Low-Rank plus Sparse Decomposition of Covariance Matrices using Neural Network Parametrization

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Abstract

This paper revisits the problem of decomposing a positive semidefinite matrix as a sum of a matrix with a given rank plus a sparse matrix. An immediate application can be found in portfolio optimization, when the matrix to be decomposed is the covariance between the different assets in the portfolio. Our approach consists in representing the low-rank part of the solution as the product $MM^T$, where $M$ is a rectangular matrix of appropriate size, parameterized by the coefficients of a deep neural network. We then use a gradient descent algorithm to minimize an appropriate loss function over the parameters of the network. We deduce its convergence speed to a local optimum from the Lipschitz smoothness of our loss function. We show that the rate of convergence grows polynomially in the dimensions of the input, output, and each of the hidden layers and hence conclude that our algorithm does not suffer from the curse of dimensionality.

1 Introduction

We present a simple yet powerful new approach to decompose a possibly large covariance matrix into the sum of a positive semidefinite low-rank matrix $L$ plus a sparse matrix $S$. Our approach consists in fixing an (upper bound for the) rank $k$ of $L$ by defining $L := MM^T$ for a suitable $M \in \mathbb{R}^{n \times k}$, where one parameterizes $M$ using a deep neural network whose coefficients are minimized using a gradient descent method.

When studying the correlation matrix, e.g., between the returns of financial assets, it is important for the design of a well-diversified portfolio to identify groups of heavily correlated assets, or more generally, to identify a few ad-hoc features that describe some dependencies between these assets. To this effect, the most natural tool is to determine the few first dominant eigenspaces of the correlation matrix and to interpret them as the dominant features driving the behavior of the assets. This procedure, generally termed Principal Component Analysis (PCA), is widely used. However, this decomposition ignores everything but the few first features of the assets. As it turns out, some coefficients in the remaining part can be relatively large with respect to the others; these indicate pairs of assets that present an ignored large correlation between themselves, beyond the dominant features revealed by PCA. Following [20], to reveal this extra structure present in $\Sigma$, we decompose it into the sum of a low-rank matrix $L$, to describe the dominant features, plus a sparse matrix $S$, to identify hidden large correlations between assets pairs.

Beyond covariance matrices, this decomposition is a procedure abundantly used in image and video processing for compression and interpretative purposes [3], but also in latent variable model selection [6], in latent semantic indexing [13], in graphical model selection [2], in graphs [19], and in gene expression [12], among others.

A rich collection of algorithms exist to compute such decomposition, see [6, 7, 9, 11, 21, 22] to name but a few, most of which are reviewed in [4] and implemented in the Matlab LRS library [5]. Among these algorithms, Principal Component Pursuit (PCP) has been proposed by [6, 7, 11] as a robust alternative to PCA. It is considered as the state-of-the-art approach that can recover the low-rank and the sparse matrices. For a given $\delta > 0$, the PCP problem is formulated as

$$
\arg \min_{L,S} \|L\|_* + \delta \|S\|_1 \quad \text{s.t. } \Sigma = L + S.
$$

where $\Sigma \in \mathbb{R}^{n \times n}$ is the observed matrix, $\|L\|_*$ is the nuclear norm of matrix $L$ (i.e. the sum of the singular values of $L$) and $\|S\|_1$ is the $l^1$-norm of matrix $S$. The numerical algorithms used for solving (1) are usually computationally expensive. Originally in [20], the (exact) Augmented Lagrange Multiplier (ALM) was used to solve (1). By incorporating the constraints of (1) into the objective multiplied by their Lagrange multiplier $Y \in \mathbb{R}^{n \times n}$, ALM is formulated as

$$
\arg \min_{L,S,Y} \|L\|_* + \delta \|S\|_1 + \langle Y, \Sigma - L - S \rangle + \frac{\delta}{2} \|\Sigma - L - S\|_F.
$$

† Keywords Correlation Matrices; Neural Network Parametrization; Low-Rank + Sparse Decomposition; Portfolio Optimization
Due to the use of the nuclear norm, this approach must perform a full singular-value decomposition (SVD) on the successive iterates, which has a negative impact on the computational performance. In [13], the authors have proposed the method of Fast Principle Component Pursuit (FPCP), which is a simple alternating minimization algorithm for solving a variation of the original PCP [1]. By incorporating the constraint into the objective, removing the costly nuclear norm term, and imposing a rank constraint on $L$, the problem (3) becomes

$$\arg\min_{L,S} \frac{1}{2}||L + S - \Sigma||_F^2 + \delta||S||_1 \quad \text{s.t. rank}(L) = t.$$ (3)

The authors apply the following alternating minimization to solve (3).

$$L_{k+1} = \arg\min_L \frac{1}{2}||L + S_k - \Sigma||_F^2 + \delta||S_k||_1 \quad \text{s.t. rank}(L) = t$$

$$S_{k+1} = \arg\min_S ||L_{k+1} + S - \Sigma||_F^2 + \delta||S||_1.$$ (4)

The sub-problem (4) can be solved by computing a partial SVD of $\Sigma - S_k$, with the only necessity of computing the $t$ first singular values and their associated eigenvectors. As the problem (5) is separable, its solution can be computed efficiently by a component-wise soft-thresholding. This solution is of comparable quality to the solution of the original PCP problem. Note that the solution to this problem depends on a hyperparameter $\delta$. In the absence of the rank constraint in (2), some guidelines to set an appropriate value for $\delta$ are known (see, e.g., [6, Theorem 1.1]). However, such considerations do not hold in the FPCP algorithm, and, to the best of our knowledge, the theoretical question of an appropriate choice for $\delta$ remains unsolved. In contrast, our method does not rely on such a hyperparameter.

In the standard PCP algorithm, neither the rank of $L$ nor its expressivity - that is, the portion of the spectrum of $\Sigma$ covered by the low-rank matrix - can be chosen in advance. In contrast, one can request that the solution of FPCP has a given expressivity. In our approach, we must first select a rank for $L$, based e.g. on a prior spectral decomposition of $\Sigma$ or based on exogenous considerations. We then apply a gradient descent method with a well-chosen loss function, using Tensorflow [1] or Pytorch [17].

2 Neural network parameterized optimization and its convergence rate

Let $\mathbb{S}^n$ be the set of $n$-by-$n$ real symmetric matrices and $\mathbb{S}^n_+ \subset \mathbb{S}^n$ be the cone of positive semidefinite matrices. Consider a matrix $\Sigma = [\Sigma_{i,j}]_{i,j} \in \mathbb{S}^n$, e.g., a covariance matrix. The matrix $\Sigma$ is to be decomposed as a sum of a positive semidefinite low-rank matrix $L = [L_{i,j}]_{i,j} \in \mathbb{S}^n$ of rank at most equal $k$ and a sparse matrix $S = [S_{i,j}]_{i,j} \in \mathbb{R}^{n \times n}$. Observe that the matrix $S$ is also a symmetric matrix. It is well-known that the matrix $L$ can be represented as $L = MM^T$, where $M = [M_{i,j}]_{i,j} \in \mathbb{R}^{n \times k}$; thus $\Sigma = MM^T + S$.

For practical purposes, we represent every symmetric $n$-by-$n$ matrix by a vector of dimension $r := n(n+1)/2$; formally, we define the operator

$$h : \mathbb{S}^n \rightarrow \mathbb{R}^{n(n+1)/2}, \quad \Sigma \mapsto h(\Sigma) := (\Sigma_{1,1}, \Sigma_{1,2}, \ldots, \Sigma_{1,n}, \Sigma_{2,2}, \Sigma_{2,3}, \ldots, \Sigma_{2,n}, \ldots, \Sigma_{n,n})^T.$$ (6)

Clearly, $h$ is an invertible linear operator with its inverse denoted by $h^{-1}$. Similarly, every vector of dimension $nk$ shall be represented by a $n$-by-$k$ matrix, and vice versa, by the linear operators

$$g : \mathbb{R}^{nk} \rightarrow \mathbb{R}^{n \times k}, \quad v := (v_1, \ldots, v_{nk})^T \mapsto g(v) := W \equiv [w_{i,j}]_{i,j} \quad \text{with} \quad w_{i,j} = v(i-1)k + j,$$

$$g^{-1} : \mathbb{R}^{n \times k} \rightarrow \mathbb{R}^{nk}, \quad M \mapsto g^{-1}(M) := (M_{1,1}, M_{1,2}, \ldots, M_{1,k}, M_{2,1}, \ldots, M_{2,k}, \ldots, M_{n,k})^T.$$ (7)
We construct a neural network with $n(n+1)/2$ inputs and $nk$ outputs; these outputs are meant to represent the coefficients of the matrix $M$ with whom we shall construct the rank $k$ matrix $L$ in the decomposition of the input matrix $X$. However, we do not use this neural network in its feed-forward mode as a heuristic to compute $M$ from an input $X$; we merely use the neural network framework as a way to parameterize a tentative solution to our decomposition problem.

We construct our neural network with $m$ layers of $\ell_i$, $i = 1, \ldots, m$ neurons, each with the same activation function $\sigma : \mathbb{R} \to [0, 1]$; we assume that the first and the second derivative of $\sigma$ are uniformly bounded from above by the constants $\sigma'_{\max} \geq 1$ and $\sigma''_{\max}$, respectively. In accordance with the standard architecture of multi-layered neural networks, for each $u = 1, \ldots, m+1$, let $A^{(u)}(\Theta) := \left[ A^{(u)}_{ij} \right]_{i,j} \in \mathbb{R}^{\ell_u \times \ell_{u-1}}$ be the weights, $b^{(u)} := \left[ b^{(u)}_i \right]_{i} \in \mathbb{R}^{\ell_u}$ be the bias, and

$$f_u^{A^{(u)}, b^{(u)}} : \mathbb{R}^{\ell_{u-1}} \to \mathbb{R}^{\ell_u}, \quad v \mapsto f_u^{A^{(u)}, b^{(u)}}(v) := A^{(u)}v + b^{(u)},$$

where we set $\ell_0 = n(n+1)/2$, $\ell_{m+1} = nk$. Moreover, for each $u, i = 1, \ldots, m$, we denote by $\sigma^{(i)} : \mathbb{R}^{\ell_i} \to \mathbb{R}^\ell_i$ $v \mapsto \sigma^{(i)}(v) := (\sigma(v_1) \ldots \sigma(v_n))^T$. Then, we denote the parameters $\Theta := (A^{(1)}, b^{(1)}, \ldots, A^{(m+1)}, b^{(m+1)})$ and define the $m$-layered neural network $\mathcal{N}^{(m, \Theta)}$ by the function

$$\mathbb{R}^n \ni x \mapsto \mathcal{N}^{(m, \Theta)}(x) := f_{m+1}^{A^{(m+1)}, b^{(m+1)}} \circ \sigma^{(m)} \circ f_m^{A^{(m)}, b^{(m)}} \circ \ldots \circ \sigma^{(1)} \circ f_1^{A^{(1)}, b^{(1)}}(x) \in \mathbb{R}^{nk}.$$ 

We therefore have to specify $\ell_m := \sum_{u=1}^{m+1} \ell_u \ell_{u-1} + \ell_u$ many parameters to describe the neural network $\mathcal{N}^{(m, \Theta)}$ completely.

Now, we are ready to define the cost function to minimize. Given $X \in \mathbb{R}^{n \times n}$, we write its 1-norm as $||X||_1 := \sum_{i=1}^n \sum_{j=1}^n |X_{ij}|$. Our objective function is, for a given $\Sigma \in \mathbb{S}^n$, the function

$$\varphi_{\text{obj}}(A^{(1)}, b^{(1)}, \ldots, A^{(m+1)}, b^{(m+1)}) := \varphi_{\text{obj}}(\Theta) := \|g(\mathcal{N}^{(m, \Theta)}(h(\Sigma))) g(\mathcal{N}^{(m, \Theta)}(h(\Sigma)))^T - \Sigma\|_1,$$

Since $M = g(\mathcal{N}^{(m, \Theta)}(h(\Sigma)))$ is our tentative solution to the matrix decomposition problem, this objective function consists in minimizing $||MM^T - \Sigma||_1 = ||S||_1$.

