every $i < \min[y_{\alpha,t}]$, mapping $g_\alpha(i) = g_\alpha(i')$ iff i and i' are in the same class.

If s isn't α-expansionary, do nothing for α.

The outcomes of α on the priority tree are:

***fin*:** Taken each time α is visited and the stage is not α-expansionary.
∞: This outcome is taken at every α-expansionary stage.

Strategy for an R^2 *node* α. This works similarly to an R^1 node. Suppose the current stage $s > 0$ is an α-expansionary stage. Let y_α be the least y

satisfying the conditions of Definition 4.2. We fix $\{c_\alpha, c'_\alpha, c''_\alpha\} \subseteq dom(f_\alpha)$ such that $f_\alpha(c_\alpha) = y_\alpha$, $f_\alpha(c'_\alpha) = y_\alpha + 1$ and $f_\alpha(c''_\alpha) = y_\alpha + 2$.

Our action for α is to enumerate y_α into S_α. We isolate every z strictly between $[y_{\alpha,t'}]$ and $[y_{\alpha,t}]$ (these are A_3-classes). Also isolate $\left[2\lfloor \frac{z}{3} \rfloor\right]_{A_2}$ for each such z. We now have $k_\alpha < y_{\alpha,t} < t$ many isolated points there. Pick k_α many elements $b_{\alpha,1}, b_{\alpha,2}, \cdots, b_{\alpha,k_\alpha}$ from $dom(f_\alpha)$ all of which are in distinct E-classes, and which are distinct from $\{c_\alpha, c'_\alpha, c''_\alpha\}$, and all previous values in $rng(h_\alpha)$. Again, this exists for the same reasons as above.

Now define $h_\alpha(\max[y_{\alpha,t'}] + i) = b_{\alpha,i}, i = 1, 2, \ldots, k_\alpha$, $h_\alpha(y_\alpha) := c_\alpha$, $h_\alpha(y_\alpha + 1) := c'_\alpha$ and $h_\alpha(y_\alpha + 2) := c''_\alpha$. At the first two α-expansionary stages, we follow the description above for R^1.

If s isn't α-expansionary, do nothing for α.

The outcomes of α on the priority tree are:

***fin*:** Taken each time α is visited and the stage is not α-expansionary.
∞: This outcome is taken at every α-expansionary stage.

Strategy for a P^1 node α. If the witness x_α is not yet defined, check if there is a least even number x_α at the current stage $s > 0$ such that:

- If α^- exists and is assigned an R^1 requirement then $x_\alpha \in S_{\alpha^-}$.
- If α^- exists and is assigned an R^2 requirement then $\frac{3x_\alpha}{2} \in S_{\alpha^-}$.
- $x_\alpha > \max S_\beta$ for all β assigned an R^1 requirement, such that $\beta <_L \alpha$ or $\beta * fin \subseteq \alpha$.
- $\frac{3x_\alpha}{2} > \max S_\beta$ for all β assigned an R^2 requirement, such that $\beta <_L \alpha$ or $\beta * fin \subseteq \alpha$.
- $x_\alpha > x_\beta$ for all P^1 nodes β of higher priority.
- $\frac{3x_\alpha}{2} > x_\beta$ for all P^2 nodes β of higher priority.
- $\#[x_\alpha]_{A_{2,s}} = 2$.

If these conditions are met, pick this number x_α as the follower. (Here φ_α refers to φ_e assigned to the node α).

Otherwise, x_α has already been picked prior to the current stage. If this is so, check if $\varphi_{\alpha,s}(x_\alpha) \downarrow= 2$; recall that P^1 wants to diagonalize against F_{A_2} being computable. If no, do nothing else. If yes, isolate all members of the class $[x_\alpha]_{A_{2,s}}$ and $\left[\frac{3x_\alpha}{2}\right]_{A_{3,s}}$. Declare diagonalization successful and never again attempt diagonalization (unless α is initialized).

The outcomes of α on the priority tree are:

***w*:** Diagonalization is not yet successful.
***s*:** Diagonalization is successful.

Strategy for a P^2 node α. If the witness x_α is not yet defined, check if there is a least number x_α divisible by 3 at the current stage $s > 0$ such that:

- If α^- exists and is assigned an R^1 requirement then $\frac{2x_\alpha}{3} \in S_{\alpha^-}$.
- If α^- exists and is assigned an R^2 requirement then $x_\alpha \in S_{\alpha^-}$.
- $\frac{2x_\alpha}{3} > \max S_\beta$ for all β assigned an R^1 requirement, such that $\beta <_L \alpha$ or $\beta * fin \subseteq \alpha$.
- $x_\alpha > \max S_\beta$ for all β assigned an R^2 requirement, such that $\beta <_L \alpha$ or $\beta * fin \subseteq \alpha$.
- $\frac{2x_\alpha}{3} > x_\beta$ for all P^1 nodes β of higher priority.

