Asset Allocation and Pension Liabilities in the Presence of a Downside Constraint

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Abstract

We revisit the question of a pension sponsor’s optimal asset allocation in the presence of a downside constraint and the possibility for the pension sponsor to contribute money to the pension plan. Using detailed data on pension plans’ asset allocations and voluntary contributions, we find a U-shaped pattern of equity weights in terms of funding status. Especially, equity weights and voluntary contribution simultaneously increase when funding status deteriorates, which is hard to reconcile with a hypothesis of risky gambling. We build a model of a pension sponsor with disutility associated with contributions, that produces observed pattern of equity weights and contribution. Relative to a common risk management strategy that a pension sponsor should fully switch to the risk-free portfolio during underfunding, implementing the optimal asset allocation and contribution policy implies a welfare gain of XX%. 
1 Introduction

A large decline in pension plans’ funding ratio motivated the creation of mandatory contribution rules and public insurance on defined benefit pension plans. For example, in the U.S. Employee Retirement Income Security Act (ERISA) in 1974 created the minimum funding contribution (MFC) and Pension Benefit Guaranty Corporation (PBGC).\(^1\) Despite of these government’s interventions to save underfunded pension plans, unfortunately large number of defined benefit pension plans are still underfunded.\(^2\) Thus, we believe that it is important to understand how underfunded pension plans can end up with funded status through the optimal asset allocation and contribution policy in the first place.

To this end, we revisit the question of a defined-benefit pension sponsor’s optimal asset allocation in the presence of a downside constraint. It is well-known (Grossman and Vila (1989)) that when markets are complete a put-based strategy is optimal by combining the unconstrained optimal portfolio and a put option on that unconstrained portfolio to hedge the downside. This analysis ignores, however, the possibility for the pension sponsor to contribute money to the pension plan over time. We analyze the joint problem of optimal investing and contribution decisions, when there is disutility associated with contributions.\(^3\) Interestingly, we find that with the possibility of costly contributions to the pension plan to satisfy the downside constraint, the optimal portfolio decision often looks like a “risky gambling” strategy where the pension sponsor increases the pension plan’s allocation to risky assets during low funding ratio.\(^4\) This is very different from the traditional prediction, where in economy downturns the pension sponsor should fully switch to the risk-free portfolio that replicates the downside constraint.

Low funding ratio affect the optimal portfolio weight in two different directions. First, the pension sponsor starts to contribute contemporaneously and keeps doing so as long as the funding ratio is low. Thus, the pension sponsor can invest more aggressively by increasing the equity weight as if the pension plan’s asset is increased by the present value of contemporaneous and

\(^1\) MFC requirements specify that sponsors of underfunded pension plans must contribute an amount equal to any unfunded liabilities. PBGC has insurance obligations to pay defined benefits to employees when pension sponsors fail to fulfill due to firms’ bankruptcy.

\(^2\) In 2013 the largest 100 corporate defined benefits pension plans in the U.S. reported 1.78 trillion USD of liabilities guaranteed with only 1.48 trillion USD of asset, which represents underfunding of more than 15%. See Milliman 2014 Corporate Pension Funding Study, www.milliman.com.

\(^3\) Rauh (2006) finds that mandatory contributions leads to a reduction in corporate investment. Thus, the disutility from contributions is a reduced form of costs of foregone investment opportunities due to a use of internal cash for contributions.

\(^4\) Funding ratio is often defined as the ratio of pension asset over the present value of future benefits.
future contributions. In other words, increased risky allocations will be hedged by contemporaneous and future contributions. Second, the pension sponsor decreases the equity weight to hedge the downside risk. If the former effect dominates the latter one, then a risky gambling behavior can be observed. However, note that this risk taking incentive is induced not by a moral hazard problem, but by a commitment to contributions.

We propose a separation approach to solve the optimal contribution and portfolio policy. The pension sponsor’s problem is cast in two separate shadow price problems. The first problem solves for the shadow price of maximizing the utility over the terminal pension plan’s asset. The second problem solves for the shadow price of minimizing the intermediate disutility from contributions. We interpret the shadow price of the utility maximization problem as the marginal benefit of increasing contributions. Similarly, the shadow price of the disutility minimization problem is the marginal cost of doing so. We show that the shadow prices for two problems are identical such that the marginal benefit and cost of increasing contributions are equal at the optimal solution.

Our approach allows us to characterize the optimal contribution, portfolio policy, and the value of put option in a simple way. Especially, the optimal contribution and the value of put option shed light on the level of minimum mandatory contributions and the premium that PBGC should charge to the pension sponsor. Also, by comparing with a case without a downside constraint, we can predict morally hazardous reactions of the pension sponsor in the presence of government insurance.

The investment behavior of pension plans has been studied by Sharpe (1976), Sundaresan and Zapatero (1997), Boulier, Trussant, and Florens (1995), and Van Binsbergen and Brandt (2007). Sharpe (1976) first recognized the value of implicit put option in pension plan’s asset to insure shortfall at the maturity. Sundaresan and Zapatero (1997) consider the interaction of pension sponsors and their employees. Given the marginal productivity of workers, the retirement date is endogenously determined. Then, pension sponsors solve the investment problem of maximizing the utility over excess assets in liabilities. Our focus is to derive the optimal contribution and portfolio policy, we model the exogenous and deterministic benefits of the pension plan.\footnote{As long as the market is complete, our model can be extended to incorporate a stochastic feature of liabilities, and the solution technique goes through.}

Our paper is closely related to Boulier, Trussant, and Florens (1995). In their problem, the investment manager chooses his portfolio weights and contribution rates to minimize the
quadratic disutility from contributions with the downside constraint. However, from the perspective of the pension sponsor the surplus at the end of the pension plan also matters since it is usually refunded to the pension sponsor and can be used to fund profitable projects. We model this motive as the utility over the terminal pension plan’s asset. Van Binsbergen and Brandt (2007) solve for the optimal asset allocation of the pension sponsor under regulatory constraints. They assume time-varying investment opportunity sets, and explore the impact of regulatory constraints on asset allocations. However, a contribution is not a control variable and a downside constraint is not explicitly specified. Instead, we assume an absence of any government regulations and derive the optimal contribution and portfolio policy. By doing this, we can have policy implications on how minimum contribution rules and premium paid to PBGC should be decided.

Our methodology is based on Karatzas, Lehoczky, and Shreve (1987) and El Karoui, Jeanblanc, and Lacoste (2005). Karatzas, Lehoczky, and Shreve (1987) solve a consumption and portfolio choice problem. They find that the initial wealth can be allocated in two problems, maximizing the utility over intermediate consumption and maximizing the utility over the terminal wealth. The optimal allocation leads to the optimal solution to the original problem. In our model, a contribution is a counterpart of consumption, but it generates the disutility and the pension sponsor’s objective is to minimize this disutility. Thus, the problem can be cast in a problem to decide how much to contribute to satisfy the downside constraint while minimizing the disutility. El Karoui, Jeanblanc, and Lacoste (2005) find a put option based solution to maximize the utility over the terminal wealth with the downside constraint. However, their solution can be applied to only initially overfunded pension plans. We allow initially underfunded pension plans to contribute in order to guarantee the terminal benefits.

There are at least three important aspects that we do not address explicitly. First, we do not incorporate time-varying investment opportunities. The expected returns of bonds and equities are predicted by macro variables, such as short rates, yield slopes, and dividend yields. This induces non-trivial hedging demands and liability risks, which drive a wedge between myopic and dynamic investment. Second, we do not consider the taxation issues. Drawing contributions from firm’s internal resources is costly for sure, however there is also a benefit from tax deductions. Third, our model do not include inflation. Depending on whether the pension sponsor’s preference is in real or nominal term, the allocation to real assets such as TIPS should be considered.

The paper is organized as follows. Section 2 describes the pension plan’s benefits and asset
return dynamics. Section 3 considers a constrained case in which there is the downside constraint, and the separation method for the optimal investment and contribution policy. Section B presents the pension sponsor’s problem without the downside constraint as a benchmark case. Section 4 presents our results and Section 5 concludes.

2 Defined Benefit (DB) Pension Model

In this section, we specify the pension plan’s benefit, the investment opportunity set, and the preferences of the pension sponsor. The next three subsections describe these three items in turn.

2.1 Pension Plan’s Benefit

We consider a finite time span that starts at 0 and finishes at a fixed and known date $T$, at which employees of the pension plan retire. Pension benefits are paid at retirement $T$ (whose value we denote by $L$). The amount of benefits usually depends on the weighted average of wages, which is uncertain. In this case, the history of wages becomes important. In our model, however, we assume that wages are deterministic and hence there is no uncertainty about future benefits. Considering deterministic benefits may seem restrictive, it is a special case that approximates the “dollar amounts formula“ used in the industry in which benefits are based on the years of service to the firm multiplied by a fixed dollar amount.

2.2 Capital Markets

The pension sponsor has two available assets, a risky stock and a risk-free money market account. Let $r$ be the risk-free rate. We assume that $r$ is constant. The stock price follows

$$dS_t = \mu S_t dt + \sigma S_t dZ_t,$$

where $\mu$ is the expected return of the stock, $\sigma$ is the volatility parameter, and $Z$ is a standard Brownian motion. We define the constant price of risk $\eta = (\mu - r)/\sigma$. Hence, in our model there is only one systematic shock and one risky asset, and the market is complete. This implies that there exists a unique pricing kernel or stochastic discount factor and any contingent claims can be replicated by constructing a dynamic portfolio consisting of the risky asset and the risk-free asset. This feature provides us with two important implications on pension sponsor’s optimal
.asset allocation and contribution policy: first, we are able to measure the value of a put option. This step is essential in determining the required amount of contributions to meet the promised obligations. Comparing the value of the put option to the initial endowment of the pension plan, we may determine the required amount of contributions and how to fund it through contributions over the horizon. Second, in turn the value of future contributions can be measured so that the pension sponsor would strategically manage the pension fund’s assets while anticipating future contributions.

