On the degree structure of equivalence relations under computable reducibility

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Abstract We study the degree structure of the $\omega$-c.e., $n$-c.e. and $\Pi_0^1$ equivalence relations under the computable many-one reducibility. In particular we investigate for each of these classes of degrees the most basic questions about the structure of the partial order. We prove the existence of the greatest element for the $\omega$-c.e. and $n$-c.e. equivalence relations. We provide computable enumerations of the degrees of $\omega$-c.e., $n$-c.e. and $\Pi_0^1$ equivalence relations. We prove that for all the degree classes considered, upward density holds and downward density fails.

1 Introduction

An important theme in the study of mathematics is the classification of mathematical structures according to various different criteria. If we can identify when two different structures are “equivalent”, then we could reduce the study of all structures in a certain class to just examining the essential properties in each equivalence class of structures. The notion of dimension in algebra is an example of such a classification. A common tool in this endeavour is to define a reduction; this is a map which reduces a (possibly) more complicated problem to a simpler question.

In the context of Borel theory, a large body of work exists on classifying problems via the Borel reducibility. These reducibilities focus on classifying equivalence relations on an uncountable domain, for example, Friedman and Stanley [7] applied this to study the isomorphism problem on the countable models of a theory.

A more recent direction of investigation has been directed towards studying equivalence relations on the domain $\omega$. By restricting the domain down to a countable set, the usual tools of computability theory can be applied to supply meaningful definitions, concepts and proof techniques. The main

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reducibility used to calibrate the relative complexity of equivalence relations on $\omega$ is computable reducibility:

**Definition 1.1** Let $E$ and $F$ be equivalence relations on $\omega$. We say that $E$ is computably reducible to $F$, denoted as $E \leq_c F$, if there exists a computable function $f : \omega \to \omega$ such that

$$\forall x, y \in \omega, \quad (x, y) \in E \Leftrightarrow (f(x), f(y)) \in F.$$ 

This definition is obviously closely related to the Borel reducibility; in the Borel case one studies equivalence relations on $2^\omega$ and we require the reductions to be Borel, or sometimes merely continuous. Indeed, Coskey, Hamkins and Miller [4] and later Miller and Ng [11] studied the countable analogues of several standard equivalence relations arising in Borel theory. This reducibility has gone by many different names in the literature, having been called $m$-reducibility in [1, 8, 2] and $FF$-reducibility in [6], in addition to a version on first-order theories which was called Turing-computable reducibility [13, 3]. Computable reducibility was also used in [9, 11]; we prefer this name since we wish to avoid confusion with the usual $m$-reducibility and Turing-reducibility between sets of numbers.

The computable reducibility was first introduced by Ershov in [5]. However his approach was much more indirect in that he phrased the notions inside the language of categories. Indeed his motivation was to study the theory of numberings, which is of fundamental interest in computable mathematics. In recent years there has been a sudden increase in interest in the computable reducibility between equivalence relations on $\omega$.

The objective of this paper is to initiate a systematic study of the degrees induced by the computable reducibility: The preordering $\leq_c$ induces an equivalence relation on the class of all equivalence relations. The class of all $R$ that are $c$-equivalent to $S$ is called the $c$-degree of $S$, denoted as $\deg_c(S)$. In this paper we shall be mainly interested in the $\alpha$-c.e. equivalence relations for $\alpha \leq \omega$ and the $\Pi^0_1$ equivalence relations; the descriptive complexity of an equivalence relation is obviously evaluated when it is viewed as a subset of $\mathbb{N} \times \mathbb{N}$. A $c$-degree is $\alpha$-c.e. if it contains an $\alpha$-c.e. member.

The concept of *degrees* has been a central notion since the dawn of computability theory, and much effort has been directed towards the understanding of the algebraic structure of the degrees of different classes of sets. For instance, the c.e. $m$-degrees, 1-degrees, (weak) truth table degrees and Turing degrees have all been well-studied. The original view that many people had of these structures were that they would be very simple and easy to classify. However, it turns out that these structures, particularly the c.e. Turing degrees, are much more complicated than originally thought. The study of the Turing degrees of larger classes of sets have also been extensively carried out, for instance, the $d$-c.e. Turing degrees and the $\Delta^0_2$ Turing degrees have both turned out to be of interest, and shown to be structurally very different from the c.e. Turing degrees. Our aim here is not to dwell upon these results as these are all readily available in the literature. Rather, we wish to point out that this earlier work serves as the primary motivation for the questions we wish to investigate in this paper.
One of the most basic questions to ask of a partial order is the existence of a greatest element. Given a class \( C \) of \( c \)-degrees, we call \( u \in C \) universal if \( u \geq a \) for every \( c \)-degree \( a \in C \). We note here that the question of a least \( c \)-degree is uninteresting, for the computable equivalence relations generate an initial segment of \( c \)-degrees of length \( \omega + 1 \).

The question of universality has already been studied for some classes of \( c \)-degrees. For instance, Lachlan [10] studied certain kinds of universal c.e. equivalence relations (ceers), called precomplete ceers. We remark that ceers are also known as positive equivalence relations, since they are closely related to positive numberings. Gao and Gerdes [8] and Andrews et al. [1] studied the c.e. \( c \)-degrees extensively, particularly in [1], where different types of universal c.e. \( c \)-degrees were identified and compared. The question of universality was then studied further in [9], where the authors showed the existence of a universal \( \Pi^0_1 \) \( c \)-degree, and that no universal degree exists for the class of \( \Delta^0_2 \) \( c \)-degrees.

The notion of a universal element of a class is closely related to the existence of a computable enumeration of the members of that class. If one can effectively list out all elements of the class, then taking the disjoint union of the members of the effective list will usually produce a universal element of the class. For instance, the Halting problem, which has universal \( m \)-degree amongst the c.e. sets, can be viewed as being the disjoint union of all c.e. sets (we can even effectively list all c.e. sets without repetition, a famous result of Friedberg). Indeed, the existence of a universal c.e. \( c \)-degree can be easily proved in the same way; first, notice that the transitive closure of any c.e. binary relation is again c.e., and hence we can easily produce an effective list containing all c.e. equivalence relations. Taking the disjoint union of this list (with no relations between different members of the list) produces a universal c.e. equivalence relation.

The same trick works to produce a universal \( \Sigma^0_n \) \( c \)-degree for each \( n > 0 \). Unfortunately, an effective listing of all \( \Pi^0_1 \) equivalence relations is less obvious. Indeed, the authors of [9] proved the existence of a universal \( \Pi^0_1 \) \( c \)-degree in an indirect way. For \( \alpha \)-c.e. equivalence relations, where \( 1 < \alpha \leq \omega \), the question of universality was not known before this article. In Section 2, we answer the question of universality in these classes and also investigate what kinds of effective enumerations are possible for these classes. The results are summarized in the table on page 4.

Another fundamental question one can ask about a partial order is whether the structure is dense. Interest in this particular property is partially motivated by results in the c.e. and \( \Delta^0_2 \) Turing degrees, where similar questions about density have led to a rich collection of knowledge and to new techniques being invented. Related questions such as those on lattice embeddings have also enriched the field. We will investigate the question of density in the classes of \( c \)-degrees we are interested in. We prove that in each of these classes (\( \alpha \)-c.e. for \( 1 \leq \alpha \leq \omega \), \( \Pi^0_1 \) and \( \Delta^0_2 \) \( c \)-degrees), downward density fails but upward density holds. We believe that this is due to the fact that it is much easier to code given information into a relation we build, compared to permitting below a given equivalence relation.
In Section 3 we consider the problem of downward density in the structure of c.e. \( c \)-degrees. It is not hard to see that the map \( A \mapsto \{(x, y) \mid x = y \text{ or } x, y \in A\} \) is an embedding (of partial orders) of the c.e. 1-degrees into the c.e. \( c \)-degrees. We call a \( c \)-degree set-induced if it contains a relation in the range of this embedding. An old result of Lachlan implies the existence of a set-induced \( c \)-degree for which downward density fails. These results and terminologies will be further elaborated upon in Section 3. We construct two different counter-examples to the downward density of the c.e. \( c \)-degrees, one which is above \( \text{deg}_c(\text{id}) \), and another which is not above \( \text{deg}_c(\text{id}) \). Both our counter-examples are, of course, not set-induced.

In Section 4 we show that downward density fails for the \( \Pi^0_1 \) \( c \)-degrees, and generalize this theorem by constructing an infinite sequence of \( \Pi^0_1 \) \( c \)-degrees \( \text{deg}_c(\text{id}) < \text{deg}_c(A_2) < \text{deg}_c(A_3) < \cdots \), each of which is a strong minimal cover over the previous. All our counter-examples are not set-induced.

In Section 5 we investigate upward density in the \( c \)-degrees. We show that in contrast to downward density, upward density holds in all the classes of interest. We do this by constructing a pair of incomparable \( \Pi^0_1 \) \( c \)-degrees above any given non-universal \( \Pi^0_1 \) \( c \)-degree.

<table>
<thead>
<tr>
<th>Degree class</th>
<th>Existence of a universal element</th>
<th>Existence of a computable enumeration</th>
<th>Downward density</th>
<th>Upward density</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Sigma^0_{n}, n \geq 1 )</td>
<td>✔ (Folklore)</td>
<td>✔ (Folklore)</td>
<td>× (Cor 3.10)</td>
<td>✔ (Cor 5.5)</td>
</tr>
<tr>
<td>( n )-c.e., ( n &gt; 1 )</td>
<td>✔ (Cor 2.2)</td>
<td>✔ (Thm 2.1)</td>
<td>× (Cor 3.10)</td>
<td>✔ (Cor 5.5)</td>
</tr>
<tr>
<td>( \omega )-c.e.</td>
<td>✔ (Cor 2.6)</td>
<td>– of all relations × (Folklore) – of all degrees ✔ (Thm 2.5)</td>
<td>× (Cor 3.10)</td>
<td>✔ (Cor 5.5)</td>
</tr>
<tr>
<td>( \Pi^0_1 )</td>
<td>✔ (Thm 3.3 of [9])</td>
<td>– of all relations × (Prop 2.10) – of all degree upper cones ✔ (Thm 2.8)</td>
<td>× (Thm 4.1)</td>
<td>✔ (Thm 5.1)</td>
</tr>
<tr>
<td>( \Delta^0_2 )</td>
<td>× (Thm 3.7 of [9])</td>
<td>× (no universal element)</td>
<td>× (Cor 3.10)</td>
<td>✔ (Thm 3.7 of [9])</td>
</tr>
</tbody>
</table>

1.1 Preliminaries. Our notation is standard, and for these and other computability theory concepts, we refer the reader to [12]. For the rest of this paper, an
(equivalence) relation $S$ always refers to a subset of ordered pairs, i.e. $S \subseteq \mathbb{N}^2$, and the domain of each relation is always $\mathbb{N}$.

If $S$ and $T$ are equivalence relations, denote $S \sqcup T$ by the equivalence relation $\{(2x,2y) \mid (x,y) \in S\} \cup \{(2x+1,2y+1) \mid (x,y) \in T\}$. Notice that $\text{deg}_e(S \sqcup T)$ is not obviously the least upper bound of the pair $\text{deg}_e(S)$ and $\text{deg}_e(T)$, since if $f$ witnesses $S \leq_c X$ and $g$ witnesses $T \leq_c X$ then $\text{rng}(f)$ and $\text{rng}(g)$ might contain elements that are $X$-related; we will not know how to identify these elements inside $S \sqcup T$.

Given a binary relation $S$ and an effective enumeration $\{S_s\}_{s \in \omega}$ of $S$, define $S \upharpoonright t := \{(x,y) \in S \mid x,y < t\}$ and $S_s \upharpoonright t := \{(x,y) \in S_s \mid x,y < t\}$. We sometimes append $[s]$ to an expression to denote the value of the expression evaluated at step or stage $s$. A number $x$ is said to be isolated with respect to an equivalence relation $T$ if $T(x,y) = 0$ for every $y \neq x$.

We fix notations for several standard equivalence relations. Let $\equiv_n$ be the computable equivalence relation with exactly $n$ many distinct equivalence classes. (Note that not every equivalence relation with finitely many classes is computable, for example, take the equivalence relation with two classes $\{A, \omega-A\}$ for any non-computable set $A$). Also fix $\text{id}$ to be the computable equivalence relation with infinitely many classes. Of course we have $$\equiv_1 <_c \equiv_2 <_c \cdots <_c \text{id},$$ and every computable equivalence relation is of the same degree as one of these. Furthermore, this chain of degrees form an initial segment of the global $c$-degrees - since if $R$ is reducible to any degree in the chain then $R$ is computable.

In order for us to work with the various classes of equivalence relations (ceers, $\Pi_1^0$ and $n$-c.e.), we wish to fix an effective enumeration of all members of the class. This is unfortunately not always possible, so we fix a computable enumeration $\{S_e\}_{e \in \omega}$ of all reflexive and symmetric (c.e., $\Pi_1^0$ or $n$-c.e., depending on the context) binary relations. In fact, we will fix the approximation to $\{S_e\}_{e \in \omega}$ such that for all $e, x, y, t$, $S_e(x, x)[t] = 1$ and $S_e(x, y)[t] = S_e(y, x)[t]$.

Notice that the reflexive and symmetric closure of a binary relation has the same descriptive complexity as the relation, while the transitive closure of a binary relation $S$ is $\Sigma_1^0$ relative to $S$. Thus, the issue of whether $\{S_e\}_{e \in \omega}$ can be turned into a computable enumeration of all members of the class comes down to the difficulty of recognizing transitivity in the class; for ceers this is trivial (the transitive closure of a c.e. binary relation is still c.e.), while for $\Pi_1^0$ and $n$-c.e. equivalence relations this question will be addressed in the respective section. The existence of an enumeration $\{S_e\}_{e \in \omega}$ for $\omega$-c.e. equivalence relations has to be defined in the right way, and we will discuss this in Section 2.2.