- $x_\alpha > x_\beta$ for all P^2 nodes β of higher priority.
- $\#[x_\alpha]_{A_{3,s}} = 3$.

If these conditions are met, pick this number x_α as the follower.

Otherwise, x_α has already been picked prior to the current stage. If this is so, check if $\varphi_\alpha(x_\alpha)\downarrow$, $\varphi_\alpha(x_\alpha+1)\downarrow$, $\varphi_\alpha(x_\alpha+2)\downarrow$, and for all $i, j \le 2$, $A_2(\varphi_\alpha(x_\alpha+i), \varphi_\alpha(x_\alpha+j)) = 1$. If no, do nothing else. If yes, isolate all members of the class $\left[\frac{2x_\alpha}{3}\right]_{A_{2,s}}$ and $[x_\alpha]_{A_{3,s}}$. Declare diagonalization successful and never again attempt diagonalization (unless α is initialized).

The outcomes of α on the priority tree are:

***w*:** Diagonalization is not yet successful.
***s*:** Diagonalization is successful.

Construction At stage s, define in the usual way the approximation to the true path δ_s of length s as the sequence of nodes eligible to act and having the correct guesses about outcomes at stage s. Initialize all nodes β to the right of δ_s, i.e., initialize the functions g_β, h_β, the set S_β and the witness x_β.

Verification Denote by δ the true path of the construction, i.e. $\delta = \liminf_{s\to\infty} \delta_s$. Now we are going to show that every node $\alpha \subset \delta$ ensures the satisfaction of the requirement assigned to it. Fix $\alpha \subset \delta$, and fix a stage s_0 such that for every $s > s_0$, $\delta_s \not<_L \alpha$.

First of all, we check that the global requirement Q is met. It is easy to see that every step in the construction isolates $[2k]_{A_2}$ iff it isolates $[3k]_{A_3}$, for each k. Thus, each A_3 class has size 1 or 3. In this case, we clearly have $A_2 \le_c A_3$ via the reduction $2k \mapsto 3k, 2k+1 \mapsto 3k+1$.

Fact 4.3 *Suppose $\alpha * \infty \subseteq \beta$. If α is an R^1-node and β is an R^2-node and $z \in S_\beta$ then $\frac{2z}{3} \in S_\alpha$. If α is an R^2-node and β is an R^1-node and $z \in S_\beta$ then $\frac{3z}{2} \in S_\alpha$. If α and β are R-nodes of the same type then $S_\beta \subseteq S_\alpha$.*

Proof Easy to check (by a straightforward induction on $|\beta|$). □

Lemma 4.4 *Each P^1 requirement succeeds.*

Proof Assume that α is assigned a P^1 requirement. Suppose α is never able to find a suitable x_α. The first two conditions do not pose a problem because if α^- exists, then as $\alpha^- * \infty \subset \delta$, there will be infinitely many elements z enumerated into S_{α^-}. Furthermore for each new z that α^- adds to S_{α^-}, we must have $\#[z]_{A_2} = 2$ and $\#\left[\frac{3z}{2}\right]_{A_3} = 3$ or vice versa. The third and fourth conditions also do not pose a problem because there are only finitely many such β, and none of them will increase S_β after stage s_0. The fifth and sixth conditions are similar, as a P-node β along the true path will only pick finitely many different x_β. Thus eventually x_α must be picked by α (and never again redefined).

Now suppose α picks its final x_α at some stage $t > s_0$. It is not hard to see that after stage t, no node other than α itself will be able to isolate $[x_\alpha]$: For an R-node β extending or to the right of α, use the fact that β is prevented from isolating anything less than the first β-expansionary stage. For an R-node β such that $\beta * \infty \subseteq \alpha$, apply Fact 4.3 and the fifth bullet of the strategy.

Therefore, if $\varphi_\alpha(x_\alpha) \neq 2$, then α never acts and so obviously $F_{A_2} \neq \varphi_e$. Otherwise if $\varphi_\alpha(x_\alpha) = 2$ we will isolate $[x_\alpha]$ making $F_{A_2}(x_\alpha) = 1$. □

Lemma 4.5 *Each P^2 requirement succeeds.*

Proof Assume that α is assigned a P^2 requirement. As in the proof of Lemma 4.4, α must eventually be able to define a final x_α. After this is picked by α, no node other than α itself is able to isolate $[x_\alpha]$. Suppose $\varphi_\alpha(x_\alpha + i)\downarrow$ for $i = 0, 1, 2$. If $A_2(\varphi_\alpha(x_\alpha + i), \varphi_\alpha(x_\alpha + j)) = 0$ for some $i, j = 0, 1, 2$, then α will never do anything to $[x_\alpha]$, which means that $A_3(x_\alpha + i, x_\alpha + j) = 1$. On the other hand, if $A_2(\varphi_\alpha(x_\alpha + i), \varphi_\alpha(x_\alpha + j)) = 1$ for every i, j, then as each A_2-class has size at most 2, this means that $\varphi_\alpha(x_\alpha + i) = \varphi_\alpha(x_\alpha + j)$ for some $i \neq j$. However α will isolate $[x_\alpha]$ which means that $A_3(x_\alpha + i, x_\alpha + j) = 0$. In either case, φ_α is not a possible reduction for $A_3 \leq_c A_2$. □