As this function is not differentiable, we shall approximate it by

$$\varphi(A^{(1)}, b^{(1)}, \ldots, A^{(m+1)}, b^{(m+1)}) := \varphi(\Theta) := \sum_{i=1}^n \sum_{j=1}^n \mu \left( \|g(\mathcal{N}^{(m, \Theta)}(h(\Sigma))) g(\mathcal{N}^{(m, \Theta)}(h(\Sigma)))^T - \Sigma\|_{i,j} \right), \quad (8)$$

where $\mu : \mathbb{R} \to [0, \infty)$ is a smooth approximation of the absolute value function with a derivative uniformly bounded by $1$ and its second derivative bounded by $\mu''_{\max}$. A widely used example of such a function is given by

$$\mu(t) := \begin{cases} \frac{t^2}{2} + \frac{\varepsilon}{2} & \text{if } |t| \leq \varepsilon \\ \frac{\varepsilon^2}{2} & \text{if } |t| > \varepsilon, \end{cases}$$

where $\varepsilon$ is a small positive constant; see [3]. With this choice for $\mu$, we have $\mu''_{\max} = 1/\varepsilon$. Another example, coming from the theory of smoothing techniques in convex optimization, is given by $\mu(t) := \varepsilon \ln(2 \cosh(t/\varepsilon))$, also with $\mu''_{\max} = 1/\varepsilon$.

We apply a gradient method to minimize the objective function $\varphi$, whose general scheme can be written as follows.

$$\text{Fix } \Theta_0$$
$$\text{For } j \geq 0$$
$$\text{Compute } \nabla \varphi(\Theta_j)$$
$$\text{Determine a step-size } h_j > 0$$
$$\text{Set } \Theta_{j+1} = \Theta_j - h_j \nabla \varphi(\Theta_j).$$

The norm we shall use in the sequel is a natural extension of the standard Euclidean norm to finite lists of matrices of diverse sizes. Specifically, for any $\gamma \in \mathbb{N}$, $m_1, \ldots, m_{\gamma}, n_1, \ldots, n_{\gamma} \in \mathbb{N}_0$, and $(X^1, \ldots, X^\gamma) \in \mathbb{R}^{m_1 \times n_1} \times \cdots \times \mathbb{R}^{m_{\gamma} \times n_{\gamma}}$, we let

$$||X^1, \ldots, X^\gamma|| := \left( \sum_{i=1}^{m_1} \sum_{j=1}^{n_1} (X^1_{ij})^2 + \cdots + \sum_{i=1}^{m_{\gamma}} \sum_{j=1}^{n_{\gamma}} (X^\gamma_{ij})^2 \right)^{1/2}. \quad (10)$$

This norm is merely the standard Euclidean norm of the vector obtained by concatenating all the columns of $X^1, \ldots, X^\gamma$. Indeed, when $\gamma = 1$, this norm coincides with the Frobenius norm.

Since the objective function in [5] is non-convex, this method can only realistically converge to one of its stationary points or to stop close enough from one, that is, at point $\Theta^*$ for which $||\Theta^*||$ is smaller than a given tolerance. The complexity of many variants of this method can be established if the function $\varphi$ has a Lipschitz continuous gradient, see [5] and references therein. We have the following convergence result.
Theorem 2.1. Let $\Sigma \in \mathbb{S}^m$ and assume that there exists $D > 0$ such that the sequence $(\Theta_j)_{j \in \mathbb{N}_0}$ of parameters constructed in \cite{9} satisfies
\[ \sup_{j \in \mathbb{N}_0} \| \Theta_j \| \leq D. \] (11)
Then, the gradient of the function $\varphi$ defined in \cite{9} is Lipschitz continuous on $D := \{ x \in \mathbb{R}^{1 \times m} : \| x \| \leq D \}$ with Lipschitz constant $L > 0$ that can be calculated by
\[
\begin{align*}
\mathcal{L}_{m-1} := & \sqrt{D^{2m-2}(1 + \| h(\Sigma) \|^2) + \sum_{j=1}^{m-1} D^{2(m-1-j)}(\ell_j + 1)}, \\
\mathcal{L}_m := & \sqrt{n} \sqrt{\ell_m} \max \left\{ D^2 \mathcal{L}_m^2, \sqrt{\ell_m} D^4 \mathcal{L}_m^2, \ell_m D^4 \mathcal{L}_m^2, \sqrt{\kappa} \mathcal{L}_m, \sqrt{\kappa \ell_m} / D^2 \right\}, \\
L := & O(\mathcal{C}(\sigma_{max}, \sigma''_{max}, \mu''_{max}) \mathcal{L}_m),
\end{align*}
\] (12)
where $\mathcal{C}(\sigma_{max}, \sigma''_{max}, \mu''_{max})$ is a constant that only depends polynomially on $\sigma_{max}, \sigma''_{max}, \mu''_{max}$. As a consequence, if for the gradient method \cite{9} there exists a constant $K > 0$ such that for all $j \geq 0$
\[ \varphi(\Theta_j) - \varphi(\Theta_{j+1}) \geq K \| \nabla \varphi(\Theta_j) \|^2, \] (13)
then for every $N \in \mathbb{N}$ we have that
\[ \min_{0 \leq j \leq N} \| \nabla \varphi(\Theta_j) \| \leq \frac{1}{\sqrt{K^2}} \left[ \frac{1}{L} \mathcal{L}(\varphi(\Theta_0) - \varphi^*) \right]^{1/2}, \] (14)
where $\varphi^* := \min_{\Theta \in \mathcal{T}} \varphi(\Theta)$. In particular, for every convergence level $\varepsilon > 0$ we have
\[ N + 1 \geq \frac{1}{\sqrt{KL}} (\varphi(\Theta_0) - \varphi^*) \implies \min_{0 \leq j \leq N} \| \nabla \varphi(\Theta_j) \| \leq \varepsilon. \]

Remark 2.2. Notice that the condition \cite{13} in Theorem 2.1 imposed on the gradient method, or more precisely on the step-size strategy $(h_j)$, is not very restrictive. We provide several examples which are frequently used.

- The sequence $(h_j)$ is chosen in advance, independently of the minimization problem. This includes, e.g., the common constant step-size strategy $h_1 = h$ or $h_j = \frac{h}{\sqrt{j}}$, for some constant $h > 0$. Indeed, one can show that
\cite{13} is satisfied for $K = 1$.
- The Goldstein-Armijo rule, which is defined as follows: given $0 < \alpha < \beta < 1$, one needs to find $(h_j)$ such that
\[ \alpha \langle \nabla \varphi(\Theta_j), \Theta_j - \Theta_{j+1} \rangle \leq \varphi(\Theta_j) - \varphi(\Theta_{j+1}) \]
\[ \beta \langle \nabla \varphi(\Theta_j), \Theta_j - \Theta_{j+1} \rangle \geq \varphi(\Theta_j) - \varphi(\Theta_{j+1}) \]
satisfies \cite{13} with $K = 2\alpha(1 - \beta)$. We refer to \cite{15} Section 1.2.3] and to \cite{16} Chapter 3] for further details.

Remark 2.3. The convergence rate \cite{14} obtained in Theorem 2.1 relies fundamentally on the Lipschitz property of the gradient of the (approximated) objective function $\varphi$ of the algorithm in \cite{9}. However, due to its structure, we see that the local Lipschitz property of $\nabla \varphi$ fails already for a single-layered neural network, as it grows polynomially of degree 4 in the parameters; see also Section 2.4. Yet, it is enough to ensure the Lipschitz property of $\nabla \varphi$ on the domain of the sequence of parameters $(\Theta_j)_{j \in \mathbb{N}_0}$ generated by the algorithm in \cite{9}, which explains the significance of assumption \cite{11}. Nevertheless, assumption \cite{11} is not very restrictive as one might expect that the algorithm \cite{9} converges and hence automatically forces assumption \cite{11} to hold true. Moreover, we empirically justify this assumption by verifying for our two main applications of the algorithm with real data coming from the S&P500 stock prices and real estate returns that indeed, empirically, assumption \cite{11} holds; see Subsection 1.4.3.

Remark 2.4. While the second part of Theorem 2.1 is standard in optimization, see, e.g., in \cite{15} Section 1.2.3], we notice that for a fixed depth $m$ of the neural network the constant $L$ in the rate of convergence of the sequence $(\min_{0 \leq j \leq N} \| \nabla \varphi(\Theta_j) \|)$ only grows polynomially in the parameters $\ell_{max} := \max \{ \ell_1, \ldots, \ell_m \}$, $\kappa$, describing the corresponding dimensions of the input, output and the hidden layers of the neural network. Therefore, we see that our algorithm overcomes the curse of dimensionality in the sense that the constants do not grow exponentially in the dimensions involved in the decomposition of a covariance matrix. Indeed, a rough estimate yields that
\[ \mathcal{L}_{m-1} \leq O\left(D^{2m-2}\left(\| h(\Sigma) \|^2 + \ell_{max}/D^2 \right) \right), \]
and hence by using \cite{12}:
\[ \mathcal{L}_m \leq O\left(n^{3/2}k^{1/2}\ell_{max}^{1/2}D^{2m+2}\| h(\Sigma) \|^2 \right). \]
We provide the proof of Theorem 2.1 in Section 4.
3 Numerical results

3.1 Numerical results based on simulated data

We start our numerical tests with a series of experiments on artificially generated data. We construct a collection of \( n \times n \) positive semidefinite matrices \( \Sigma \) that can be written as \( \Sigma = L_0 + S_0 \) for a known matrix \( L_0 \) of rank \( k_0 \leq n \) and a known matrix \( S_0 \) of given sparsity \( s_0 \). We understand by sparsity the number of null elements of \( S_0 \) divided by the number of coefficients of \( S_0 \), when a sparse matrix is determined by an algorithm, we consider that every component smaller in absolute value than \( \varepsilon = 0.01 \) is null. To construct one matrix \( L_0 \), we first sample \( nk_0 \) independent standard normal random variables that we arrange into an \( n \times k_0 \) matrix \( M \). Then \( L_0 \) is simply taken as \( MM^T \). To construct a symmetric positive semidefinite sparse matrix \( S_0 \) with \( 2N \) non-zero off-diagonal elements, we first select uniformly randomly \( N \) distinct pairs \((i, j)\) with \( 1 \leq i < j \leq N \). For each selected pair \((i, j)\), we construct an \( n \times n \) matrix \( A \) that has only four non-zero coefficients: its off-diagonal elements \((i, j)\) and \((j, i)\) are set to a number \( b \) drawn uniformly randomly in \([-1, 1]\), whereas the diagonal elements \((i, i)\) and \((j, j)\) are set to a number \( a \) drawn uniformly randomly in \([0, 1]\). This way, the matrix \( A \) is positive semidefinite. We take for \( S_0 \) the sum of \( N \) such matrices \( A \), each corresponding to a different pair \((i, j)\), so that \( S_0 \) is a positive semidefinite matrix with \( 2N \) non-zero off-diagonal elements.

Given an artificially generated matrix \( \Sigma = L_0 + S_0 \), where \( L_0 \) has a prescribed rank \( k_0 \) and \( S_0 \) a sparsity \( s_0 \), we run our algorithm to construct a matrix \( M \in \mathbb{R}^{n \times k} \). With \( L := MM^T \) and \( S := \Sigma - L \), we determine the approximated rank \( r(L) \) of \( L \) by counting the number of eigenvalues of \( L \) that are larger than \( \varepsilon = 0.01 \). We also determine the sparsity \( s(S) \) as specified above, by taking as null every coefficient smaller than \( \varepsilon = 0.01 \) in absolute value. We compute the discrepancy between the calculated low-rank part \( L \) and the correct one \( L_0 \) by \( \text{rel.error}(L) := ||L - L_0||_F/||L_0||_F \) and between \( S \) and the true \( S_0 \) by \( \text{rel.error}(S) := ||S - S_0||_F/||S_0||_F \). Table 1 reports the average of these quantities over ten runs of our algorithm DNN (short for Deep Neural Network), as well as their standard deviation (in parenthesis). We carried our experiments on various values for the dimension \( n \) of the matrix \( \Sigma \), for the given rank \( r(L_0) \) of \( L_0 \), for the given sparsity \( s(S_0) \) of \( S_0 \) and for the chosen forced (upper bound for the) rank \( k \) in the construction of \( L \) introduced in Section 2.

We have decided to compare our algorithm with FPCP \(^{18}\) because it is derived from the state-of-the-art Robust Principle Component (see \(^4\) for extensive comparative tests) especially among those methods where the user can pre-specify the rank of the low-rank matrix \( L \), as in our algorithm.

When choosing \( n = 100 \), our algorithm indeed achieves the maximal rank \( k \) for the output matrix \( L \), which fails to hold for FPCP. Moreover, the corresponding sparsities are comparable when \( n = 100 \), but as the FPCP algorithm applies a shrinkage by replacing every matrix entries \( S_{ij} \) by \( \text{sign}(S_{ij}) ||S_{ij}|| - 1/\sqrt{n} \), we see that FPCP forces a higher sparsity.