1.2 Organization of the paper The paper is organized as follows. We shall be primarily interested in the class of $\Pi_1^0$ equivalence relations, the class of $\omega$-c.e. equivalence relations and the class of $n$-c.e. equivalence relations for each $n \geq 1$.

In Section 2 we consider the problem of producing a computable enumeration of each class as well as the existence of a universal equivalence relation of
the class. In Theorem 2.1 we provide a computable enumeration of all n-c.e. equivalence relations and deduce the existence of a universal n-c.e. equivalence relation for each \( n \geq 1 \). In Theorem 2.5 we provide a computable enumeration of \( \omega \)-c.e. equivalence relations in which every degree is represented, and deduce the existence of a universal \( \omega \)-c.e. equivalence relation. Then in Theorem 2.8 we provide a computable enumeration of \( \Pi_1^0 \) equivalence relations in which the upper cone of every degree is represented, and deduce the existence of a universal \( \Pi_1^0 \) equivalence relation. Note that the existence of a universal \( \Pi_1^0 \) equivalence relation is proved in [9], while a universal \( \alpha \)-c.e. equivalence relation (for \( 1 < \alpha \leq \omega \)) was not known before. We also show in Proposition 2.10 that a computable enumeration of all \( \Pi_1^0 \) equivalence relations is impossible.

In Section 3 we consider the problem of downward density in the structure of the c-degrees of c.e. equivalence relations. In Theorem 3.7 we construct a c.e. c-degree which is not set-induced, minimal modulo the computable c-degrees and not above \( \text{deg}_c(\text{id}) \). In Theorem 3.9 we construct another c.e. c-degree which is not set-induced and minimal, and this time above \( \text{deg}_c(\text{id}) \).

In Section 4 we show that downward density fails for the \( \Pi_1^0 \) c-degrees. Next we generalize this theorem by constructing an infinite sequence of \( \Pi_1^0 \) c-degrees \( \text{deg}_c(\text{id}) < \text{deg}_c(A_2) < \text{deg}_c(A_3) < \cdots \), each of which is a strong minimal cover over the previous. All our counter-examples are not set-induced.

Finally in Section 5 we prove that in contrast to downward density, upward density holds in all the classes of interest.

2 Enumerations of a class and universality

2.1 n-c.e. equivalence relations In this section we address the question of the existence of a computable enumeration of all n-c.e. equivalence relations, and the existence of a universal n-c.e. equivalence relation. We fix some \( n \in \omega \), \( n \geq 1 \).

Theorem 2.1 There is a computable enumeration of all n-c.e. equivalence relations. More specifically, there exists a computable function \( f(e, x, y, s) \) such that:

- For all \( e, x, y, \lim_{s \to \infty} f(e, x, y, s) = E_e(x, y) \).
- For all \( e, E_e \) is a n-c.e. equivalence relation.
- For each n-c.e. equivalence relation \( S \) there exists some \( e \) such that \( S = E_e \).
- For all \( e, x, y, f(e, x, y, 0) = 0 \) and \( \# \{ s \mid f(e, x, y, s) \neq f(e, x, y, s+1) \} \leq n \).

Proof As mentioned in Section 1.1 we fix \( \{ S_e \}_{e \in \omega} \) to be a computable enumeration of all n-c.e. binary relations which are reflexive and symmetric. For each \( e \), we construct (an approximation to) \( E_e \) uniformly in \( e \).

At stage \( v = 0 \), define \( u_0 := 0 \) and \( E_{e,0} := \{ (x, x) : x \in \omega \} \), and do nothing else. At stage \( v > 0 \), let \( t \) be largest such that \( u_t \downarrow \), and assume that \( E_{e,v-1} \) has also been defined (note that \( t \) always exists). Check if \( S_{e,v} \uparrow t + 1 \) is transitive. If it is, define \( u_{t+1} = v \) and \( E_{e,v} := S_{e,v} \uparrow t + 1 \). If not, define \( E_{e,v} := E_{e,v-1} \) and leave \( u_{t+1} \uparrow \).

We now verify that \( E_e \) has the properties we want.
The existence of a computable enumeration of all $\omega$-c.e. equivalence relation immediately yields:

Corollary 2.2 There exists a universal $\omega$-c.e. equivalence relation for any fixed $n \geq 1$.

However, if one considers varying $n$ over all of $\omega$, then there will not be a universal element of the class:

Proposition 2.3 The collection

$$\{S : S \text{ is an } \omega\text{-c.e. equivalence relation for some } n \in \omega\}$$

has no universal element.

Proof Given an $\omega$-c.e. equivalence relation $S$, the obvious thing to do is to consider the following equivalence relation (generated by) $E$:

$$E := \{(\langle e, 0 \rangle, \langle e, 1 \rangle) \mid \varphi_e(\langle e, 0 \rangle) \downarrow = x \land \varphi_e(\langle e, 1 \rangle) \downarrow = y \land S(\langle x, y \rangle) = 0\}.$$ 

By the definition of $E$, we have for each $e$ such that $\varphi_e$ is total, $E(\langle e, 0 \rangle, \langle e, 1 \rangle) \not= S(\varphi_e(\langle e, 0 \rangle), \varphi_e(\langle e, 1 \rangle))$, which implies $E \not\leq C S$ via $\varphi_e$.

Now for each $e$, $E(\langle e, 0 \rangle, \langle e, 1 \rangle)$ initially starts off taking value 0, before $\varphi_e$ converges on the two inputs. Once $\varphi_e(\langle e, 0 \rangle)$ and $\varphi_e(\langle e, 1 \rangle)$ both converge, we might see $S(x, y) = 0$ where we now have to let $E(\langle e, 0 \rangle, \langle e, 1 \rangle) = 1$, and subsequently we follow the changes in $S(x, y)$. Hence $E$ changes at most one more time than $S$, and so $E$ is $n + 1$-c.e.

2.2 $\omega$-c.e. equivalence relations Now we turn to $\omega$-c.e. equivalence relations. In particular, we wish to investigate the question of whether there exists a universal $\omega$-c.e. equivalence relation, and what kind of enumerations are possible of this class. The best kind of enumeration one could ask for is an effective listing $\langle \langle S_e, f_e \rangle \rangle_{e \in \omega}$ of all $\omega$-c.e. equivalence relations $S_e$ along with a computable bound $f_e$ for the number of changes in $S_e$. This is of course impossible, due to general computability theoretic reasons. Therefore, we will fix an enumeration $\langle \langle S_e, h_e \rangle \rangle_{e \in \omega}$ in the following sense:

Lemma 2.4 There exists an effective listing $\langle \langle S_e, h_e \rangle \rangle_{e \in \omega}$ of all $\omega$-c.e. equivalence relations $S_e$ such that $h_e$ is partial computable. Here every $S_e$ is an $\omega$-c.e. equivalence relation and every $\omega$-c.e. equivalence relation $S$ is equal to
\(S_e\) for some \(e\) such that \(h_e\) is total and bounds the number of mind changes of \(S_e\).

**Proof** \(S_e\) is taken to be a slowed down version of the \(e^{th}\) possible \(\omega\)-c.e. binary relation \(T_e\). We follow the proof of Theorem 2.1. We start with \(S_e(x, y)[0] = 0\) for all \(x \neq y\). For some \(t\), if we discover that \(\varphi_e(x, y)\) has converged for all \(x, y < t\), we proceed to wait for convergence of all \(x, y < t + 1\). Before that happens, we keep \(S_e(x, y) = 0\) for all \((x, y)\) such that \(x > t\) or \(y > t\), and update \(S_e\) to copy changes in \(T_e \upharpoonright t\) only when \(T\) looks transitive. When \(\varphi(x, y)\) converges for all \(x, y < t + 1\), if ever, and when \(T_e \upharpoonright t + 1\) again looks transitive, we extend the domain of \(h_e\) and then set \(S_e\) to now copy \(T_e \upharpoonright t + 1\), again updating only if \(T_e \upharpoonright t + 1\) looks transitive. This ensures that \(S_e\) is always an equivalence relation, and the domain of \(f_e\) is \(\{(x, y) | x, y < t\}\) for some \(t \leq \omega\), and where \(S_e = T_e\) and \(h(e)\) is total in the event that \(T_e\) turns out to be \(\omega\)-c.e. and transitive.

Note that if \(h_e\) is partial, then \(S_e - \{(x, x) | x \in \omega\}\) is finite and certainly \(\omega\)-c.e.

Unlike Theorem 2.1 for \(n\)-c.e. equivalence relations, Lemma 2.4 is not sufficient to deduce the existence of a universal \(\omega\)-c.e. equivalence relation, because the effective disjoint union of all \(S_e\) is not \(\omega\)-c.e. However, recall that every \(\omega\)-c.e. \(m\)-degree (of subsets of \(\omega\)) contains an \(\omega\)-c.e. set with an approximation that changes at most identity bounded many times. If we could prove, for example, the analogous statement that there exists a computable enumeration \(\{E_e\}_{e \in \omega}\) of equivalence relations with identity bounded changes, and which contains an equivalence relation of every \(\omega\)-c.e. degree, then the disjoint union of \(E_e\) will provide a universal element of the class. This is provided by the following theorem:

**Theorem 2.5** For each \(e\), there exists (uniformly in \(e\)) an approximation to an equivalence relation \(E_e\) such that:

- For every \(x, y\), the number of changes in the approximation of \(E_e(x, y)[s]\) is bounded by \(\max \{x, y\}\).
- If \(h_e\) is total then \(S_e \equiv_c E_e\).

**Proof** Fix \(e\). Let \(\hat{h}(y) = \max_{x, x' \leq y} h_e(x, x')\); then \(\hat{h}\) is a partial computable function whose domain is an initial segment of \(\omega\). Without loss of generality, we assume that \(\hat{h}\) is strictly increasing on its domain. Define \(h^*(x) = \max \{z - 1, 0\}\) where \(z\) is least such that \(\hat{h}(z) \geq x\). Now define

\[E_e(x, y) = \begin{cases} S_e(h^*(x), h^*(y)), & \text{if } h^*(x), h^*(y) \text{ both exist}, \\ 1, & \text{if } x = y, \\ 0, & \text{otherwise.} \end{cases}\]

Now \(E_e(x, y)\) has the following obvious approximation for \(x \neq y\): Begin with \(E_e(x, y) = 0\) and wait for both \(h^*(x)\) and \(h^*(y)\) to become defined. When this happens, copy \(S_e(h^*(x), h^*(y))\). Thus, the number of changes in the approximation of \(E_e(x, y)\) is at most that of \(S_e(h^*(x), h^*(y))\), which is in turn bounded by \(h_e(h^*(x), h^*(y))\). Suppose \(x < y\) (the other case \(y < x\) follows symmetrically). Therefore \(h^*(x) \leq h^*(y)\) by the definition of \(h^*\). Let \(z_y\) be
least such that $\hat{h}(z_y) \geq y$ (this is defined as $h^*(y) \downarrow$). If $h^*(y) = 0$ then $h^*(x) = 0$ and so $E_e(x, y)$ changes at most once. So suppose that $h^*(y) > 0$, and in particular, $h^*(y) \leq z_y - 1$. Since $h^*(x), h^*(y) \leq z_y - 1$, this means that $h_c(h^*(x), h^*(y)) \leq \hat{h}(z_y - 1)$. However $\hat{h}(z_y - 1) < y$ by the minimality of $z_y$, and so $S_e(h^*(x), h^*(y))$ changes no more than $\max\{x, y\}$ many times. This verifies the first property claimed in the theorem.

It is also clear that $E_e$ is an equivalence relation, because $S_e$ is. Finally, suppose that $h_c$ is total. We wish to see that $S_e \equiv E_e$. Now $E_e \leq_c S_e$ is given by $h^*$, which must be total since $h_c$ is total and $\hat{h}$ is strictly increasing. For the direction $S_e \leq E_e$, we simply take the map $z \mapsto \hat{h}(z) + 1$. Notice that $h^*(\hat{h}(z) + 1) = z$. 

\begin{corollary}
There exists a universal $\omega$-c.e. equivalence relation.
\end{corollary}
\begin{proof}
The effective disjoint union $\sqcup_{e \in \omega} E_e$ from Theorem 2.5 is clearly an $\omega$-c.e. equivalence relation. Every $\omega$-c.e. equivalence relation is equal to $S_e$ for some $e$ where $h_e$ is total, which has the same degree as $E_e$.
\end{proof}

### 2.3 $\Pi^0_1$ equivalence relations

We recall Theorem 3.3 of [9], where it is shown that there is a universal $\Pi^0_1$ equivalence relation:

\begin{theorem}[Theorem 3.3]
There exists a universal $\Pi^0_1$ equivalence relation.
\end{theorem}

The proof in [9] does not mention enumerations of the class, nor does it explicitly show how to uniformly transform a $\Pi^0_1$ binary relation $S_e$ into a transitive equivalence relation. For the sake of completeness and to fit $\Pi^0_1$ into the context of our paper, we will give a slightly different presentation in this section.

First, we fix $\{S_e\}_{e \in \omega}$ as in Section 1.1. We first show that it is possible to construct an effective enumeration $\{E_e\}_{e \in \omega}$ of $\Pi^0_1$ equivalence relations in which the upper cone of every $\Pi^0_1$ degree is represented:

\begin{theorem}
There exists an effective $\Pi^0_1$ enumeration of a sequence $\{E_e\}_{e \in \omega}$ such that:

- For every $e$, $E_e$ is a $\Pi^0_1$ equivalence relation.
- For any $\Pi^0_1$ equivalence relation $S$, there exists some $e$ such that $S \leq E_e$.
\end{theorem}

\begin{proof}
Fix $e$, and we shall build a $\Pi^0_1$ enumeration of $E_e$ uniformly in $e$ such that if $S_e$ is transitive, then $S_e \leq E_e$. The issue that we must keep $E_e$ transitive, while $S_e$ might not be, and we have to do it in a way where we can extend our reduction $f_e$ witnessing $S_e \leq E_e$ whenever $S_e$ looks transitive on a larger fragment.