Now, denote by $A_t$ the value of the assets (at time $t$) of the pension fund. We assume that the sponsor is endowed with an initial level of assets equal to $A_0 = \lambda_0 Le^{-rT}$, $\lambda_0 > 0$. Thus, $\lambda_0$ represents the initial funding ratio. Then, the pension plan’s asset value $A$ follows

$$dA_t = [(r + \pi_t(\mu - r)) A_t + Y_t] dt + \pi_t \sigma A_t dZ_t,$$

where $\pi$ denotes a fraction of the asset invested in the risky stock, and $Y$ denotes the pension sponsor’s contribution to the pension plan’s asset. Note that sponsor’s contribution is the only inflows to the pension plan we consider.

### 2.3 Pension Sponsor’s Preferences

In this section, we specify the objective function of the pension sponsor. We explicitly impose an ex post downside constraint such that the pension plan’s asset value at time $T$ should be greater than or equal to the pension benefits: $A_T \geq L$. In other words, the cost of not meeting the pension benefits is infinite. While we recognize that this assumption is extreme, it is possible to extend our results to a case with finite penalties for underfunding by specifying the downside constraint such that $A_T \geq lL$, where $l$ is a positive constant less than 1. We therefore assume that the sponsor’s first objective is to maximize an utility which is a function of the pension’s asset at the end of the investment horizon. We also assume that the sponsor suffers disutility from contributions in the form of costly withdrawal of internal resource. Thus, the sponsor’s second objective is to minimize this disutility which is a function of contributions made along the horizon. The overall utility function of the sponsor is given by:

$$\max_{\pi,Y} \mathbb{E} \left[ e^{-\beta T} u(A_T) - \int_0^T e^{-\beta t} \phi(Y_t) dt \right]$$

subject to $A_T \geq L$,

where $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ and $\phi(x) = k \frac{x^\theta}{\theta}$. The first term in equation (1) is a standard power utility with a relative risk aversion of $\gamma$ over the final pension plan’s asset. The motivation for this
utility is that the sponsor ultimately maximizes the plan’s asset since the sponsor has a claim on any pension plan’s surplus, which can be used to finance profitable investment project and will be valuable especially when internal financing is scarce or external financing is too costly.\footnote{Petersen (1992) uses plan-level data to find evidence in support of the financing motives.}

The second term in equation (1) represents the pension sponsor’s disutility from contributing to the pension plan. The pension sponsor has limited internal resources for profitable projects which might be foregone if the pension sponsor uses the internal cash to contribute to the pension plan.\footnote{Rauh (2006) finds that capital expenditures decline with mandatory contributions to DB pension plans.} We capture the cost of foregone projects due to contributions as the separable disutility function. A parameter $\theta$ will capture a desire to smooth contributions over time. To have convex disutility, we assume that $\theta > 1$. A parameter $k$ captures the sponsor’s tradeoff between the utility over the final pension plan’s asset and the disutility from contributions. For example, if the pension sponsor is financially healthy (sufficiently high internal resources), the impact of contributing to the pension plan is relatively small and thus the value of $k$ is low. Conversely, if the sponsor has insufficient internal resources, $k$ is high. Finally, $\beta$ is the subjective discount rate of the pension sponsor.

Our utility specification can be also interpreted as a portfolio choice problem with additively separable utility from intermediate consumption and bequest utility. The only difference is that, in our model, contribution is negative of consumptions and derives disutility, not utility. Karatzas, Lehoczky, and Shreve (1987) approaches this kind of portfolio choice problem by considering separately the two problems of maximizing utility of consumption only and of maximizing utility of bequest only, and then appropriately composing them. This motivates us to decompose the pension sponsor’s problem into two separate ones:

- **Utility Maximization Problem**

  The pension sponsor cannot contribute over time. The pension sponsor manages the initial endowment $W_0$ to maximize the expected utility over the final pension plan’s asset given the downside constraint:

  $$\max_{\pi^u} \mathbb{E} \left[ e^{-\beta T} u (W_T) \right]$$

  subject to:

  $$W_0 \geq \mathbb{E}^Q \left[ e^{-rT} W_T \right]$$

  $$W_T \geq L,$$

  where $\pi^u$ is a fraction of $W$ invested in the risky stock, the superscript $Q$ on the expectation operator represents that the expectation is calculated under the unique risk-neutral
measure $\mathbb{Q}$.

- **Disutility Minimization Problem**

  The pension sponsor minimizes the expected disutility from contributions while satisfying that the present value of contributions is at least $X_0$:

  $$
  \min_Y \mathbb{E} \left[ \int_0^T e^{-\beta t} \phi(Y_t) \, dt \right] \tag{3}
  $$

  subject to:

  $$
  X_0 \leq \mathbb{E}^\mathbb{Q} \left[ \int_0^T e^{-rt} Y_t \, dt \right] .
  $$

  At time zero, the sponsor simply augments the initial endowment by $X_0$:

  $$
  A_0 + X_0 = W_0. \tag{4}
  $$

  For the augmented endowment $W_0$, the sponsor will face an optimization problem (2) with utility coming only from terminal asset. Similarly, for the amount $X_0$, the sponsor will face an optimization problem (3) with disutility coming only from contributions. We will show how $X_0$ should be determined in order for the composed one of solutions to two separate problems to be optimal to the original problem (1). Whenever there is the downside constraint, we consider the problem as a constrained case. When there is no downside constraint and it serves as a benchmark case that we solve in the Appendix B.

### 3 Characterizing the Optimal Policies

In this section, we investigate the sponsor’s optimal asset allocation and contribution policies when there is the downside constraint. To this end, we solve two problems, (2) and (3) separately and discuss how the optimality to the original problem (1) can be achieved by choosing the proper $X_0$.

#### 3.1 Utility Maximization Problem

First, we solve the utility maximization problem. The budget constraint (4) implies that the initial endowment for the first problem $W_0$ is greater than the original endowment, $A_0$, and that the difference $W_0 - A_0$ is the required present value of contributions. That is, the pension sponsor anticipates the future contributions and thus, at time zero the pension sponsor can behave as if the required present value of contributions is borrowed against the future contributions. The
required amount of contributions will be determined later by taking into account both the utility over the final asset and the disutility from contributions. If the initial endowment for the first problem is less than the present value of the benefits, \( W_0 \leq Le^{-rT} \), there is no solution that guarantees the benefits for sure at the maturity. This implies that the required present value of contributions \( X_0 = W_0 - A_0 \) should be greater than the (if any) deficit \( \max(Le^{-rT} - A_0, 0) \). For example, if the pension plan is initially underfunded, the present value of contributions should be greater than the initial shortfall, \( Le^{-rT} - A_0 \). The dynamic budget constraint for the first problem is

\[
dW_t = (r + \pi_t^a(\mu - r))W_t dt + \pi_t^a W_t \sigma dZ_t,
\]

where \( \pi_t^a \) is the fraction of \( W \) invested in the risky stock. Note that there’s no contribution flow since it’s already reflected in the increased initial endowment \( W_0 \).

**Put-based Strategy**

It is well-known (Grossman and Vila (1989)) that when the market is complete the optimal strategy of the first problem consists in investing a fraction of asset in the unconstrained optimal portfolio and using the remaining fraction of asset to purchase a put option on that unconstrained portfolio to hedge the downside. We call this strategy a put-based strategy. Suppose that we construct the put-based strategy as follows:

\[
W_T = \underbrace{I_u(y_0 \xi_T)}_{\text{Unconstrained optimal portfolio}} + \underbrace{(L - I_u(y_0 \xi_T))^+}_{\text{Put option}} \geq L,
\]

where \( I_u(\cdot) \) is the inverse function of marginal utility \( u'(\cdot) \), \( \xi_t \) is (subjective) marginal rate of substitution, and \( (x)^+ = \max(x, 0) \) is max operator. The marginal rate of substitution is evolving according to

\[
\frac{d\xi_t}{\xi_t} = -(r - \beta)dt - \eta dZ_t.
\]

Without loss of generality, we assume that the initial value of the marginal rate of substitution is normalized to one, \( \xi_0 = 1 \). The first part is the unconstrained optimal portfolio and the second part is the put option on that with a strike price \( L \). This implies that the sponsor is able to meet the promised obligation always. Notice that the unconstrained optimal portfolio value is chosen such that the marginal utility is proportional to the marginal rate of substitution of the economy at the terminal date. It will be shown that the parameter \( y_0 \) is a shadow price, i.e. a marginal increase in the utility when the initial endowment \( W_0 \) for the first problem is marginally increased (or the required present value of contributions is marginally increased).
Now, the question is how to decide the optimal \( y_0 \). For this, we define the following function for any \( 0 < y_0 < \infty \):

\[
W_u(y_0) = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}I_u(y_0\xi_T)\right] + \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}(L - I_u(y_0\xi_T))^+\right],
\]

PV of unconstrained optimal portfolio

The function \( W_u(y_0) \) calculates the cost of constructing the put-based strategy. Proposition 1 explicitly computes this function.

