Appendix

Proof of Theorem 1

Theorem 1. *The* bestO-D *problem is NP-hard.*

Proof. **Reduction from Set-Cover Problem to the *bestO-D* Problem:** We convert an arbitrary instance of the set cover problem into an instance of the *bestO-D* problem. Given a set U with n elements, let $\mathcal{Q} \subseteq 2^U$ be a collection of subsets of U. The set cover problem concerns that given an integer k, whether U can be covered with k subsets in $\mathcal{Q}$. We convert an arbitrary set cover problem into a *bestO-D* problem as follows. First, we construct the network of the terrorists $G = \langle V, E\rangle$. For each element $i \in U$, we add a vertex v labeled as $\{i\}$ into the network. For each non-singleton subset $Q \in \mathcal{Q}$, we add a vertex v labeled as Q into the network. We call this set the $label\ set$ of the corresponding vertex v, denoted as $l(v)$. For each $i \in U, Q \in \mathcal{Q}$, we add an edge between the vertex labeled as $\{i\}$ and the vertex labeled as Q if $i \in Q$. Second, we construct the support set of the attacker's mixed strategy, i.e., the set of the pure strategies which are used with non-zero probabilities. For each element $i \in U$, we construct an corresponding pure attacker strategy, denoted as $A_i = \{\cup_{i\in Q, Q\in\mathcal{Q}} l^{-1}(Q)\}$, where $l^{-1}(Q)$ represents the vertex in the graph whose label set is Q. We then add this strategy into the support set. Thus the support set of the attacker's mixed strategy includes $|U|$ pure strategies, each pure strategy corresponds to an element in U.

For example, assume that $U = \{1, 2, 3\}$ and $\mathcal{Q} = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\}$. The constructed graph is shown in Figure 1.

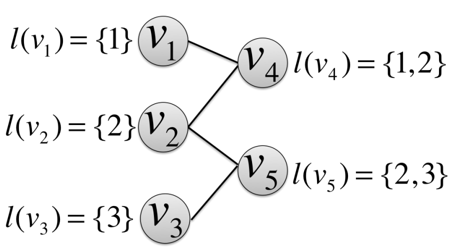

Figure 1: Constructed graph

The support set of the attacker's mixed strategy includes 3 pure strategies as is shown in Table 1.

pure strategy	corresponding element in U	vertices
A_1	1	$\{v_1, v_4\}$
A_2	2	$\{v_2, v_4, v_5\}$
A_3	3	$\{v_3, v_5\}$

Table 1: Constructed attacker strategies

We then show that set U can be covered with k subsets in $\mathcal{Q}$ if and only if the defender can monitor all of the attacker's strategies with k resources in the corresponding *bestO-D* problem.

The 'if' direction: Assume that the defender can monitor all the attacker strategies by allocating k resources on k vertices of the graph. Let B represent the set of the k vertices. We show that set U can be covered by k sets $l(v), \forall v \in B$.

Assume that the set U cannot be covered by the k sets $l(v), \forall v \in B$. It indicates that there exists at least an element $i \in U$ such that $i \notin l(v), \forall v \in B$. Note that attacker strategy A_i is used with nonzero probability. Given the definition of A_i, it holds that $\forall v \in A_i, i \in l(v)$. Thus, no vertex in A_i is included in B, and thus the attacker's strategy A_i is not monitored by the k resources, which contradicts our assumption.

The 'only if' direction: If the set U can be covered by k subsets $Q \in \mathcal{Q}$, then the defender can monitor all attacker strategies with k resources by allocating them on vertices whose label sets are the k subsets Q. This is because for each $i \in U$, if there exists a vertex v with $i \in l(v)$ which is covered, then the attacker strategy A_i will be monitored. Given that U can be covered by the k subsets Q, it holds that for each $i \in U$, it is a member of at least one of the k subsets. Therefore, all attacker's strategies can be monitored by k resources. $\square$

Proof of Theorem 2

Theorem 2. *Let* $U' = \max_{S\in\mathcal{S}_{better}} U_d(S, \mathbf{y})$*, where* $\mathcal{S}_{better}$ *is the solution returned by betterO-D. Let* $U^* = \max_{S\in\mathcal{S}} U_d(S, \mathbf{y})$*. Then* $\frac{U'-U_d(\emptyset,\mathbf{y})}{U^*-U_d(\emptyset,\mathbf{y})} \geq 1 - \frac{1}{e}$*.*