Define $f_e(x)$ inductively by setting $f_e(0) = 0$, and $f_e(x + 1)$ to be the least stage larger than $f_e(x)$ such that $S_e[f_e(x + 1)] \upharpoonright x + 1$ looks like an equivalence relation. Then $f_e$ is partial computable, and is total if and only if $S_e$ is an equivalence relation. Now define $E_e(x, y)$ (for $x \neq y$) by the following. If $x \not\in \text{rng}(f_e)$ or $y \not\in \text{rng}(f_e)$ then $E_e(x, y) = 0$, otherwise let $a, b$ be such that $f_e(a) = x$ and $f_e(b) = y$, and define $E_e(x, y) = 0$ if and
only if \( S_e(a, b)[f_e(z)] = 0 \) for some \( z \) in \( \text{dom}(f_e) \). Obviously if \( x = y \) we set \( E_e(x, y) = 1 \).

There is an obvious \( \Pi^0_1 \) enumeration of \( E_e \): First, note that \( \text{rng}(f_e) \) is computable (uniformly in \( e \)), since \( s \in \text{rng}(f_e) \) if and only if this has happened before querying \( S_e[s + 1] \) (this is true even if \( \text{rng}(f_e) \) is finite). Furthermore, the condition “\( S_e(a, b)[f_e(z)] = 0 \) for some \( z \) in \( \text{dom}(f_e) \)” is c.e. Hence \( E_e \) is \( \Pi^0_1 \), with an enumeration effectively found in \( e \).

It is obvious that \( E_e \) is reflexive and symmetric (since \( S_e \) is at every stage). For transitivity, suppose \( E(x, y) = E(y, w) = 1 \) and \( E(x, w) = 0 \) for distinct \( x, y, w \). Then clearly there must exist \( a, b, c \) such that \( f_e(a) = x, f_e(b) = y \) and \( f_e(c) = w \). Also fix \( z \in \text{dom}(f_e) \) such that \( S_e(a, c)[f_e(z)] = 0 \) triggering us to define \( E_e(x, w) = 0 \). Now observe that \( S_e[f_e \max \{a, b, c, z\}] \) cannot be transitive, contradicting the properties of \( f_e \).

Finally suppose that \( S_e \) is an equivalence relation, hence \( f_e \) is total. Then \( f_e \) witnesses that \( S_e \leq_c E_e \).

\( \square \)

We deduce immediately [9, Theorem 3.3]:

**Corollary 2.9**

There exists a universal \( \Pi^0_1 \) equivalence relation.

We now turn to the interesting question of whether there exists an effective enumeration of all \( \Pi^0_1 \) equivalence relations, or even an enumeration representing all \( \Pi^0_1 \) degrees. First of all, it is easy to see that an enumeration of all \( \Pi^0_1 \) equivalence relations is impossible:

**Proposition 2.10**

There does not exist an effective sequence \( \{E_e\}_{e \in \omega} \) of \( \Pi^0_1 \) equivalence relations such that for every \( \Pi^0_1 \) equivalence relation \( S \), there is some \( e \) such that \( S = E_e \). (We can even replace “\( S = E_e \)” by “\( S \leq_c E_e \)” via a primitive recursive function”).

Note that Proposition 2.10 does not contradict Theorem 2.8. In fact, it shows that the proof of Theorem 2.8 is sharp in the sense that the reductions \( f_e \) constructed there cannot all be total.

**Proof of Proposition 2.10**

Fix the sequence \( \{E_e\}_{e \in \omega} \) as above, and we shall construct a \( \Pi^0_1 \) equivalence relation \( S \) such that \( S \neq E_e \) for every \( e \). In fact, we shall ensure that for every \( e, p_e \) does not witness \( S \leq_c E_e \), where \( p_e \) is the \( e^\text{th} \) primitive recursive function.

At the beginning we define \( S(\langle e, i \rangle, \langle e', j \rangle)[0] = 0 \) for all \( e, e', i, j \) such that \( e \neq e' \), and \( S(\langle e, i \rangle, \langle e, j \rangle)[0] = 1 \) for all \( e, i, j \). For each \( e \), we do the following uniformly. We begin our strategy for \( e \) by first setting \( S(\langle e, 0 \rangle, \langle e, 1 \rangle) = 0 \), and wait for a stage \( s \) such that \( E_e(p_e(\langle e, 0 \rangle), p_e(\langle e, 1 \rangle)) = 0 \). For every stage \( t \) where \( E_e(p_e(\langle e, 0 \rangle), p_e(\langle e, 1 \rangle)) \) is still \( 1 \), we set \( S(\langle e, t + 2 \rangle, \langle e, i \rangle) = 0 \) for every \( i < s + 2 \) (that is, declare \( \langle e, t + 2 \rangle \) to be an isolated point).

Let \( s \) be the first stage such that \( E_e(p_e(\langle e, 0 \rangle), p_e(\langle e, 1 \rangle)) \) is \( 0 \) (note that \( s \) may not exist). At this point, we have made \( \langle e, i \rangle \) an isolated point for all \( 2 \leq i < s + 2 \). However \( S(\langle e, s + 2 \rangle, \langle e, i \rangle) = 1 \) for all \( i \). We wait for either \( E_e(p_e(\langle e, 0 \rangle), p_e(\langle e, s + 2 \rangle)) = 0 \) or \( E_e(p_e(\langle e, 1 \rangle), p_e(\langle e, s + 2 \rangle)) = 0 \); at least one of these must hold because \( E_e \) is transitive. Suppose \( E_e(p_e(\langle e, 0 \rangle), p_e(\langle e, s + 2 \rangle)) = 0 \).

Then we declare \( \langle e, 1 \rangle \) to be isolated and do nothing else for \( e \); otherwise declare \( \langle e, 0 \rangle \) to be isolated.
We now verify that $S$ has the properties we want. Firstly, it is clear that the above describes a $\Pi^0_1$ approximation to $S$, as $E_e$ is transitive and $p_e$ is total. Now if $s$ is never found by the strategy for $e$ then the $e^{th}$ column of $S$ will consist only of isolated points. If $s$ is found by the strategy then the $e^{th}$ column $S[e] = \{\langle e, i \rangle \mid i \in \omega \}$ will consist of $s+1$ many isolated points, while the rest of the points belong to a single equivalence class. In either case, $S[e]$ is an equivalence relation. Hence $S$ is an equivalence relation. Finally, it is straightforward to check that for every $e$, $p_e$ does not witness $S \leq c E_e$.

Given Proposition 2.10 we can ask if there is an enumeration of $\Pi^0_1$ equivalence relations in which every degree is represented, like in the case of $\omega$-c.e. equivalence relations. Part of the difficulty of obtaining such an enumeration is that the reductions from $S_e$ to $E_e$ must be allowed to be partial. We leave open the question:

**Question 2.11** Is there an effective enumeration of $\Pi^0_1$ equivalence relations $\{E_e\}_{e \in \omega}$ such that if $S$ is a $\Pi^0_1$ equivalence relation, then there is some $e$ such that $S \equiv_c E_e$?

### 3 Downward density and c.e. equivalence relations

We now investigate the structure of the c.e. $c$-degrees, in particular, the problem of downward density. Unlike the $\Pi^0_1$ $c$-degrees (see Section 4), it is not the case that every c.e. $c$-degree is comparable with $\deg_c(\mathsf{id})$. Given any c.e. set $A \subseteq \omega$, define the ceer $R_A = \{(x, y) \mid x = y \text{ or } x, y \in A\}$. Then observe that $\mathsf{id} \leq_c R_A$ if and only if $A$ is not simple. Thus, even though the computable $c$-degrees $\deg_c(\equiv_1) < \deg_c(\equiv_2) < \cdots < \deg_c(\mathsf{id})$ is an initial segment of the c.e. $c$-degrees, we now get non-trivial c.e. $c$-degrees which are incomparable with this initial segment.

This situation is very similar to that of the c.e. 1-degrees, where it had been observed that the “bottom” degree, $0_1 = \deg_1(\emptyset \oplus \omega)$, isn’t really the least element of the c.e. 1-degrees: The simple c.e. sets all have 1-degree incomparable with $0_1$. Nevertheless, much of the earlier studies of the c.e. 1-degrees have been focused on the interval $[0_1, \deg_1(K)]$. For example, Lachlan [10] showed that every finite initial segment in this interval is a distributive lattice, and hence the first-order theory of the c.e. 1-degrees is undecidable. Young [14] showed that this interval is neither an upper semilattice nor a lower semillatice. Part of the reason for this is due to the fact that results about $m$-degrees of c.e. sets can often be transferred to the interval $[0_1, \deg_1(K)]$, as the non-simplicity of the c.e. sets in this interval often allow $m$-reductions to be turned into 1-reductions.

It was observed in [1] that the interval $[0_1, \deg_1(K)]$ embeds into the c.e. $c$-degrees; in fact, the interval $[0_1, \deg_1(K)]$ of c.e. 1-degrees was shown to be order-isomorphic with the interval $[\deg_c(\mathsf{id}), \deg_c(R_K)]$ of c.e. $c$-degrees. Note that $\deg_c(R_K)$ is not the top c.e. $c$-degree; in fact, $R_K$ is far from being universal amongst the c.e. $c$-degrees.

**Definition 3.1** We call a $c$-degree $a$ set-induced if there exists an infinite and co-infinite set $A \subseteq \omega$ such that $R_A \in a$. 
It was shown in [1] that every $c$-degree in the interval $[\text{deg}_c(\text{id}), \text{deg}_c(R_K)]$ is set-induced. In fact, the same proof shows a little more; for completeness, we reproduce the argument here:

**Lemma 3.1** Suppose $R$ is a non-computable equivalence relation and $A \subseteq \omega$ is any set such that $R \leq_c R_A$. Then there exists a set $B \subseteq \omega$ such that $R\equiv_c R_B$.

**Proof** Fix a computable function $f$ witnessing $R \leq_c R_A$. Since $R$ is not computable, $\text{rng}(f)$ is an infinite c.e. set. We let $\{d_n\}_{n \in \omega}$ be a one-one computable enumeration of $\text{rng}(f)$. Let $B = \{n \in \omega \mid d_n \in A\}$. Notice that this gives $B \leq_1 A$, as expected.

To see that $R \leq_c R_B$, use the reduction $(d_n)^{-1} \circ f$. For the other direction $R_B \leq_c R$, we map $n \mapsto \text{the least } x \text{ such that } f(x) = d_n$.

Lemma 3.1 shows not only that every $c$-degree in the interval $[\text{deg}_c(\text{id}), \text{deg}_c(R_K)]$ is set-induced, but also that:

**Corollary 3.2** A non-computable c.e. $c$-degree $a$ is set-induced if and only if $a \leq \text{deg}_c(R_K)$.

Before examining the question of downward density for $c$-degrees, we mention one related result about c.e. 1-degrees. Recall the following definition:

**Definition 3.2** If $a < b$ are two elements of a partial order, then $b$ is said to be a strong minimal cover of $a$ if the following is true of the partial order: $\forall c \leq b (c = b \lor c \leq a)$.

Lachlan [10] proved the existence of a minimal element of the interval $[0_1, \text{deg}_1(K)]$. In fact, a closer examination of the proof reveals that there exists a c.e. 1-degree which is a strong minimal cover of $0_1$ within the structure of all c.e. 1-degrees:

**Theorem 3.3 ([10, Lachlan])** There exists a non-simple and non-computable c.e. set $A$ such that for every set $B \leq_1 A$, we have $A \equiv_1 B$ or $B$ is computable.

We now turn to the matter at hand in this section. The statement of downward density for $c$-degrees says that given any non-computable c.e. $c$-degree $a$, there exists a non-computable (c.e.) $c$-degree $b < a$. We show that this fails by constructing two counter-examples: Theorem 3.7 provides a “minimal” c.e. $c$-degree $a \not\geq \text{deg}_c(\text{id})$, while Theorem 3.9 provides a strong minimal cover $a$ of $\text{deg}_c(\text{id})$. However the existence of a strong minimal cover of $\text{deg}_c(\text{id})$ in the c.e. $c$-degrees can already be deduced from Theorem 3.3:

**Proposition 3.4** There is a c.e. set-induced $c$-degree $a > \text{deg}_c(\text{id})$ such that $a$ is a strong minimal cover of $\text{deg}_c(\text{id})$.

**Proof** Consider $R_A >_c \text{id}$ where $A$ is from Theorem 3.3. Now if $R \leq_c R_A$ then by Lemma 3.1 either $R$ is computable, or else $R \equiv_c R_B$ for some $B \leq_1 A$. As $B$ is not computable we have $B \geq_1 A$, which means that $R \geq C R_A$.

Thus our counter-examples to downward density in Theorems 3.7 and 3.9 will be c.e. $c$-degrees that are not set-induced. It shall be convenient to isolate
properties that imply that a certain c-degree is not set-induced. We identify two conditions sufficient for a degree to be not set-induced.

The first and probably the easiest way to produce an equivalence relation \( R \) that does not have set-induced c-degree is to make every \( R \)-class finite:

**Lemma 3.5** If \( R \) is any non-computable equivalence relation where every class is finite, then \( \deg_c(R) \) is incomparable with every non-computable set-induced c-degree.