Lemma 4.6 *Each R^1 requirement succeeds.*

Proof Assume that α is assigned an R^1 requirement, and that f_n witnesses $E_m \leq_c A_2$ (otherwise there is nothing to prove). In particular, f_n is total, $E_m(x, y) = A_2(f_n(x), f_n(y))$ for every x, y and $l(\alpha, s) \to \infty$.

Suppose that $\alpha * fin \subset \delta$, hence there is a final α-expansionary stage. Then we claim that E_m is computable. Since there are only finitely many expansionary stages, hence either $rng(f_\alpha)$ is finite, or the number y cannot be found after the final α-expansionary stage. Obviously if $rng(f_\alpha)$ is finite, then E_m is computable. So assume that y cannot be found. Since S_{α^-} is infinite, and $\max S_\beta$ and x_β in the second through fifth conditions for y in Definition 4.1 are all eventually finite, the only thing preventing y from being found is if for almost every $z, z' \in rng(f_\alpha)$ we have $A_2(z, z') = 0$. (Recall that α^- will isolate every $z \notin S_{\alpha^-}$). But this of course means that E is computable (in fact, $E \equiv_c \mathtt{id}$).

Now suppose that $\alpha * \infty \subset \delta$. Hence, S_α is infinite. Suppose $s_1 > s_0$ is the first α-expansionary stage after the final initialization to α. We claim that no strategy can isolate any $z < y_{\alpha,s_1}$ after stage s_1. (Note that y_{α,s_1} is the smallest element of S_α). Again we apply Fact 4.3 and the fifth bullet of the strategy, and note that every node of lower priority than β is also initialized when β is initialized. This means that for every $z < y_{\alpha,s_1}$, we have $[z]_{s_1} = [z]$.

We shall argue that g_α is total and $A_2(x, y) = E_m(g_\alpha(x), g_\alpha(y))$ for every x, y. Clearly g_α is total because S_α is infinite. There are three possibilities for x. First, if $\max[y_{\alpha,t'}] < x < \min[y_{\alpha,t}]$ at some α-expansionary stage s, where $t' \geq s_1$, then the construction will make x isolated. The construction at stage s defines $g_\alpha(x) = b_{\alpha,j}$ for some b_{α_j} found at the α-expansionary stage s, and it is clear that $g_\alpha^{-1}([b_{\alpha,j}]) = \{x\}$. Therefore, $A_2(x, y) = E_m(g_\alpha(x), g_\alpha(y)) = 0$ no matter what y is.

Let's assume now that $x < y_{\alpha,s_1}$. Now at the second expansionary stage, we will define $g_\alpha(x) = b_{\alpha,j}$ for some b_{α_j} found at the second α-expansionary stage. Clearly if $y \geq y_{\alpha,s_1}$ then $y \notin [x]$, and we have $A_2(x, y) = E_m(g_\alpha(x), g_\alpha(y)) = 0$. On the other hand if $y < y_{\alpha,s_1}$ then the construction at the second α-expansionary stage defines $g_\alpha(y) = b_{\alpha,k}$ where $k = j$ iff $[y]_{s_1} = [x]_{s_1}$. However, by our comments above, $[x]_{s_1} = [x]$ and $[y]_{s_1} = [y]$ and so we certainly have $A_2(x, y) = E_m(g_\alpha(x), g_\alpha(y))$.

Finally let's assume that $x = y_{\alpha,s}$ or $x = y_{\alpha,s}+1$ for some α-expansionary stage s. Without loss of generality, assume $x = y_{\alpha,s}$. The construction defines $g_\alpha(x) = c_\alpha$ (for the value found at s), in particular, $f_\alpha(g_\alpha(x)) = x$. If $y \neq y_{\alpha,s}$ or $y_{\alpha,s}+1$ then of course $A_2(x,y) = E_m(g_\alpha(x), g_\alpha(y)) = 0$. If $y = y_{\alpha,s}+1$ then $f_\alpha(g_\alpha(y)) = y$ as well, and so $A_2(x,y) = A_2\left(f_\alpha(g_\alpha(x)), f_\alpha(g_\alpha(y))\right) = E_m(g_\alpha(x), g_\alpha(y))$. □

Lemma 4.7 *Each R^2 requirement succeeds.*

Proof Assume that α is assigned an R^2 requirement, and that f_α witnesses $E_i \leq_c A_3$. In particular, f_α is total, $E_i(x,y) = A_3(f_\alpha(x), f_\alpha(y))$ for every x, y and $L(\alpha, s) \to \infty$.