We note that our algorithm achieves a higher accuracy than FPCP in all the different dimensions \( n \) we tested, in terms of relative errors for \( L \) and \( S \), especially when the forced rank \( k \) matches the actual rank \( r_0 \) of \( L_0 \). Unsurprisingly, when the forced rank \( k \) differs from \( r_0 \), these relative errors are higher for both methods. However, our method achieves much better results than FPCP. This is particularly fortunate since when one is interested in obtaining a low-rank plus sparse decomposition for a given matrix, for example for the correlation matrix \( \Sigma \) of stock prices, one cannot assume to know a priori the rank of the output matrix \( L \), which corresponds to the number of explanatory factors of \( \Sigma \).

In the second part of the table, we consider the forced rank \( k \) to be equal to the actual rank \( r_0 \) of \( L_0 \) and choose it to be equal to 2.55 \% of the size \( n \) of the matrix. We set the sparsity \( s_0 \) of \( S_0 \) to be equal to 95 \%. For dimensions \( n \geq 200 \), we see that both FPCP and DNN return a matrix \( L \) with a lower rank than the target rank \( k \). However, our algorithm achieves a higher rank for \( L \) than FPCP, where the difference grows in the dimension \( n \). Therefore, it seems that our algorithm is more suitable for high-dimensional matrices than FPCP as it recovers more explanatory factors. Additionally, our algorithm also achieves slightly better performance than FPCP in terms of sparsity.

Various network architectures have been tested. They only marginally influence the results.

3.2 Application on a five hundred S&P500 stocks portfolio

In this section, we evaluate our algorithm on real market data and compare it to FPCP to demonstrate its capability also when the low-rank plus sparse matrix decomposition is not known. A natural candidate for our experiment is the correlation matrix of stocks in the S&P500, due to its relatively large size and the abundant, easily available data. Five hundred S&P500 stocks were part of the index between 2017 and 2018. To make the representation more readable, we have sorted these stocks in eleven sectors according to the global industry classification standard\(^{11}\). We have constructed the correlation matrix \( \Sigma \) from the daily returns of these 500 stocks during 250 consecutive trading days (see Figure 1). As the data used to construct \( \Sigma \) are available at an identical frequency, the matrix \( \Sigma \) is indeed positive semidefinite.

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\(^{11}\)First, we have those belonging to the energy sector, second, those from the materials sector, then, in order, those from industrials, real estate, consumer discretionary, consumer staples, health care, financials, information technology, communication services, and finally utilities. We may notice that utilities seem almost uncorrelated to the other sectors, and that real estate and health care present a significantly lower level of correlation to the other sectors than the rest.
matrix. The original matrix $\Sigma$ is not positive semidefinite. In contrast, the FPCP Algorithm might output a non-positive semidefinite $L$ matrix has some negative eigenvalues. This phenomenon can happen in empirical correlation matrices when the data of the different variables are either not sampled over the same time frame not with the same frequency; we refer to [10] for more details.

In the right figure, we plot the first 50 eigenvalues where the forced rank is set to $k=5$ and $0.95$ for a better visualization. The sparse matrix of FPCP is sparser than the one returned by our DNN algorithm. The low-rank matrices $L$ are almost identical.

In Figure 3 we plot the eigenvalues of the matrix $L$ returned by FPCP and DNN, as well as the eigenvalues of the original matrix $\Sigma$. In the left figure, we see the first 17 eigenvalues of the matrix $L$ where the forced rank is set to $k=15$. In the right figure, we plot the first 50 eigenvalues where the forced rank is set to $k=88$. Notice that the correlation matrix has some negative eigenvalues. This phenomenon can happen in empirical correlation matrices when the data of the different variables are either not sampled over the same time frame not with the same frequency; we refer to [10] for a further discussion on this issue. The DNN algorithm, setting $L := MM^T$, avoids negative eigenvalues, although the original matrix $\Sigma$ is not positive semidefinite. In contrast, the FPCP Algorithm might output a non-positive semidefinite matrix.

### 3.3 Application on real estate return

We have computed the low-rank plus sparse decomposition of the real estate return matrix for 44 countries. The correlation matrix contains 88 returns, alternating the residential returns and the corporate returns of each country. Note that the scale of values for the correlation matrix ranges between $-0.2$ and $0.9$. Notice that the sparse matrix from DNN is slightly less sparse than the FPCP matrix due to the much cruder shrinkage method applied in FPCP.

### Table 1: Comparison between the FPCP and the DNN algorithm

<table>
<thead>
<tr>
<th>$n$</th>
<th>$r(L_0)$</th>
<th>$k$</th>
<th>$s(S_0)$</th>
<th>Algo</th>
<th>$r(L)$</th>
<th>$s(S)$</th>
<th>rel.error($S$)</th>
<th>rel.error($L$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>10</td>
<td>5</td>
<td>0.6</td>
<td>FPCP</td>
<td>5.0 (0.0)</td>
<td>0.01 (0.0)</td>
<td>0.54 (0.08)</td>
<td>0.86 (0.01)</td>
</tr>
<tr>
<td>100</td>
<td>10</td>
<td>5</td>
<td>0.95</td>
<td>FPCP</td>
<td>5.0 (0.0)</td>
<td>0.04 (0.0)</td>
<td>6.56 (0.21)</td>
<td>0.54 (0.01)</td>
</tr>
<tr>
<td>100</td>
<td>10</td>
<td>10</td>
<td>0.6</td>
<td>FPCP</td>
<td>8.2 (0.4)</td>
<td>0.06 (0.01)</td>
<td>0.51 (0.03)</td>
<td>0.77 (0.01)</td>
</tr>
<tr>
<td>100</td>
<td>10</td>
<td>10</td>
<td>0.95</td>
<td>FPCP</td>
<td>9.0 (0.0)</td>
<td>0.07 (0.01)</td>
<td>5.78 (0.2)</td>
<td>0.74 (0.01)</td>
</tr>
<tr>
<td>100</td>
<td>20</td>
<td>20</td>
<td>0.6</td>
<td>FPCP</td>
<td>11.5 (0.5)</td>
<td>0.14 (0.01)</td>
<td>0.35 (0.03)</td>
<td>0.57 (0.01)</td>
</tr>
<tr>
<td>100</td>
<td>20</td>
<td>20</td>
<td>0.95</td>
<td>FPCP</td>
<td>12.0 (0.0)</td>
<td>0.19 (0.01)</td>
<td>4.09 (0.2)</td>
<td>0.53 (0.01)</td>
</tr>
<tr>
<td>200</td>
<td>5</td>
<td>5</td>
<td>0.95</td>
<td>FPCP</td>
<td>5.0 (0.0)</td>
<td>0.06 (0.0)</td>
<td>2.84 (0.15)</td>
<td>0.78 (0.01)</td>
</tr>
<tr>
<td>400</td>
<td>10</td>
<td>10</td>
<td>0.95</td>
<td>FPCP</td>
<td>7.4 (0.49)</td>
<td>0.02 (0.0)</td>
<td>3.1 (0.07)</td>
<td>0.86 (0.0)</td>
</tr>
<tr>
<td>800</td>
<td>20</td>
<td>20</td>
<td>0.95</td>
<td>FPCP</td>
<td>9.0 (0.0)</td>
<td>0.01 (0.0)</td>
<td>3.29 (0.04)</td>
<td>0.92 (0.0)</td>
</tr>
<tr>
<td>1600</td>
<td>40</td>
<td>40</td>
<td>0.95</td>
<td>FPCP</td>
<td>10.0 (0.0)</td>
<td>0.01 (0.0)</td>
<td>3.42 (0.02)</td>
<td>0.96 (0.0)</td>
</tr>
</tbody>
</table>

The 70 largest eigenvalues account for 90% of $\Sigma$'s trace, that is, the sum of all its 500 eigenvalues.

The countries are ordered by continent and subcontinents: Western Europe (Belgium, Luxembourg, Netherlands, France, Germany, Switzerland, Austria, Denmark, Norway, Sweden, Finland, United Kingdom, Ireland, Italy, Spain, Portugal), Eastern Europe (Croatia, Estonia, Latvia, Lithuania, Russia, Poland, Bulgaria, Hungary, Romania, Slovak Republic, Czech Republic), Near East (Turkey, Saudi Arabia), Southern America (Brazil, Chile, Colombia, Peru, Mexico), Northern America (United States, Canada), Eastern Asia (India, China, Hong Kong, Singapore, Japan) Oceania (Australia, New Zealand) and Africa (Republic South Africa).

We thank Eric Schaanning for providing us this correlation matrix.
Figure 1: Decomposition into a low-rank plus a sparse matrix of the correlation matrix of 500 stocks among the S&P500 stocks. The forced rank is set to $k = 3$.

Figure 2: Decomposition into a low-rank plus a sparse matrix of the correlation matrix of the real estate returns. The forced rank is set to $k = 3$. 
3.4 Empirical verification of bounded parameters

To verify empirically our assumption (11) in Theorem 2.1 that the parameters \((\Theta_j)_{j \in \mathbb{N}_0}\) generated by our algorithm \[9\] remain in a compact set, we plotted the running maximum \(\max_{0 \leq j \leq J} \|\Theta_j\|\) as a function of the number of iterations \(J\) for both examples on the S&P500 and the real estate data used in the previous section. For both cases, we observe, as desired, that the running maximum \(\max_{0 \leq j \leq J} \|\Theta_j\|\) converges, which means that at least empirically, \((\Theta_j)_{j \in \mathbb{N}_0}\) remains in a compact set.

4 Proof of convergence

A vast majority of first-order methods for minimizing locally a non-convex function with provable convergence rate are meant to minimize \(L\)-smooth functions, that is, differentiable functions with a Lipschitz continuous gradient. Also, the size of the Lipschitz constant with respect to a suitable norm plays a prominent role in this convergence rate. As
a critical step in the convergence proof for the minimization procedure of the function $\varphi$, we compute explicitly the Lipschitz constant of its gradient.

To calculate the Lipschitz constant of the gradient of the function $\varphi$ we first consider the special case of a single layer neural network. Then, due to the recursive compositional structure of a multi-layer neural network, we can also calculate the Lipschitz constant of the gradient of $\varphi$ in the multi-layer case. To simplify the notation consistently with Section 2, we consider a neural network with a single layer of $\ell$ neurons, each with the same activation function $\sigma : \mathbb{R} \rightarrow [0, 1]$ having the first and the second derivative of $\sigma$ uniformly bounded from above by the constants $\sigma_{\text{max}}^1 \geq 1$ and $\sigma_{\text{max}}^2$, respectively. Moreover, let $A = \{A_{i,j}\}_{i,j} \in \mathbb{R}^{f \times r}$ be the coefficients, $b = \{b_i\} \in \mathbb{R}^f$ be the bias on the input and

$$f_1^{A,b} : \mathbb{R}^r \rightarrow \mathbb{R}^f, \quad w \mapsto f_1^{A,b}(w) = Aw + b.$$ 

The coefficients on the output are denoted by $C = \{C_{i,j}\}_{i,j} \in \mathbb{R}^{nk \times \ell}$ and the bias by $d = \{d_i\} \in \mathbb{R}^n$. As above, we define

$$f_\ell^{2,c,d} : \mathbb{R}^\ell \rightarrow \mathbb{R}^nk \quad v \mapsto f_\ell^{2,c,d}(v) = Cv + d.$$ 

With $\tilde{\sigma} : \mathbb{R}^\ell \rightarrow \mathbb{R}^\ell$, $u \mapsto \tilde{\sigma}(u) = (\sigma(u_1) \cdots \sigma(u_\ell))^T$, the single layer neural network $f^{A,b,C,d} : \mathbb{R}^d \rightarrow \mathbb{R}^n$ is then the composition of these three functions, that is: $f^{A,b,C,d} = f_\ell^{2,c,d} \circ \tilde{\sigma} \circ f_1^{A,b}$. For a given $\Sigma \in \mathbb{S}^n$, our approximated objective function with respect to the above single-layer neural network is defined by

$$\tilde{\varphi}(A,b,C,d) = \sum_{i=1}^n \sum_{j=1}^m \mu \left( g \left( f^{A,b,C,d}(h(\Sigma)) \right) g \left( f^{A,b,C,d}(h(\Sigma)) \right)^T - \Sigma \right)_{i,j}, \quad 1 \leq i, j \leq n$$

where $\mu : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth approximation of the absolute value function with a derivative uniformly bounded by 1 and its second derivative bounded by $\mu''_{\text{max}}$.