**Proof** If \( R_A \leq_c R \) via a computable function \( f \), then \( f(A) \) is a finite set and hence \( A \) is computable. On the other hand if \( R \leq_c R_A \) and \( R \) is non-computable, then by Lemma 3.1 there is a \( B \) such that \( R \equiv_c R_B \). Since \( R_B \leq_c R \), \( B \) is computable, a contradiction.

Unfortunately we will not use Lemma 3.5 to ensure that the ceers constructed in Theorems 3.7 and 3.9 are not of set-induced degree. This is because it is generally a messy task to construct non-trivial c.e. equivalence relations where every class is finite. However, we will be able to easily do this for \( \Pi^1_0 \) equivalence relations. Therefore Lemma 3.5 will be used in the proof of Theorems 4.1, 4.2 and 4.8 in the next section.

To ensure that the c.e. equivalence relations constructed in Theorems 3.7 and 3.9 are not of set-induced degree, we will introduce a second sufficient condition:

**Lemma 3.6** Let \( R \) be any non-computable c.e. equivalence relation with the property that given any c.e. set \( W \), if infinitely many \( R \)-classes contain an element of \( W \), then every \( R \)-class contains an element of \( W \). Then \( R \) is a counter-example to downward density. If additionally each \( R \)-class contains at least two elements of each \( W \) which intersects infinitely many \( R \)-classes, then there is no set-induced degree below \( \deg_c(R) \).

**Proof** Fix \( R \) as above. Let \( S \leq_c R \) via a computable function \( f \). If \( \text{rng}(f) \) intersects only finitely many \( R \)-classes then as \( R \) is c.e., \( S \) is computable. Otherwise each \( R \)-class must contain an element of \( \text{rng}(f) \). Since \( R \) is c.e., this obviously allows us to show that \( R \leq_c S \).

Now suppose \( R_A \leq_c R \) via \( f \) for some co-infinite set \( A \). Then \( \text{rng}(f) \) must intersect infinitely many different \( R \)-classes. Hence each \( R \)-class will contain at least two elements of \( \text{rng}(f) \). Since \( R \) is not computable, there are at least two \( R \)-classes. This means that \( f(x) R f(y) \) for some pair \( x, y \not\in A \) and \( x \neq y \), a contradiction.

In Theorem 3.7 we will use Lemma 3.6 to ensure that the degree of the constructed ceer is not above a set-induced degree.

**Theorem 3.7** There is a non-computable c.e. c-degree \( a \nleq \deg_c(\text{id}) \) such that \( a \) is not set-induced and for every \( b \leq a \), either \( b \) is computable or \( b \geq a \).

**Proof** We are going to build a c.e. equivalence relation \( A \) and ensure the following requirements:

\( R_{2c}: A \neq \varphi_c \).
If this is true, collapse the classes containing $R_{2e+1}$: $W_e$ intersects infinitely many $A$-classes ⇒ every $A$-class contains at least two elements of $W_e$.

It should be clear that these requirements ensure the properties needed for $A$. The $R_{2e}$ requirements ensure that $A$ is not computable. Lemma 3.6 and the $R_{2e+1}$ requirements together ensure that $\deg_e(A)$ is a counterexample to downward density and does not bound a set-induced degree. Since $\text{id} = e \in R_{\omega \oplus \emptyset}$, it follows that $\deg_e(A) \not\geq \deg_e(\text{id})$.

During the construction, the classes of $A$ are ordered according to the magnitude of the least element of the class. The $n$-th class of $A_n$ is denoted by $C_{n,s}$. Since $A$ is c.e., the basic operation we will perform at each step $s$ is to collapse $C_n$ and $C_m$ for certain classes $C_n$ and $C_m$. This has the obvious meaning, i.e. define $C_{n,s+1} = C_{n,s} \cup C_{m,s}$ if $C_n$ is the smaller class, and rearrange the names of all affected classes. To say that a class $C_n$ is larger than $m$ (or another class $C_m$) means that $n > m$.

The reader might expect the construction of $A$ to be carried out on a priority tree using infinite injury, since one might want to guess if a certain c.e. set $W$ intersects infinitely many different $A$-classes or not. However, the actions of the different requirements are sufficiently independent from each other that we will construct $A$ in an “injury-free” way.

We will always prevent $R_e$ from collapsing two classes smaller than $e$. Of course, $R_e$ is still allowed to collapse two or more classes if at most one class being collapsed is smaller than $e$.

We now describe the basic strategy for $R_{2e+1}$. We act for $R_{2e+1}$ at a stage $s > 2e + 1$ if at least $2s$ many different $A_n$-classes contain an element of $W_{e,s}$, where $s$ is the largest stage $< s$ where we had acted for $R_{2e+1}$. (Obviously, if $s$ does not exist, set it to be equal to $2e + 1$). At such a stage $s$, we act for $R_{2e+1}$ by doing the following. Let $N$ be the least such that $C_{N,s}$ does not yet contain at least two elements of $W_e$. If $N > s$ do nothing else. Otherwise find the least pair $M > M' > \max\{N, 2e + 1\}$ such that $C_{M,s}$ and $C_{M',s}$ each contains an element of $W_e$; such a pair $M' < M$ exists by assumption (note that $\max\{N, 2e + 1\} \leq s$). Collapse the classes $C_{N,s}, C_{M,s}$, and $C_{M'}$.

We now describe the basic strategy for $R_{2e}$. At a stage $s > 2e$, we check to see if:

- For every $\langle i', j' \rangle < s$, $\varphi(i', j') \downarrow = 0$ implies that $i'$ and $j'$ are in different classes, and
- There exists some pair of numbers $\langle i, j \rangle < s$ such that $\varphi_e(i, j) \downarrow = 0$ and $i$ and $j$ currently belong to different $A$-classes which are both larger than $2e$.

If this is true, collapse the classes containing $i$ and $j$. Otherwise do nothing else at this stage.

Construction of $A$: At the beginning, let $A_0 = \text{id}$, i.e. $C_{n,0} = \{n\}$ for all $n$. At stage $s > 0$, let all requirements $R_e$ for $e < s$ act (if necessary) according to the basic strategy above.

We now verify that all the requirements are met. First of all, note that each class $C_n$ can be collapsed with another class $C'_m$ infinitely often, and so each class can be infinite. However, each pair of classes $C_n$ and $C_m$ are collapsed with each other only finitely often. We now verify this fact. More
specifically, suppose there is a pair $n < m$ such that there are infinitely many steps $s$ in the construction in which $C_{n,s}$ and $C_{m,s}$ are collapsed with each other. Fix a pair such that $(n, m)$ is the least; recall that the standard pairing function $\langle \cdot, \cdot \rangle$ has the property that $\langle x, y \rangle < \langle x + 1, y \rangle$ and $\langle x, y \rangle < \langle x, y + 1 \rangle$ for all $x, y$.

Now by examining the construction, only finitely many requirements (specifically, only those $R_e$ for $e < m$) can cause the current $C_n$ and the current $C_m$ to be collapsed. Therefore, we can fix an $e$ where $R_e$ does this infinitely often. This means that $e$ cannot be even, because if $R_e$ ever collapses any two classes, it must be that $\varphi(i, j) = 0$ and that $i, j$ are put the same $A$-class, which will block $R_e$ from doing anything else. Therefore, $e$ has to be odd.

By the minimality of the pair $\langle n, m \rangle$, there is a time after which every class $C_k$ for $k \leq n$ is eventually growing monotonically (as eventually no two classes smaller or equal to $n$ is collapsed). Formally, this means that there exists a stage $t_0$ such that for all $s \geq t_0$, and $k \leq n$, we have $C_{k,s} \subseteq C_{k,s+1}$. If $R_e$ collapses $C_n$ and $C_m$ after stage $t_0$, then some class less than or equal to $C_n$ will now contain at least two elements of $W$. By monotonicity, eventually every class less than or equal to $C_n$ will contain at least two elements of $W$, and so makes it impossible for $R_e$ to collapse $C_n$ and $C_m$ (in fact, any class larger than $C_n$) again.

This means that every $A$-class $C_n$ eventually grows monotonically. This implies that $A$ will (at the end) consist of infinitely many different classes. We now apply this to show that every requirement is satisfied. To show that $R_{2e}$ is met, suppose that $\varphi_e$ is total and correctly computes $A$. Fix any pair of elements $i, j$ belonging to the $2e + 1$-th and the $2e + 2$-th classes respectively. Then $\varphi_e(i, j)$ must be equal to 0. But this means that at all large enough stages, $R_{2e}$ would want to act to collapse these two classes. Since this is impossible, it must be because at all large enough stages, the first clause in the basic strategy for $R_{2e}$ fails. This is also impossible if $\varphi_e = A$.

Now we argue that $R_{2e+1}$ is met. Suppose that $W_e$ intersects infinitely many $A$-classes, and let $C_N$ be the smallest $A$-class which contains at most one element of $W_e$. Eventually from some point on, the classes $C_0, C_1, \ldots, C_N$ will be growing monotonically. Since $W_e$ intersects infinitely many $A$-classes, there must be infinitely many stages where we will act for $R_{2e+1}$. Eventually $s$ will be larger than $N$, and we will act to put two elements of $W_e$ into $C_N$. \qed

We now turn to our second counter-example to downward density, this time constructing a c.e. $c$-degree which is above $\deg_c(\langle 1d \rangle)$. Unlike in the proof of Theorem 3.7, we cannot directly use Lemma 3.6 to show that the constructed $c$-degree is not set-induced, because we have to now make the constructed degree bound $\deg_c(\langle 1d \rangle)$. A modification is needed:

**Lemma 3.8 (Modified Lemma 3.6)** Let $R$ be any non-computable c.e. equivalence relation with the property that $(2n, 2m) \notin R$ for every $n \neq m$. We call an $R$-class even if it contains an even number, otherwise we call the $R$-class odd.

Suppose $R$ has the property that given any c.e. set $W$, if infinitely many odd $R$-classes contain an element of $W$, then every $R$-class contains at least
two elements of $W$. Then $R$ is a counter-example to downward density and the only set-induced degree below $\deg_c(R)$ is $\deg_c(\text{id})$.

**Proof** Fix $R$ as above. Let $S \leq_c R$ via a computable function $f$. If $\text{rng}(f)$ is in only finitely many odd $R$-classes then as $R$ is c.e., $S$ is computable. Otherwise each $R$-class must contain an element of $\text{rng}(f)$. Since $R$ is c.e., this allows us as before to show that $R \leq_c S$.

Now suppose $R_A \leq_c R$ via $f$ for some co-infinite set $A$. Then $\text{rng}(f)$ must intersect infinitely many different odd $R$-classes (otherwise $R_A \equiv_c \text{id}$). Hence each $R$-class will contain at least two elements of $\text{rng}(f)$. Since $R$ is not computable, there are at least two $R$-classes. This means that $f(x) R f(y)$ for some pair $x, y \not\in A$ and $x \neq y$, a contradiction.

In Theorem 3.9 we will use Lemma 3.8 to ensure that the degree of the constructed c.e. relation with only finitely many distinct classes is not of set-induced degree. Notice that the equivalence relation with only finitely many different odd classes is not of set-induced degree.

**Theorem 3.9** There is a c.e. c-degree $a > \deg_c(\text{id})$ such that $a$ is not set-induced and for every $b \leq a$, either $b$ is computable or $b \geq a$. In other words, $a$ is a strong minimal cover of $\deg_c(\text{id})$.

**Proof** We will modify the proof of Theorem 3.7. We are going to build a c.e. equivalence relation $A$ and ensure that $(2n, 2m) \not\in A$ for every $n \neq m$. We will ensure the following requirements:

- **$R_{2e}$**: $A \neq \varphi_c$.
- **$R_{2e+1}$**: $W_e$ intersects infinitely many odd $A$-classes $\Rightarrow$ every $A$-class contains at least two elements of $W_e$.

It should be clear that these requirements ensure the properties needed for $A$. The $R_{2e}$ requirements ensure that $A$ is not computable. Lemma 3.8 and the $R_{2e+1}$ requirements together ensure that $\deg_c(A)$ is a counter-example to downward density and is not of set-induced degree. Clearly $\text{id} \leq_c A$ by sending $x \mapsto 2x$.

We remark that we can satisfy $R_{2e+1}$ (trivially) by ensuring that $A$ only has finitely many odd classes at the end; however it is for the sake of $R_{2e}$ that we will deliberately ensure that we end up with infinitely many odd classes at the end.

We follow closely the proof of Theorem 3.7. Let $o(n, s)$ be the $n$-th odd class at stage $s$, i.e. $C_{o(n,s),s}$ is an odd class and there are exactly $n$ many odd classes amongst $C_{0,s}, C_{1,s}, \ldots, C_{o(n,s),s}$.

We now describe the basic strategy for $R_{2e+1}$. We act for $R_{2e+1}$ at a stage $s > 2e + 1$ if at least $2\bar{s}$ many different odd $A_0$-classes contain an element of $W_{e,s}$, where $\bar{s}$ is the largest stage $< s$ where we had acted for $R_{2e+1}$. (Obviously, if $\bar{s}$ does not exist, set it to be equal to $2e + 1$). At such a stage $s$, we act for $R_{2e+1}$ by doing the following. Let $N$ be the least such that $C_{N,s}$ does not yet contain at least two elements of $W_e$ (note that $C_{N,s}$ might be an even class). If $N > \bar{s}$ do nothing else. Otherwise find the least pair $M > M' > \max\{o(N,s), o(2e+1,s)\}$ such that $C_{M,s}$ and $C_{M',s}$ each contains an element of $W_e$ and are both odd classes; such a pair $M' < M$ exists by assumption (note that $\max\{N, 2e + 1\} \leq \bar{s}$). Collapse the classes $C_N, C_{M'}$ and $C_M$. 