Proof. The defender utilities are always negative, therefore an approximation ratio in terms of utility is meaningless. To facilitate analysis, we first define a non-negative function f for each defender allocation S such that $|S| \leq R$:

$$f_{\mathbf{y}}(S) = U_d(S, \mathbf{y}) - U_d(\emptyset, \mathbf{y}) = \sum_{A\in\mathcal{A}'|A\cap S\neq\emptyset} y_A P(A) \quad (1)$$

Thus f gives the marginal benefit of allocation S over the non-defending defender strategy.

Given the attacker's mixed strategy $\mathbf{y}$, we first prove that $f_{\mathbf{y}}(S)$ is sub-modular, i.e., $f_{\mathbf{y}}(S_1\cup\{v\}) - f_{\mathbf{y}}(S_1) \geq f_{\mathbf{y}}(S_2 \cup \{v\}) - f_{\mathbf{y}}(S_2)$, for each $S_1 \subseteq S_2 \subseteq V, v \in V \setminus S_2$.

Let $\overline{\mathcal{A}}_1 = \{A \in \mathcal{A}'|A \cap S_1 = \emptyset\}$, i.e., the set of attacker strategies that are *not* monitored by S_1. Similarly, let $\overline{\mathcal{A}}_2 = \{A \in \mathcal{A}'|A \cap S_2 = \emptyset\}$. Since $S_1 \subseteq S_2$, every subgraph that is monitored by S_1 is also monitored by S_2, i.e., we have $\overline{\mathcal{A}}_2 \subseteq \overline{\mathcal{A}}_1$.

Now we will show that $f_{\mathbf{y}}(S_1\cup\{v\}) - f_{\mathbf{y}}(S_1) \geq f_{\mathbf{y}}(S_2 \cup \{v\}) - f_{\mathbf{y}}(S_2)$. According to Eq.(1), we have:

$$\begin{aligned} & f_{\mathbf{y}}(S_1 \cup \{v\}) - f_{\mathbf{y}}(S_1) \\ &= \sum_{A\in\mathcal{A}'|A\cap(S_1\cup\{v\})\neq\emptyset} y_A P(A) - \sum_{A\in\mathcal{A}'|A\cap S_1\neq\emptyset} y_A P(A) \\ &= \textstyle\sum_{A\in\overline{\mathcal{A}}_1|A\cap\{v\}\neq\emptyset} y_A P(A) \qquad (2) \end{aligned}$$

and similarly,

$$f_{\mathbf{y}}(S_2 \cup \{v\}) - f_{\mathbf{y}}(S_2) = \textstyle\sum_{A\in\overline{\mathcal{A}}_2|A\cap\{v\}\neq\emptyset} y_A P(A) \quad (3)$$

Since $\overline{\mathcal{A}}_2 \subseteq \overline{\mathcal{A}}_1$ and $y_A P(A) \geq 0$, we conclude that $f_{\mathbf{y}}(S_1 \cup \{v\}) - f_{\mathbf{y}}(S_1) \geq f_{\mathbf{y}}(S_2 \cup \{v\}) - f_{\mathbf{y}}(S_2)$. Hence, $f_{\mathbf{y}}(S)$ is a non-negative sub-modular function.

Observe that, *betterO-D* (Algorithm 2) repeatedly starts from each single vertex $v \in V$ and iteratively maximizes $U_d(S, \mathbf{y})$ by adding a node v^* to the current defender allocation S. Consider the best starting vertex $v_g = \arg\max_{v \in V} U_d(\{v\}, \mathbf{y})$ and the solution S^{v_g} obtained from this vertex v_g. Because $U_d(\emptyset, \mathbf{y})$ is a constant, it is equivalent to say S^{v_g} is an solution obtained by greedily maximizing the non-negative sub-modular function $f_{\mathbf{y}}(S)$. Let $S^* = \arg\max_{S \in \mathcal{S}} U_d(S, \mathbf{y})$ and $S' = \arg\max_{S \in \mathcal{S}_{better}} U_d(S, \mathbf{y})$, according to (Nemhauser, Wolsey, and Fisher 1978), we have:

$$f_{\mathbf{y}}(S') \geq f_{\mathbf{y}}(S^{v_g}) \geq (1 - \frac{1}{e}) f_{\mathbf{y}}(S^*) \tag{4}$$