To be able to derive the Lipschitz constant of its gradient, we first need to identify the partial derivatives of $\tilde{\varphi}$. For abbreviating some lengthy expressions, we will use the following notation throughout this section, where we set for a fixed given $\Sigma \in \mathbb{S}^n$:

$$\omega_{i,j} := \left[ g \left( f^{A,b,C,d}(h(\Sigma)) \right) g \left( f^{A,b,C,d}(h(\Sigma)) \right)^T - \Sigma \right]_{i,j}, \quad 1 \leq i, j \leq n$$

$$X := f^{A,b,C,d}(h(\Sigma)) \in \mathbb{R}^n$$

$$Y := \tilde{\sigma} \circ f_1^{A,b}(h(\Sigma)) \in \mathbb{R}^\ell$$

$$Z := f_\ell^{2,c,d}(h(\Sigma)) \in \mathbb{R}^\ell.$$ 

**Lemma 4.1.** Let $\tilde{\varphi}$ be the function defined in (15). Then, for every $1 \leq i \leq \ell$ we have that

$$\frac{\partial \tilde{\varphi}(A,b,C,d)}{\partial h_i} = \sum_{\eta=1}^\ell \sum_{\kappa=1}^n \mu' \left( \omega_{i,\eta} \right) \sigma' \left( Z_{\eta,\kappa} \right) C_{i,\eta} X_{\eta,\kappa} + X_{i,\kappa} C_{\eta,\kappa}.$$ 

Moreover, for every $1 \leq i \leq \ell$, $1 \leq \eta \leq \kappa \leq r$ we have that

$$\frac{\partial \tilde{\varphi}(\varphi)}{\partial A_{i,\eta}} = h(\Sigma)_{i,\eta} \frac{\partial \tilde{\varphi}(\varphi)}{\partial h_i}.$$ 

**Proof.** Let $1 \leq i \leq \ell$. Then by definition of $\tilde{\varphi}$, we have that

$$\frac{\partial \tilde{\varphi}(A,b,C,d)}{\partial h_i} = \sum_{\eta=1}^\ell \sum_{\kappa=1}^n \mu' \left( \omega_{i,\eta} \right) \sigma' \left( Z_{\eta,\kappa} \right) C_{i,\eta} X_{\eta,\kappa} + X_{i,\kappa} C_{\eta,\kappa}.$$ 

In particular, we need to evaluate

$$\frac{\partial}{\partial h_i} g \circ f_\ell^{2,c,d} \circ \tilde{\sigma} \circ f_1^{A,b}(h(\Sigma)) = [\nabla_X g(X)] \left[ \nabla_Z \left( f_\ell^{2,c,d} \circ \tilde{\sigma} \right) \right](Z) \frac{\partial f_1^{A,b}(h(\Sigma))}{\partial h_i}.$$ 

To that end, observe that

$$\frac{\partial f_1^{A,b}(h(\Sigma))}{\partial h_i} = \frac{\partial (Ah(\Sigma) + b)}{\partial h_i} = \begin{pmatrix} 1_{s=1} \\ \vdots \\ 1_{s=\ell} \end{pmatrix},$$

$$\left[ \nabla_Z \left( f_\ell^{2,c,d} \circ \tilde{\sigma} \right) \right]_i = \left[ \nabla_Z (C\tilde{\sigma}(Z) + d) \right]_i = \begin{pmatrix} C_{1,\kappa} \sigma'(Z) \\ \vdots \\ C_{n,\kappa} \sigma'(Z) \end{pmatrix},$$

$$\left[ \nabla_X g(X) \right]_i = \begin{pmatrix} g_1(X) \\ \vdots \\ g_n(X) \end{pmatrix}.$$
Moreover, the third order tensor $\nabla_x g(X)$, as a $nk$-dimensional vector of $n$-by-$k$ matrices, has for $(i-1)k+j$-th element the matrix whose only nonzero element is a 1 at position $(i,j)$, namely

$$\left[ \nabla_x g(X) \right]_{(i-1)k+j} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & 1_{(i,j)} & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{n \times k}. \quad (20)$$

Therefore, applying (19) and (20) in (18) implies that

$$\frac{\partial g}{\partial b_i} \left( f^{A,b,C,d}(h(\Sigma)) \right) = \sigma'(Z_i) \begin{pmatrix} C_{1,i} & \cdots & C_{k,i} \\ \vdots & \ddots & \vdots \\ C_{(n-1)k+1,i} & \cdots & C_{nk,i} \end{pmatrix}. \quad (21)$$

This, together with (17), ensures that

$$\frac{\partial \tilde{\varphi}(A,b,C,d)}{\partial b_i} = \sum_{s=1}^{n} \sum_{k=1}^{n} \mu'(\omega_{i,j}) \sigma'(Z_s) \begin{pmatrix} C_{1,s} & \cdots & C_{k,s} \\ \vdots & \ddots & \vdots \\ C_{(n-1)k+1,s} & \cdots & C_{nk,s} \end{pmatrix} + \begin{pmatrix} X_1 & \cdots & X_k \\ \vdots & \ddots & \vdots \\ X_{(n-1)k+1} & \cdots & X_{nk} \end{pmatrix} \begin{pmatrix} C_{1,i} & \cdots & C_{(n-1)k+1,i} \\ \vdots & \ddots & \vdots \\ C_{nk,i} & \cdots & C_{(n-1)k+1,i} \end{pmatrix} \right]_{i,j}$$

which proves the first part. For the second part, observe that for every $1 \leq i \leq \ell$, $1 \leq \eta \leq r$ we have that

$$\frac{\partial f^{A,b}(h(\Sigma))}{\partial A_{i,\eta}} = \frac{\partial (Ah(\Sigma) + b)}{\partial A_{i,\eta}} = \begin{pmatrix} h(\Sigma)_{i,1} \\ \vdots \\ h(\Sigma)_{i,\ell} \end{pmatrix} = h(\Sigma)_{i,\ell} \frac{\partial f^{A,b}(h(\Sigma))}{\partial b_i}. \quad (22)$$

Therefore, using the same identities derived in (17) and (18) shows that indeed

$$\frac{\partial \tilde{\varphi}(A,b,C,d)}{\partial A_{i,\eta}} = h(\Sigma)_{i,\ell} \frac{\partial \tilde{\varphi}(A,b,C,d)}{\partial b_i}. \quad (23)$$

**Lemma 4.2.** Let $\tilde{\varphi}$ be the function defined in (15). Then, for every $\nu := (\alpha-1)k + \beta$ with $1 \leq \alpha \leq n$ and $1 \leq \beta \leq k$ we have that

$$\frac{\partial \tilde{\varphi}(A,b,C,d)}{\partial d_{\nu}} = 2 \sum_{j=1}^{n} \mu'(\omega_{i,j}) X_{(j-1)k+\beta}. \quad (24)$$

Moreover, for every $1 \leq i \leq \ell$, $\nu := (\alpha-1)k + \beta$ with $1 \leq \alpha \leq n$ and $1 \leq \beta \leq k$ we have that

$$\frac{\partial \tilde{\varphi}(A,b,C,d)}{\partial C_{i,\nu}} = Y_i \frac{\partial \tilde{\varphi}(A,b,C,d)}{\partial d_{\nu}}. \quad (25)$$

**Proof.** First, notice that

$$\frac{\partial \tilde{\varphi}(A,b,C,d)}{\partial d_{\nu}} = \sum_{i=1}^{n} \sum_{j=1}^{n} \mu'(\omega_{i,j}) \left[ \frac{\partial g(f^{C,d}(Y))}{\partial d_{\nu}} (g(X))^T + (g(X)) \left( \frac{\partial g(f^{C,d}(Y))^T)}{\partial d_{\nu}} \right) \right]_{i,j}. \quad (26)$$

Moreover, the same calculation as in (19) and (20), using that $\nu := (\alpha-1)k + \beta$, ensures that

$$\frac{\partial g}{\partial d_{\nu}} = \left[ \nabla_x g(X) \right] \frac{\partial (CY + d)}{\partial d_{\nu}} = \left[ \nabla_x g(X) \right] \begin{pmatrix} 1_{\nu=1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1_{(\nu=m_k)} \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{n \times k}. \quad (27)$$
This, the definition of the function $g$, and \[22\] ensure that

$$
\frac{\partial \tilde{\varphi}(A, b, C, d)}{\partial d_{\nu}} = \begin{pmatrix}
\frac{\partial \tilde{\varphi}(A, b, C, d)}{\partial d_{\nu}} = \sum_{i=1}^{n} \sum_{j=1}^{n} \mu'(\omega_{i,j}) \begin{pmatrix}
0 \\
\vdots \\
1_{(i,j)} \\
0 \\
\vdots \\
X_k \\
\vdots \\
X_{(n-1)k+1} \\
X_k \\
\vdots \\
0 \\
\vdots \\
0
\end{pmatrix} + \begin{pmatrix}
X_1 \\
\vdots \\
X_k \\
\vdots \\
X_{(n-1)k+1} \\
X_k \\
\vdots \\
0 \\
\vdots \\
0
\end{pmatrix}
\end{pmatrix}_{i,j}
$$

Therefore, using that $\omega_{i,j} = \omega_{k,i}$ for every $1 \leq i, j \leq n$ we obtain indeed that

$$
\frac{\partial \tilde{\varphi}(A, b, C, d)}{\partial d_{\nu}} = 2\mu'(\omega_{1,\alpha})X_{(n-1)k+1} + \sum_{j=1}^{n} \mu'(\omega_{1,j})X_{j-1} + \sum_{i=1}^{n} \mu'(\omega_{1,i})X_{i-1}
$$

which proves the first part. For the second part, observe that for $1 \leq i \leq \ell$ and $\nu = (\alpha - 1)k + \beta$ as above, we can use \[22\] and the calculation as in \[22\] to see that

$$
\frac{\partial g\left(f_{C,d}^{X,Y}(Y)\right)}{\partial C_{\nu,i}} = \left[\nabla_X g(X)\right]\frac{\partial (CY + d)}{\partial C_{\nu,i}} = \left[\nabla_X g(X)Y\right] \begin{pmatrix}
1_{\nu=1} \\
\vdots \\
1_{\nu=nk}
\end{pmatrix} = Y_i \frac{\partial g\left(f_{C,d}^{X,Y}(Y)\right)}{\partial d_{\nu}}.
$$

Using the same identity as in \[22\], we therefore conclude that indeed

$$
\frac{\partial \tilde{\varphi}(\vartheta)}{\partial C_{\nu,i}} = Y_i \frac{\partial \tilde{\varphi}(\vartheta)}{\partial d_{\nu}}.
$$

Having derived the partial derivatives of the objective function $\tilde{\varphi}$, we are now able to start deriving the Lipschitz constant of its gradient. One of the key tools in its derivation is the following lemma, which shows how to infer the Lipschitz constant of some functions from the Lipschitz constant of other, simpler Lipschitz-continuous functions.