If $\alpha * \infty \subset \delta$ then we follow the proof of Lemma 4.6 to show that h_α is total and witnesses that $A_3 \leq_c E_i$. So we assume that $\alpha * fin \subset \delta$. Hence there is a final α-expansionary stage. We claim that $E_i \leq_c A_2$. Again since there are only finitely many expansionary stages, either $rng(f_\alpha)$ is finite, or the number y cannot be found after the final α-expansionary stage. Obviously if $rng(f_\alpha)$ is finite, then E_i is computable. So assume that y cannot be found. As before, the only thing preventing y from being found is if for almost every number z, we have that $\#\left([z]_{A_3} \cap rng(f_\alpha)\right) \leq 2$.

We first argue that $E_i \leq_c A_2 \sqcup \mathtt{id}$. Recall that given equivalence relations S and T we defined $S \sqcup T = \{(2x, 2y) \mid S(x,y) = 1\} \cup \{(2x+1, 2y+1) \mid T(x,y) = 1\}$. Fix y_0 such that $\#\left([z]_{A_3} \cap rng(f_\alpha)\right) \leq 2$ for every $z \geq y_0$, and consider a stage t after which $A_3 \upharpoonright y_0$ is stable. Define H to be the following partial computable function. Assume that at every stage $s > t$, exactly one new element z_s enters $rng(f_\alpha)$ (this set was assumed to be infinite). If $z_s < y_0$ and H is not yet defined on any element of $[z_s]$, then take $H(z_s) = 2s+1$. Otherwise take $H(z_s) = H(z_{s'})$ where $t < s' < s$ is such that $[z_s] = [z_{s'}]$. (Note that it is not important at what stage this is measured, as we assumed that $[z]$ is stable after stage t if $z < y_0$). Suppose $z_s \geq y_0$. If $H(z)$ is not yet defined for any z such that $\lfloor \frac{z}{3} \rfloor = \lfloor \frac{z_s}{3} \rfloor$ then define $H(z_s) = 4\lfloor \frac{z_s}{3} \rfloor$. If $H(z)$ is already defined for exactly one z such that $\lfloor \frac{z}{3} \rfloor = \lfloor \frac{z_s}{3} \rfloor$, define $H(z_s) = 4\lfloor \frac{z_s}{3} \rfloor + 2$. Otherwise $H(z)$ is already defined for two z such that $\lfloor \frac{z}{3} \rfloor = \lfloor \frac{z_s}{3} \rfloor$, and in this case we define $H(z_s) = 2s+1$.

Now we claim that $H \circ f_\alpha$ is total and witnesses $E_i \leq_c A_2 \sqcup \mathtt{id}$. It is clearly total since $dom(H) = rng(f_\alpha)$. Now fix x, y. We shall argue that $E_i(x,y) = (A_2 \sqcup \mathtt{id})\left(H(f_\alpha(x)), H(f_\alpha(y))\right)$. First suppose that $f_\alpha(x) < y_0$, then we would have defined $H(f_\alpha(x)) = 2s+1$ for some s. If $f_\alpha(y)$ is in a different class than $f_\alpha(x)$, then obviously $E_i(x,y) = 0$, and $H(f_\alpha(y))$ is either even or of the form $2v+1$ for some $v \neq s$. In either case, $E_i(x,y) = (A_2 \sqcup \mathtt{id})\left(H(f_\alpha(x)), H(f_\alpha(y))\right) = 0$. On the other hand if $f_\alpha(y) < y_0$ and is in the same class as $f_\alpha(x)$, then $E_i(x,y) = (A_2 \sqcup \mathtt{id})\left(H(f_\alpha(x)), H(f_\alpha(y))\right) = 1$.

Now if $f_\alpha(x), f_\alpha(y)$ are both $\geq y_0$ and $\lfloor \frac{f_\alpha(x)}{3} \rfloor \neq \lfloor \frac{f_\alpha(y)}{3} \rfloor$ then it is easy to check that $E_i(x,y) = (A_2 \sqcup \mathtt{id})\left(H(f_\alpha(x)), H(f_\alpha(y))\right) = 0$. On the other hand suppose $\lfloor \frac{f_\alpha(x)}{3} \rfloor = \lfloor \frac{f_\alpha(y)}{3} \rfloor$. If $H(f_\alpha(x))$ and $H(f_\alpha(y))$ are both even then we are okay, because recall that $[2k]_{A_2}$ is isolated iff $[3k]_{A_3}$ is isolated by the construction, for every k. On the other hand if $H(f_\alpha(x))$ is odd, then this means that $3\lfloor \frac{f_\alpha(x)}{3} \rfloor, 3\lfloor \frac{f_\alpha(x)}{3} \rfloor + 1$ and $3\lfloor \frac{f_\alpha(x)}{3} \rfloor + 2$ are all in $rng(f_\alpha)$, which means that these elements must be isolated by the construction. This means that $E_i(x,y) = (A_2 \sqcup \mathtt{id})\,(H(f_\alpha(x)), H(f_\alpha(y))) = 0$. This shows that $E_i \leq_c A_2 \sqcup \mathtt{id}$.