**Proposition 1.** The function \( W_u(y_0) \) is given by

\[
W_u(y_0) = y_0^{-\frac{1}{\gamma}} e^{-\alpha_u T} N(\delta_1(y_0, T)) + Le^{-rT} N(-\delta_2(y_0, T)),
\]

where \( \alpha_u = \frac{\beta}{\gamma} + \left(1 - \frac{1}{\gamma}\right)\left(r + \frac{\eta^2}{2}\right) \). \( \delta_1 \) and \( \delta_2 \) can be found in Appendix A. Also, the first derivative of \( W_u(y_0) \) is given by

\[
W_u'(y_0) = -\frac{1}{\gamma} y_0^{-\frac{1}{\gamma}-1} e^{-\alpha_u T} N(\delta_1(y_0, T)) < 0.
\]

Since the market is complete and the put-based strategy consists in the underlying asset and the put option, we can interpret this valuation as Black and Scholes (1973) formula for the put-based strategy on an underlying asset with a dividend yield of \( \alpha_u \) and the initial underlying asset value of \( y_0^{-1/\gamma} \). The first part is the present value of the terminal unconstrained optimal portfolio value multiplied by the probability that the downside constraint is met at the maturity under the forward measure. Note that the final unconstrained optimal portfolio value is discounted with the dividend yield \( \alpha_u \) which is a weighted average of the pension sponsor’s subjective discount rate and subjective risk-adjusted expected return. Suppose that the pension sponsor is extremely risk averse. Then, the pension sponsor will allocate all pension plan’s asset in the risk-free asset, and thus the terminal unconstrained optimal portfolio value can be discounted with the risk-free rate: \( \lim_{\gamma \to \infty} \alpha_u = r \). The second part is the present value of the benefits multiplied by the probability that the put option is in-the-money under the risk-neutral measure.

Since we have the concave utility function, a higher shadow price implies a lower cost of constructing the put-based strategy. Thus, we can see that \( W_u(y_0) \) is decreasing in the shadow price \( y_0 \), which implies that \( W_u(y_0) \) is invertible. Let \( Y_u \) denote the inverse of this function. Then, the following theorem shows that it is optimal to set \( y_0 \) equal to \( Y_u(W_0) \), i.e. the cost of the put-based strategy is exactly equal to the initial endowment \( W_0 \) to the utility maximization problem.
**Theorem 2.** For any $W_0 \geq L e^{-rT}$, $y_0 = \mathcal{Y}_u(W_0)$ is optimal for the problem (2), and the optimal portfolio weight is given by

$$\pi^u_t = \frac{\eta}{\gamma \sigma} (1 - \varphi_t),$$

where $\varphi_t = \frac{L e^{-rT}}{W_t} N (-\delta_2(y_t, \tau)) < 1$, $\tau = T - t$, and $y_t = y_0 \xi_t$.

Theorem 2 states that the optimal shadow price should be $y_0 = \mathcal{Y}_u(W_0)$ such that the cost of constructing the put-based strategy is exactly same as the initial endowment for the first problem, $W_0$. Then, the optimal portfolio weight is a weighted average of the mean-variance efficient portfolio and zero investment in the equity. The weight on the mean-variance efficient portfolio is denoted by $1 - \varphi_t$. The parameter $\varphi_t$ measures the probability that the terminal asset value is lower than the benefits, i.e. the put option is in-the-money. The closer the asset is to the present value of the benefits, the less fraction of the asset is invested in the equity. This is intuitive since the sponsor is simultaneously holding the put option whose delta is close to $-1$ during in-the-money and one share of the underlying asset whose delta is always one so the net delta is zero.

Now, we compute the value function of the first problem and relate its first derivative to the shadow price. Let $J(W_0)$ be the value function of the first problem:

$$J(W_0) = \mathbb{E} \left[ e^{-\beta T} u \left( I_u (y_0 \xi_T) + (L - I_u (y_0 \xi_T))^+ \right) \right].$$

This function computes the expected utility when the put-based strategy is employed with the shadow price of $y_0 = \mathcal{Y}_u(W_0)$. Proposition 3 explicitly computes the value function and states that the first derivative of the value function, i.e. the shadow price is indeed $y_0 = \mathcal{Y}_u(W_0)$.

**Proposition 3.** The value function $J(W_0)$ is given by

$$J(W_0) = \frac{y_0^{1-\frac{1}{\gamma}}}{1 - \gamma} e^{-\alpha u T} N \left( \delta_1(y_0, T) \right) + e^{-\beta T} \frac{L^{1-\gamma}}{1 - \gamma} N \left( -\delta_3(y_0, T) \right),$$

where $\delta_3$ can be found in Appendix A. Also, we have $J'(W_0) = y_0$.

### 3.2 Disutility Minimization Problem

The second problem is to decide how to contribute along the horizon to minimize the expected disutility while satisfying the required present value of contributions. Alternatively, the problem can also be stated that the pension sponsor has the initial endowment $X_0$ in a separate account to fund future contributions and decides how to manage this fund. The usual assumption is that
the pension sponsor considers only self-financing strategies. Let $X_t$ be the time $t$ value of this fund. Then, the dynamic budget constraint of the second problem is given by

$$dX_t = \left[ \left( r + \pi_t^\phi (\mu - r) \right) X_t - Y_t \right] dt + \pi_t^\phi \sigma X_t dZ_t,$$

(11)

where $\pi_t^\phi$ is a fraction of $X_t$ invested in the equity. Note that contribution to the pension plan is outflow from this fund.

Now, the problem becomes a standard portfolio choice problem with intermediate outflow (contribution) and no bequest. However, there are two important differences. First, contribution does not increase the pension sponsor’s utility, but increase the disutility so that the objective is to minimize it. Second, the static budget constraint states that the present value of contributions should be greater than or equal to the initial endowment. At the optimal solution, the static budget constraint is binding and thus the terminal value of this fund will be zero, $X_T = 0$.

Similar to the utility maximization problem, we consider a contribution policy such that the marginal disutility is proportional to the marginal rate of substitution of the economy at each time:

$$Y_t = I_\phi (z_0 \xi_t),$$

where $I_\phi (\cdot)$ be the inverse function of $\phi'(\cdot)$. It will be shown that the parameter $z_0$ is a shadow price, i.e. a marginal increase in the disutility when the required present value of contributions $X_0$ is marginally increased. To determine the optimal $z_0$, we define the following function for any $0 < z_0 < \infty$:

$$\mathcal{W}_\phi (z_0) = \mathbb{E}^Q \left[ \int_0^T e^{-rt} I_\phi (z_0 \xi_t) dt \right],$$

The function $\mathcal{W}_\phi (z_0)$ computes the present value of contributions from time zero to the terminal date when intermediate contribution is set to be $I_\phi (z_0 \xi_t)$. Proposition 4 explicitly computes this function.

**Proposition 4.** The function $\mathcal{W}_\phi (z_0)$ is given by

$$\mathcal{W}_\phi (z_0) = \left( \frac{z_0}{k} \right)^{1/\theta} \frac{1 - e^{-\alpha \phi T}}{\alpha \phi},$$

(12)

where $\alpha_\phi = \frac{\theta}{\theta - 1} \left( r - \frac{\eta^2}{2(\theta - 1)} \right) - \frac{\rho}{\theta - 1}$. Also, the first derivative of $\mathcal{W}_\phi (z_0)$ is given by

$$\mathcal{W}'_\phi (z_0) = \frac{1}{z_0 (\theta - 1)} \mathcal{W}_\phi (z_0) > 0.$$  

(13)
The present value of contributions has a form of annuity with a rate of return \( \alpha \), which is a weighted average of the pension sponsor’s subjective discount rate and the subjective risk adjusted expected return. A strong incentive to smooth contributions over time (high \( \theta \)) implies that the contribution stream can be discounted with a rate \( r \): \( \lim_{\theta \to \infty} \alpha = r \). Since we have the convex disutility function, the present value of contributions would be higher if a marginal disutility (shadow price) is higher. Thus, we can see that \( W_\phi(z_0) \) is increasing, which implies that \( W_\phi(z_0) \) is invertible. Let us denote by \( \gamma_\phi \) the inverse of the function \( W_\phi \). Then, the following theorem shows that setting \( z_0 \) equal to \( \gamma_\phi(X_0) \) is optimal for the problem (3).

**Theorem 5.** For any \( X_0 > 0 \), setting \( z_0 = \gamma_\phi(X_0) \) is optimal for the problem (3), and the optimal hedging policy is

\[
\pi_t^\phi = -\frac{\eta}{(\theta - 1)\sigma}.
\]

By setting the marginal disutility of contribution to be proportional to the marginal rate of substitution of the economy at each time, the minimum disutility can be achieved. The shadow price is determined such that the present value of contributions is identical with the required present value of contributions \( X_0 \). The optimal hedging policy is to short the equity, since the optimal contribution is increasing in the marginal rate of substitution or decreasing in the stock return. Whenever the stock price decreases, the pension sponsor should increase contribution flow which can be hedged with profits from short positions in the equity. If the pension sponsor has a strong desire to smooth the contribution (higher \( \theta \)), the pension sponsor would decrease short positions in the equity since contribution flow is not sensitive to the market risk.

Finally, we compute the value function of the second problem. Let \( C(X_0) \) be the value function of the second problem:

\[
C(X_0) = \mathbb{E} \left[ \int_0^T e^{-\beta t} \phi(I_\phi(z_0\xi_t)) \, dt \right]. \tag{14}
\]

This function computes the expected disutility when contribution is set to be \( I_\phi(z_0\xi_t) \) where \( z_0 = \gamma_\phi(X_0) \). Proposition 6 explicitly computes the value function and states that the first derivative of \( C(X_0) \) (shadow price) is indeed \( z_0 = \gamma_\phi(X_0) \).