We now describe the basic strategy for \( R_{2e} \). At a stage \( s > 2e \), we check to see if:

- For every \( \langle i', j' \rangle < s \), \( \varphi(i', j') \downarrow = 0 \) implies that \( i' \) and \( j' \) are in different classes, and
- There exists some pair of numbers \( \langle i, j \rangle < s \) such that \( \varphi_e(i, j) \downarrow = 0 \) and \( i \) and \( j \) currently belong to different \( A \)-classes which are both larger than \( o(2e, s) \) and are both odd classes.

If this is true, collapse the classes containing \( i \) and \( j \). Otherwise do nothing else at this stage.

**Construction of \( A \):** At the beginning, let \( A_0 = \text{id} \), i.e. \( C_n, 0 = \{ n \} \) for all \( n \).

At stage \( s > 0 \), let all requirements \( R_e \) for \( e < s \) act (if necessary) according to the basic strategy above.

We now verify that all the requirements are met. First of all, note that no two even classes are ever collapsed together, and thus \( (2n, 2m) \notin A \) for every \( n \neq m \).

Next, we observe the following facts:

- Every \( A \)-class \( C_n \) eventually grows monotonically. This is because \( o(n, s) \geq n \) for every \( n \) and \( s \), and so the same argument from Theorem 3.7 applies.
- For each \( n \), there are only finitely many steps in the construction where we collapse \( C_{m,s} \) and \( C_{o(n,s),s} \) for some \( m < o(n, s) \). To see this, note that this can only be done by some requirement \( R_e \) for \( e < n \). Fix some odd \( R_e \) which does this infinitely often. Since each class eventually grows monotonically, this means that eventually \( R_e \) will be collapsing the three classes \( N < M' < M \) where \( N > n \). This of course means that \( M' \) and \( M \) are both larger than \( o(n, s) \) which is a contradiction.
- At the end, \( A \) must contain infinitely many odd classes. This follows because of the preceding fact and also that if \( C_{n,s} \) is collapsed with \( C_{m,s} \) and \( n < m \), then \( C_{m,s} \) must be an odd class.

Since \( A \) contains infinitely many odd classes, the same argument from before works to show that \( R_e \) is met for every \( e \).

Notice that if \( R \leq_c S \) and \( S \) is a c.e., then so is \( R \). Therefore we can immediately conclude that any class of equivalence relations containing the c.e. equivalence relations is not downwards dense:

**Corollary 3.10** The class of \( \Sigma^0_n \) equivalence relations, the n.c.e. equivalence relations and the \( \omega \)-c.e. equivalence relations are not downwards dense for \( n \geq 1 \).

### 4 Downward density and \( \Pi^0_1 \) equivalence relations

We now turn to the structure of the \( \Pi^0_1 \) \( c \)-degrees. In particular, we will show in this section that downward density fails in the structure of \( \Pi^0_1 \) \( c \)-degrees. Before making our statements precise, we first make a few trivial observations about the \( \Pi^0_1 \) \( c \)-degrees. Recall that the computable \( c \)-degrees form an initial segment of the \( \Pi^0_1 \) \( c \)-degrees. In fact, each \( \Pi^0_1 \) equivalence relation has comparable \( c \)-degree with \( \text{id} \): If \( R \) is \( \Pi^0_1 \) and has infinitely many
classes, then we can compute an infinite set of elements which are pairwise \( R \)-inequivalent, allowing us to reduce \( \text{id} \) to \( R \). On the other hand, if \( R \) is \( \Pi^0_1 \) with finitely many distinct classes then every element is eventually \( R \)-inequivalent with all but one class, so it is computable.

Since every \( \Pi^0_1 \) \( c \)-degree is comparable with \( \text{deg}_c(\text{id}) \), we will construct a strong minimal cover of \( \text{deg}_c(\text{id}) \) which is not of set-induced degree.

**Theorem 4.1** There exists a \( \Pi^0_1 \) \( c \)-degree \( a_2 > \text{deg}_c(\text{id}) \) such that \( a_2 \) is a strong minimal cover of \( \text{deg}_c(\text{id}) \), and where \( a_2 \) is not of set-induced degree.

As we will later formally prove Theorem 4.2, a stronger version of this theorem, we not prove Theorem 4.1. Instead, we give some relevant definitions.

For any integer \( x \in \omega \) and equivalence relation \( E \), \( [x]_E \) is defined to be the equivalence class of \( E \) containing \( x \); where the context is clear we simply write \( [x] \). (Note that \( [s] \) also used as a stage notation earlier, but there should be no confusion). An equivalence relation \( E \) is said to be size-\( n \) if for every \( x, y \) such that \( E(x, y) = 1 \), we have \([\frac{x}{n}] = [\frac{y}{n}] \). In other words, each size-\( n \) equivalence relation has \( \# [x] \leq n \) as well as an upper bound on the set \( [x] \) for every \( x \).

Notice that \( \text{id} \) is a size-1 equivalence relation.

It is easy to see that a size-\( n \) \( \Pi^0_1 \) equivalence relation \( E \) is computable iff the function \( F_E : x \mapsto \# [x]_E \) is computable.

Our goal is to extend Theorem 4.1 to an infinite sequence \( \text{deg}_c(\text{id}) = a_1 \prec_c a_2 \prec_c a_3 \prec_c \cdots \) such that \( a_{j+2} \) is a strong minimal cover of \( a_{j+1} \) for every \( j \in \omega \). In the next theorem, Theorem 4.2, we give the formal proof for the existence of \( a_2 \) and \( a_3 \). The infinite sequence \( \text{deg}_c(\text{id}) = a_1 \prec_c a_2 \prec_c a_3 \prec_c \cdots \) is a straightforward generalization of Theorem 4.2 and there are no new mathematical ideas needed apart from increased notational complexity.

**Theorem 4.2** There exist \( \Pi^0_1 \) \( c \)-degrees \( a_2 \) and \( a_3 \) which are not set-induced such that \( \text{deg}_c(\text{id}) < a_2 < a_3 \), \( a_2 \) is a strong minimal cover of \( \text{deg}_c(\text{id}) \), and \( a_3 \) is a strong minimal cover of \( a_2 \).

**Proof** We shall construct a size-2 \( \Pi^0_1 \) equivalence relation \( A_2 \) and a size-3 \( \Pi^0_1 \) equivalence relation \( A_3 \) satisfying the following requirements.

\[
Q: A_2 \preceq_c A_3.
\]

\[
R^1_{(m,n)}: E_m \preceq_c A_2 \text{ via } f_n \Rightarrow \exists \text{ computable function } g_{(n,m)} \text{ s.t. } A_2 \preceq_c E_m \text{ or } E_m \text{ is computable.}
\]

\[
R^2_{(i,j)}: E_i \preceq_c A_3 \text{ via } f_j \Rightarrow \exists \text{ computable function } h_{(i,j)} \text{ s.t. } A_3 \preceq_c E_i \text{ via } h_{(i,j)} \text{ or } E_i \preceq_c A_2.
\]

\[
P^1_2: F_{A_2} \neq \varphi_c.
\]

\[
P^2_2: A_3 \preceq_c A_2 \text{ via } \varphi_c.
\]

Here, we think of \( E_m \) and \( E_i \) as the “pull-back” of \( A_2 \) and \( A_3 \) under \( f_n \) and \( f_j \) respectively, and enumerate \( E_m \) and \( E_i \) in the obvious way.

By Lemma 3.5, \( \text{deg}_c(A_2) \) and \( \text{deg}_c(A_3) \) are not set-induced. We fix the starting values \( A_{2,0} = \{ (i, i), (2i, 2i + 1), (2i + 1, 2i) \mid i \in \omega \} \) and \( A_{3,0} = \{ (i, i), (3i, 3i+1), (3i+1, 3i+2), (3i+2, 3i+3) \mid i \in \omega \} \).

\( Q \) is a global requirement and will be satisfied by ensuring that for every \( z \), \([2z]_{A_2} \) is isolated if and only if \([3z]_{A_3} \) is isolated for every \( z \).
Tree of strategies} Effectively order the requirements (of order type \(\omega\)). We will carry out the construction on a finitely branching priority tree, where every node at the same (next) level is assigned the same (next) requirement. The outcomes of each node will be given later.

**Notation** For any node \(\alpha\), define \(\alpha^\frown\) to be \(\beta\) of the longest length such that \(\beta \neq \infty \subseteq \alpha\). Of course, \(\alpha^\frown\) need not exist. During the construction, we define \(\delta_s\) to be the stage \(s\) approximation to the true path.

The length function for \(R_{(m,n)}\) is defined as \(l((m,n),s) := \) the largest \(k\) such that for every \((x,y) < k\), we have \(f_{n,s}(x) \downarrow, f_{n,s}(y) \downarrow\) and \(E_{m,s}(x,y) = A_{2,s}(f_{n,s}(x), f_{n,s}(y))\).

The length function for \(R_{(i,j)}\) is defined similarly as \(L((i,j),s) := \) the largest \(k\) such that for every \((x,y) < k\), we have \(f_{j,s}(x) \downarrow, f_{j,s}(y) \downarrow\), and \(E_{i,s}(x,y) = A_3,f_{j,s}(x), f_{j,s}(y))\).

**Definition 4.1** (\(R^1\)-expansionary stage) If \(\alpha\) is assigned an \(R^1\) strategy, then we call a stage \(s > 0\) an \(\alpha\)-expansionary stage if the following hold.

- \(\alpha \subseteq \delta_s\).
- \(l(\alpha,s) \neq l(\alpha,t), \) where \(t\) is the largest \(\alpha\)-expansionary stage less than \(s\) (if it exists).
- \(\#(\text{rng}(f_{\alpha,s}) - \{0,1,\ldots,t\}) > 2t\).
- There exists an even number \(y\) such that:
  - If \(\alpha^\frown\) exists and \(\alpha^\frown\) is assigned an \(R^1\) requirement then \(y \in S_{\alpha^\frown}\) (this parameter \(S_{\alpha^\frown}\) will be defined later). If \(\alpha^\frown\) is assigned an \(R^2\) requirement then \(\frac{2y}{3} \in S_{\alpha^\frown}\).
  - \(y > \max S_{\beta}\) for all \(\beta\) assigned an \(R^1\) requirement such that \(\beta \neq \infty \subseteq \alpha\) or \(\beta \neq \infty \subseteq \alpha\).
  - \(\frac{2y}{3} > \max S_{\beta}\) for all \(\beta\) assigned some \(R^2\) requirement such that \(\beta \neq \infty \subseteq \alpha\) or \(\beta \neq \infty \subseteq \alpha\).
  - \(y > x_\beta\) (the follower assigned to the positive strategy \(\beta\)) for each \(\beta\) of higher priority than \(\alpha\) assigned some \(P^1\) requirement.
  - \(y > \frac{2x_\beta}{3}\) for each higher priority \(\beta\) assigned some \(P^2\) requirement.
- \(\|y\|_{A_2,s} \subseteq \text{rng}(f_{\alpha,s})\).
- \(\#(\|y\|_{A_2,s} = 2\).
- \(f_{\alpha,s}\) looks like a reduction on all the numbers above.

**Definition 4.2** (\(R^2\)-expansionary stage) If \(\alpha\) is assigned an \(R^2\) strategy, then we call a stage \(s > 0\) an \(\alpha\)-expansionary stage if the following hold.

- \(\alpha \subseteq \delta_s\).
- \(L(\alpha,s) \neq L(\alpha,t), \) where \(t\) is the largest \(\alpha\)-expansionary stage less than \(s\) (if it exists).
- \(\#(\text{rng}(f_{\alpha,s}) - \{0,1,\ldots,t\}) > 3t\).
- There exists a number \(y\) divisible by 3 such that:
  - If \(\alpha^\frown\) exists and \(\alpha^\frown\) is assigned an \(R^1\) requirement then \(\frac{2y}{3} \in S_{\alpha^\frown}\) (this parameter \(S_{\alpha^\frown}\) will be defined later). If \(\alpha^\frown\) is assigned an \(R^2\) requirement then \(y \in S_{\alpha^\frown}\).
  - \(\frac{2y}{3} > \max S_{\beta}\) for all \(\beta\) assigned an \(R^1\) requirement such that \(\beta \neq \infty \subseteq \alpha\) or \(\beta \neq \infty \subseteq \alpha\).
  - \(y > \max S_{\beta}\) for all \(\beta\) assigned some \(R^2\) requirement such that \(\beta \neq \infty \subseteq \alpha\) or \(\beta \neq \infty \subseteq \alpha\).
\[ \frac{2y}{3} > x_\beta \] (the follower assigned to the positive strategy \( \beta \)) for each \( \beta \) of higher priority than \( \alpha \) assigned some \( P^2 \) requirement.

- \( y > x_\beta \) for each higher priority \( \beta \) assigned some \( P^2 \) requirement.

- \( \lceil y \rceil_{A_{1,s}} \subseteq \mathrm{rng}(f_{\alpha,s}) \).

- \( \# \lceil y \rceil_{A_{1,s}} = 3 \).

- \( f_{\alpha,s} \) looks like a reduction on all the numbers above.

**Strategies of individual nodes** We now describe the formal strategy for each requirement. Clearly, all the parameters and values described below are to be interpreted relative to the last stage at which a node is initialized.

**Strategy for an \( R^1 \) node \( \alpha \).** Suppose the current stage \( s > 0 \) is an \( \alpha \)-expansionary stage. Let \( y_0 \) be the least \( y \) satisfying the conditions of Definition 4.1. We fix \( \{c_\alpha, c'_\alpha\} \subseteq \mathrm{dom}(f_\alpha) \) such that \( f_\alpha(c_\alpha) = y_0 \) and \( f_\alpha(c'_\alpha) = y_0 + 1 \).