**Lemma 4.3.** Let $\phi_1 : \mathbb{R}^m \to \mathbb{R}$ be Lipschitz continuous with Lipschitz constant $L_1$. Let $\phi_2 : \mathbb{R}^n \to \mathbb{R}^m$ be a function for which there exists a function $L_2 : \mathbb{R}^n \times \mathbb{R}^n \to (0, \infty)$ such that for all $x, y \in \mathbb{R}^n$ we have

$$
\|\phi_2(x) - \phi_2(y)\| \leq L_2(x, y)\|x - y\|.
$$

Assume that $\phi_1 \circ \phi_2$ is bounded by a constant $B_{12}$. Finally, let $\phi_3 : \mathbb{R}^n \to \mathbb{R}$ be a function for which

1. $|\phi_3(y)| \leq B_3\|y\|$ for all $y \in \mathbb{R}^n$ for some positive function $B_3 : \mathbb{R}^n \to (0, \infty)$;
2. there exist three functions $L_{3,1} : \mathbb{R}^n \times \mathbb{R}^n \to (0, \infty)$, $L_{3,2} : \mathbb{R}^n \times \mathbb{R}^n \to (0, \infty)$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ such that for all $x, y \in \mathbb{R}^n$ we have

$$
\|\phi_3(x) - \phi_3(y)\| \leq L_{3,1}(x, y)\|x - y\| + L_{3,2}(x, y)\|f(x) - f(y)\|.
$$

Then, the function $\Phi := (\phi_3 \circ \phi_2)\phi_3$ satisfies for all $x, y \in \mathbb{R}^n$ that

$$
|\Phi(x) - \Phi(y)| \leq B_{12}(L_{3,1}(x, y)\|x - y\| + L_{3,2}(x, y)\|f(x) - f(y)\|) + B_3(y)L_1L_2(x, y)\|x - y\|.
$$

**Proof.** For all $x, y \in \mathbb{R}^n$, we can write

$$
|\Phi(x) - \Phi(y)| = |\phi_1(\phi_3(x))\phi_3(x) - \phi_1(\phi_3(y))\phi_3(y)|
\leq |\phi_1(\phi_3(x)) - \phi_1(\phi_3(y))| + |\phi_1(\phi_3(x)) - \phi_1(\phi_3(y))|\phi_3(y)|
\leq B_{12}(\|\phi_3(x) - \phi_3(y)\| + B_3\|\phi_3(x) - \phi_3(y)\|) + B_3(y)L_1𝐿_2(\|\phi_3(x) - \phi_3(y)\|)
\leq B_{12}(L_{3,1}(x, y)\|x - y\| + L_{3,2}(x, y)\|f(x) - f(y)\|) + B_3(y)L_1L_2(x, y)\|x - y\|.\]

\[\square\]
Recall the norm which for \( \gamma \in \mathbb{N}, m_1, \ldots, m_\gamma, n_1, \ldots, n_\gamma \in \mathbb{N}_0 \), and \((X^1, \ldots, X^\gamma) \in \mathbb{R}^{m_1 \times n_1} \times \cdots \times \mathbb{R}^{m_\gamma \times n_\gamma}\), is defined by
\[
\|(X^1, \ldots, X^\gamma)\| := \left( \sum_{i=1}^{m_1} \sum_{j=1}^{n_1} (X^1_{ij})^2 + \cdots + \sum_{i=1}^{m_\gamma} \sum_{j=1}^{n_\gamma} (X^\gamma_{ij})^2 \right)^{1/2},
\]
and hence for \( \gamma = 1 \) coincides with the Frobenius norm, denoted here indifferently as \( \|\cdot\|_F \) or \( \|\cdot\|_2 \). For symmetric matrices \(X\) we also use a dedicated norm defined as \( \|X\|_S := \|h(X)\| \), where \( h : S^n \to \mathbb{R}^{n(n+1)/2} \) is the function defined in \[16\].

Note also that \( 2\|X\|_S^2 = \|X\|_F^2 + \|\text{diag}(X)\|_F^2 \). To obtain the Lipschitz constant of the gradient of \( \tilde{\varphi} \) we start with the following lemma.

**Lemma 4.4.** Let \( \varphi \) be the function defined in \[15\] with respect to a given \( \Sigma \in S^r \) and let \( D > 0 \) be any constant. Then for any \( 1 \leq i \leq \ell \) the function
\[
D := \{(A, b, C, d) \in \mathbb{R}^{r \times r} \times \mathbb{R}^{r \times n \times t} \times \mathbb{R}^{nk} : \|(A, b, C, d)\| \leq D\} \ni (A, b, C, d) \mapsto \frac{\partial \tilde{\varphi}(A, b, C, d)}{\partial b_i} \in \mathbb{R}
\]
is Lipschitz continuous. Moreover, there is a constant \( L_b > 0 \) satisfying for \( L_Z := \sqrt{1 + \|\Sigma\|^2} \) that
\[
L^2_b = O(C_b(\sigma'_{\max}, \sigma''_{\max}; \mu'_{\max}) n^2 D^2 \max\{D^2L^2_Z, \ell^2D^2L^2_Z, \ell^4D^4, n\ell^4\},
\]
where \( C_b(\sigma'_{\max}, \sigma''_{\max}, \mu'_{\max}) \) is a constant that only depends polynomially on \( \sigma'_{\max}, \sigma''_{\max}, \mu'_{\max} \), such that for all \((A, b, C, d), (\bar{A}, \bar{b}, \bar{C}, \bar{d}) \in D\) we have that
\[
\sum_{i=1}^\ell \left( \frac{\partial \tilde{\varphi}(A, b, C, d)}{\partial b_i} - \frac{\partial \tilde{\varphi}(\bar{A}, \bar{b}, \bar{C}, \bar{d})}{\partial b_i} \right)^2 \leq L^2_b \|(A, b, C, d) - (\bar{A}, \bar{b}, \bar{C}, \bar{d})\|^2.
\]

**Proof.** We divide the proof into several steps.

1. Let \( 1 \leq i \leq \ell \). The function \((A, b) \mapsto [f^A_{i,b}(h(\Sigma))]_i\) is Lipschitz continuous with constant \( L_Z := \sqrt{1 + \|\Sigma\|^2} \).

   Indeed, for any \((A, \bar{b}), (A, b) \in \mathbb{R}^{r \times r} \times \mathbb{R}^{r \times r}\) we have by the Cauchy-Schwarz inequality that
   \[
   \left( \left[ f^A_{i,b}(h(\Sigma)) \right]_i - \left[ f^A_{i,\bar{b}}(h(\Sigma)) \right]_i \right)^2 = \left( (A_{ii} - \bar{A}_{ii})h(\Sigma) + b_i - \bar{b}_i \right)^2 \leq \left\| A_{ii} - \bar{A}_{ii} \right\|^2 \leq \frac{1}{b_i - \bar{b}_i} \left\| h(\Sigma) \right\| \leq \frac{L^2_Z}{b_i - \bar{b}_i} \left\| h(\Sigma) \right\|^2.
   \]

2. The function \((A, b) \mapsto f^A_{i,b}(h(\Sigma))\) is Lipschitz continuous with Lipschitz constant \( \sigma'_{\max} L_Z \). Moreover, the map \((A, b) \mapsto \| \tilde{\sigma} \circ f^A_{i,b}(h(\Sigma)) \|\) is bounded by \( \sqrt{\ell} \).

   Indeed, using the notation of Lemma 4.3 for the functions \( \phi_1 := \sigma, \phi_2 := [f^A_{i,b}(h(\Sigma))]_i\) and \( \phi_3 := 1\), we have that \( L_{A,1} = L_{A,2} = 0, L_{\Sigma} = 1, L_{1} = \sigma'_{\max}, \) and \( L_{2} = L_{Z} \). Therefore, we directly obtain from Lemma 4.3 the Lipschitz continuity of the analyzed function with Lipschitz constant \( \sigma'_{\max} L_Z \). Moreover, the bound on \((A, b) \mapsto \| \tilde{\sigma} \circ f^A_{i,b}(h(\Sigma)) \|\) follows directly by definition of \( \tilde{\sigma} \), as \( \sigma' \) is bounded by \( 1 \).

3. By the same argument as in Step 2, we see that the function \((A, b) \mapsto \left[ \tilde{\sigma} \circ f^A_{i,b}(h(\Sigma)) \right]_i\) is Lipschitz continuous with Lipschitz constant \( L_{\sigma' \circ \tilde{\sigma}, i} := \sigma''_{\max} L_Z \) and is bounded by \( \sigma''_{\max} \).

4. Let \( 1 \leq \nu \leq nk \). The function \((A, b, C, d) \mapsto [f^C_{i,d}(\tilde{\sigma} \circ f^A_{i,b}(h(\Sigma)))]_{\nu i}\) satisfies for all \((A, b, C, d), (\bar{A}, \bar{b}, \bar{C}, \bar{d}) \) that
\[
\left[ f^C_{j,d}(\tilde{\sigma} \circ f^A_{i,b}(h(\Sigma))) \right]_j - \left[ f^C_{j,d}(\tilde{\sigma} \circ f^A_{i,\bar{b}}(h(\Sigma))) \right]_j \leq \left( \left| C_{\nu, i} \right| \left( \sigma''_{\max} \right)^2 L^2_Z + \ell + 1 \right) \left\| (A_{ii} - \bar{A}_{ii})h(\Sigma) + b_i - \bar{b}_i \right\| \leq \left( \left| C_{\nu, i} \right| \left( \sigma''_{\max} \right)^2 L^2_Z + \ell + 1 \right) \left\| h(\Sigma) \right\| \leq \left( \left| C_{\nu, i} \right| \left( \sigma''_{\max} \right)^2 L^2_Z + \ell + 1 \right) \left\| h(\Sigma) \right\|^2
\]
and that
\[
\left[ f^C_{j,d}(\tilde{\sigma} \circ f^A_{i,b}(h(\Sigma))) \right]_j \leq \left\| C_{\nu, i} \right\| \left\| h(\Sigma) \right\| \leq \sqrt{\ell + 1} \sqrt{\left\| C_{\nu, i} \right\|^2 + \left\| d_\nu - d_{\bar{\nu}} \right\|^2}.
\]

Indeed, to that end, consider first the function \((C, d, Y) \mapsto \phi_1(C_{\nu, i}, d_\nu, Y) := C_{\nu, i} Y + d_\nu \). Notice that
\[
\left[ \phi_1 (C, d, Y) - \phi_1 (\bar{C}, \bar{d}, \bar{Y}) \right] \leq \left( \sum_{i=1}^\ell \left| C_{\nu, i} \right| \left| Y_i - \bar{Y}_i \right| + \left\| C_{\nu, i} \right\| \left\| Y \right\| + \left| d_\nu - d_{\bar{\nu}} \right| \right)^2.
\]
Therefore, by choosing $Y := \tilde{\sigma} \circ f_{1}^{A,h}(h(\Sigma))$ and $\tilde{Y} := \tilde{\sigma} \circ f_{1}^{A,h}(h(\Sigma))$, using the constants established in Step 2, and the Cauchy-Schwarz inequality, we get as desired that
\[
\left| \left| f_{2}^{C,d} \circ \tilde{\sigma} \circ f_{1}^{A,h}(h(\Sigma)) \right|^{} - \left| f_{2}^{C,d} \circ \tilde{\sigma} \circ f_{1}^{A,h}(h(\Sigma)) \right|^{} \right|^2 \\
= \left( \phi \left( C, d, Y \right) - \phi \left( \tilde{C}, \tilde{d}, Y \right) \right)^2 \\
\leq \left( \sum_{i=1}^{\ell} C_{i} \nu \left| \sigma_{\text{max}} \right|^L \left( f_{i} \right) \right) \cdot \nu \left( A_{i} - \tilde{A}_{i} \right) \left\| \nu \right\| \left\| Y \right\| + \left| d_{\nu} - \tilde{d}_{\nu} \right|^2 \\
\leq \left( \sum_{i=1}^{\ell} C_{i} \nu \left| \sigma_{\text{max}} \right|^L \left( f_{i} \right) \right) \cdot \left\| A - \tilde{A} \right\|^2 + \left\| b - \tilde{b} \right\|^2 + \left| C_{\nu} - \tilde{C}_{\nu} \right|^2 + \left| d_{\nu} - \tilde{d}_{\nu} \right|^2 \\
\leq \left( \|C_{\nu}\|^2 \left( \sigma_{\text{max}} \right)^L \left( f_{i} \right) \right) \cdot \left\| (A, b, C_{\nu}, d_{\nu}) - (\tilde{A}, \tilde{b}, \tilde{C}_{\nu}, \tilde{d}_{\nu}) \right\|^2. \\
\tag{26}
\]

In addition, using the bound on the function analyzed in Step 2, we see that
\[
\left| f_{2}^{C,d} \circ \tilde{\sigma} \circ f_{1}^{A,h}(h(\Sigma)) \right|^{} \leq \left| C_{\nu} \right|^L \left\| Y \right\| + \left| d_{\nu} \right| \leq \sqrt{\ell + 1} \cdot \sqrt{\left| C_{\nu} \right|^2 + \left| d_{\nu} \right|^2}. \\
\]

5. Let $1 \leq i, j \leq n$. For the function $(A, b, C, d) \mapsto \left[ g \left( f_{A,b,C,d}^{A,h}(h(\Sigma)) \right) \cdot g \left( f_{A,b,C,d}^{A,h}(h(\Sigma)) \right)^T - \Sigma \right]_{i,j} \equiv \omega_{i,j}$, there exists a function $L_{\nu,i,j} : \mathbb{R}^{nk} \times \mathbb{R}^{nk} \times \mathbb{R}^{nk} \times \mathbb{R}^{nk} \rightarrow (0, \infty)$ satisfying
\[
\left| \omega_{i,j} - \tilde{\omega}_{i,j} \right|^2 \leq L_{\nu,i,j}(C, d, \tilde{C}, \tilde{d}) \left\| \theta - \tilde{\theta} \right\|^2 \\
\]
for all $\theta \equiv (A, b, C, d)$, $\tilde{\theta} \equiv (\tilde{A}, \tilde{b}, \tilde{C}, \tilde{d})$.