**Proposition 6.** The value function \( C(X_0) \) is given by

\[
C(X_0) = \frac{k}{\theta} \left( \frac{z_0}{k} \right)^{\frac{\theta}{\theta - 1}} \frac{1 - e^{-\alpha_\phi T}}{\alpha_\phi},
\]

where \( z_0 = \gamma_\phi(X_0) \) and satisfies \( C'(X_0) = z_0 \).
3.3 Optimality of Separation

So far, we derive the solutions for the utility maximization problem and the disutility minimization problem while taking the required present value of contributions as given. We now show that the optimal choice of the required present value of contributions $X_0$ leads us to the solution for the original problem. The next theorem shows that how the required present value of contributions $X_0$ is determined to achieve the optimality of the original problem.

**Theorem 7.** Consider an arbitrary portfolio and contribution policy pair $(\tilde{\pi}, \tilde{Y})$ satisfying the downside constraint. Then, there exists a pair $(\pi, Y)$ dominating $(\tilde{\pi}, \tilde{Y})$. In particular, the value function of the original problem $V(A_0)$ is given by

$$V(A_0) = \max_{X_0} J(A_0 + X_0) - C(X_0). \quad (15)$$

The intuition is as follows. For an arbitrary portfolio and contribution policy pair, we can take the present value of that contribution stream, $X_0 = \mathbb{E}^Q \left[ \int_0^T e^{-rt} \tilde{Y}_t dt \right]$. Then, for $X_0$, $\tilde{\pi}$ is a feasible strategy to the utility maximization problem (2), and $\tilde{Y}$ is a feasible strategy to the disutility minimization problem (3). We can find the optimal solutions to each problem and they will (weakly) dominate $(\tilde{\pi}, \tilde{Y})$. Thus, finding the optimal solution to the original problem (1) can be translated into the problem to find the required present value of contributions $X_0$ to maximize the difference between two value functions of (2) and (3), $J(A_0 + X_0) - C(X_0)$.

Suppose that (15) has an interior solution. This implies that the FOC with respect to $X_0$ equals zero:

$$J'(A_0 + X_0) = C'(X_0).$$

This condition states that at the optimal solution, the marginal increase in the value function of the utility maximization problem should be identical with the marginal increase in the value function of the disutility minimization problem. Thus, we can interpret LHS as the marginal benefit of increasing the required present value of contributions, and RHS as the marginal cost of increasing the required present value of contributions. Recall that the shadow prices of both problems are obtained when the static budget constraints hold with equality. Hence, we have

$$J'(A_0 + X_0) = C'(X_0)$$

$\Leftrightarrow \quad x = y_0 = z_0$

$\Leftrightarrow \quad x = \mathcal{Y}_u(A_0 + X_0) = \mathcal{Y}_\phi(X_0)$

$\Leftrightarrow \quad A_0 = \mathcal{W}_u(x) - \mathcal{W}_\phi(x). \quad (16)$
Define the following function for $0 < x < \infty$:

$$W(x) = W_u(y) - W_\phi(y).$$

This function computes the initial endowment required to have the shadow price of $x$ for both problems. Then, the marginal benefit and cost of increasing $X_0$ are identical so that the optimality can be achieved. Proposition 8 shows that there exists a unique $x$ solving $W(x) = A_0$, and thus we obtain the optimal solution to the original problem. Also, we can express the optimal asset allocation to the original problem composing the optimal portfolio weights to the first and second problems.

**Proposition 8.** The function $W(x)$ is decreasing in $x$, $\lim_{x \to 0} W(x) = \infty$ and $\lim_{x \to \infty} W(x) = -\infty$. Hence, there exists a unique $x$ satisfying $W(x) = A_0$. Then, the optimal path of pension’s asset of the original problem is given by $A_t = W_t - X_t$ and the optimal portfolio weight is given by

$$\pi_t = \pi_u^t + \pi_\phi^t (1 - \rho_t),$$

where $\rho_t = 1 + \frac{X_t}{A_t}$. Finally, the optimal contribution rate is given by

$$\frac{Y_t}{W_t} = (\rho_t - 1) \frac{\alpha_\phi}{1 - e^{-\alpha_\phi (T-t)}}.$$

Intuitively, the time $t$ pension plan’s asset can be expressed as $A_t = W_t - X_t$ since future contributions can be interpreted as liability for the sponsor. This implies that the pension plan might be underfunded along the horizon: $A_t < Le^{-r(T-t)}$. However, in this case the sponsor contributes substantially at time $t$ and also in the future as long as the pension plan is still underfunded. Thus, the present value of future contributions $X_t$ is large so that the asset value taking into account future contributions will be always greater than or equal to the present value of the benefits: $W_t = A_t + X_t \geq Le^{-r(T-t)}$.

The optimal portfolio weight is a weighted average of two weights, $\pi_u^t$ and $\pi_\phi^t$. The weight is the ratio of the present value of the terminal pension plan’s asset over the current pension plan’s asset. Note that because a possibility of future contributions, this ratio is generally not equal to one. When the expected contribution is small, then the weight $\rho$ is close to one. Also, $\pi_u^t$ becomes the mean-variance efficient portfolio ($\frac{\mu}{\sigma}$) since it is more likely that the downside constraint is not binding. Thus, the optimal portfolio weight, $\pi_t$ is close to the mean-variance efficient portfolio.

As the economy gets worse (the equity price drops), the pension plan’s asset gets close to the downside constraint. There are two effects of economic downturns on the optimal portfolio
weight. First, the pension sponsor will have high $X_t$ due to short positions of $\pi^\phi$, which can be used to hedge large contemporaneous and future contributions. This indicates an increase in $\rho_t$. Thus, the pension sponsor will increases the equity weight, which is hedged by contemporaneous and future contributions. Second, the optimal equity weight for the utility maximization problem $\pi^u$ will decrease, since the present value of the terminal pension plan’s asset, $W_t$ approaches to the present value of the benefits. If the latter effect dominates the former one, then a risk management behavior can be observed, i.e. a decrease in the equity weight as the economy gets worse. On the other hand, if the former effect dominates, we can see a risk taking behavior. However, note that this risk taking incentive is induced not by a moral hazard problem, but by a commitment to contributions in the future.

The optimal contribution policy as a fraction of the current pension plan’s asset also depends on the pension plan’s leverage ratio $\rho_t$ and time-to-maturity $T - t$. The pension sponsor contributes more when the pension plan’s asset return is low so that $\rho$ is high. For the same $\rho_t$, the ratio of the contribution to the pension plan’s asset is higher when time-to-maturity is short. Since the pension plan’s objective is to minimize the expected disutility, the pension plan would defer a contribution as much as it can.

4 Quantitative Analysis

We now turn to quantitative analysis of the model. We use 10-year for the pension plan’s maturity $T$. According to Bureau of Labor Statistics, as of 2014 the median years of tenure with current employer for workers with age over 65 years is 10.3-year. Also, we use $\eta = 0.4$ for the market price of risk, $\sigma = 20\%$ for the volatility of the equity, $r = 2\%$ for the risk-free rate, and $\beta = 1\%$ for the pension sponsor’s subjective discount rate. These numbers are standard assumptions in the literature. The expected excess return of the equity is $\mu - r = \sigma \eta = 8\%$. We use $\gamma = 5$, which implies the equity weight of the mean-variance efficient portfolio is $\frac{\eta}{\gamma \sigma} = 40\%$. For the disutility function, we use $k = 7000$ and $\theta = 2.6$. We choose these disutility parameters intentionally to capture relatively high opportunity costs of contributions. Suppose that there is no downside constraint. The sponsor still has an incentive to contribute to increase the terminal pension asset while sacrificing some of disutility due to contributions. However, with chosen disutility parameters, $k = 7000$ and $\theta = 2.6$, in the benchmark case that we solve in Appendix B (without the downside constraint), the amount of contributions is small (less than 2\% in terms of pension’s asset) since disutility of contributions is relatively high to utility.
Table 1: Summary of key variables and parameters

<table>
<thead>
<tr>
<th>Variable</th>
<th>Symbol</th>
<th>Parameters</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Terminal benefits</td>
<td>(L)</td>
<td>Pension plan’s investment horizon</td>
<td>(T)</td>
<td>10-year</td>
</tr>
<tr>
<td>Asset (Original)</td>
<td>(A)</td>
<td>Price of Risk</td>
<td>(\eta)</td>
<td>0.4</td>
</tr>
<tr>
<td>Asset (Util. Maximization)</td>
<td>(W)</td>
<td>Risk-free rate</td>
<td>(r)</td>
<td>2%</td>
</tr>
<tr>
<td>Asset (Disutil. Minimization)</td>
<td>(X)</td>
<td>Subjective discount rate</td>
<td>(\beta)</td>
<td>1%</td>
</tr>
<tr>
<td>Shadow price</td>
<td>(x)</td>
<td>Pension sponsor’s risk aversion</td>
<td>(\gamma)</td>
<td>5</td>
</tr>
<tr>
<td>Subjective marginal rate of substitution</td>
<td>(\xi)</td>
<td>Elasticity of disutility</td>
<td>(\theta)</td>
<td>2.6</td>
</tr>
<tr>
<td>Portfolio weight of equity</td>
<td>(\pi)</td>
<td>Relative importance of disutility</td>
<td>(k)</td>
<td>7000</td>
</tr>
<tr>
<td>Contribution flow</td>
<td>(Y)</td>
<td>Initial funding ratio</td>
<td>(\lambda_0)</td>
<td>80% or 120%</td>
</tr>
</tbody>
</table>

This table summarizes the symbols for the key variables used in the model and the parameter values in the baseline case.

of a higher value of pension asset. Thus, if we see more contributions in the model with the downside constraint, it is the only incentive for the sponsor to contribute more. Finally, we use two values for the initial funding ratio, \(\lambda_0 = 80\% \) or 120\% and normalize the initial pension asset to one: \(A_0 = 1\). We will vary preference parameters, \((\gamma, k, \theta)\), and the price of risk to see the impacts on the optimal present value of contributions, portfolio and contribution policy. Table 1 summarizes all the key variables and parameters in the model.