Our action for \( \alpha \) is to enumerate \( y_0 \) into \( S_\alpha \), which is a collection of all such \( y \)s chosen by \( \alpha \) in this way. Definition 4.1 ensures that at \( s \), there are at least \( t + 1 \) many distinct equivalence classes in \( \mathrm{rng}(f_\alpha) \) which are larger than \( t \). Here we let \( t' < t \) be the previous two \( \alpha \)-expansionary stages before \( s \) (if \( s \) is the first or second expansionary stage, we give a separate description below). We know there are at most \( y_{\alpha,t} + 1 \) many elements strictly between the class \( [y_{\alpha,t}] \) and \( [y_{\alpha,t}] \) , where \( y_{\alpha,t}, y_{\alpha,t} \) are the elements enumerated into \( S_\alpha \) at stage \( t' \) and \( t \).

We isolate every \( z \) strictly between \( [y_{\alpha,t}] \) and \( [y_{\alpha,t}] \). That is, for every \( z \) with \( \max[y_{\alpha,t}, y_{\alpha,t}] < z < \min[y_{\alpha,t}, y_{\alpha,t}] \), we make \( z \) isolated. Also isolate \( [3/2]_{A_{\alpha}} \) for each \( z \). We now have \( k_\alpha < y_{\alpha,t} < t \) many isolated points there. Pick \( k_\alpha \) many elements \( b_{\alpha,1}, b_{\alpha,2}, \ldots, b_{\alpha,k_\alpha} \) from \( \mathrm{dom}(f_\alpha) \) all of which are in distinct \( E \)-classes, and which are distinct from \( \{c_\alpha, c'_\alpha\} \), and all previous values in \( \mathrm{rng}(g_\alpha) \). This exists because \( f_\alpha \) looks like a reduction, and \( k_\alpha < t \). Note that if \( b' \) is a previous value in \( \mathrm{rng}(g_\alpha) \) then \( f_\alpha(b') \) converges at stage \( t \) and hence \( f_\alpha(b') \) < \( t \) (by the usual convention).

Now define \( g_\alpha(\max[y_{\alpha,t'}]) = b_{\alpha,i}, i = 1, 2, \ldots, k_\alpha, g_\alpha(y_{\alpha}) := c_\alpha \) and \( g_\alpha(3/2) := c'_\alpha \). If \( s \) isn’t \( \alpha \)-expansionary, do nothing for \( \alpha \).

The outcomes of \( \alpha \) on the priority tree are:

- **fin:** Taken each time \( \alpha \) is visited and the stage is not \( \alpha \)-expansionary.
- **\( \infty \):** This outcome is taken at every \( \alpha \)-expansionary stage.

**Strategy for an \( R^2 \) node \( \alpha \).** This works similarly to an \( R^1 \) node. Suppose the current stage \( s > 0 \) is an \( \alpha \)-expansionary stage. Let \( y_0 \) be the least \( y \)
satisfying the conditions of Definition 4.2. We fix \( \{c_\alpha, c_\alpha', c_\alpha''\} \subseteq \text{dom}(f_\alpha) \) such that \( f_\alpha(c_\alpha) = y_\alpha, f_\alpha(c_\alpha') = y_\alpha + 1 \) and \( f_\alpha(c_\alpha'') = y_\alpha + 2 \).

Our action for \( \alpha \) is to enumerate \( y_\alpha \) into \( S_\alpha \). We isolate every \( z \) strictly between \([y_\alpha, t]\) and \([y_\alpha, t)\) (these are \( A_3 \)-classes). Also isolate \([2\lfloor \frac{z}{3} \rfloor, 2\rfloor\) for each such \( z \). We now have \( k_\alpha < y_\alpha, t < t \) many isolated points there. Pick \( k_\alpha \) many elements \( b_{\alpha, 1}, b_{\alpha, 2}, \ldots, b_{\alpha, k_\alpha} \) from \( \text{dom}(f_\alpha) \) all of which are in distinct \( E \)-classes, and which are distinct from \( \{c_\alpha, c_\alpha', c_\alpha''\} \), and all previous values in \( \text{rng}(h_\alpha) \). Again, this exists for the same reasons as above.

Now define \( h_\alpha(\max[y_\alpha, t] + i) = b_{\alpha, i}, i = 1, 2, \ldots, k_\alpha, h_\alpha(y_\alpha) := c_\alpha, h_\alpha(y_\alpha + 1) := c_\alpha' \) and \( h_\alpha(y_\alpha + 2) := c_\alpha'' \). At the first two \( \alpha \)-expansionary stages, we follow the description above for \( R^3 \).

If \( s \) isn’t \( \alpha \)-expansionary, do nothing for \( \alpha \).

The outcomes of \( \alpha \) on the priority tree are:

- **fin:** Taken each time \( \alpha \) is visited and the stage is not \( \alpha \)-expansionary.
- **\( \infty \):** This outcome is taken at every \( \alpha \)-expansionary stage.

**Strategy for a \( P_1 \) node \( \alpha \):** If the witness \( x_\alpha \) is not yet defined, check if there is a least even number \( x_\alpha \) at the current stage \( s > 0 \) such that:

- If \( \alpha^- \) exists and is assigned an \( R^1 \) requirement then \( x_\alpha \in S_\alpha^- \).
- If \( \alpha^- \) exists and is assigned an \( R^2 \) requirement then \( \frac{2\alpha}{3} \in S_\alpha^- \).
- \( x_\alpha > \max S_\beta \) for all \( \beta \) assigned an \( R^1 \) requirement, such that \( \beta <_L \alpha \) or \( \beta \ast \text{fin} \subseteq \alpha \).
- \( \frac{2\alpha}{3} > \max S_\beta \) for all \( \beta \) assigned an \( R^2 \) requirement, such that \( \beta <_L \alpha \) or \( \beta \ast \text{fin} \subseteq \alpha \).
- \( x_\alpha > x_\beta \) for all \( P_1 \) nodes \( \beta \) of higher priority.
- \( \frac{2\alpha}{3} > x_\beta \) for all \( P_2 \) nodes \( \beta \) of higher priority.
- \#\{x_\alpha\}_{A_2, 3} = 2.

If these conditions are met, pick this number \( x_\alpha \) as the follower. (Here \( \varphi_\alpha \) refers to \( \varphi_\alpha \) assigned to the node \( \alpha \)).

Otherwise, \( x_\alpha \) has already been picked prior to the current stage. If this is so, check if \( \varphi_{\alpha, \varphi}(x_\alpha) \downarrow = 2 \); recall that \( P^1 \) wants to diagonalize against \( F_{A_3} \) being computable. If no, do nothing else. If yes, isolate all members of the class \([x_\alpha]_{A_2, 3}\) and \([\frac{2\alpha}{3}]_{A_2, 3}\). Declare diagonalization successful and never again attempt diagonalization (unless \( \alpha \) is initialized).

The outcomes of \( \alpha \) on the priority tree are:

- **w:** Diagonalization is not yet successful.
- **s:** Diagonalization is successful.

**Strategy for a \( P_2 \) node \( \alpha \):** If the witness \( x_\alpha \) is not yet defined, check if there is a least number \( x_\alpha \) divisible by 3 at the current stage \( s > 0 \) such that:

- If \( \alpha^- \) exists and is assigned an \( R^1 \) requirement then \( \frac{2\alpha}{3} \in S_\alpha^- \).
- If \( \alpha^- \) exists and is assigned an \( R^2 \) requirement then \( x_\alpha \in S_\alpha^- \).
- \( \frac{2\alpha}{3} > \max S_\beta \) for all \( \beta \) assigned an \( R^1 \) requirement, such that \( \beta <_L \alpha \) or \( \beta \ast \text{fin} \subseteq \alpha \).
- \( x_\alpha > \max S_\beta \) for all \( \beta \) assigned an \( R^2 \) requirement, such that \( \beta <_L \alpha \) or \( \beta \ast \text{fin} \subseteq \alpha \).
- \( \frac{2\alpha}{3} > x_\beta \) for all \( P_1 \) nodes \( \beta \) of higher priority.
$$x_\alpha > x_\beta$$ for all $P^2$ nodes $\beta$ of higher priority.
$$\#[x_\alpha]_{A_{3,s}} = 3.$$

If these conditions are met, pick this number $x_\alpha$ as the follower.

Otherwise, $x_\alpha$ has already been picked prior to the current stage. If this is so, check if $\varphi_\alpha(x_\alpha) \downarrow, \varphi_\alpha(x_\alpha + 1) \downarrow, \varphi_\alpha(x_\alpha + 2) \downarrow$, and for all $i, j \leq 2$, $A_2(\varphi_\alpha(x_\alpha + i), \varphi_\alpha(x_\alpha + j)) = 1$. If no, do nothing else. If yes, isolate all members of the class $[2z_\alpha]_{A_{2,s}}$ and $[x_\alpha]_{A_{3,s}}$. Declare diagonalization successful and never again attempt diagonalization (unless $\alpha$ is initialized).

The outcomes of $\alpha$ on the priority tree are:

- $w$: Diagonalization is not yet successful.
- $s$: Diagonalization is successful.

**Construction** At stage $s$, define in the usual way the approximation to the true path $\delta_s$ of length $s$ as the sequence of nodes eligible to act and having the correct guesses about outcomes at stage $s$. Initialize all nodes $\beta$ to the right of $\delta_s$, i.e., initialize the functions $g_\beta, h_\beta$, the set $S_\beta$ and the witness $x_\beta$.

**Verification** Denote by $\delta$ the true path of the construction, i.e. $\delta = \liminf \delta_s$.

Now we are going to show that every node $\alpha \subset \delta$ ensures the satisfaction of the requirement assigned to it. Fix $\alpha \subset \delta$, and fix a stage $s_0$ such that for every $s > s_0, \delta_s \not< L \alpha$.

First of all, we check that the global requirement $Q$ is met. It is easy to see that every step in the construction isolates $[2k]_{A_2}$ iff it isolates $[3k]_{A_3}$, for each $k$. Thus, each $A_3$ class has size 1 or 3. In this case, we clearly have $A_2 \leq \alpha, A_3$ via the reduction $2k \mapsto 3k, 2k + 1 \mapsto 3k + 1$.

**Fact 4.3** Suppose $\alpha \ast \infty \subseteq \beta$. If $\alpha$ is an $R^1$-node and $\beta$ is an $R^2$-node and $z \in S_\beta$ then $\frac{2z}{3} \in S_\alpha$. If $\alpha$ is an $R^2$-node and $\beta$ is an $R^1$-node and $z \in S_\beta$ then $\frac{3z}{2} \in S_\alpha$. If $\alpha$ and $\beta$ are $R$-nodes of the same type then $S_\beta \subseteq S_\alpha$.

**Proof** Easy to check (by a straightforward induction on $|\beta|$). \hfill $\square$

**Lemma 4.4** Each $P^1$ requirement succeeds.

**Proof** Assume that $\alpha$ is assigned a $P^1$ requirement. Suppose $\alpha$ is never able to find a suitable $x_\alpha$. The first two conditions do not pose a problem because if $\alpha^-$ exists, then as $\alpha^+ \ast \infty \subset \delta$, there will be infinitely many elements $z$ enumerated into $S_{\alpha^-}$. Furthermore for each new $z$ that $\alpha^-$ adds to $S_{\alpha^-}$, we must have $\#[z]_{A_2} = 2$ and $\# \left[ \frac{2z}{3} \right]_{A_3} = 3$ or vice versa. The third and fourth conditions also do not pose a problem because there are only finitely many such $\beta$, and none of them will increase $S_\beta$ after stage $s_0$. The fifth and sixth conditions are similar, as a $P$-node $\beta$ along the true path will only pick finitely many different $x_\beta$. Thus eventually $x_\alpha$ must be picked by $\alpha$ (and never again redefined).

Now suppose $\alpha$ picks its final $x_\alpha$ at some stage $t > s_0$. It is not hard to see that after stage $t$, no node other than $\alpha$ itself will be able to isolate $[x_\alpha]$: For an $R$-node $\beta$ extending or to the right of $\alpha$, use the fact that $\beta$ is prevented from isolating anything less than the first $\beta$-expansionary stage. For an $R$-node $\beta$ such that $\beta \ast \infty \subseteq \alpha$, apply Fact 4.3 and the fifth bullet of the strategy.
Therefore, if \( \varphi_\alpha(x_\alpha) \neq 2 \), then \( \alpha \) never acts and so obviously \( F_{A_2} \neq \varphi_\alpha \). Otherwise if \( \varphi_\alpha(x_\alpha) = 2 \) we will isolate \([x_\alpha]\) making \( F_{A_2}(x_\alpha) = 1 \). \( \square \)

**Lemma 4.5** Each \( P^2 \) requirement succeeds.

**Proof** Assume that \( \alpha \) is assigned a \( P^2 \) requirement. As in the proof of Lemma 4.4, \( \alpha \) must eventually be able to define a final \( x_\alpha \). After this is picked by \( \alpha \), no node other than \( \alpha \) itself is able to isolate \([x_\alpha]\). Suppose \( \varphi_\alpha(x_\alpha + i) \downarrow \) for \( i = 0, 1, 2 \). If \( A_2(\varphi_\alpha(x_\alpha + i), \varphi_\alpha(x_\alpha + j)) = 0 \) for some \( i, j = 0, 1, 2 \), then \( \alpha \) will never do anything to \([x_\alpha]\), which means that \( A_3(x_\alpha + i, x_\alpha + j) = 1 \). On the other hand, if \( A_2(\varphi_\alpha(x_\alpha + i), \varphi_\alpha(x_\alpha + j)) = 1 \) for every \( i, j \), then as each \( A_2 \)-class has size at most 2, this means that \( \varphi_\alpha(x_\alpha + i) = \varphi_\alpha(x_\alpha + j) \) for some \( i \neq j \). However \( \alpha \) will isolate \([x_\alpha]\) which means that \( A_3(x_\alpha + i, x_\alpha + j) = 0 \). In either case, \( \varphi_\alpha \) is not a possible reduction for \( A_3 \preceq_c A_2 \). \( \square \)

**Lemma 4.6** Each \( R^1 \) requirement succeeds.

**Proof** Assume that \( \alpha \) is assigned an \( R^1 \) requirement, and that \( f_\alpha \) witnesses \( E_m \preceq_c A_2 \) (otherwise there is nothing to prove). In particular, \( f_\alpha \) is total, \( E_m(x, y) = A_2(f_\alpha(x), f_\alpha(y)) \) for every \( x, y \) and \( l(\alpha, s) \to \infty \).

