Proof of the prime power conjecture for projective planes of order $n$ with abelian collineation groups of order $n^2$

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Abstract
Let $G$ be an abelian collineation group of order $n^2$ of a projective plane of order $n$. We show that $n$ must be a prime power and that the $p$-rank of $G$ is at least $b + 1$ if $n = p^b$ for an odd prime $p$.

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1 Introduction

The purpose of this note is a surprisingly elementary proof of the following result.

**Theorem 1.1** Let $G$ be an abelian collineation group of order $n^2$ of a projective plane of order $n$. Then $n$ is a prime power, say $n = p^b$. If $p > 2$, then the $p$-rank of $G$ is at least $b + 1$.

Theorem 1.1 is the most conclusive known result in the context of the prime power conjecture for projective planes. Let us consider some background. Among other things, Dembowski and Piper [3] showed that there are only three possible types of projective planes of order $n$ with abelian collineation groups $G$ of order $n^2$. These are translation planes, dual translation planes and the so-called type (b) planes. By a classical result of André [1], in the case of translation planes and dual translation planes, the collineation group $G$ is always an elementary abelian $p$-group. Following [3], a projective plane of order $n$ is called a type (b) plane if it has an abelian collineation group of order $n^2$ whose orbits on the point set $\mathcal{P}$ are $\{p\}, L \setminus \{p\}$ and $\mathcal{P} \setminus L$ where $(p, L)$ is a suitable incident point-line pair. In this case, we call $G$ a group of type (b). Such groups exist for all prime powers $n$, see [2] or [7]. As a consequence of the prime power conjecture for projective planes, it has been conjectured that groups of type (b) only exist for prime powers $n$. Combining the results of André and Dembowski and Piper, we have the following.

**Result 1.2** [1, 3] Let $\Pi$ be a projective plane of order $n$ with an abelian collineation group $G$ of order $n^2$. Then one of the following hold.
(a) $\Pi$ is a translation plane or its dual, $n$ is a prime power and $G$ is elementary abelian.
(b) $\Pi$ is a plane of type (b).

Groups of type (b) are closely related to planar functions. Let $H$ and $K$ be groups of order $n$. A planar function of degree $n$ is a map $f : H \to K$ such that for every $h \in H \setminus \{1\}$ the induced map $f_h : x \mapsto f(hx)f(x)^{-1}$ is bijective. If a planar function from $H$ to $K$ exists, then $H \times K$ is a group of type (b), see [2, 7]. Thus Theorem 1.1 implies the following.

**Corollary 1.3** If there is a planar function of degree $n$ between abelian groups, then $n$ is a prime power.

The prime power conjecture for planar functions has been studied in many papers. The best result previous to Corollary 1.3 is due to S.L. Ma [6].
2 The result

A good way to talk about collineation groups of type (b) is to use the group ring. We first introduce the necessary notation. Let $G$ be a multiplicatively written finite group with identity element $1$. For $X = \sum a_g g \in \mathbb{Z}[G]$ we write $|X| = \sum a_g$, $X^{(i)} = \sum a_g g^i$ and $[X]_1 = a_1$ (the coefficient of 1 in $X$). For $r \in \mathbb{Z}$ we write $r$ for the group ring element $r \cdot 1$, and for $S \subset G$ we write $S$ instead of $\sum_{g \in S} g$. It is well known [5] that an abelian group $G$ of order $n^2$ is a group of type (b) on a suitable projective plane of order $n$ if and only if there is a subgroup $N$ of order $n$ of $G$ and an $n$-subset $D$ of $G$ such that

$$DD^{(-1)} = n + G - N$$

in $\mathbb{Z}[G]$. The set $D$ in is called an $(n, n, 1)$ difference set in $G$ relative to $N$.

We prepare the proof of our main result with two lemmas. Let $G$ be a finite abelian group, and let $p$ be a prime. By $r_p(G)$ we denote the $p$-rank of $G$, i.e. the minimum number of generators of the Sylow $p$-subgroup of $G$.