4.1 Determination of Required Present Value of Contributions

Figure 1 plots the determination of \(X_0\) by equating the shadow prices of the first and second problem: \(\mathcal{V}_u(A_0 + X_0) = \mathcal{V}_\phi(X_0)\). The initial pension plan’s asset is normalized to one \(A_0 = 1\) and thus the required present value of contributions can be interpreted as a fraction of the initial pension plan’s asset. Panel A is a case when the pension plan is initially underfunded, \(\lambda_0 = 80\%\), and Panel B is a case when overfunded, \(\lambda_0 = 120\%\). We also plot the benchmark case. The marginal benefit of increasing \(X_0\) or the shadow price of the utility maximization problem is decreasing in the required present value of contributions since the utility function is concave. Also, the marginal cost of increasing \(X_0\) for the constrained case is always above that of the benchmark case. One dollar is more valuable for the constrained case since it can be used to construct the pub-based strategy and avoid infinite penalty of not meeting the benefits.

We can see that the required present value of the contribution is \(X_0 = 1.90\%\) and the shadow price is \(y = 0.20\) for the benchmark case. This indicates that along the horizon the pension sponsor contributes 1.90\% of the initial asset even though there is no downside constraint. Even after taking into disutility of contributions, a small amount of contributions is still optimal since
Figure 1: Determination of Present Value of Contribution

Panel A: Initially Underfunded Pension

Panel B: Initially Overfunded Pension

This figure plots shadow prices of first and second problem as a function of present value of contribution. Panel A is for an initially underfunded pension with 80% funding ratio, and Panel B is for an initially overfunded pension with 120% funding ratio.
the sponsor can be exposed to higher utility of terminal asset through contributions.

For the underfunded pension plan, the marginal benefit curve \( Y_u(A_0 + X_0) \) has the left asymptote line at \( X_0 = Le^{-rT} - A_0 = 25\% \).\(^8\) Since the plan is initially underfunded, to guarantee the benefits for sure the sponsor should contribute 25\% of the initial asset at least over the horizon. The marginal cost curve will determine whether the sponsor should contribute more than that or not. It turns out that the required present value of contributions is \( X_0 = 25\% \) and the shadow price is \( y = 12.08 \), i.e. the sponsor will contribute the minimum amount to meet the benefits. This is because contribution is too costly given our choices of disutility parameters and thus there is no incentive to contribute more than the minimum amount required to guarantee the benefits. Two things are worth mentioning. First, compared to the benchmark case, the pension sponsor contributes substantially. Even if contribution is too costly, the sponsor has no choice but to contribute the minimum amount for meeting the benefits. Second, the fact that the sponsor contributes no more than the minimum amount does not imply that the sponsor should switch the entire pension’s portfolio to the risk-free asset. If all contributions were made at time zero, zero investment in the equity would be the optimal. However, the sponsor will strategically choose when and how much contributions should be made while making sure that the present value of contributions is equal to the predetermined \( X_0 \). Thus, the optimal way to manage the fund would be to increase risky allocation and contribution at the same time during economic downturns (equity returns are negative). In this way, the impact of contribution will be amplified since the sponsor can enjoy the upside potential of equity returns more. This will be explained in more details in the next section.

For the overfunded pension plan, the present value of contributions is \( X_0 = 2.10\% \) and the shadow price is \( y = 0.23 \). This implies that relative to the benchmark case additional contributions of 0.20\% are required to guarantee the benefits. Since the fund is initially overfunded, with the small amount of additional contributions the put-based strategy can be constructed.

4.2 Portfolio Weight and Contribution Policy Along the Horizon

4.2.1 Initially Underfunded Plan

Figure 2 plots times series of funding ratio, equity weight, and contribution rate for the initially underfunded plan \((\lambda_0 = 80\%)\). We use the required present value of contributions determined

\(^8\) Since we normalize the initial asset to one, the initial funding ratio of 80\% implies that the deficit is 25\% of the initial asset.
in the previous section. We use historical S&P 500 index from Sep. 1, 2007 to Sep. 1, 2017 at monthly frequency to compute model implied policies. In Panel A, the evolution of S&P 500 index is plotted. NBER recession is shaded. Over the past 10-year, the stock price earned high risk premium except during the recent financial crisis. By doing this exercise, we can see how the sponsor should behave to meet the benefits especially during recession.

First, we can see that the equity weight of the benchmark case is not sensitive to equity returns as much as the constrained case. This is because the required present value of contributions is low and also the optimal contribution policy is not sensitive to the state of the economy for the benchmark case. This also can be seen in Panel D. Contribution rates along the horizon are almost flat and generally lower than 1% of asset. Recall that equity weight will be higher when the sponsor is contributing huge amount contemporaneously and anticipating more contributions coming in the future. Thus, stable and low contributions imply stable equity weight as well. Finally, the equity weight of the benchmark case is approaching to the mean-variance efficient portfolio, which is \( \frac{n}{\gamma \sigma} = 40\% \) as time approaches to the maturity since no more contributions are expected.

On the other hand, the level of contribution rate for the constrained case is higher and it fluctuates more than the benchmark case. Especially, the sponsor increases contributions when there was a huge negative shock to equity returns. Also, the sponsor increases equity weight at the same time. The intuition is that the pension sponsor expects that future contributions will be made, and thus can take more risks by increasing equity weight. Put differently, if the stock price recovers after negative returns, the positive return will be amplified due to a risky strategy by increasing contributions and equity weight. If the stock price performs badly, the loss can be hedged by future contributions. Indeed, during the financial crisis, the funding ratio of the constrained case decreased, but recovered following the economic recovery. As expected, the performance of the constrained case is much better than the benchmark case in this period due to increased positions in the equity. We can see that as the funding ratio improves and time passes the sponsor starts to decrease equity weight. This is because the remaining contributions and the current deficit are balanced so that there’s no incentive to deviate from the risk-free asset to meet the benefits. Interestingly, the benchmark case is slightly underfunded at the maturity but the constrained case is just funded. This is obvious result since the required present value of contributions is exactly same as the initial amount of deficit so that it is impossible to have a funding ratio higher than one.
Figure 2: Historical Evolution of Model Implied Policies (Initially Underfunded)

Panel A: S&P 500 Index

Panel B: Funding Ratio

Panel C: Equity Weight

Panel D: Contribution Rate
4.2.2 Initially Overfunded Plan

Figure 3 plots times series of funding ratio, equity weight, and contribution rate for the initially overfunded plan ($\lambda_0 = 120\%$). The level of contribution rate for the constrained case is slightly higher than the benchmark case. Recall that the difference in the required present value of contributions is just 0.20%. However, equity weights are very different. During the crisis, the equity weight of the benchmark case is slightly higher due to increased contributions. The sponsor with the downside constraint, on the other hand, is decreasing equity weight to almost 10%. Since contributions are too costly, the initially overfunded sponsor is first engaged in the risk management policy by decreasing equity weight to defer contributions. If equity returns are negative consecutively, then the sponsor can’t defer contributions anymore. From that point, larger contributions will be made and the sponsor will take more risks by increasing equity weight to exploiting possible positive shocks to equity returns. In the past 10-year, this scenario did not happen and the sponsor successfully managed the funding ratio greater than one just by risk management. However, this does not come at free. We can see that the funding ratio at the terminal date is slightly lower than the benchmark case. This is because the sponsor couldn’t enjoy high equity returns following the crisis due to the risk management strategy.

4.2.3 Performance of Model Implied Policies

In this section, we analyze the performance of our model. Using monthly level of S&P 500 index from Jan. 1, 1950 to Sep. 1, 2017, we have 693 number of overlapping 10-year periods. With this price data, we simulate the value of terminal asset under the benchmark and constrained case while assuming that the initial funding ratio is 100%. Figure 4 plots the histogram of two terminal asset values.

Clearly, we can see benefit and cost of the constrained case relative to the benchmark case. By employing the put-based strategy, the sponsor is giving up the upside potential. However, due to the put-based strategy, the terminal asset is always insured and thus the funding ratio is greater than or equal to one. On the other hand, the terminal asset under the benchmark case can have higher funding ratios and also can be underfunded. The net effect depends on the sponsor’s preferences parameters, which we examine in the next section.
Figure 3: Historical Evolution of Model Implied Policies (Initially Overfunded)

Panel A: Funding Ratio

Panel B: Equity Weight

Panel C: Contribution Rate
4.3 Effect of Initial Funding Ratio

In Figure 5, we vary the initial funding ratio by changing the benefits, $L$, and find the required present value of contributions (Panel A). Also, based on that, we plot equity weights at time zero (Panel B), contribution rate at time zero (Panel C), and monetary costs of the constrained case compared to the benchmark case (Panel D). We calculate monetary costs of the downside constraint as follows:

$$ J(A_0 + c) = J^{BC}(A_0), $$

where $c$ is the monetary cost of the downside constraint and $J^{BC}(A_0)$ is the value function of the benchmark case. Thus, $c$ measures the amount of additional pension asset required for the constrained case to have the same level of the value function as the benchmark case.

We can see that for the benchmark case, the required present value of contribution, equity weight, and contribution rate do not depend on the initial funding ratio. The purpose of contributions in the benchmark case is purely to increase the terminal pension asset while taking into account disutility of contributions. Thus, the level of benefits will not change the required present value of contributions. Also, the equity weight is slightly higher than the mean-variance efficient one, $\frac{\mu}{\sigma} = 40\%$ since future contributions can hedge higher risky positions.

