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A SOLUTION TO THE MULTIDIMENSIONALITY IN OPTION PRICING

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A Solution to the Multidimensionality in Option Pricing

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Abstract: We provide an accurate, simple formula for pricing multidimensional European options. The formula is as simple as the Black-Scholes formula. Therefore, the (costly) computational methods are needless. Moreover, our method allows the calculation of the implied volatility of the underlying asset of a multidimensional option.

1 Introduction

Given the increase in financial innovations, multidimensional options are becoming popular. Recently, multidimensional options dominate both the literature and financial practice. Also, the dimensionality of the options is increasing. Moreover, an increase in dimensionality will increase the complexity and cost of pricing options.

Consequently, approximations using numerical/computational methods are used to price multi-asset options. For example, Deelstra *et al.* (2004). Chen *et al.* (2008), Dhaene *et al.* (2020), Shirzadi *et al.* (2020), Kim *et al.* (2020), and Ikamari *et al.* (2020) adopted these methods. Also, previous literature typically relied on the comonotonicity theory. It is well known that using the existing methods, the computational cost of pricing such options can be high. Therefore, there is an urgent need to devise an efficient method for pricing multidimensional to circumvent the complex and costly procedures.

In accordance with financial innovations, we note that the multi-asset underlying asset such as a stock index or market basket can be viewed as a distinct asset. This is realistic and consistent with the market reality (see, for example, Gorton and Pennacchi (1993)) because composite assets can be traded in the market as individual assets. This is echoed by other sources such as dmiralmarkets.com (see [2] [3]) that verify the fact that even stock indices can be traded as an individual asset using, for example, a Contract for Difference CFD. Given these recent financial innovations, we note that the Black-Scholes approach becomes directly applicable to multidimensional options.

In this paper, we consider a European option whose underlying is composed of multiple assets (a weighted sum of assets). In doing so, we devise a method to circumvent the complexity that arises from the multidimensionality of option pricing. That is, we transform the model to make it as simple as the one-dimensional case. Furthermore, the assumption of comonotonicity and other assumptions regarding the structure of the underlying asset can be relaxed. In doing so, we devise a method to avoid the assumption of a geometric Brownian motion of the composite underlying. We derive a price formula for a European option whose underlying is a portfolio of assets. The price formula is as simple as the Black-Scholes formula. Furthermore, our method allows the calculation of the implied volatility for a portfolio of assets; this is novel and useful to practitioners and investors.

2 A Simple Solution

A simple solution to multidimensionality is to recognize that the multiple-asset underlying can be tradable as an individual asset, and assume that it is a geometric Brownian motion. That is, the dynamics of the value of the underlying composite asset, Y_u, (under risk neutrality) are given by

$$dY_u = Y_u \left[r du + \sigma dZ_u \right], \tag{1}$$

where r is the interest rate, σ is the volatility of returns of the underlying asset, u is time, and Z is a Gaussian variable. We note that the dynamics hold irrespective of the dimensionality or the composition of Y_u, for example, Y_u can be an index or a basket of assets, etc.

Proposition 1. The price of the European call is given by

$$C\left(t,Y_t\right)=e^{-r(T-t)}E\left[Y\left(T\right)-K\right]^{+}, \tag{2}$$

where K is the strike price, T is the expiry date and $T-t$ is the time to expiration.

Proof. Using Ito's rule, the dynamics of the options price are given by

$$\begin{aligned} dC\left(t,Y_t\right) &= C_t dt + C_Y dY + \frac{1}{2}C_{YY}\left(dY\right)^2 \\ &= \left(C_t + rYC_Y + \frac{1}{2}\sigma^2 Y^2 C_{YY}\right)dt + \sigma Y C_Y dZ, \end{aligned} \tag{3}$$

where the subscripts denote partial derivatives. Therefore, using the Black-Scholes assumptions, we obtain its partial differential equation PDE as shown in the following equation:

$$C_t + r\left(YC_Y - C\right) + \frac{1}{2}\sigma^2 Y^2 C_{YY} = 0, C\left(T, Y(T)\right) = g\left(Y_t\right), \tag{4}$$

where g is the payoff of the option. Thus, the price of the call option is

$$C\left(t,Y_t\right) = Y_t N\left(d_1\right) - e^{-r(T-t)} K N\left(d_2\right), \tag{5}$$

where $d_1 = \frac{\ln(Y_t/K)+\left(r+\sigma^2/2\right)(T-t)}{\sqrt{\sigma^2(T-t)}}$ and $d_2 = d_1 - \sqrt{\sigma^2 (T-t)}$. □

Corollary 1. The price of the put option is

$$P(t, Y_t) = e^{-r(T-t)} K N(-d_2) - Y_t N(-d_1), \tag{6}$$

According to the classical Black-Scholes model, the volatility parameter needs to be estimated, and the method of estimation is arbitrary. Similarly, σ can be estimated by using similar methods, for example, by using historical data of Y_u. Another example is to use the method proposed by Alghalith *et al.* (2020) in which (given that $(dY_u)^2 = Y_u^2 \sigma^2 du$) the volatility can be estimated (by using the current value and previous value of Y_u) as $\sigma = \sqrt{\frac{(Y_{t+1}-Y_t)^2}{Y_t^2 \Delta u}}$.

However, in the next section, we show that the assumption of a geometric Brownian motion of the underlying can be avoided.

3 An Alternative Method

In this section, we will propose a method to avoid the assumption of a geometric Brownian motion of the underlying Y. This is relevant if the individual stock price is assumed to be log-normal, and thus, the underlying basket price, in general, is not log-normal.