Thus, $\frac{U' - U_d(\emptyset, \mathbf{y})}{U^* - U_d(\emptyset, \mathbf{y})} \geq 1 - \frac{1}{e}$. □

Proof of Theorem 3

Theorem 3. *Let* $(\mathbf{x}^*, \mathbf{y}^*)$ *be the optimal solution of a TPD, and let* $(\mathbf{x}', \mathbf{y}')$ *be the solution computed by Algorithm 1 with Line 5 skipped. Then* $\frac{U_d(\mathbf{x}', \mathbf{y}') - U_d(\emptyset, \mathbf{y}')}{U_d(\mathbf{x}^*, \mathbf{y}^*) - U_d(\emptyset, \mathbf{y}')} \geq 1 - \frac{1}{e}$.

Proof. Firstly, given that $(\mathbf{x}^*, \mathbf{y}^*)$ is a minimax solution,

$$U_d(\mathbf{x}^*, \mathbf{y}^*) \geq U_d(\mathbf{x}, \mathbf{y}^*) \quad \forall \mathbf{x} \tag{5}$$

$$U_d(\mathbf{x}^*, \mathbf{y}) \geq U_d(\mathbf{x}^*, \mathbf{y}^*) \quad \forall \mathbf{y} \tag{6}$$

Furthermore, we define

$$f_{\mathbf{y}}(\mathbf{x}) = \sum_{S \in \mathcal{S}} x_S f_{\mathbf{y}}(S) \tag{7}$$

According to Eqs.(1), (5), (6) and (7), we have:

$$\begin{aligned} f_{\mathbf{y}^*}(\mathbf{x}^*) &= \sum_{S \in \mathcal{S}} x^*_S f_{\mathbf{y}^*}(S) \\ &= \sum_{S \in \mathcal{S}} x^*_S (U_d(S, \mathbf{y}^*) - U_d(\emptyset, \mathbf{y}^*)) \\ &= U_d(\mathbf{x}^*, \mathbf{y}^*) - U_d(\emptyset, \mathbf{y}^*) \\ &\geq U_d(\mathbf{x}, \mathbf{y}^*) - U_d(\emptyset, \mathbf{y}^*) \\ &= f_{\mathbf{y}^*}(\mathbf{x}) \quad \forall \mathbf{x} \end{aligned} \tag{8}$$

$$\begin{aligned} f_{\mathbf{y}}(\mathbf{x}^*) &= \sum_{S \in \mathcal{S}} x^*_S f_{\mathbf{y}}(S) \\ &= \sum_{S \in \mathcal{S}} x^*_S (U_d(S, \mathbf{y}) - U_d(\emptyset, \mathbf{y})) \\ &= U_d(\mathbf{x}^*, \mathbf{y}) - U_d(\emptyset, \mathbf{y}) \\ &\geq U_d(\mathbf{x}^*, \mathbf{y}^*) - U_d(\emptyset, \mathbf{y}) \\ &= f_{\mathbf{y}^*}(\mathbf{x}^*) + U_d(\emptyset, \mathbf{y}^*) - U_d(\emptyset, \mathbf{y}) \quad \forall \mathbf{y} \end{aligned} \tag{9}$$

Let $S' = \arg\max_{S \in \mathcal{S}_{better}} U_d(S, \mathbf{y}')$, i.e., S' is the pure defender strategy with maximal defender utility, when using *betterO-D* against attacker mixed strategy $\mathbf{y}'$. By definition of $\mathbf{y}'$, we have $U_d(S', \mathbf{y}') = U_d(\mathbf{x}', \mathbf{y}')$. Let $S^* = \arg\max_{S \in \mathcal{S}} U_d(S, \mathbf{y}')$, it holds that $U_d(S^*, \mathbf{y}') \geq U_d(\mathbf{x}^*, \mathbf{y}')$. Using Eqs.(4) and (9),

$$\begin{aligned} & f_{\mathbf{y}'}(\mathbf{x}') = f_{\mathbf{y}'}(S') \\ & \geq (1 - \frac{1}{e}) f_{\mathbf{y}'}(S^*) \\ & \geq (1 - \frac{1}{e}) f_{\mathbf{y}'}(\mathbf{x}^*) \\ & \geq (1 - \frac{1}{e}) [f_{\mathbf{y}^*}(\mathbf{x}^*) + U_d(\emptyset, \mathbf{y}^*) - U_d(\emptyset, \mathbf{y}')] \end{aligned} \tag{10}$$