Now we check that $A_2 \sqcup \mathtt{id} \leq_c A_2$. By Lemma 4.4 there are infinitely many z such that z is A_2-isolated. Fix an infinite computable set U containing only A_2-isolated points, let $U = \{u_0 < u_1 < \cdots\}$. Define G by $G(2n+1) = u_{2n+1}$ for each n, and $G(2n) = u_{2n}$ if $n \in U$, and $G(2n) = n$ otherwise. First of all G is clearly injective. It is also easy to check that G witnesses that $A_2 \sqcup \mathtt{id} \leq_c A_2$. □

This ends the proof of Theorem 4.2. □

As mentioned earlier, our goal is to extend Theorem 4.2 to an infinite sequence $deg_c(\mathtt{id}) = \boldsymbol{a_1} <_c \boldsymbol{a_2} <_c \boldsymbol{a_3} <_c \cdots$:

Theorem 4.8 *There exist Π^0_1 c-degrees $deg_c(\mathtt{id}) = \boldsymbol{a_1} <_c \boldsymbol{a_2} <_c \boldsymbol{a_3} <_c \cdots$ such that $\boldsymbol{a_{k+2}}$ is a strong minimal cover of $\boldsymbol{a_{k+1}}$ for every k. All constructed degrees are not set-induced.*

Proof This is a straightforward generalization of Theorem 4.2. We want to construct, for each $k > 1$, a size-k Π^0_1 equivalence relation A_k. By Lemma 3.5, $deg_c(A_k)$ is not set-induced.

We again begin with fix $A_{k,0} = \{(ki+j_1, ki+j_2) \mid 0 \leq j_1, j_2 < i\}$. Naturally we wish to satisfy the requirements

- $\boldsymbol{Q^k}$: $A_{k-1} \leq_c A_k$, for $2 \leq k \leq n$.
- $\boldsymbol{R^k_{\langle m,n\rangle}}$: $E_m \leq_c A_k$ via $f_n \Rightarrow \exists$ computable function $g^{(k)}_{\langle n,m\rangle}$ s.t. $A_k \leq_c E_m$ via $g^{(k)}_{\langle n,m\rangle}$ or $E_m \leq_c A_{k-1}$, for $2 \leq k \leq n$.
- $\boldsymbol{P^2_e}$: $f_{A_2} \neq \varphi_e$.
- $\boldsymbol{P^k_{e'}}$: $A_k \not\leq_c A_{k-1}$ via $\varphi_{e'}$, for $3 \leq k \leq n$.

The strategy for Q^k is similar as before. We ensure that for every x, k and k', we isolate $[kx]_{A_k}$ iff we isolate $[k'x]_{A_{k'}}$, and we keep $\#[x]_{A_k} = 1$ or k. This obviously ensures the global requirements are met. R^k_α works similarly as before. It measures $rng(f_\alpha)$, putting numbers y into S_α whenever it finds a new y such that $\#[y] = k$ and $[y] \subseteq rng(f_\alpha)$ (amongst other conditions). It then isolates every class not in S_α (except for those blocked by higher priority requirements), as well as the corresponding classes in the other $A_{k'}$, as required by the Q requirements. The P^k-requirements uses a class $[x_\alpha]$ where $\#[x_\alpha] = k$ to diagonalize. Since classes of A_{k-1} have size at most $k-1$, this is achieved by preventing $[x_\alpha]$ from breaking up while waiting for φ_α to converge. There are no additional difficulties, and we leave the formal verification of the proof to the reader. □

5 Upward density

We now turn to the question of upward density. In Sections 3 and 4 we showed that downward density fails in the classes of c-degrees being considered in this paper. In this section, we will show that in contrast, upward density holds in these classes. We will first prove this for the Π^0_1 c-degrees.

Theorem 5.1 *Let $\boldsymbol{u}$ be the universal Π^0_1 c-degree, and $\boldsymbol{a}$ be any Π^0_1 c-degree such that $\boldsymbol{a} < \boldsymbol{u}$. Then there exists two incomparable Π^0_1 c-degrees $\boldsymbol{b_0}$ and $\boldsymbol{b_1}$ such that $\boldsymbol{a} < \boldsymbol{b_0}, \boldsymbol{b_1} < \boldsymbol{u}$.*

Proof Fix $U \in \boldsymbol{u}$ and $A \in \boldsymbol{a}$. We will build B_0 and B_1 satisfying the following requirements:

$$P_{2e} : B_0 \not\leq_c A \sqcup B_1 \text{ via } \varphi_e,$$
$$P_{2e+1} : B_1 \not\leq_c A \sqcup B_0 \text{ via } \varphi_e.$$

Obviously we will take $\boldsymbol{b_0} = deg_c(A \sqcup B_0)$ and $\boldsymbol{b_1} = deg_c(A \sqcup B_1)$.