Lemma 1. The value of the underlying composite asset Y_u is given by

$$Y_u = Y_t e^{c+V_u Z_u} \ , \tag{7}$$

where Z_u is normal.

Proof. Since $Y_u = e^{\ln Y_u} = \frac{Y_t}{Y_t} e^{\ln Y_u} = Y_t e^{c+\ln Y_u}$, where c is a constant, we use the following transformation

$$Y_u = Y_t e^{c+\ln Y_u} = Y_t e^{c+\frac{Z_u}{Z_u}\ln Yu} = Y_t e^{c+V_u Z_u} \square \tag{8}$$

Thus, $Y_T = Y_t e^{c+V_T Z_T}$.

Now, we are ready to derive the price of the European call.

Proposition 2. The price of the European call is given by

$$C\left(t, Y_t\right) = Y_t N\left(d_1\right) - e^{-r(T-t)} K N\left(d_2\right), \tag{9}$$

where $d_1 = \frac{\ln(Y_t/K)+\left(r+\frac{\hat{v}^2}{2}\right)(T-t)}{\sqrt{\hat{v}^2(T-t)}}$ and $d_2 = d_1 - \sqrt{\hat{v}^2\left(T-t\right)}$.

Proof. The option price can be expressed as a weighted average of the

Black-Scholes prices conditional on V as follows

$$C(t, Y_t) = \int_v E\left[e^{-r(T-t)} g(Y_T) / V_T = v\right] dF(v) = \int_v C_{BS}(v)\, dF(v), \quad (10)$$

where F is the cumulative density of V and C_{BS} is the classical Black-Scholes price. By the continuity, the expected value is a specific value of C_{BS} denoted by $\hat{C}_{BS} = C_{BS}(\hat{v})$, where $\hat{v}$ is a value (outcome) of V. Thus,

$$C(t, Y_t) = \int_v C_{BS}(v)\, dF(v) = C_{BS}(\hat{v}). \quad (11)$$

Therefore, the price of the European call is

$$C(t, Y_t) = Y_t N(d_1) - e^{-r(T-t)} K N(d_2), \quad (12)$$

where $d_1 = \frac{\ln(Y_t/K) + \left(r + \frac{\hat{v}^2}{2}\right)(T-t)}{\sqrt{\hat{v}^2 (T-t)}}$ and $d_2 = d_1 - \sqrt{\hat{v}^2 (T-t)}$.$\square$

The current value of the underlying Y_t is known. In the classical Black-Scholes model, the volatility parameter needs to be estimated. Similarly, $\hat{v}$ can be estimated by using similar methods. Moreover, the implied value of $\hat{v}$ can be computed by using the formula. The implied values can also be used to estimate $\hat{v}$.

A Verification:

A simple way to verify the result is to let $\tilde{C}(t, Y_t)$ be the true option price, and $\bar{C}(r, s, \sigma_i, T-t)$ be the classical Black-Scholes price of the European option, where s is the current underlying price and σ_i is the volatility parameter. By the continuity of $\bar{C}$, there are specific values of the parameters s and σ such as $\hat{s}$ and $\hat{\sigma}$, so that $\tilde{C}(t, Y_t) = \bar{C}(r, \hat{s}, \breve{\sigma}, T-t)$. Therefore, the true option price can be expressed using the Black-Scholes formula.

4 Illustrations and Comparisons

In this section, we provide two practical examples to demonstrate the advantages of our approach. We then outline the contributions of our paper to the literature and compare them to previous literature.

Example 1

If $Y_t = K = 100, r = .03$, $\hat{v} = .2$ and $T - t = .5$, then $C(t, Y_t) = 6.37$.

Furthermore and very importantly, this method allows us to calculate the implied volatility of the stock index and other composite assets. The importance of implied volatility is well known. For example, it can be used to forecast movements in the portfolio. We believe that finding the value of the implied volatility for a portfolio of assets is very useful.

Example 2

Using data for options based on S&P 500 in March 2021, $Y(t) = K = 3725$, $r = .02$, $T - t = 1$, and $C(t, Y_t) = 470.55$, then the implied volatility of the S&P index is 29.51%.

An Outline of the Contributions:

1. We offered a simple, Black-Scholes formula-like for calculating the price of multidimensional options. The formula is as simple as the classical Black-Scholes formula.

2. We avoided the (costly) computational methods. According to previous methods, the higher the dimensionality, the higher is computational cost.

3. We were able to calculate the implied volatility under multidimensional options. This is important since the use of the implied volatility is very popular in the industry.

4. We avoided the assumptions regarding the structure of the underlying asset.

Comparisons with the Other Studies:

- Other studies did not offer a (simple) price formula.
- Other studies used approximations. We use exact analyses.
- Other studies used (costly) computational/numerical methods. Our model is as simple as the Black-Scholes model.
- Other studies made assumptions about the structure and distribution of the underlying asset, such as the comonotonicity and log-normality of the underlying asset. We relax these assumptions.
- The implied volatility of the underlying asset cannot be calculated using other studies. It is easy to calculate the implied volatility using our method.

5 Concluding Remarks

Given the increase in financial innovations, research on multidimensional options became very important in finance, because the dimensionality of the options is increasing, which, in turn, will increase the complexity and cost of pricing options. Consequently, numerical/computational methods are used to price multi-asset options. This paper bridges the gap in the literature by introducing a simple, accurate formula for pricing multidimensional options,

We expect that our approach will open new grounds in derivative pricing, volatility, and other areas in finance. Future studies may extend our approach to pricing other options, such as Asian and American options. In this paper, we assume investors are risk neutral. Thus, a possible future extension of our paper could be relaxing this assumption. Our methods will be useful to academics, practitioners, and policymakers.

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