Indeed, by the definition of the notion $X$ introduced in [10] and the Cauchy-Schwarz inequality we have that
\[
\left| \omega_{i,j} - \tilde{\omega}_{i,j} \right|^2 \\
\leq \left( \sum_{k=1}^{k} \left| X_{(i-1)k+s} \cdot |X_{(j-1)k+s} - \tilde{X}_{(j-1)k+s}| + |X_{(j-1)k+s} - \tilde{X}_{(i-1)k+s}| \right|^2 \right) \\
\leq 2 \left( \sum_{k=1}^{k} \left| X_{(i-1)k+s} \right|^2 \right) \cdot \left( \sum_{k=1}^{k} \left| X_{(j-1)k+s} - \tilde{X}_{(j-1)k+s} \right|^2 \right) + 2 \left( \sum_{k=1}^{k} \left| \tilde{X}_{(j-1)k+s} \right|^2 \right) \cdot \left( \sum_{k=1}^{k} \left| X_{(i-1)k+s} - \tilde{X}_{(i-1)k+s} \right|^2 \right) \\
\leq 2 \left( \sum_{k=1}^{k} \left( \ell + 1 \right) \left( \left| C_{(i-1)k+s,} \right|^2 + \left| d_{(i-1)k+s} \right|^2 \right) \right) \cdot \left( \sum_{k=1}^{k} \left( \left| C_{(j-1)k+s,} \right|^2 \left( \sigma_{\text{max}} \right)^2 \left( f_{i} \right) \right) \cdot \left( \ell + 1 \right) \left( \left| \theta - \tilde{\theta} \right|_{(j-1)k+s} \right)^2 \right) \\
\]
and
\[
\left| \omega_{i,j} - \tilde{\omega}_{i,j} \right|^2 \leq 2 \left( \sum_{k=1}^{k} \left| C_{(j-1)k+s,} \right|^2 \left( \sigma_{\text{max}} \right)^2 \left( f_{i} \right) \right) \cdot \left( \ell + 1 \right) \left( \left| \theta - \tilde{\theta} \right|_{(j-1)k+s} \right)^2. \\
\tag{27}
\]

Moreover, observe that by definition of the function $g$ we have that $\sum_{s=1}^{k} \left| d_{(j-1)k+s} \right|^2 = \left| g_{(j, x)} \right|^2$ as well as
\[
\sum_{s=1}^{k} \left| C_{(j-1)k+s,} \right|^2 = \sum_{s=1}^{k} \sum_{s=1}^{s} \left| C_{(j-1)k+s,} \right|^2 = \sum_{s=1}^{s} \sum_{s=1}^{k} \left| C_{(j-1)k+s,} \right|^2 = \sum_{s=1}^{s} \left| g_{(j, x)} \right|^2. \\
\tag{28}
\]

In addition, we have that
\[
\sum_{s=1}^{k} \left( \left| C_{(j-1)k+s,} \right|^2 \left( \sigma_{\text{max}} \right)^2 \left( f_{i} \right) \right) \cdot \left( \ell + 1 \right) \left( \left| \theta - \tilde{\theta} \right|_{(j-1)k+s} \right)^2 \\
= \sum_{s=1}^{k} \left( \left| C_{(j-1)k+s,} \right|^2 \left( \sigma_{\text{max}} \right)^2 \left( f_{i} \right) \right) \cdot \left( \ell + 1 \right) \sum_{s=1}^{k} \left( \left| \theta - \tilde{\theta} \right|_{(j-1)k+s} \right)^2 \\
\leq \sum_{s=1}^{k} \left( \left| C_{(j-1)k+s,} \right|^2 \left( \sigma_{\text{max}} \right)^2 \left( f_{i} \right) \right) \cdot \left( \ell + 1 \right) \left| \theta - \tilde{\theta} \right|^2. \\
\]
Plugging this and (28) into (27) hence implies that
\[ |\omega_{i,j} - \hat{\omega}_{i,j}|^2 \leq 2(\ell + 1) \left( \sum_{i=1}^{\ell} ||g(C_{i,j})||^2 + ||g(d_{j})||^2 \right) \cdot \left( \sum_{i=1}^{\ell} ||g(C_{i,j})||^2 \left( \sigma'_{\max} \right)^2 L_Z^2 + \ell + 1 \right) ||\vartheta - \vartheta'||^2 + 2(\ell + 1) \left( \sum_{i=1}^{\ell} ||g(C_{i,j})||^2 + ||g(d_{j})||^2 \right) \cdot \left( \sum_{i=1}^{\ell} ||g(C_{i,j})||^2 \left( \sigma'_{\max} \right)^2 L_Z^2 + \ell + 1 \right) ||\vartheta - \vartheta'||^2 \] (29)

6. For any 1 ≤ i, j ≤ n and 1 ≤ ℓ ≤ ℓ, consider the function
\[ (A,b,C,d) \mapsto \mu' \left( g \left( f^{A,b,C,d}_h(h(\Sigma)) \right) g \left( f^{A,b,C,d}_h(h(\Sigma)) \right)^T - \Sigma \right) \sigma'_{\max} \]
and let \( L_{\mu'\sigma',i,j} : R^{nk \times 1} \to R^n \times R^{nk \times 1} \to R^{nk} \to (0,\infty) \) be defined by
\[ L_{\mu'\sigma',i,j}(C,d,\bar{C},\bar{d}) = \sigma'_{\max} L_Z + \sigma'_{\max} L_{\omega,\lambda,i,j}(C,d,\bar{C},\bar{d}), \]
where \( L_{\omega,\lambda,i,j}(C,d,\bar{C},\bar{d}) \) is the function defined in Step 5. Then for every \( (A,b,C,d), (A',\bar{b},\bar{C},\bar{d}) \) we have that
\[ |\mu'(\omega_{i,j})\sigma'(Z) - \mu'(\hat{\omega}_{i,j})\sigma'(\bar{Z})| \leq L_{\mu'\sigma',i,j}(C,d,\bar{C},\bar{d})(A,b,C,d) - (A',\bar{b},\bar{C},\bar{d})| \]
and \( \mu'(\omega_{i,j})\sigma'(Z) \) is bounded by \( B_{\mu'\sigma',i,j} \equiv B_{\mu'\sigma} \equiv \sigma'_{\max}\).
Indeed, while the boundedness follows directly by the definition of the function, we can for the regularity property apply Lemma 4.3 using \( \phi_1 \equiv \mu' \), \( \phi_2 \equiv \omega_{i,j} \), and \( \phi_3 \equiv \sigma'(Y) \) using the notation of Lemma 4.3 and 10, for which we have that \( L_1 = \sigma'_{\max}, B_1 = 1 \) as \( \mu_{\max} \leq 1, L_2 = L_{\omega,\lambda,i,j}(C,d,\bar{C},\bar{d}), L_3,1 = \sigma'_{\max} L_Z \) by Step 3, \( L_{3,2} = 0 \), and \( B_3 = \sigma'_{\max}\).

7. For every 1 ≤ i, j ≤ n and 1 ≤ ℓ ≤ ℓ, the function
\[ (A,b,C,d) \mapsto \vartheta \mapsto V_{i,j,\vartheta}(\vartheta) := \sum_{s=1}^{k} C_{(i-1)k+s,i} f^{A,b,C,d}_h(h(\Sigma)) \]

satisfies for any \( \vartheta \equiv (A,b,C,d), \bar{\vartheta} \equiv (\bar{A},\bar{b},\bar{C},\bar{d}) \) that
\[ |V_{i,j,\vartheta}(\vartheta) - V_{i,j,\bar{\vartheta}}(\bar{\vartheta})|^2 \leq L_{C_{i,j,i},\vartheta}(C)||\vartheta - \bar{\vartheta}||^2 + L_{\bar{C}_{i,j,i},\vartheta}(\bar{C},\bar{d})||g(C_{i,j}) - g(\bar{C}_{i,j})||^2, \]
where
\[ L_{\bar{C}_{i,j,i},\vartheta}(\bar{C},\bar{d}) := 2(\ell + 1) \left( \sum_{j=1}^{\ell} ||g(C_{i,j})||^2 \left( \sigma'_{\max} \right)^2 L_Z^2 + \ell + 1 \right), \]
\[ L_{\bar{C}_{i,j,i},\vartheta}(\bar{C},\bar{d}) := 2(\ell + 1) \left( \sum_{j=1}^{\ell} ||g(C_{i,j})||^2 + ||g(d_{j})||^2 \right), \]
and is bounded by
\[ B_{C_{i,j,i},\vartheta}(C,d) = ||g(C_{i,j})|| \sqrt{\ell + 1} \sqrt{\sum_{j=1}^{\ell} ||g(C_{i,j})||^2 + ||g(d_{j})||^2}. \]
Indeed, by the Cauchy-Schwarz inequality, we have that
\[ |V_{i,j,\vartheta}(\vartheta) - V_{i,j,\bar{\vartheta}}(\bar{\vartheta})|^2 \leq \left( \sum_{s=1}^{k} C_{(i-1)k+s,i} \cdot |X_{(j-1)k+s} - \bar{X}_{(j-1)k+s}| + |X_{(j-1)k+s}| \cdot |C_{(i-1)k+s,i} - \bar{C}_{(i-1)k+s,i}| \right)^2 \]
\[ \leq 2 \left( \sum_{s=1}^{k} C_{(i-1)k+s,i}^2 \cdot \left( \sum_{s=1}^{k} |X_{(j-1)k+s} - \bar{X}_{(j-1)k+s}|^2 \right)^{1/2} \right. \]
\[ + 2 \left( \sum_{s=1}^{k} |X_{(j-1)k+s}|^2 \cdot \left( \sum_{s=1}^{k} |C_{(i-1)k+s,i} - \bar{C}_{(i-1)k+s,i}|^2 \right)^{1/2} \right). \]
Therefore, using (25) and the same argument as for (29) implies that

$$\left| V_{i,j,s} (\bar{\vartheta}) - V_{i,j,s} (\bar{\vartheta}) \right|^2 \leq 2 \left| g(C_{\omega, \mu, \lambda}) \right|^2 \left( \sum_{s=1}^{\ell} \left| g(C_{\omega, \mu, \lambda}) \right|^2 (\sigma'_{\text{max}})^2 L^2 + (\ell + 1) \right) \left| \bar{\vartheta} - \bar{\vartheta} \right|^2 + 2(\ell + 1) \left( \sum_{s=1}^{\ell} \left| g(C_{\omega, \mu, \lambda}) \right|^2 + \left| g(\bar{d}) \right|^2 \right) \left| g(C_{\omega, \mu, \lambda}) \right| - g(C_{\omega, \mu, \lambda})^2 \right|^2$$

$$= L^2_{\text{Cauchy}} (C_{\omega, \mu, \lambda}) \left| \vartheta - \bar{\vartheta} \right|^2 + L^2_{\text{Xac}, C_{\omega, \mu, \lambda}} (\bar{C}, \bar{d}) \left| g(C_{\omega, \mu, \lambda}) \right| - g(C_{\omega, \mu, \lambda})^2 \right|^2.$$

Moreover, to calculate an upper bound, we apply the Cauchy-Schwarz inequality, the bound obtained in (25), and (28) to see that indeed

$$\left| V_{i,j,s} (\bar{\vartheta}) \right| \leq \sqrt{\sum_{s=1}^{k} \left| C_{\omega, \mu, \lambda} \right|^2 \left( \sum_{s=1}^{\ell} \left| g(C_{\omega, \mu, \lambda}) \right|^2 + \left| g(d) \right|^2 \right)^2 \left( \sum_{s=1}^{\ell} \left| g(C_{\omega, \mu, \lambda}) \right|^2 + \left| g(d) \right|^2 \right)^2 }.$$

8. Finally, we are able to prove the desired result. To that end, recall from Lemma 4.1 that for every $1 \leq \iota \leq \ell$ we have that

$$\frac{\partial \ddot{\varphi} (A, b, C, d)}{\partial b_{\iota}} = \sum_{s=1}^{n} \sum_{s=1}^{n} \mu' (\omega_{s}) \sigma' (Z_{s}) \left( \sum_{s=1}^{k} \left( C_{\omega, \mu, \lambda} \right) \right) \left( C_{\omega, \mu, \lambda} \right) \left( \sum_{s=1}^{\ell} \left| g(C_{\omega, \mu, \lambda}) \right|^2 + \left| g(d) \right|^2 \right)^2 \left( \sum_{s=1}^{\ell} \left| g(C_{\omega, \mu, \lambda}) \right|^2 + \left| g(d) \right|^2 \right)^2 }.$$