**Lemma 2.1** Let $G$ be a finite abelian group, let $N$ be a subgroup of $G$, and let $p$ be a prime. Then

$$[G^{(p)}]_1 = p^{r_p(G)}$$
$$[G^{(p)} N]_1 = p^{r_p(G/N)} |N|.$$  

**Proof** Straightforward checking. □

**Lemma 2.2** Let $G$ be an abelian group, let $D \in \mathbb{Z}[G]$ with $|D| = k$ and

$$DD^{(-1)} = k + X,$$
$$DX = aG$$

for some integer $a$ and $X \in \mathbb{Z}[G]$. Furthermore, let $p \geq 3$ be a prime dividing $k$. Then

$$(p - 1)k^2 \leq k [X + X^{(p)}]_1 + [XX^{(p)}]_1$$

with equality if and only if $D^{(-1)} D^{(p)}$ has coefficients 0 and $p$ only.

**Proof** Write $A := D^{(-1)} D^{(p)} = \sum a_g g$. Then $\sum a_g = k^2$. Since $G$ is abelian, we have $D^{(p)} = D^p$ in $\mathbb{Z}_p[G]$. As $k$ is divisible by $p$, we get

$$A = (k + X) D^{p-1} = XD^{p-1} = aGD^{p-2} = akGD^{p-3} = 0$$

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in $\mathbb{Z}_p[G]$. Hence all $a_g$ are divisible by $p$ and thus
\[ \sum a_g^2 \geq p \sum a_g = pk^2 \]
with equality if and only if $a_g \in \{0,p\}$ for all $g$. On the other hand, we have
\[ AA^{(-1)} = (k + X)(k + X^{(p)}) = k^2 + k(X + X^{(p)}) + XX^{(p)} \]
and thus
\[ \sum a_g^2 = [AA^{(-1)}]_1 = k^2 + k[X + X^{(p)}]_1 + [XX^{(p)}]_1. \]
This proves the lemma. \( \square \)

Now we are ready to prove our main result.

**Theorem 2.3** Let $D$ be the relative difference set satisfying (1), and let $p \geq 3$ be a prime divisor of $n$. Then
\[ (p - 2)n \leq p^r_s(G) - p^r_s(N) - p^r_s(G/N). \]

**Proof** Since $|D| = n$, (1) implies that $D$ contains exactly one element of each coset of $N$ in $G$, i.e.
\[ DN = G. \quad (2) \]
Because of (1) and (2), we can apply Lemma 2.2 with $X = G - N$ and $k = n$. Note that $[X + X^{(p)}]_1 = p^r_s(G) - p^r_s(N)$ using Lemma 2.1. Furthermore,
\[ [XX^{(p)}]_1 = [(n^2 - n)G - G^{(p)}N + nN]_1 = n^2 - np^r_s(G/N) \]
again using Lemma 2.1. Thus Lemma 2.2 gives us
\[ (p - 1)n^2 \leq n(p^r_s(G) - p^r_s(N)) + n^2 - np^r_s(G/N). \]
Subtracting $n^2$ and dividing by $n$ gives the assertion. \( \square \)

**Proof of Theorem 1.1**
In view of Result 1.2, we can assume that $G$ is a group of type (b). It is shown in [4] that $n$ must be a power of 2 if $n$ is even. Thus we can assume that $n$ is odd. If $n$ is not a prime power, then there is a prime divisor $p \geq 3$ of $n$ such that the Sylow $p$-subgroup $S$ of $G$ has order less than $n$. But then $p^r_s(G) \leq |S| < n$ contradicting Theorem 2.3. Thus $n$ is a prime power, say $n = p^b$ where $p$ is an odd prime. Theorem 2.3 shows $p^r_s(G) > n$, and so $G$ must have rank at least $b + 1$. \( \square \)
References