Next, consider the constrained case. The required present value of contributions is de-
Figure 5: Effect of Initial Funding Ratio

Panel A: Present Value of Contribution

Panel B: Portfolio Weights at Time Zero

Panel C: Contribution Rate at Time Zero

Panel D: Monetary Cost

This figure plots the required present value of contributions (Panel A), equity weights at time zero (Panel B), contribution rate at time zero (Panel C), and monetary cost (Panel D) as a function of initial funding ratio. We fix the initial asset at one and vary the benefit level, $L$. 

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creasing in the initial funding ratio and approaching to a benchmark case. This is intuitive since for higher level of benefits the minimum amount of contributions for meeting the benefits is also higher. For a \( \lambda_0 = 70\% \) funded pension, \( X_0 = 42.86\% \) of the initial asset should be made by contributions along the horizon. This is same as the amount of deficit \( Le^{-rT} - A_0 \). In the first year, \( Y_0/A_0 = 3.21\% \) should be made. This number can be translated into \( Y_0/(Le^{-rT} - A_0) = 7.5\% \) of the deficit in the first year. We also plot in Panel A the value of put option which is also decreasing in the initial funding ratio. We can see that the value of put option is generally greater than the required present value of contributions. For example, a \( \lambda_0 = 70\% \) funded pension has to contribute 42.86\% of the initial asset along the horizon, but the value of put option is 105.28\%. This implies that the pension sponsor should use 105.28 – 42.86 = 62.42\% of the initial asset additionally to construct the put option. The rest of the initial wealth will be allocated to the unconstrained optimal portfolio. Thus, this will induce substantial utility loss for the constrained sponsor relative to the benchmark case. In Panel D, we measure this utility loss due to the downside constraint in terms of additional amount of asset needed to have the same level of utility as the benchmark case. As expected, monetary cost is decreasing in the initial funding ratio. For a 70\% funded pension, the pension sponsor with the downside constraint needs additional asset of 36.99\%.

Next, the equity weight of the constrained case exhibits an U-shaped pattern. The initially overfunded sponsor defers contributions and employs the risk management policy first, i.e. decreases the equity weight, since the pension sponsor wants to avoid costly contributions as much as it can. On the other hand, the initially underfunded pension sponsor starts to contribute at time zero. Since contemporaneous and future contributions can hedge high equity positions, the pension sponsor takes more risks.

4.4 Effects of Relative Importance of Disutility

Figure 6 plots the required present value of contributions and the put option value at time zero for the initially underfunded (Panel A) and overfunded (Panel B) pension plan as we vary the tradeoff between utility and disutility, \( k \). Since drawing contributions from the pension sponsor’s internal resources is more costly when \( k \) is large, we can see that the required present value of the contribution decreases as \( k \) increases for both the benchmark and constrained cases. However, there is a key difference between the initially underfunded and overfunded pension plans. The initially overfunded pension plan has a put-based strategy even without a contribution. Hence, when \( k \) is sufficiently large, the pension sponsor won’t contribute at all and just
This figure plots the present value of the contribution and the put option value for the initially underfunded (Panel A) and overfunded (Panel B) pension plan as a function of the relative importance of the disutility ($k$). The initially underfunded pension plan has 80% funding ratio, and the initially overfunded pension plan has 120% funding ratio.
use the put-based strategy without any contribution. However, the initially underfunded pension plan can not construct a put-based strategy without contribution. Thus, we can see that even if $k$ is sufficiently large, the underfunded pension plan takes the required present value of contributions, which is equal to the time zero shortfall $L e^{-rT} - W_0 = 25\%$.

The put option value at time zero increases as $k$ increases for both underfunded and overfunded pension plans. When contributing is more costly, the pension sponsor decreases the required present value of contributions and allocations in the unconstrained optimal portfolio, which makes the overall pension plan’s asset less risky and increases the put option value. For low $k$, contributing more than the put option value is optimal since contributing is less costly and the pension sponsor can hold more unconstrained optimal portfolio. However, when $k$ is high, the opposite happens. A fraction of the put option is funded by the initial pension plan’s asset.

### 4.5 Effects of Risk Aversion

Now, we investigate effects of the risk aversion. We report the required present value of contributions and the put option value at time zero for the initially underfunded (Panel A) and overfunded (Panel B) pension plan as we vary the relative risk aversion, $\gamma$ in Figure 7. We also plot the required present value of the contribution for the benchmark case. When the risk aversion is high, the mean-variance efficient portfolio holds less equity. The risk of the underlying asset is reduced, and thus the put option value decreases, which implies that less contributions are required. For the initially overfunded pension plan, when the risk aversion is very high, the put option value becomes worthless since the pension sponsor holds zero equity position and the downside constraint is always satisfied (note that it is initially overfunded). Some contributions are still optimal since the pension sponsor can achieve higher utility even taking into account the disutility from contributions.

### 5 Conclusion

We develop the separation approach to analyze the pension sponsor’s contribution and portfolio policy in the presence of the downside constraint at the terminal date. The problem can be cast in two separate shadow price problems, the utility maximization problem and the disutility minimization problem. At the optimal solution, two shadow prices are identical. We show that while guaranteeing the benefits, both risk management and risk taking behaviors can emerge. When
This figure plots the present value of the contribution and the put option value for the initially underfunded (Panel A) and overfunded (Panel B) pension plan as a function of the relative risk aversion ($\gamma$). The initially underfunded pension plan has 80% funding ratio, and the initially overfunded pension plan has 120% funding ratio.
the pension plan’s asset decreases, the pension sponsor first decreases the equity weight and de-
fers contributions as much as it can to avoid costly contributions. Then, only when the pension
plan’s asset is significantly deteriorated, the pension sponsor starts to contribute and increases
the equity weight, which is hedged by large contemporaneous and future contributions. In our
model, the pension sponsor’s risk taking behavior is induced not by a moral hazard problem,
but by commitment to contributions. We hope to extend our analysis to include time-varying
expected returns, and stochastic benefits in future research.
Appendix

A Proof

Proof of Proposition 1. By Girsanov’s Theorem, there exists a unique equivalent measure \( Q \) in which all traded
assets earn the risk-free rate, and under \( Q \) measure the following stochastic process is a standard Brownian motion.

\[
dZ_t^Q = dZ_t + \eta dt.
\]

To compute \( \mathcal{W}_u(y_0) \), we can derive the dynamics of \( \xi_t \) under \( Q \) measure.

\[
\frac{d\xi_t}{\xi_t} = (\beta - r)dt - \eta dZ_t
\]

\[
= (\beta - r + \eta^2)dt - \eta dZ_t^Q.
\]

The random variable \( I_u(y_0 \xi_T) \) can be expressed as

\[
I_u(y_0 \xi_T) = y_0^{-\frac{1}{2}} \exp \left( -\frac{T}{\gamma} (\beta - r + \frac{1}{2} \eta^2) + \frac{\eta}{\gamma} (Z_T^Q - Z_0^Q) \right).
\]

given that \( \xi_0 = 1 \). Let \( A \) be the event in which \( L > I_u(y_0 \xi_T) \). The event \( A \) is equivalent to \( x < -\delta_2(y_0, T) \),
where \( x \) is a standard normal random variable, and \( \delta_2(y_0, T) \) is given by

\[
\delta_2(y_0, T) = \frac{\log \frac{y_0}{1} + \frac{T}{\gamma} (r - \beta - \frac{\eta^2}{2})}{\eta \sqrt{T}}.
\]

since \( Z_T^Q - Z_0^Q \) is normally distributed with zero mean and variance of \( T \). Then, \( \mathcal{W}_u(y_0) \) can be expressed as

\[
\mathcal{W}_u(y_0) = \mathbb{E}^Q \left[ e^{-rT} I_u(y_0 \xi_T)(1 - 1(A)) \right] + Le^{-rT} N(-\delta_2(y_0, T)),
\]

where \( 1(A) \) is an indicator function of event \( A \) and \( N(\cdot) \) is a cumulative distribution function of standard normal
random variable. The first part can be easily computed:

\[
\mathbb{E}^Q \left[ e^{-rT} I_u(y_0 \xi_T)(1 - 1(A)) \right] = \exp \left( -\left( r + \frac{1}{\gamma} (\beta - r + \frac{1}{2} \eta^2) \right) T \right) y_0^{-\frac{1}{2}}
\]

\[
\int_{-\infty}^{\infty} \exp \left( \frac{\eta \sqrt{T}}{\gamma} x \right) n(x) dx
\]

\[
= \frac{1}{y_0^\frac{1}{2}} e^{-\alpha_u T} \int_{-\delta_2(y_0, T)}^{\infty} n \left( x - \frac{\eta \sqrt{T}}{\gamma} \right) dx
\]

\[
= \frac{1}{y_0^\frac{1}{2}} e^{-\alpha_u T} N(\delta_1(y_0, T)),
\]

where \( n(\cdot) \) is a probability distribution function of a standard normal random variable, \( \alpha_u \) and \( \delta_1(y_0, T) \) are given by

\[
\alpha_u = \frac{\beta}{\gamma} + \left( 1 - \frac{1}{\gamma} \right) \left( r + \frac{\eta^2}{2\gamma} \right)
\]

\[
\delta_1(y_0, T) = \delta_2(y_0, T) + \frac{\eta \sqrt{T}}{\gamma}.
\]