Suppose that \( \alpha \ast \text{fin} \subset \delta \), hence there is a final \( \alpha \)-expansionary stage. Then we claim that \( E_m \) is computable. Since there are only finitely many expansionary stages, hence either \( \text{rng}(f_\alpha) \) is finite, or the number \( y \) cannot be found after the final \( \alpha \)-expansionary stage. Obviously if \( \text{rng}(f_\alpha) \) is finite, then \( E_m \) is computable. So assume that \( y \) cannot be found. Since \( S_\alpha^- \) is infinite, and \( \text{max} S_\beta \) and \( x_\beta \) in the second through fifth conditions for \( y \) in Definition 4.1 are all eventually finite, the only thing preventing \( y \) from being found is if for almost every \( z, z' \in \text{rng}(f_\alpha) \) we have \( A_2(z, z') = 0 \). (Recall that \( \alpha^- \) will isolate every \( z \notin S_\alpha^- \). But this of course means that \( E \) is computable (in fact, \( E \equiv_c \text{id} \)).

Now suppose that \( \alpha \ast \infty \subset \delta \). Hence, \( S_\alpha \) is infinite. Suppose \( s_1 > s_0 \) is the first \( \alpha \)-expansionary stage after the final initialization to \( \alpha \). We claim that no strategy can isolate any \( z < y_{\alpha,s_1} \) after stage \( s_1 \). (Note that \( y_{\alpha,s_1} \) is the smallest element of \( S_\alpha \).) Again we apply Fact 4.3 and the fifth bullet of the strategy, and note that every node of lower priority than \( \beta \) is also initialized when \( \beta \) is initialized. This means that for every \( z < y_{\alpha,s_1} \), we have \([z]_{s_1} = [z]\).

We shall argue that \( g_\alpha \) is total and \( A_2(x, y) = E_m(g_\alpha(x), g_\alpha(y)) \) for every \( x, y \). Clearly \( g_\alpha \) is total because \( S_\alpha \) is infinite. There are three possibilities for \( x \). First, if \( \max[y_{\alpha,t'}] < x < \min[y_{\alpha,t}] \) at some \( \alpha \)-expansionary stage \( s \), where \( t' \geq s_1 \), then the construction will make \( x \) isolated. The construction at stage \( s \) defines \( g_\alpha(x) = b_{\alpha,j} \) for some \( b_{\alpha,j} \) found at the \( \alpha \)-expansionary stage \( s \), and it is clear that \( g_{\alpha}^{-1}([b_{\alpha,j}]) = \{x\} \). Therefore, \( A_2(x, y) = E_m(g_\alpha(x), g_\alpha(y)) = 0 \) no matter what \( y \) is.

Let’s assume now that \( x < y_{\alpha,s_1} \). Now at the second expansionary stage, we will define \( g_\alpha(x) = b_{\alpha,k} \) for some \( b_{\alpha,k} \) found at the second \( \alpha \)-expansionary stage. Clearly if \( y \geq y_{\alpha,s_1} \) then \( y \notin [x] \), and we have \( A_2(x, y) = E_m(g_\alpha(x), g_\alpha(y)) = 0 \). On the other hand if \( y < y_{\alpha,s_1} \) then the construction at the second \( \alpha \)-expansionary stage defines \( g_\alpha(y) = b_{\alpha,k} \) where \( k = j \) if \( [y]_{s_1} = [x]_{s_1} \). However, by our comments above, \([x]_{s_1} = [x]\) and \([y]_{s_1} = [y]\) and so we certainly have \( A_2(x, y) = E_m(g_\alpha(x), g_\alpha(y)) \).
Finally let’s assume that \( x = y_{a,s} \) or \( x = y_{a,s} + 1 \) for some \( \alpha \)-expansionary stage \( s \). Without loss of generality, assume \( x = y_{a,s} \). The construction defines \( g_{\alpha}(x) = c_{a} \) (for the value found at \( s \)), in particular, \( f_{\alpha}(g_{\alpha}(x)) = x \). If \( y \neq y_{a,s} \) or \( y_{a,s} + 1 \) then of course \( A_{2}(x, y) = E_{m}(g_{\alpha}(x), g_{\alpha}(y)) = 0 \). If \( y = y_{a,s} + 1 \) then \( f_{\alpha}(g_{\alpha}(y)) = y \) as well, and so \( A_{2}(x, y) = A_{2}(f_{\alpha}(g_{\alpha}(x)), f_{\alpha}(g_{\alpha}(y))) = E_{m}(g_{\alpha}(x), g_{\alpha}(y)) \).

\[ \]

Lemma 4.7 \quad Each \( R^{2} \) requirement succeeds.

Proof \quad Assume that \( \alpha \) is assigned an \( R^{2} \) requirement, and that \( f_{\alpha} \) witnesses \( E_{i} \leq c A_{3} \). In particular, \( f_{\alpha} \) is total, \( E_{i}(x, y) = A_{3}(f_{\alpha}(x), f_{\alpha}(y)) \) for every \( x, y \) and \( L(\alpha, s) \to \infty \).

If \( \alpha \neq \infty \) then we follow the proof of Lemma 4.6 to show that \( h_{\alpha} \) is total and witnesses that \( A_{3} \leq c E_{i} \). So we assume that \( \alpha \neq fin \in \delta \). Hence there is a final \( \alpha \)-expansionary stage. We claim that \( E_{i} \leq c A_{2} \). Again since there are only finitely many expansionary stages, either \( rng(f_{\alpha}) \) is finite, or the number \( y \) cannot be found after the final \( \alpha \)-expansionary stage. Obviously if \( rng(f_{\alpha}) \) is finite, then \( E_{i} \) is computable. So assume that \( y \) cannot be found. As before, the only thing preventing \( y \) from being found is if for almost every number \( z \), we have that \( \#([z]_{A_{3}} \cap rng(f_{\alpha})) \leq 2 \).

We first argue that \( E_{i} \leq c A_{2} \sqcup id \). Recall that given equivalence relations \( S \) and \( T \) we defined \( S \sqcup T = \{(2x, 2y) \mid S(x, y) = 1\} \sqcup \{(2x+1, 2y+1) \mid T(x, y) = 1\} \). Fix \( y_{0} \) such that \( \#([z]_{A_{3}} \cap rng(f_{\alpha})) \leq 2 \) for every \( z \geq y_{0} \), and consider a stage \( t \) after which \( A_{3} \upharpoonright y_{0} \) is stable. Define \( H \) to be the following partial computable function. Assume that at every stage \( s > t \), exactly one new element \( z_{s} \) enters \( rng(f_{\alpha}) \) (this set was assumed to be infinite). If \( z_{s} < y_{0} \) and \( H \) is not yet defined on any element of \([z_{s}]_{A_{3}} \), then \( H(z_{s}) = 2s + 1 \). Otherwise take \( H(z_{s}) = H(\bar{z}_{s}) \) where \( t < s' < s \) is such that \([z_{s}]_{A_{3}} = [z_{s'}]_{A_{3}} \).

(Recall that it is not important at what stage this is measured, as we assumed that \( z_{s} \) is reachable after stage \( t \) if \( z < y_{0} \)). Suppose \( z_{s} \geq y_{0} \). If \( H(z) \) is not yet defined for any \( z \) such that \([\frac{z_{s}}{2}] = [\frac{z}{2}] \) then define \( H(z_{s}) = 4[\frac{z}{2}] + 3 \). If \( H(z) \) is already defined for exactly one \( z \) such that \([\frac{z}{2}] = [\frac{z_{s}}{2}] \), define \( H(z_{s}) = 4[\frac{z}{2}] + 2 \). Otherwise \( H(z) \) is already defined for two \( z \) such that \([\frac{z}{2}] = [\frac{z_{s}}{2}] \), and in this case we define \( H(z_{s}) = 2s + 1 \).

Now we claim that \( H \circ f_{\alpha} \) is total and witnesses \( E_{i} \leq c A_{2} \sqcup id \). It is clearly total since \( dom(H) = rng(f_{\alpha}) \). Now fix \( x, y \). We shall argue that \( E_{i}(x, y) = (A_{2} \sqcup id)(H(f_{\alpha}(x)), H(f_{\alpha}(y))) \). First suppose that \( f_{\alpha}(x) < y_{0} \), then we would have defined \( H(f_{\alpha}(x)) = 2s + 1 \) for some \( s \). If \( f_{\alpha}(y) \) is of a different class than \( f_{\alpha}(x) \), then obviously \( E_{i}(x, y) = 0 \), and \( H(f_{\alpha}(y)) \) is either even or of the form \( 2v + 1 \) for some \( v \neq s \). In either case, \( E_{i}(x, y) = (A_{2} \sqcup id)(H(f_{\alpha}(x)), H(f_{\alpha}(y))) = 0 \). On the other hand if \( f_{\alpha}(y) < y_{0} \) and is in the same class as \( f_{\alpha}(x) \), then \( E_{i}(x, y) = (A_{2} \sqcup id)(H(f_{\alpha}(x)), H(f_{\alpha}(y))) = 1 \).

Now if \( f_{\alpha}(x), f_{\alpha}(y) \) are both \( \geq y_{0} \) and \( [\frac{f_{\alpha}(x)}{3}] \neq [\frac{f_{\alpha}(y)}{3}] \) then it is easy to check that \( E_{i}(x, y) = (A_{2} \sqcup id)(H(f_{\alpha}(x)), H(f_{\alpha}(y))) = 0 \). On the other hand suppose \( [\frac{f_{\alpha}(x)}{3}] = [\frac{f_{\alpha}(y)}{3}] \). If \( H(f_{\alpha}(x)) \) and \( H(f_{\alpha}(y)) \) are both even then we are okay, because recall that \( [2k]_{A_{2}} \) is isolated iff \( [3k]_{A_{3}} \) is isolated by the construction, for every \( k \). On the other hand if \( H(f_{\alpha}(x)) \) is odd, then this means that \( 3[\frac{f_{\alpha}(x)}{3}] \neq [\frac{f_{\alpha}(x)}{3}] + 1 \) and \( 3[\frac{f_{\alpha}(x)}{3}] + 2 \) are all in \( rng(f_{\alpha}) \),
which means that these elements must be isolated by the construction. This means that \( E_i(x, y) = (A_2 \cup \text{id}) (H_f(x), H_f(y)) = 0 \). This shows that \( E_i \leq_c A_2 \cup \text{id} \).

Now we check that \( A_2 \cup \text{id} \leq_c A_2 \). By Lemma 4.4 there are infinitely many \( z \) such that \( z \) is \( A_2 \)-isolated. Fix an infinite computable set \( U \) containing only \( A_2 \)-isolated points, let \( U = \{ u_0 < u_1 < \cdots \} \). Define \( G \) by \( G(2n + 1) = u_{2n+1} \) for each \( n \), and \( G(2n) = u_{2n} \) if \( n \in U \), and \( G(2n) = n \) otherwise. First of all \( G \) is clearly injective. It is also easy to check that \( G \) witnesses that \( A_2 \cup \text{id} \leq_c A_2 \).

This ends the proof of Theorem 4.2.

As mentioned earlier, our goal is to extend Theorem 4.2 to an infinite sequence \( \text{deg}_c(\text{id}) = a_1 <_c a_2 <_c a_3 <_c \cdots \):

**Theorem 4.8** There exist \( \Pi^1_1 \) \( c \)-degrees \( \text{deg}_c(\text{id}) = a_1 <_c a_2 <_c a_3 <_c \cdots \) such that \( a_{k+2} \) is a strong minimal cover of \( a_{k+1} \) for every \( k \). All constructed degrees are not set-induced.

**Proof** This is a straightforward generalization of Theorem 4.2. We want to construct, for each \( k > 1 \), a size-\( k \) \( \Pi^1_1 \) equivalence relation \( A_k \). By Lemma 3.5, \( \text{deg}_c(A_k) \) is not set-induced.

We again begin with fix \( A_{k,0} = \{(ki+j_1, ki+j_2) \mid 0 \leq j_1, j_2 < i \} \). Naturally we wish to satisfy the requirements

\[
Q^k: A_{k-1} \leq_c A_k, \text{ for } 2 \leq k \leq n.
\]

\[
R^k_{(m,n)}: E_m \leq_c A_k \text{ via } f_n \Rightarrow \exists \text{ computable function } g^{(k)}_{(n,m)} \text{ s.t. } A_k \leq_c E_m
\]

\[
\text{via } g^{(k)}_{(n,m)}, \text{ or } E_m \leq_c A_{k-1}, \text{ for } 2 \leq k \leq n.
\]

\[
P^2_k: f_{A_2} \neq \varphi_c.
\]

\[
P^k_{\alpha^2}: A_k \not\leq_c A_k-1 \text{ via } \varphi_{\alpha'}, \text{ for } 3 \leq k \leq n.
\]

The strategy for \( Q^k \) is similar as before. We ensure that for every \( x, k \) and \( k' \), we isolate \([kx]_{A_k}\) iff we isolate \([k'x]_{A_{k'}}\), and we keep \( \#[x]_{A_k} = 1 \) or \( k \).