The basic strategy for P_{2e} will be a version of the Sacks coding strategy. The idea is that if $\varphi_e(x)$ goes inside the even integers for infinitely many x, then by coding U inside B_0 and on the set of these xs, we can force a disagreement to appear, showing that $B_0 \not\leq_c A$ via φ_e. On the other hand if infinitely many $\varphi_e(x)$ go inside the odd integers, then it will be easy for us to ensure that $B_0 \not\leq_c B_1$ via φ_e, since we are building both sets B_0 and B_1.

We now describe the basic strategy of P_{2e}. We begin by taking $B_{i,0} = \{(\langle e, x\rangle, \langle e, y\rangle) \mid e, x, y \in \omega\}$ for $i = 0, 1$. Denote $C_e = \{\langle e, x\rangle \mid x \in \omega\}$; thus our starting value of B_i is the disjoint union of $C_e \times C_e$ for all e. As the construction proceeds we will refine B_i on C_e.

At stage s of the construction, we act for P_{2e} by taking the following steps. If P_{2e} has just been initialized we pick a fresh value for $i(2e)$ and begin work for B_0 inside $C_{i(2e)}$, and set $M = \langle i(2e), 0\rangle$. Note that at this point as $i(2e)$ is freshly picked, we have $B_0 \upharpoonright C_{i(2e)} = \left\{(x, y) \in B_0 \mid x, y \in C_{i(2e)}\right\} = C_{i(2e)} \times C_{i(2e)}$.

Now suppose that $i(2e)$ is defined and k is least such that $f_{2e}(k) \uparrow$. Check if $\varphi_e(M) \downarrow$ and φ_e currently looks like a reduction on the elements $\leq M$. If not, we keep $f_{2e}(k) \uparrow$ and isolate the next element of $C_{i(2e)}$ larger than M with respect to B_0. If the conditions are met, then our action depends on the following cases:

- *M is marked.* First of all check to see if φ_e is injective on $C_{i(2e)}$ so far, and if not, we get an easy win by isolating every element of $C_{i(2e)}$ with respect to B_0, and do nothing else for this requirement.
  Otherwise we set $B_0\left(f_{2e}(k-1), M\right) = 1 - (A \sqcup B_1)\left(\varphi_e\left(f_{2e}(k-1)\right), \varphi_e(M)\right)$. Note that we can do this because M is marked means that we have kept $B_0\left(f_{2e}(k-1), M\right) = 1$ up till now. Initialize all lower priority requirements, and isolate all the rest of $C_{i(2e)}$ with respect to B_0. Do nothing else for this requirement.
- *The following three conditions hold:*
  - *M is not marked, and*
  - *$\varphi_e(M)$ is odd, and*
  - *either $P \notin C_{i(e')}$ for any $e' < 2e$, or P is isolated, or $\varphi_e(M) = \varphi_e(M')$ for some $M' < M$ and $M' \in C_{i(2e)}$, and where $P = \frac{\varphi_e(M)-1}{2}$.*

  In this case we set $f_{2e}(k) = M$ and redefine M to be the next element of $C_{i(2e)}$ larger than $f_{2e}(k)$ and not yet isolated. We mark the new M and isolate it from every element except for $f_{2e}(k)$.
- *M is not marked and $\varphi_e(M)$ is even.* Then we set $f_{2e}(k) = M$ and redefine M to be the next element of $C_{i(2e)}$ larger than $f_{2e}(k)$ and not yet isolated. We update the coding of U into B_0 by

setting $U_s(i,j) = B_{0,s}(f_{2e}(i), f_{2e}(j))$ for all $i,j \leq k$, and set $U_s(i,k+1) = B_{0,s}(f_{2e}(i), M) = B_{0,s}(M, f_{2e}(i))$ for all $i \leq k$ and for the new M. Again we can do this because the new value of M is B_0-related to $f_{2e}(i)$ for all $i \leq k$ just before this step.

- *Otherwise*. We isolate M and redefine it to be the next element of $C_{i(2e)}$ not yet isolated. Follow the previous case to update the coding of U into B_0 up to and including this new M (this new M is now the intended image for $k+1$).

The basic strategy for P_{2e+1} is similar to the one for P_{2e}, except we reverse the roles of B_0 and B_1. When a requirement is initialized, we reset the parameters f, i and M associated with the requirement. We isolate all remaining elements of the previous column C_i associated with the requirement.