Therefore, $\frac{U_d(\mathbf{x}', \mathbf{y}') - U_d(\emptyset, \mathbf{y}')}{U_d(\mathbf{x}^*, \mathbf{y}^*) - U_d(\emptyset, \mathbf{y}')} \geq 1 - \frac{1}{e}$. □

Proof of Theorem 4

Theorem 4. *The bestO-A problem is NP-hard, even when network externality is zero, i.e.,* $\delta = 0$.

Proof. We reduce a set-cover optimization problem to a *bestO-A* problem without the network externality effect. Given a set U with n elements and a set $\mathcal{Q}$ of subsets of U such that $\cup_{Q \in \mathcal{Q}} Q = U$, the task for the set-cover optimization problem is to find a minimum subset $\mathcal{Q}' \subseteq \mathcal{Q}$, such that each element of U is contained in at least one set in $\mathcal{Q}'$. Denote a set-cover optimization problem as a pair $\langle U, \mathcal{Q} \rangle$. For arbitrary $\langle U, \mathcal{Q} \rangle$, we construct a *bestO-A* problem as follows.

Step 1: We create a vertex u_i for each element $u_i \in U$, and set the capability value τ_{u_i} of each vertex u_i to 1.

Step 2: For each $Q_j \in \mathcal{Q}$, we create a new vertex Q_j and set the capability value τ_{Q_j} of each vertex Q_j to 0. Then we create an edge between each node u_i and Q_j such that $u_i \in Q_j$.

Step 3: We create an auxiliary vertex H which is connected to each vertex Q_j and set the capability value τ_H to 1. Then, we create R auxiliary vertices where R is the number of defender resources and set the capability value τ_{R_k} of each vertex R_k to 0. For each vertex $R_k, k = 2, \cdots, R$, we create an edge between R_k and R_{k-1}, then we create an edge between R_1 and H[1].

Step 4: We then create $|\mathcal{Q}| + 1$ pure strategies for the defender. First, for each vertex Q_j, we create a pure strategy $S_j = \{R_1, \cdots, R_{R-1}, Q_j\}$ and set the corresponding possibility x_j to $\frac{1}{|\mathcal{Q}|(|U|+1)}$. Then, we create the last pure defender strategy $S_{|\mathcal{Q}|+1} = \{R_1, \cdots, R_{R-1}, R_R\}$ and put the remaining possibility $\frac{|U|}{|U|+1}$ on it. Thus, the marginal cover probabilities for each vertex u_i and the auxiliary vertex H are 0, that for each vertex Q_j is $\frac{1}{|\mathcal{Q}|(|U|+1)}$. The auxiliary vertices $R_k, k = 1, \cdots, R - 1$ are covered by all the defender pure strategies, the last auxiliary vertex R_R is only covered by $S_{|\mathcal{Q}|+1}$.

Obviously, the above steps are executed in polynomial time. For a given set-cover instance $\langle U, \mathcal{Q} \rangle$ and the number of defender resources R, the above steps create a weighted graph $G = (V, E)$ with $|U| + |\mathcal{Q}| + R + 1$ vertices and a mixed defender strategy with $|\mathcal{Q}| + 1$ pure strategies for the

[1]We use such a construction for simplicity, this connection is not essential for the NP-hardness reduction. Indeed, these R auxiliary vertices can be arbitrarily connected to other vertices.

corresponding *bestO-A* problem instance. For example, for a set-cover instance with $U = \{u_1, u_2, u_3\}$ and $\mathcal{Q} = \{Q_1 = \{u_1, u_2\}, Q_2 = \{u_1\}, Q_3 = \{u_2\}, Q_4 = \{u_2, u_3\}\}$, suppose $R = 2$, the corresponding *bestO-A* problem instance is shown in Figure 2.