This obviously ensures the global requirements are met. \( R^k \) works similarly as before. It measures \( \text{rng}(f_n) \), putting numbers \( y \) into \( S_\alpha \) whenever it finds a new \( y \) such that \( \#[y] = k \) and \( [y] \subseteq \text{rng}(f_n) \) (amongst other conditions). It then isolates every class not in \( S_\alpha \) (except for those blocked by higher priority requirements), as well as the corresponding classes in the other \( A_{k'} \), as required by the \( Q \) requirements. The \( P^k \)-requirements uses a class \([x_\alpha]\) where \( \#[x_\alpha] = k \) to diagonalize. Since classes of \( A_{k-1} \) have size at most \( k - 1 \), this is achieved by preventing \([x_\alpha]\) from breaking up while waiting for \( \varphi_\alpha \) to converge. There are no additional difficulties, and we leave the formal verification of the proof to the reader.

### 5 Upward density

We now turn to the question of upward density. In Sections 3 and 4 we showed that downward density fails in the classes of \( c \)-degrees being considered in this paper. In this section, we will show that in contrast, upward density holds in these classes. We will first prove this for the \( \Pi^1_1 \) \( c \)-degrees.
Theorem 5.1 Let \( u \) be the universal \( \Pi^1_1 \) c-degree, and \( a \) be any \( \Pi^1_1 \) c-degree such that \( a < u \). Then there exists two incomparable \( \Pi^1_1 \) c-degrees \( b_0 \) and \( b_1 \) such that \( a < b_0, b_1 < u \).

Proof Fix \( U \in u \) and \( A \in a \). We will build \( B_0 \) and \( B_1 \) satisfying the following requirements:

\[
P_{2e} : B_0 \nleq_c A \cup B_1 \text{ via } \varphi_e,
\]

\[
P_{2e+1} : B_1 \nleq_c A \cup B_0 \text{ via } \varphi_e.
\]

Obviously we will take \( b_0 = deg_e(A \cup B_0) \) and \( b_1 = deg_e(A \cup B_1) \).

The basic strategy for \( P_{2e} \) will be a version of the Sacks coding strategy. The idea is that if \( \varphi_e(x) \) goes inside the even integers for infinitely many \( x \), then by coding \( U \) inside \( B_0 \) and on the set of these \( x \), we can force a disagreement to appear, showing that \( B_0 \nleq_c A \) via \( \varphi_e \). On the other hand if infinitely many \( \varphi_e(x) \) go inside the odd integers, then it will be easy for us to ensure that \( B_0 \nleq_c B_1 \) via \( \varphi_e \), since we are building both sets \( B_0 \) and \( B_1 \).

We now describe the basic strategy of \( P_{2e} \). We begin by taking \( B_{i,0} = \{(e, x), (e, y) \mid e, x, y \in \omega \} \) for \( i = 0, 1 \). Denote \( C_e = \{(e, x) \mid x \in \omega \} \) for any \( e \). As the construction proceeds we will refine \( B_i \) on \( C_e \).

At stage \( s \) of the construction, we act for \( P_{2e} \) by taking the following steps. If \( P_{2s} \) has just been initialized we pick a fresh value for \( i(2e) \) and begin work for \( B_0 \) inside \( C_{i(2e)} \), and set \( M = \langle i(2e), 0 \rangle \). Note that at this point as \( i(2e) \) is freshly picked, we have \( B_0 \upharpoonright C_{i(2e)} = \{(x, y) \in B_0 \mid x, y \in C_{i(2e)} \} = C_{i(2e)} \times C_{i(2e)} \).

Now suppose that \( i(2e) \) is defined and \( k \) is such that \( f_{2e}(k) \uparrow \). Check if \( \varphi_e(M) \downarrow \) and \( \varphi_e \) currently looks like a reduction on the elements \( \leq M \). If not, we keep \( f_{2e}(k) \uparrow \) and isolate the next element of \( C_{i(2e)} \) larger than \( M \) with respect to \( B_0 \). If the conditions are met, then our action depends on the following cases:

- **\( M \) is marked.** First of all check to see if \( \varphi_e \) is injective on \( C_{i(2e)} \) so far, and if not, we get an easy win by isolating every element of \( C_{i(2e)} \) with respect to \( B_0 \), and do nothing else for this requirement.
  
  Otherwise we set \( B_0(f_{2e}(k-1), M) = 1 \). We can do this because \( M \) is marked means that we have kept \( B_0(f_{2e}(k-1), M) = 1 \) up till now. Initialize all lower priority requirements, and isolate all the rest of \( C_{i(2e)} \) with respect to \( B_0 \). Do nothing else for this requirement.

- **The following three conditions hold:**
  - \( M \) is not marked, and
  - \( \varphi_e(M) \) is odd, and
  - either \( P \in C_{i(e')} \) for any \( e' < 2 \epsilon \), or \( P \) is isolated, or \( \varphi_e(M) = \varphi_e(M') \) for some \( M' < M \) and \( M' \in C_{i(2e)} \), and where \( P = \varphi_e(M')^{-1} \).

  In this case we set \( f_{2e}(k) = M \) and redefine \( M' \) to be the next element of \( C_{i(2e)} \) larger than \( f_{2e}(k) \) and not yet isolated. We mark the new \( M \) and isolate it from every element except for \( f_{2e}(k) \).

- **\( M \) is not marked and \( \varphi_e(M) \) is even.** Then we set \( f_{2e}(k) = M \) and redefine \( M \) to be the next element of \( C_{i(2e)} \) larger than \( f_{2e}(k) \) and not yet isolated. We update the coding of \( U \) into \( B_0 \) by
setting \( U_s(i,j) = B_{0,s}(f_{2e}(i), f_{2e}(j)) \) for all \( i,j \leq k \), and set \( U_s(i,k+1) = B_{0,s}(f_{2e}(i), M) = B_{0,s}(M, f_{2e}(i)) \) for all \( i \leq k \) and for the new \( M \). Again we can do this because the new value of \( M \) is \( B_0 \)-related to \( f_{2e}(i) \) for all \( i \leq k \) just before this step.

- **Otherwise.** We isolate \( M \) and redefine it to be the next element of \( C_{i(2e)} \) not yet isolated. Follow the previous case to update the coding of \( U \) into \( B_0 \) up to and including this new \( M \) (this new \( M \) is now the intended image for \( k+1 \)).

The basic strategy for \( P_{2e+1} \) is similar to the one for \( P_{2e} \), except we reverse the roles of \( B_0 \) and \( B_1 \). When a requirement is initialized, we reset the parameters \( f, i \) and \( M \) associated with the requirement. We isolate all remaining elements of the previous column \( C_i \) associated with the requirement.

**Construction of \( B_0 \) and \( B_1 \):** As indicated above, at the beginning we begin by taking \( B_{1,0} = \{(e,x), (e,y) \mid e, x, y \in \omega \} \) for \( i = 0, 1 \). At stage \( s > 0 \), let all requirements \( R_e \) for \( e < s \) act in order according to the basic strategy above.

**Verification:** We now verify the construction works. It is clear that each \( R_e \) is initialized only finitely often. We first check that the constructed objects are equivalence relations:

**Lemma 5.2** \( B_0 \) and \( B_1 \) are equivalence relations.

**Proof** The only way for something to go wrong is for some column to be not transitive. So let’s check that every column \( C_k \) is transitive with respect to \( B_0 \) (similarly for \( B_1 \)).

We initially start with the full relation on \( C_k \). If \( C_k \) is never picked by any requirement then it stays full. Otherwise some requirement \( P_{2e} \) begins work in \( C_k \). Then \( P_{2e} \) will assign \( f_{2e}(0), f_{2e}(1), \ldots \) and begin isolating every other element until initialized. Additionally it also copies \( U \upharpoonright dom f_{2e} \) into \( B_0 \upharpoonright rng f_{2e} \) (unless the last element is marked). Since for every \( m \), there are cofinitely many \( s \) such that \( U_s \upharpoonright m \) is transitive, we can assume that \( B_0 \upharpoonright rng f_{2e} \) is always transitive.

At the end, we have the following possibilities:

- \( f_{2e} \) is total. Then \( C_k \) will contain \( rng f_{2e} \) and infinitely many isolated elements, where \( rng f_{2e} \equiv_{e} U \).
- \( dom f_{2e} \) is finite, and we eventually go through infinitely many different \( M \). Then \( C_k \) contains a (copy of a) finite initial segment of \( U \) as well as infinitely many isolated elements.
- \( dom f_{2e} \) is finite, and we eventually settle on a final \( M \), where \( k \) is the largest element of \( dom f_{2e} \). If \( M \) is marked then \( M \) is in the same class as \( f_{2e}(k) \) and \( C_k \) contains the finite initial segment of \( U \upharpoonright k \). Otherwise if \( M \) is not marked then \( C_k \) contains the finite initial segment of \( U \upharpoonright k + 1 \). Finally if \( R_{2e} \) is initialized or if we act one last time under the first case of the basic strategy, then \( C_k \) is also transitive.

**Lemma 5.3** Each \( R_e \) acts only finitely often, that is, it extends \( dom f_e \) and redefines \( M \) only finitely often.
Proof. We proceed for $R_{2e}$, the other case is symmetric. Since $R_{2e}$ is initialized finitely often, fix the final $C_i(2e)$ assigned to $R_{2e}$. Suppose $R_e$ acts infinitely often. Hence no element is ever marked by $R_{2e}$.

Suppose the third case is taken infinitely often by $R_{2e}$. Hence $f_{2e}$ is total. Furthermore, as $R_{2e}$ never gets stuck waiting forever, this means that $\varphi_e \circ f_{2e}$ is total, and provides a reduction from $U$ to $A \cup B_1$. Since the range of this reduction is contained in the even integers, this means that $U \leq_e A$, a contradiction.

Now suppose the fourth case is taken cofinitely often by $R_{2e}$. In this case we will discover, for some $e' < 2e$, an infinite sequence $M_0 < M_1 < \cdots$ of elements inside $C_i(2e)$ such that for every $j \neq j'$,

- $\varphi_e(M_j)$ is odd,
- $\varphi_e(M_j)^{-1} \in C_{i(e')}$,
- $\varphi_e(M_j)^{-1}$ is picked by $R_{e'}$ to be in $\text{rng} f_{e'}$ or a value of $M$,
- $\varphi_e(M_j) \neq \varphi_e(M_{j'})$.

However this is impossible since by the induction hypothesis, $R_{e'}$ only goes through finitely many different values of $M$ and $\text{dom} f_{e'}$. 

\[ \square \]

Lemma 5.4 Each $R_e$ is satisfied.

Proof. Assume that $\varphi_e$ witnesses that $B_0 \leq_e A \cup B_1$. By Lemma 5.3 $R_{2e}$ eventually stops acting, therefore, the final action taken by $R_{2e}$ must be under the first case. If the final action was due to the fact that we discovered that $\varphi_e$ is not injective on $C_i(2e)$, then clearly we get a contradiction. Thus we can assume that the final action under the first case was to set $B_0(f_{2e}(k-1), M) = 1 - (A \cup B_1)(\varphi_e(f_{2e}(k-1)), \varphi_e(M))$. Suppose $s_1$ is the stage where we took the final action, and $s_0 < s_1$ was the earlier stage where $R_{2e}$ marked $M$. It remains to verify that $(A \cup B_1)(\varphi_e(f_{2e}(k-1)), \varphi_e(M))$ will never change after $s_1$, and we get the desired contradiction. This can only be possible if $(\varphi_e(f_{2e}(k-1)), \varphi_e(M)) \in A_{s_1} \cup B_{1,s_1}$.

At the earlier stage where we marked $M$ we must have discovered that $\varphi_e(f_{2e,s_1}(k-1))$ is odd and that either $\varphi_e(f_{2e,s_1}(k-1))^{-1} \notin C_{i(e')}$ for any $e' < 2e$, or $\varphi_e(f_{2e,s_1}(k-1))^{-1}$ is isolated. If the latter holds then we must in fact have $\varphi_e(f_{2e,s_1}(k-1)) = \varphi_e(M)$.

So we may suppose that $\varphi_e(f_{2e,s_1}(k-1))^{-1} \notin C_{i(e')}$ for any $e' < 2e$ at stage $s_0$. At stage $s_1$, if $v \neq i(e')$ for any $e'$, then $C_v$ will never be assigned to any requirement and so is never modified by the construction, where $\varphi_e(f_{2e,s_1}(k-1))^{-1} \in C_v$. Therefore we can assume that $v = i(e')$ at stage $s_1$.

Since no higher priority requirement is initialized between $s_0$ and $s_1$, we must have $e' \geq 2e$. We do not have to worry about $e' = 2e$ because $C_{i(2e)}$ must be the full relation with respect to $B_1$, since it is already associated with $R_{2e}$ working for $B_0$. Thus the last case to consider is when $e' > 2e$. However $R_{e'}$ is initialized at stage $s_1$, which means that there can be no more changes to $C_v$ with respect to $B_1$ after stage $s_1$. 

\[ \square \]

This ends the proof of Theorem 5.1.
It is easy to modify the proof of Theorem 5.1 for the other classes being studied. Therefore, we have:

Corollary 5.5  The classes of $\Pi^0_1$, $\Sigma^0_n$, $n$-c.e. and $\omega$-c.e. equivalence relations are upwards dense, for $n \geq 1$.

Proof  In each class, there is a universal member $U$. We apply an appropriate version of the Sacks coding strategy, and leave the proof to the reader.

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Note
1. Note that $S$ and $E_e$ are actually equal instead of merely $\equiv_c$. This is the best kind of enumeration one can hope for.

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