Now, the first derivative of \( \mathcal{W}_u(y_0) \) can be computed as

\[
\mathcal{W}_u'(y_0) = \frac{1}{\gamma} y_0^{-\frac{1}{2}} \left( -e^{-\alpha_u T} N(\delta_1(y_0, T)) + y_0^{-\frac{1}{2}} e^{-\alpha_u T} n(\delta_1(y_0, T)) \frac{\partial \delta_1(y_0, T)}{\partial y_0} \right)
\]

\[
- Le^{-rT} n(-\delta_2(y_0, T)) \frac{\partial \delta_2(y_0, T)}{\partial y_0}.
\]
Note that $\frac{\partial \delta_1(y_0,T)}{\partial y_0} = \frac{\partial \delta_2(y_0,T)}{\partial y_0}$ and

$$y_0^{-\frac{1}{2}} e^{-\alpha_u T} n(\delta_1(y_0,T)) = y_0^{-\frac{1}{2}} e^{-\alpha_u T} n \left( \delta_2(y_0,T) + \frac{\eta \sqrt{T}}{\gamma} \right) = y_0^{-\frac{1}{2}} \exp \left( -\alpha_u T - \frac{\eta \sqrt{T}}{\gamma} \delta_2(y_0,T) - \frac{\eta^2 T}{2\gamma^2} \right) n(\delta_2(y_0,T)) = Le^{-rT} n(-\delta_2(y_0,T)).$$

Hence, last two terms cancel out.

\[ \qed \]

**Proof of Theorem 2.** Consider any random variable $\widetilde{W}_T \geq K$, which is feasible by a self financing trading strategy and the initial endowment $W_0$. This implies that

$$W_0 \geq \mathbb{E}^Q \left[ e^{-rT} \widetilde{W}_T \right].$$

Then, we want to show that

$$\mathbb{E} \left[ e^{-\beta T} u(W_T) \right] \geq \mathbb{E} \left[ e^{-\beta T} u(\widetilde{W}_T) \right].$$

Since $u$ is a concave utility, we have

$$u(W_T) - u(W_T) \geq u' (W_T) \left( W_T - \widetilde{W}_T \right). \quad \text{(A.1)}$$

We can compute $u' (W_T)$:

$$u' (W_T) = u' (\max (I_u (\gamma_u (W_0) \xi_T), L)) = \min (\gamma_u (W_0) \xi_T, u' (K)) = \gamma_u (W_0) \xi_T - (\gamma_u (W_0) \xi_T - u' (L))^+. \quad \text{(A.2)}$$

Substitute in (A.1), we have

$$u(W_T) - u(\widetilde{W}_T) \geq \gamma_u (W_0) \xi_T \left( W_T - \widetilde{W}_T \right) + (\gamma_u (W_0) \xi_T - u' (K))^+ \left( \widetilde{W}_T - L \right).$$

The second term is due to that $W_T = L$ corresponds to $\gamma_u (W_0) \xi_T > u' (L)$. The second term is always greater or equal to zero since $\widetilde{W}_T \geq L$. Multiplying $e^{-rT}$ and taking expectation under the physical measure of the first term of RHS yields

$$\gamma_u (W_0) \mathbb{E} \left[ e^{-\beta T} \xi_T \left( W_T - \widetilde{W}_T \right) \right] = \gamma_u (W_0) \mathbb{E}^Q \left[ e^{-rT} \left( W_T - \widetilde{W}_T \right) \right] \geq \gamma_u (W_0) \left( W_0 - \mathbb{E}^Q \left[ e^{-rT} \widetilde{W}_T \right] \right) \geq 0.$$

Hence, we obtain the desired inequality. Now, the optimal portfolio weight can be obtained by matching volatility of (5) and $W_t = \mathbb{E}^Q_t \left[ e^{-r(T-t)} W_T \right]$. By Proposition 1, we can easily compute the latter:

$$W_t = y_t^{-\frac{1}{2}} e^{-\alpha_u (T-t)} N (\delta_1(y_t, T-t)) + Le^{-r(T-t)} N (-\delta_2(y_t, T-t)), \quad \text{where } y_t = \gamma_u (W_0) \xi_t.$$

The diffusion part of the above is

$$\text{diff} \left( dW_t \right) = \eta y_t^{-\frac{1}{2}} e^{-\alpha_u (T-t)} N (\delta_1(y_t, T-t)).$$

This should be equal to the diffusion part of (5), $\pi^u \pi^u W_t \sigma$. Hence, we have

$$\pi^u = \frac{\eta}{\gamma \sigma} \left( 1 - \varphi_t \right),$$

where $\varphi_t = \frac{K e^{-r(T-t)}}{W_t} N (-\delta_2 (y_t, T-t)) < 1$. \[ \qed \]
Proof of Proposition 3. We first compute $J(W_0)$:

$$J(W_0) = \mathbb{E} \left[ e^{-\beta T} I_u(y_0\xi_T)^{1-\xi_T} \left( 1 - 1(A) \right) + e^{-\beta T} I_u'(y_0\xi_T)^{1-\xi_T} \left( 1 - 1(A) \right) \right]$$

Note that the expectation is under the physical measure. The random variable $I_u(y_0\xi_T)$ can be expressed as

$$I_u(y_0\xi_T) = y_0^{-\frac{1}{\gamma}} \exp \left( -\frac{T}{\gamma} (\beta - r - \frac{1}{2}\gamma^2) + \frac{\eta}{\gamma} (Z_T - Z_0) \right).$$

The event $A$ is equivalent to $x < -\delta_3(y_0, T)$, where $x$ is a standard normal random variable, and $\delta_3(y_0, T)$ is given by $\delta_3(y_0, T) = \delta_2(y_0, T) + \eta \sqrt{T}$, since $Z_T - Z_0$ is normally distributed with zero mean and variance of $T$. If we follow similar steps as Proposition 1, we can obtain (10). Take the first derivative of (10), then we have

$$J'(W_0) = \mathbb{E} \left[ e^{-\beta T} u' \left( I_u(y_0\xi_T) \right) I_u'(y_0\xi_T) \xi_T \left( 1 - 1(A) \right) \right] Y_u'(W_0)$$

$$= y_0 \mathbb{E}^Q \left[ e^{-\beta T} I_u'(y\xi_T) \xi_T \left( 1 - 1(A) \right) \right] Y_u'(W_0)$$

$$= y_0 \mathcal{W}_u(y) Y_u'(W_0)$$

$$= y_0.$$

\[ \square \]

Proof of Proposition 4. We can interchange the integral and expectation:

$$\mathcal{W}_\phi(z_0) = \int_0^T e^{-rt} \mathbb{E}^Q[I_\phi(z_0\xi_t)] dt,$$

where

$$I_\phi(z_0\xi_t) = \left( \frac{z_0}{k} \right)^{\frac{1}{\theta-1}} \exp \left( t \left( \beta - r + \frac{1}{2}\gamma^2 \right) - \frac{\eta}{\gamma} (Z_t^Q - Z_0^Q) \right).$$

The inner expectation is

$$\mathbb{E}^Q[I_\phi(z_0\xi_t)] = \left( \frac{z_0}{k} \right)^{\frac{1}{\theta-1}} \exp \left( \frac{t}{\theta-1} \left( \beta - r + \frac{\theta\gamma^2}{2(\theta-1)} \right) \right).$$

Now, we can express $\mathcal{W}_\phi(z_0)$ as

$$\mathcal{W}_\phi(z_0) = \left( \frac{z_0}{k} \right)^{\frac{1}{\theta-1}} \int_0^T e^{-\alpha_\phi t} dt$$

$$= \left( \frac{z_0}{k} \right)^{\frac{1}{\theta-1}} \frac{1 - e^{-\alpha_\phi T}}{\alpha_\phi},$$

where $\alpha_\phi = \frac{\theta}{\theta-1} \left( r - \frac{\gamma^2}{2(\theta-1)} \right).$ The first derivative is straightforward. \[ \square \]

Proof of Theorem 5. Consider arbitrary contribution policy $\{\bar{Y}\}$, whose present value is greater than $X_0$. This implies that

$$\mathbb{E}^Q \int_0^T e^{-rt} \bar{Y}_t dt \geq X_0.$$ 

Then, we want to show that

$$\mathbb{E} \left[ \int_0^T e^{-\beta t} \phi(Y_t) \right] \leq \mathbb{E} \left[ \int_0^T e^{-\beta t} \phi(\bar{Y}_t) \right].$$
Since $\phi$ is a convex disutility, we have

$$\phi(Y_t) \leq \phi(\tilde{Y}) + \phi'(Y_t)(Y_t - \tilde{Y}).$$

Multiplying $e^{-\beta t}$ and taking integral and expectation under the physical measure of the second term of RHS yields

$$Y_t \phi(\tilde{Y}) + \mathbb{E}^Q \left[ \int_0^T e^{-r_t Y_t} dt \right] \leq Y_t \phi(\tilde{Y}) + \mathbb{E}^Q \left[ \int_0^T e^{-r_t \tilde{Y}} dt \right] \leq 0.$$

Hence, we obtain the desired inequality. Now, the optimal hedging of contributions can be obtained by matching volatility of (11) and $X_t = \mathbb{E}^Q \left[ \int_t^T e^{-r(s-t)} Y_s ds \right]$. By Proposition 4, we can easily compute the latter:

$$X_t = \left( \frac{z_t}{k} \right)^\theta \frac{1 - e^{-\alpha \phi(T-t)}}{\alpha \phi}.$$ (A.2)

where $z_t = z_0 \xi_t$. The diffusion part of $X_t$ is

$$\text{diff}(dX_t) = -\frac{\eta}{\theta - 1} X_t.$$

This should be equal to the diffusion part of (11), $\pi_t^\phi X_t \sigma$. Hence, we have

$$\pi_t^\phi = -\frac{\eta}{(\theta - 1)\sigma}.$$

\[\square\]