Therefore, for all $1 \leq \iota \leq \ell$, using the regularity property derived in Step 6 and Step 7, we can apply Lemma 4.3 with $\phi_1 \equiv \phi_{b, \omega, \mu, \lambda}$, $\phi_2 \equiv \sum_{s=1}^{n} \sum_{s=1}^{n} \mu' (\omega_{s}) \sigma' (Z_{s})$, and $\phi_3 \equiv \sum_{s=1}^{k} \left( C_{\omega, \mu, \lambda} \right) \left( C_{\omega, \mu, \lambda} \right)$, for which we have $B_{12} = B_{\lambda, \omega, \mu, \lambda}$, $L_1 = 1$, $L_2 = L_{\text{Xac}, C_{\omega, \mu, \lambda}} (C, d, \lambda)$, $B_3 = B_{\text{Xac}, C_{\omega, \mu, \lambda}} (C, d) + B_{\text{Xac}, C_{\omega, \mu, \lambda}} (C, d)$, $L_{3,1} = L_{\text{Cauchy}, C_{\omega, \mu, \lambda}} (C)$, and $L_{3,2} = 0$ to obtain the following regularity for the function

$$\left( A, b, C, d \right) \rightarrow \left( \mu' (\omega_{s}) \sigma' (Z_{s}) \right) \left( \sum_{s=1}^{k} \left( C_{\omega, \mu, \lambda} \right) \right) \left( C_{\omega, \mu, \lambda} \right) \left( \sum_{s=1}^{\ell} \left| g(C_{\omega, \mu, \lambda}) \right|^2 + \left| g(d) \right|^2 \right)^2 \left( \sum_{s=1}^{\ell} \left| g(C_{\omega, \mu, \lambda}) \right|^2 + \left| g(d) \right|^2 \right)^2 }.$$

This and (30) hence ensures that

$$\sum_{s=1}^{\ell} \left( \frac{\partial \ddot{\varphi} (\vartheta)}{\partial b_{\iota}} - \frac{\partial \ddot{\varphi} (\bar{\vartheta})}{\partial b_{\iota}} \right)^2 \leq 6 n^2 \sum_{s=1}^{\ell} \left( \sum_{s=1}^{k} \left( C_{\omega, \mu, \lambda} \right) \right) \left( \sum_{s=1}^{\ell} \left| g(C_{\omega, \mu, \lambda}) \right|^2 + \left| g(d) \right|^2 \right)^2 \left( \sum_{s=1}^{\ell} \left| g(C_{\omega, \mu, \lambda}) \right|^2 + \left| g(d) \right|^2 \right)^2 }.$$

This and (30) hence ensures that

$$\sum_{s=1}^{\ell} \left( \frac{\partial \ddot{\varphi} (\vartheta)}{\partial b_{\iota}} - \frac{\partial \ddot{\varphi} (\bar{\vartheta})}{\partial b_{\iota}} \right)^2 \leq 6 n^2 \sum_{s=1}^{\ell} \left( \sum_{s=1}^{k} \left( C_{\omega, \mu, \lambda} \right) \right) \left( \sum_{s=1}^{\ell} \left| g(C_{\omega, \mu, \lambda}) \right|^2 + \left| g(d) \right|^2 \right)^2 \left( \sum_{s=1}^{\ell} \left| g(C_{\omega, \mu, \lambda}) \right|^2 + \left| g(d) \right|^2 \right)^2 }.$$
Using their definition, the first term can be estimated using the following bounds
\[
\sum_{i,j=1}^{\ell} \left( \sum_{i=1}^{n} L^2_{\mathcal{C}dX_{i,j},(C)} \right) \leq 2(\ell + 1) \left( \sigma_{\max}'' L^2_{\mathcal{Z}} \right)^2 \sum_{i=1}^{\ell} \sum_{i,j=1}^{n} \left| |g(C_{i,j})_i'||2 + |g(d)_j'||2 \right|
+ 2(\ell + 1) \left( \sigma_{\max}'' \mu_{\max}'' \right)^2 \sum_{i=1}^{\ell} \sum_{i,j=1}^{n} L^2_{\mathcal{C}dX_{i,j},(C,d,C,\bar{d})} \left| |g(C_{i,j})_i'||2 + |g(d)_j'||2 \right|
\leq 2(\ell + 1) \left( \sigma_{\max}'' L^2_{\mathcal{Z}} \right)^2 \left| |C||2 + |d||2 \right|^2 \left( \left| |C||2 + |d||2 \right|^2 \right) \left( \left| |C||2 + \sigma_{\max}'' L^2_{\mathcal{Z}} + \ell + 1 \right) \right.
\]

By symmetry, the same estimate holds true for the second term. To estimate the third and fourth term, notice that
\[
\left( \sum_{i=1}^{\ell} \sum_{i,j=1}^{n} L^2_{\mathcal{C}dX_{i,j},(C)} \right) = 2\sum_{i=1}^{\ell} \sum_{i,j=1}^{n} \left| |g(C_{i,j})_i'||2 + \left| |g(d)_j'||2 \right|^2 \left( \left| |g(C_{i,j})_i'||2 + \left| |g(d)_j'||2 \right|^2 \right) \left( \left| |g(C_{i,j})_i'||2 + \left| |g(d)_j'||2 \right|^2 \right) \left( \left| |C||2 + \sigma_{\max}'' L^2_{\mathcal{Z}} + n(\ell + 1) \right) \right.
\]

For the last two terms, note that
\[
\sum_{i=1}^{\ell} \sum_{i,j=1}^{n} L^2_{\mathcal{C}dX_{i,j},(C,\bar{d})} \left| |g(C_{i,j})_i'||2 - \left| |g(\bar{C}_{i,j})_i'||2 \right|^2 \left( \left| |g(C_{i,j})_i'||2 + \left| |g(\bar{C}_{i,j})_i'||2 \right|^2 \right) \left( \left| |C||2 + \sigma_{\max}'' L^2_{\mathcal{Z}} + n(\ell + 1) \right) \right.
\]

Therefore, using all the estimates for the six terms, that \( \left| |C||2 + |d||2 \leq \left| |\bar{d}||2 \right| \) and \( \left| |C - \bar{d}||2 \leq \left| |\bar{d} - \bar{\bar{d}}||2 \right| \) and that \( \bar{d} \equiv (A,b,C,d), \bar{d} \equiv (A,b,\bar{C},\bar{d}) \in D \) we get that
\[
\sum_{i=1}^{\ell} \left( \frac{\partial \bar{d}(\bar{d})}{\partial b_{\bar{d}}} - \frac{\partial \bar{d}(\bar{d})}{\partial b_{\bar{d}}} \right)^2 \leq 24n^2 \left| |\bar{d}||2 \left( \left( \ell + 1 \right) \left( \sigma_{\max}'' L^2_{\mathcal{Z}} \right)^2 \left| |\bar{d}||2 + 4(\ell + 1)^2 \left( \mu_{\max}'' \right)^2 \left( \sigma_{\max}'' \right)^4 \left| |\bar{d}||2 \right|^4 \left( \left| |\bar{d}||2 \right|^4 \left( \left| |\bar{d}||2 + \left| |\bar{d}||2 \right|^2 \right) \left| |\bar{d}||2 \right| \right|^2 \left( \left| |\bar{d}||2 + \left| |\bar{d}||2 \right|^2 \right) \left| \bar{d} - \bar{\bar{d}}\right| \right)^2 \right.
\]

Therefore, we conclude that indeed
\[
L^2 = O \left( n^2 D^2 \max\{ L^2_{\mathcal{Z}} (\sigma'_{\max}'' L^2_{\mathcal{Z}}), \ell^3 D^4 (\sigma'_{\max}'' L^2_{\mathcal{Z}}), \ell^2 D^6 (\sigma'_{\max}'' L^2_{\mathcal{Z}})^2, D^2 (\sigma'_{\max}'' L^2_{\mathcal{Z}})^2, n(\ell + 1)^2 \left( \sigma_{\max}'' L^2_{\mathcal{Z}} \right)^2 \left| |\bar{d}||2 + 4(\ell + 1)^2 \left( \mu_{\max}'' \right)^2 \left( \sigma_{\max}'' \right)^4 \left| |\bar{d}||2 \right|^4 \left( \left| |\bar{d}||2 \right|^4 \left( \left| |\bar{d}||2 + \left| |\bar{d}||2 \right|^2 \right) \left| |\bar{d}||2 \right| \right|^2 \left( \left| |\bar{d}||2 + \left| |\bar{d}||2 \right|^2 \right) \left| \bar{d} - \bar{\bar{d}}\right| \right)^2 \right.
\]

where \( C_6(\sigma'_{\max}'' \sigma''_{\max}'' \mu'_{\max}'' \mu''_{\max}) \) is a constant that only depends polynomially on \( \sigma'_{\max}'' \sigma''_{\max}'' \mu'_{\max}'' \mu''_{\max} \).

We continue with another lemma analyzing the Lipschitz property of a particular function which will be useful for the calculation of the Lipschitz constant of the gradient of \( \varphi \).

\]

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Lemma 4.5. Let $\hat{\varphi}$ be the function defined in (15) with respect to a given $\Sigma \in S^n$. Moreover, denote $L_Z := \sqrt{1 + \|\Sigma\|^2}$ and let $D > 0$ be any constant. Then for any $1 \leq \nu \leq nk$ the function

$$D := \{(a, b, c, d) \in \mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R}^{nk \times l} \times \mathbb{R}^{nk} : \|(a, b, c, d)\| \leq D\} \ni (a, b, c, d) \rightarrow \frac{\partial \hat{\varphi}(a, b, c, d)}{\partial d_\nu} \in \mathbb{R}$$

is Lipschitz continuous. Moreover, there exists a constant $L_{\nu, \alpha, j}$ satisfying

$$L_{\nu, \alpha, j}^2 = O \left( C_\nu(\sigma_{\max}'(\alpha, \beta), \mu_{\max}''(\alpha, \beta)) n^2 \max\{D^2 L_Z^2, \ell n, \ell^2 D^6 L_Z^2, \ell^3 D^4\} \right),$$

where $C_\nu(\sigma_{\max}', \mu_{\max}'')$ is a constant only depending polynomially on $\sigma_{\max}'$, $\mu_{\max}''$, so that for all $(a, b, c, d), (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) \in D$ we have that

$$\sum_{\nu=1}^{nk} \left( \frac{\partial \hat{\varphi}(a, b, c, d)}{\partial d_\nu} - \frac{\partial \hat{\varphi}(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})}{\partial d_\nu} \right)^2 \leq L_{\nu, \alpha, j}^2 \| (a, b, c, d) - (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) \|^2.$$

Proof. By Lemma 4.2 we know that for every $\nu := (\alpha - 1)k + \beta$ with $1 \leq \alpha \leq n$ and $1 \leq \beta \leq k$ we have that

$$\frac{\partial \hat{\varphi}(a, b, c, d)}{\partial d_\nu} = 2 \sum_{j=1}^{n} \mu'(\omega_{\alpha, j}) X_{(j-1)k+\beta}. \tag{31}$$

Therefore, setting $\phi_1 \equiv \mu'$, $\phi_2 \equiv \omega_{\alpha, j}$, and $\phi_3 \equiv X_{(j-1)k+\beta}$ with $\nu := (\alpha - 1)k + \beta$, we have that $B_{12} = \mu_{\max}' \leq 1$, $L_1 = \mu_{\max}''$, $L_2(C, d) = L_{\nu, \alpha, j}(C, d, \tilde{c}, \tilde{d})$ by Step 5 of the proof of Lemma 4.4. $L_{3,1}(C) = \sqrt{\|C_v\|^2 (\sigma_{\max}')^2 L_Z^2 + \ell + 1}$ by (24), $L_{3,2} = 0$, $B_3 = \mu_{\max}'' \leq 1$, and $B_4(C, d) = \sqrt{\|C_v\|^2 + \|d_v\|^2} L_{\nu, \alpha, j}(C, d, \tilde{c}, \tilde{d})$ by (25) in the notation of Lemma 4.3. By Lemma 4.3 we can hence conclude that the function

$$(a, b, c, d) \rightarrow \Gamma_{\alpha, \beta, j}(a, b, c, d) := \mu'(\omega_{\alpha, j}) X_{(j-1)k+\beta}$$

satisfies for all $(a, b, c, d), (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$ that

$$|\Gamma_{\alpha, \beta, j}(a, b, c, d) - \Gamma_{\alpha, \beta, j}(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})| \leq \left( \sqrt{\|C_v\|^2 (\sigma_{\max}')^2 L_Z^2 + \ell + 1} + \|C_v\|^2 [ \|d_v\|^2 L_{\nu, \alpha, j}(C, d, \tilde{c}, \tilde{d}) ] \right) \|(a, b, c, d) - (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})\|, \tag{32}$$

where $\nu = (\alpha - 1)k + \beta$. Moreover, notice from (29) and using that $\|C\|^2 + \|d\|^2 \leq D^2$ we see that

$$L_{\nu, \alpha, j}(C, d, \tilde{c}, \tilde{d}) \leq \sqrt{2(\ell + 1)(\|C\|^2 + \|d\|^2)} \left[ (\|C\|^2 (\sigma_{\max}')^2 L_Z^2 + \ell + 1) + 2(\ell + 1)(\|C\|^2 + \|d\|^2) \right] \left[ (\|C\|^2 (\sigma_{\max}')^2 L_Z^2 + \ell + 1) \right] \leq \sqrt{4(\ell + 1)D^2 (D^2 (\sigma_{\max}')^2 L_Z^2 + \ell + 1)} \tag{33}$$