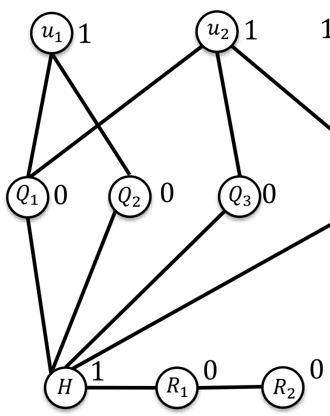

Figure 2: A *bestO-A* problem instance corresponding to the set-cover instance with $U = \{u_1, u_2, u_3\}$ and $\mathcal{Q} = \{Q_1 = \{u_1, u_2\}, Q_2 = \{u_1\}, Q_3 = \{u_2\}, Q_4 = \{u_2, u_3\}\}$. Here, the mixed defender strategy uses five allocations: $\{R_1, Q_1\}$, $\{R_1, Q_2\}$, $\{R_1, Q_3\}$, $\{R_1, Q_4\}$, $\{R_1, R_2\}$, and the corresponding possibilities are $\frac{1}{16}$, $\frac{1}{16}$, $\frac{1}{16}$, $\frac{1}{16}$, $\frac{3}{4}$, respectively.

The task for the *bestO-A* problem is to find a subset A of V such that the induced subgraph $G[A]$ is connected and the attacker can obtain the maximum expected utility playing against the defender's mixed strategy $\mathbf{x}$. We now show that any set of Q_j contained in a maximum-utility connected subgraph of G is a minimum cover with respect to the set-cover instance and vice versa.

We first show that any optimal solution $G[A]$ for the *bestO-A* problem does not contain any auxiliary vertex $R_k, k = 1, \cdots, R$. Observe that the marginal cover probability of each vertex $R_k, k = 1, \cdots, R-1$ is 1, then the expected utility of any subgraph containing a vertex $R_k, k = 1, \cdots, R-1$ is 0. We can easily find a subgraph (e.g., $G[\{H\}]$) with utility 1, so any vertex $R_k, k = 1, \cdots, R-1$ is not included in a optimal solution. Besides, the capability value τ_{R_R} is 0, so the expected utility of $G[\{R_R\}]$ is 0. and vertex R_R is only connected to R_{R-1}, thus any larger connected subgraph containing R_R must contain R_{R-1}, resulting that its expected utility is also 0. Therefore, R_R is not included in any optimal solution.

Then, we show that any optimal solution $G[A]$ must contains H and all the vertices u_i. First, we can find a feasible solution $A = V \setminus \{R_k, R_k, k = 1, \cdots, R\}$. The payoff of this solution is $|U| + 1$ and the cover probability of $G[A]$ is $\frac{1}{|U|+1}$, thus the expected utility of $G[A]$ is $(|U| + 1)(1 - \frac{1}{|U|+1}) = |U|$. It is obvious that any solution not containing all the vertices u_i and H cannot achieve this expected utility, thus is not optimal solution.

To this end, any optimal solution $G[A]$ contains H and all vertices u_i, and does not contain any auxiliary vertex $R_k, k = 1, \cdots, R$. Thus, any optimal solution can be identified by the set of vertices Q_j contained in it. Since H is included in $G[A]$, the connectivity of $G[A]$ is equivalent to the requirement that, for each vertex u_i, there is at least one vertex $Q_j \in A$ such that $u_i \in Q_j$. Thus, maximizing the expected utility of $G[A]$ is equivalent to minimizing of the Q_j vertices, which, in turn, is equivalent to minimizing the set cover represented by the Q_j. For example, in the instance shown in Figure 2, the set-cover instance has two optimal solutions of size 2 (i.e., $\mathcal{Q}' = \{Q_1, Q_4\}$ or $\{Q_2, Q_4\}$.), corresponding to the two optimal solutions with utility $4 * (1 - \frac{2}{16}) = 3.5$ of the *bestO-A* problem (i.e., $A = \{Q_1, Q_4, H, u_1, u_2, u_3\}$ or $\{Q_2, Q_4, H, u_1, u_2, u_3\}$). $\square$