Construction of B_0 and B_1: As indicated above, at the beginning we begin by taking $B_{i,0} = \{(\langle e,x\rangle, \langle e,y\rangle) \mid e,x,y \in \omega\}$ for $i = 0,1$. At stage $s > 0$, let all requirements R_e for $e < s$ act in order according to the basic strategy above.

Verification: We now verify the construction works. It is clear that each R_e is initialized only finitely often. We first check that the constructed objects are equivalence relations:

Lemma 5.2 *B_0 and B_1 are equivalence relations.*

Proof The only way for something to go wrong is for some column to be not transitive. So let's check that every column C_k is transitive with respect to B_0 (similarly for B_1):

We initially start with the full relation on C_k. If C_k is never picked by any requirement then it stays full. Otherwise some requirement P_{2e} begins work in C_k. Then P_{2e} will assign $f_{2e}(0), f_{2e}(1), \cdots$ and begin isolating every other element until initialized. Additionally it also copies $U \restriction dom f_{2e}$ into $B_0 \restriction rng f_{2e}$ (unless the last element is marked). Since for every m, there are cofinitely many s such that $U_s \restriction m$ is transitive, we can assume that $B_0 \restriction rng f_{2e}$ is always transitive.

At the end, we have the following possibilities:

- f_{2e} is total. Then C_k will contain $rng f_{2e}$ and infinitely many isolated elements, where $rng f_{2e} \equiv_c U$.
- $dom f_{2e}$ is finite, and we eventually go through infinitely many different M. Then C_k contains a (copy of a) finite initial segment of U as well as infinitely many isolated elements.
- $dom f_{2e}$ is finite, and we eventually settle on a final M, where k is the largest element of $dom f_{2e}$. If M is marked then M is in the same class as $f_{2e}(k)$ and C_k contains the finite initial segment of $U \restriction k$. Otherwise if M is not marked then C_k contains the finite initial segment of $U \restriction k+1$. Finally if R_{2e} is initialized or if we act one last time under the first case of the basic strategy, then C_k is also transitive. □

Lemma 5.3 *Each R_e acts only finitely often, that is, it extends $dom f_e$ and redefines M only finitely often.*

Proof We proceed for R_{2e}, the other case is symmetric. Since R_{2e} is initialized finitely often, fix the final $C_{i(2e)}$ assigned to R_{2e}. Suppose R_e acts infinitely often. Hence no element is ever marked by R_{2e}.

Suppose the third case is taken infinitely often by R_{2e}. Hence f_{2e} is total. Furthermore, as R_{2e} never gets stuck waiting forever, this means that $\varphi_e \circ f_{2e}$ is total, and provides a reduction from U to $A \sqcup B_1$. Since the range of this reduction is contained in the even integers, this means that $U \leq_c A$, a contradiction.

Now suppose the fourth case is taken cofinitely often by R_{2e}. In this case we will discover, for some $e' < 2e$, an infinite sequence $M_0 < M_1 < \cdots$ of elements inside $C_{i(2e)}$ such that for every $j \neq j'$,

- $\varphi_e(M_j)$ is odd,
- $\frac{\varphi_e(M_j)-1}{2} \in C_{i(e')}$,
- $\frac{\varphi_e(M_j)-1}{2}$ is picked by $R_{e'}$ to be in $rng f_{e'}$ or a value of M,
- $\varphi_e(M_j) \neq \varphi_e(M_{j'})$.

However this is impossible since by the induction hypothesis, $R_{e'}$ only goes through finitely many different values of M and $dom f_{e'}$. □

Lemma 5.4 *Each R_e is satisfied.*

Proof Assume that φ_e witnesses that $B_0 \leq_c A \sqcup B_1$. By Lemma 5.3 R_{2e} eventually stops acting, therefore, the final action taken by R_{2e} must be under the first case. If the final action was due to the fact that we discovered that φ_e is not injective on $C_{i(2e)}$, then clearly we get a contradiction. Thus we can assume that the final action under the first case was to set $B_0\left(f_{2e}(k-1), M\right) = 1 - (A \sqcup B_1)\left(\varphi_e\left(f_{2e}(k-1)\right), \varphi_e(M)\right)$. Suppose s_1 is the stage where we took the final action, and $s_0 < s_1$ was the earlier stage where R_{2e} marked M. It remains to verify that $(A \sqcup B_1)\left(\varphi_e\left(f_{2e}(k-1)\right), \varphi_e(M)\right)$ will never change after s_1, and we get the desired contradiction. This can only be possible if $\left(\varphi_e\left(f_{2e}(k-1)\right), \varphi_e(M)\right) \in A_{s_1} \sqcup B_{1,s_1}$.