Therefore, using the estimates (32) and (33) in (31) yields for $\nu = (\alpha - 1)k + \beta$ that

$$\left| \frac{\partial \hat{\varphi}(a, b, c, d)}{\partial d_\nu} - \frac{\partial \hat{\varphi}(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})}{\partial d_\nu} \right| \leq 2n \left[ \sqrt{\|C_v\|^2 (\sigma_{\max}')^2 L_Z^2 + \ell + 1} + \mu_{\max}' \sqrt{\ell + 1} \|C_v\|^2 |d_v|^2 \sqrt{4(\ell + 1)D^2 (D^2 (\sigma_{\max}')^2 L_Z^2 + \ell + 1)} \right] \|(a, b, c, d) - (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})\|.$$

This ensures that

$$\left| \frac{\partial \hat{\varphi}(a, b, c, d)}{\partial d_\nu} - \frac{\partial \hat{\varphi}(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})}{\partial d_\nu} \right|^2 \leq 8n^2 \left[ (\|C_v\|^2 (\sigma_{\max}')^2 L_Z^2 + \ell + 1) + 4(\mu_{\max}')^2 (\|C_v\|^2 + |d_v|^2) D^2 (D^2 (\sigma_{\max}')^2 L_Z^2 + \ell + 1) \right] \|(a, b, c, d) - (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})\|^2.$$
Summing up in $\nu$, we hence obtain that
\[
\sum_{i=1}^{nk} \left( \frac{\partial \tilde{\phi}(A,b,C,d)}{\partial d_i} - \frac{\partial \phi(A,b,C,d)}{\partial d_i} \right)^2 
\leq 8n^2 \left[ \left| |C| \right|^2 (\sigma'_{\max})^2 L_Z^2 + nk(\ell + 1) \right] 
+ 4(\mu''_{\max})^2(\ell + 1)(\|C\|^2 + \|d\|^2) \left( (2\sigma''_{\max})^2 L_Z^2 + \ell + 1 \right) \| (A, b, C, d) - (A, b, C, d) \|^2.
\]
This and using that $\|C\|^2 \leq (\|C\|^2 + \|d\|^2) \leq D^2$ indeed implies that
\[
L_d^2 = O \left( n^2 \max \{ D^2 (\sigma'_{\max})^2 L_Z^2, tnk, \ell^2 D^6 L_Z^2 (\mu''_{\max})^2 (\sigma''_{\max})^2, \ell^2 D^4 (\mu''_{\max})^2 \} \right)
= O \left( \mathcal{C}_d(\sigma'_{\max}, \mu''_{\max}) n^2 \max \{ D^2 L_Z^2, tnk, \ell^2 D^6 L_Z^2, \ell^2 D^4 \} \right),
\]
where $\mathcal{C}_d(\sigma'_{\max}, \mu''_{\max})$ is a constant that only depends polynomially on $\sigma'_{\max}, \mu''_{\max}$.

Now we are able to prove the Lipschitz property of $\nabla \tilde{\phi}$.

**Lemma 4.6.** Let $\tilde{\phi}$ be the function defined in (15) with respect to a given $\Sigma \in \mathbb{R}^n$. Moreover, denote $L_Z := 1 + \| \Sigma \|_2^2$ and let $D \geq 1$ be any constant. Then the function
\[
\mathcal{D} =: \{(A, b, C, d) \in \mathbb{R}^{n \times r} \times \mathbb{R}^r \times \mathbb{R}^{nk \times \ell} \times \mathbb{R}^{nk} : \| (A, b, C, d) \| \leq D \} \ni (A, b, C, d) \mapsto (\nabla \tilde{\phi})(A, b, C, d)
\]
is Lipschitz continuous with Lipschitz constant $L_\varphi > 0$ satisfying
\[
L_\varphi^2 = O(\mathcal{C}_\varphi(\sigma'_{\max}, \sigma''_{\max}, \mu''_{\max}) n^2 \max \{ D^2 L_Z^2, \ell D^6 L_Z^2, \ell^2 D^6 L_Z^2, kn, \ell^2 D^4 \} ),
\]
where $\mathcal{C}_\varphi(\sigma'_{\max}, \sigma''_{\max}, \mu''_{\max})$ is a constant that only depends polynomially on $\sigma'_{\max}, \sigma''_{\max}, \mu''_{\max}$.

**Proof.** Using the notation $\tilde{\vartheta} \equiv (A, b, C, d)$ and the ones introduced in (16), we can apply Lemma 4.1 Lemma 4.2 that
\[
\Sigma_{\tilde{\vartheta}} = \| \Sigma \|_2^2,
\]
and that $Y_{\tilde{\vartheta}}^2 \leq 1$ to see that
\[
\| \nabla \tilde{\phi}(\vartheta) - \nabla \tilde{\phi}(\tilde{\vartheta}) \|^2 = \sum_{i=1}^\ell \left( \frac{\partial \tilde{\phi}(\vartheta)}{\partial b_i} - \frac{\partial \tilde{\phi}(\tilde{\vartheta})}{\partial b_i} \right)^2 \left( 1 + \sum_{j=1}^r h(\Sigma) \right) + nk \sum_{i=1}^\ell \left( \frac{\partial \tilde{\phi}(\vartheta)}{\partial d_i} - \frac{\partial \tilde{\phi}(\tilde{\vartheta})}{\partial d_i} \right)^2 \left( 1 + \sum_{i=1}^\ell \right) \bigg| \bigg| \bigg| \bigg| \bigg| \bigg| \bigg|
\leq \sum_{i=1}^\ell \left( \frac{\partial \tilde{\phi}(\vartheta)}{\partial b_i} - \frac{\partial \tilde{\phi}(\tilde{\vartheta})}{\partial b_i} \right)^2 \left( 1 + \| \Sigma \|_2^2 \right) + nk \sum_{i=1}^\ell \left( \frac{\partial \tilde{\phi}(\vartheta)}{\partial d_i} - \frac{\partial \tilde{\phi}(\tilde{\vartheta})}{\partial d_i} \right)^2 (1 + \ell) \bigg| \bigg| \bigg| \bigg| \bigg| \bigg| \bigg|
\leq (L_\varphi^2 (1 + \| \Sigma \|_2^2) + L_\varphi^2 (1 + \ell)) \| \vartheta - \tilde{\vartheta} \|^2 \bigg| \bigg| \bigg| \bigg| \bigg| \bigg| \bigg|
= (L_\varphi^2 L_Z^2 + L_\varphi^2 (1 + \ell)) \| \vartheta - \tilde{\vartheta} \|^2 .
\]
Therefore, we deduce from Lemma 4.4 and Lemma 4.5 that
\[
L_\varphi^2 = O(\mathcal{C}_\varphi(\sigma'_{\max}, \sigma''_{\max}, \mu''_{\max}) n^2 \ell \max \{ D^2 L_Z^2, \ell D^6 L_Z^2, \ell^2 D^6 L_Z^2, kn, \ell^2 D^4 \} ),
\]
where $\mathcal{C}_\varphi(\sigma'_{\max}, \sigma''_{\max}, \mu''_{\max})$ is a constant that only depends polynomially on $\sigma'_{\max}, \sigma''_{\max}, \mu''_{\max}$.

Finally we are able to provide the proof of Theorem 2.1

**Proof of Theorem 2.1.** Notice that by the structure of an $m$-layered neural network, we can write for each $t = 1, \ldots, m$
\[
N_t^{\theta_t, \theta_t}(h(\Sigma)) = A^{(t+1)}(\sigma^{(t)} \circ N^{t-1, \theta_{t-1}}(h(\Sigma)) + b^{(t+1)},
\]
(34)
where $N^{0, \theta_0}(h(\Sigma)) = A^{(1)}(h(\Sigma) + b^{(1)})$. The Lipschitz constant of $\nabla \phi(\Theta_m)$ has been calculated in Lemma 4.6 for the case $m = 1$. Now for $m > 1$, observe that by the recursive representation (34), we can apply the same argument presented in this section to calculate the Lipschitz constant of $\nabla \phi(\Theta_t)$ for the single-layered neural network, but with the Lipschitz constant of the map
\[
\Theta_{t-1} \mapsto N^{t-1, \Theta_{t-1}}(h(\Sigma))
\]
instead of the Lipschitz constant of the map
\[
(A, b) \mapsto A^{(1)}(h(\Sigma),
\]
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which by Step 1 of the proof of Lemma 4.4 is equal to \( L_Z = \sqrt{1 + \| h(\Sigma) \|^2} \). More precisely, using the representation and the Lipschitz constant in (26), one can calculate the square of the Lipschitz constant of \( \nabla N \) using a recursion for \( t = 1, \ldots, m - 1 \). We then obtain, when relegating any dependence on the parameters \( \sigma'_{\text{max}}, \sigma''_{\text{max}}, \) and \( \mu''_{\text{max}} \) in a hidden constant, that for \( t = 1, \ldots, m - 1 \)
\[
\begin{align*}
\mathcal{L}_t^2 &:= 1 + \| h(\Sigma) \|^2, \\
\mathcal{L}_t^4 &:= D^2 \mathcal{L}_{t-1}^2 + \ell_t + 1.
\end{align*}
\]

For the last layer, we can apply Lemma 4.6 but with \( \mathcal{L}_{m-1} \) instead of \( L_Z \), to see that
\[
\mathcal{L}_m^2 := n^2 \ell_m \max \{ D^4 \mathcal{L}_{m-1}^4, \ell_m D^4 \mathcal{L}_{m-1}^4, \ell_m^2 D^6 \mathcal{L}_{m-1}^2, nD^2 \mathcal{L}_{m-1}^2, kn\ell_m, \ell_m^3 D^4 \}.
\]
Hence we get that
\[
L := O(\mathcal{C}(\sigma'_{\text{max}}, \sigma''_{\text{max}}, \mu''_{\text{max}}) \mathcal{L}_m),
\]
where \( \mathcal{C}(\sigma'_{\text{max}}, \sigma''_{\text{max}}, \mu''_{\text{max}}) \) is a constant that only depends polynomially on \( \sigma'_{\text{max}}, \sigma''_{\text{max}}, \mu''_{\text{max}} \). Resolving the recursion, we estimate
\[
\mathcal{L}_{m-1}^2 = D^{2m-2}(1 + \| h(\Sigma) \|^2) + \sum_{j=1}^{m-1} D^{2(m-j-1)}(\ell_j + 1) \leq D^{2m-2}\left(1 + \| h(\Sigma) \|^2 + \frac{\ell_{\text{max}} + 1}{D^2 - 1}\right),
\]
where \( \ell_{\text{max}} := \max \{ \ell_1, \ldots, \ell_m \} \).

The second part of Theorem 2.1 is well-known in optimization theory; see, e.g., [15] Section 1.2.3, but we still present the argument for the sake of completeness. Indeed, using \([11]\) and \([13]\), we obtain that
\[
\left( \min_{0 \leq j \leq N} \| \nabla \varphi(\Theta_j) \|^2 \right) \leq \frac{1}{N+1} \sum_{j=0}^N \| \nabla \varphi(\Theta_j) \|^2 \leq \frac{L}{(N+1)K} \sum_{j=0}^N \left[ \varphi(\Theta_j) - \varphi(\Theta_{j+1}) \right] \\
= \frac{L}{(N+1)K} \left[ \varphi(\Theta_0) - \varphi(\Theta_{N+1}) \right] \\
\leq \frac{L}{(N+1)K} \left[ \varphi(\Theta_0) - \min_{\theta \in \Theta} \varphi(\theta) \right],
\]
which finishes the proof. \( \square \)

References