**Proof of Proposition 6.** We first compute $C(X_0)$:

$$C(X_0) = \mathbb{E} \left[ \int_0^T e^{-\beta t} \frac{k}{\theta} \left( \frac{z_0 \xi_t}{k} \right)^\theta dt \right] = k \left( \frac{z_0}{k} \right)^\theta \int_0^T e^{-\beta t} \mathbb{E} \left[ \xi_t^{\theta - 1} \right] dt = k \left( \frac{z_0}{k} \right)^\theta \int_0^T e^{-\alpha \phi t} dt = k \left( \frac{z_0}{k} \right)^\theta \frac{1 - e^{-\alpha \phi T}}{\alpha \phi}.$$

Take the first derivative of (14):

$$C'(X_0) = \mathbb{E} \left[ \int_0^T e^{-\beta t} \phi' \left( I_\phi (z_0 \xi_t) \right) I_\phi' (z_0 \xi_t) \xi_t dt \right] \mathcal{Y}'(X_0) = z_0 \mathbb{E} \left[ \int_0^T e^{-r_t I_\phi' (z_0 \xi_t) \xi_t dt} \right] \mathcal{Y}'(X_0) = z_0 W_0(z_0) \mathcal{Y}'(X_0) = z_0$$

\[\square\]
Proof of Theorem 7. We can compute the present value of arbitrary contribution policy:

\[ \tilde{X}_t = \mathbb{E}^Q_t \left[ \int_t^T e^{-r(T-t)} \tilde{Y}_s ds \right]. \]

Also, let \( \tilde{W}_t \) denote \( \tilde{W}_t = \tilde{A}_t + \tilde{X}_t \), where \( \tilde{A} \) is a corresponding asset process to \( (\tilde{\pi}, \tilde{Y}) \) with \( \tilde{A}_0 = A_0 \). Then, \( \tilde{\pi} \) satisfies the following static budget constraint

\[ \tilde{W}_0 \geq \mathbb{E}^Q \left[ e^{-rT} \tilde{W}_T \right], \]

and the downside constraint: \( \tilde{W}_T \geq L \) since \( \tilde{W}_T = \tilde{A}_T \) due to \( \tilde{X}_T = 0 \) by definition and \( \tilde{A}_T \geq L \) by assumption. Hence, \( \tilde{\pi} \) is a feasible trading strategy to the second problem with the initial wealth \( \tilde{W}_0 \), and \( \tilde{Y} \) is a feasible contribution policy to the second problem with the required present value of contributions \( X_0 \). Let \( \pi_t^a \) and \( \pi_t^\phi \) be the optimal trading strategy to the first and the optimal hedging strategy to the second problem, respectively. Also, let \( W_t \) and \( X_t \) be the optimal path of asset value to the first problem, and the optimal path of portfolio value to the second problem, respectively. Finally, let \( Y_t \) denote the optimal contribution policy to the second problem. Then, we can construct the following asset allocation policy, and path of the pension plan’s asset:

\[
\begin{align*}
\pi_t &= \frac{\pi_t^a W_t - \pi_t^\phi X_t}{W_t - X_t} \\
A_t &= \frac{W_t - X_t}.
\end{align*}
\]

We need to prove that these policies are feasible for the original problem. Consider the discounted pension plan’s asset:

\[
\begin{align*}
e^{-rt} A_t &= e^{-rt} W_t - e^{-rt} X_t \\
&= \tilde{W}_0 + \int_0^t e^{-rs} \pi^a_s \sigma X_s dZ^Q_s - \tilde{X}_0 + \int_0^t e^{-rs} Y_s ds - \int_0^t e^{-rs} \pi^\phi_s \sigma X_s dZ^Q_s \\
&= A_0 + \int_0^t e^{-rs} Y_s ds + \int_0^t e^{-rs} \pi^\phi_s \sigma X_s dZ^Q_s \\
&= \mathbb{E}^Q_t \left[ e^{-rT} A_T \right] - \mathbb{E}^Q_t \left[ \int_t^T e^{-rT} Y_s ds \right],
\end{align*}
\]

since \( A_T = W_T - X_T = W_T \). Also, the downside constraint is satisfied: \( A_T = W_T \geq L \). Hence, \((\pi, Y)\) is an admissible portfolio and contribution policy to the original problem. Then, we have

\[
\begin{align*}
\mathbb{E} \left[ e^{-\beta T} u(A_T) \right] &\geq \mathbb{E} \left[ e^{-\beta T} u(\tilde{A}_T) \right] \\
\mathbb{E} \left[ \int_0^T e^{-\beta t} \phi(Y_t) dt \right] &\leq \mathbb{E} \left[ \int_0^T e^{-\beta t} \phi(\tilde{Y}_t) dt \right].
\end{align*}
\]

Hence, we have a desired inequality. Then, (15) is straightforward. \( \square \)

Proof of Proposition 8. From (6), we can easily see that \( \mathcal{W}_u(x) \) is decreasing, \( \lim_{x \to 0} \mathcal{W}_u(x) = \infty \), and \( \lim_{x \to \infty} \mathcal{W}_u(x) = Le^{-rT} \). Also, from (12) we can see that \( \mathcal{W}_\phi(x) \) is increasing, \( \lim_{x \to 0} \mathcal{W}_\phi(x) = 0 \), and \( \lim_{x \to \infty} \mathcal{W}_\phi(x) = \infty \).

Suppose that we find \( x \) solving (16), i.e. the required present value of contributions, \( X_0 \). Then, we can set the optimal portfolio and contribution policy, and the optimal path of the pension plan’s asset to the original problem as (A.3), (A.4) using solutions to the first and second problems. Then, the optimal portfolio weight is straightforward. Note that by (A.2) the optimal path of the fund for the disutility minimization problem is

\[ X_t = \left( \frac{x \xi_t}{k} \right)^{\frac{1}{\alpha_{\phi}}} \frac{1 - e^{-\alpha_{\phi}(T-t)}}{\alpha_{\phi}} = Y_t \frac{1 - e^{-\alpha_{\phi}(T-t)}}{\alpha_{\phi}}. \]

Hence, the optimal contribution rate is

\[ \frac{Y_t}{A_t} = \frac{X_t}{A_t} \frac{\alpha_{\phi}}{1 - e^{-\alpha_{\phi}(T-t)}}. \]
B Benchmark Case

Here, we consider the benchmark case. There’s no downside constraint and the pension sponsor contributes purely for maximizing the terminal pension plan’s asset while taking into account the disutility from contributions. The pension sponsor’s problem becomes

\[
\max_{\pi,Y} \mathbb{E} \left[ e^{-\beta T} u(W_T) - \int_0^T e^{-\beta t} \phi(Y_t) \, dt \right].
\]

Everything we derive for the constrained case goes through, except for the first problem. Now, let \( W_u^{BC}(y_0) \) be the counterpart of \( W_u(y_0) \) in the constrained case:

\[
W_u^{BC}(y_0) = \mathbb{E}^Q \left[ e^{-rT} I_u(y_0 \xi_T) \right].
\]

Note that the pension plan holds just the unconstrained optimal portfolio since there’s no downside constraint. Similarly, we can consider \( Y_u^{BC}(W_0) \), and \( J^{BC}(W_0) \) as the benchmark version of \( Y_u(W_0) \), and \( J(W_0) \). Proposition 9 summarizes the results for the first problem in the benchmark case.

**Proposition 9.** The function \( W_u^{BC}(y_0) \) is given by

\[
W_u^{BC}(y_0) = y_0^{-\frac{1}{2}} e^{-\alpha_u T}.
\]

Also, the first derivative is given by

\[
W_u^{BC}(y_0)' = -\frac{1}{\gamma} y_0^{-\frac{1}{2} - 1} e^{-\alpha_u T} < 0.
\]

For a given \( y_0 \), we have \( W_u^{BC}(y_0) < W_u(y_0) \). For any \( W_0, y_0 = Y_u^{BC}(W_0) \) is optimal for the utility maximization problem, and the optimal portfolio weight is given by \( \pi_u^{BC} = \frac{\alpha}{\gamma} \). The value function \( J^{BC}(W_0) \) is given by

\[
J^{BC}(W_0) = y_0^{1 - \frac{1}{\gamma}} e^{-\alpha_u T},
\]

and satisfies \( J^{BC}(W_0)' = y_0 \).

**Proof of Proposition 9.** The first part of (6) is the present value of the terminal pension plan’s asset, \( I_u(y_0 \xi_T) \) if \( I_u(y_0 \xi_T) > L \), otherwise zero. Hence, \( W_u^{BC}(y_0) \) can be easily computed from that. The first derivative is straightforward. Now, we can express \( W_u(y_0) \) as

\[
W_u(y_0) = y_0^{-\frac{1}{2}} e^{-\alpha_u T} + L e^{-rT} N(-\delta_2(y_0, T)) - y_0^{-\frac{1}{2}} e^{-\alpha_u T} N(-\delta_1(y_0, T)) > W_u^{BC}(y_0).
\]

The last two terms are the present value of \( (L - I_u(y_0 \xi_T))^+ \), and thus positive. The remaining part can be proved following similar procedures as in Theorem 2 and Proposition 3.

Without the downside constraint, the present value of the terminal pension plan’s asset is smaller than the constrained case. To achieve the same level of marginal utility, the benchmark case requires smaller initial asset since the put option doesn’t have to be purchased. As we expect, the optimal portfolio weight is the mean-variance efficient portfolio. Now, Theorem 7 and Proposition 8 can be stated for the benchmark case by substituting corresponding counterparts with \( J^{BC}(W_0), W_u^{BC}(y), Y_u^{BC}(W_0), \) and \( \pi_u^{BC} \).
References