At the earlier stage where we marked M we must have discovered that $\varphi_e\left(f_{2e,s_1}(k-1)\right)$ is odd and that either $\frac{\varphi_e\left(f_{2e,s_1}(k-1)\right)-1}{2} \notin C_{i(e')}$ for any $e' < 2e$, or $\frac{\varphi_e\left(f_{2e,s_1}(k-1)\right)-1}{2}$ is isolated. If the latter holds then we must in fact have $\varphi_e\left(f_{2e,s_1}(k-1)\right) = \varphi_e(M)$.

So we may suppose that $\frac{\varphi_e\left(f_{2e,s_1}(k-1)\right)-1}{2} \notin C_{i(e')}$ for any $e' < 2e$ at stage s_0. At stage s_1, if $v \neq i(e')$ for any e', then C_v will never be assigned to any requirement and so is never modified by the construction, where $\frac{\varphi_e\left(f_{2e,s_1}(k-1)\right)-1}{2} \in C_v$. Therefore we can assume that $v = i(e')$ at stage s_1. Since no higher priority requirement is initialized between s_0 and s_1, we must have $e' \geq 2e$. We do not have to worry about $e' = 2e$ because $C_{i(2e)}$ must be the full relation with respect to B_1, since it is already associated with R_{2e} working for B_0. Thus the last case to consider is when $e' > 2e$. However $R_{e'}$ is initialized at stage s_1, which means that there can be no more changes to C_v with respect to B_1 after stage s_1. □

This ends the proof of Theorem 5.1. □

It is easy to modify the proof of Theorem 5.1 for the other classes being studied. Therefore, we have:

Corollary 5.5 *The classes of* Π^0_1, Σ^0_n, *n-c.e. and* ω*-c.e. equivalence relations are upwards dense, for* $n \geq 1$.

Proof In each class, there is a universal member U. We apply an appropriate version of the Sacks coding strategy, and leave the proof to the reader. □

Acknowledgments

The first author is partially supported by the grants MOE2015-T2-2-055 and RG131/17.

Note

1. Note that S and E_e are actually equal instead of merely $\equiv_c$. This is the best kind of enumeration one can hope for.

References

[1] U. Andrews, S. Lempp, J. Miller, and K.M. Ng. Universal computably enumerable equivalence relations. *Journal of Symbolic Logic*, 79(1):60 – 88, 2014. 2, 3, 11, 12

[2] C. Bernardi and A. Sorbi. Classifying positive equivalence relations. *Journal of Symbolic Logic*, 48(3):529–538, 1983. 2

[3] W. Calvert and J.F. Knight. Classification from a computable viewpoint. *Bulletin of Symbolic Logic*, 12(2):191–218, 2006. 2

[4] S. Coskey, J.D. Hamkins, and R. Miller. The hierarchy of equivalence relations on the natural numbers under computable reducibility. *Computability*, 1(1):15–38, 2012. 2

[5] Yu. L. Ershov. Theory of numberings. 1977. 2

[6] E.B. Fokina and S.-D. Friedman. Equivalence relations on classes of computable structures. In *In Proceedings of the 5th Conference on Computability in Europe: Mathematical Theory and Computational Practice*, Springer-Verlag. 2

[7] H. Friedman and L. Stanley. A borel reducibility theory for classes of countable structures. *Journal of Symbolic Logic*, 54(3):894–914, 1989. 1

[8] S. Gao and P. Gerdes. Computably enumerable equivalence relations. *Studia Logica*, 67(1):27–59, 2001. 2, 3

[9] E. Ianovski, R. Miller, K.M. Ng, and A. Nies. Complexity of equivalence relations and preorders from computability theory. *Journal of Symbolic Logic*, 79(3):859, 2014. 2, 3, 4, 6, 9, 10

[10] A. H. Lachlan. Initial segments of one-one degrees. *Pacific Journal of Mathematics*, 29:351–366, 1969. 3, 11, 12

[11] R. Miller and K.M. Ng. Finitary reducibilities on equivalence relations. *Journal of Symbolic Logic*, 81(4):1225–1254, 2016. 2

[12] R.I. Soare. *Recursively enumerable sets and degrees : a study of computable functions and computably generated sets.* Perspectives in mathematical logic. Berlin ; New York : Springer-Verlag, c1987., 1987. 4

[13] J.F. Knight W. Calvert, D. Cummins and S. Miller. Comparing classes of finite structures. *Algebra and Logic*, 43(6):374–392, 2004. 2

[14] P. R. Young. Notes on the structure of recursively enumerable sets. *Notices of the American Mathematical Society*, 10:586, 1963. 11

Ng
School of Physical and Mathematical Sciences
Nanyang Technological University
Singapore 637371
Republic of Singapore kmng@ntu.edu.sg
http://www.ntu.edu.sg/home/kmng/

Yu
School of Physical and Mathematical Sciences
Nanyang Technological University
Singapore 637371
Republic of Singapore