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NEW DEVELOPMENT ON THE THIRD ORDER STOCHASTIC DOMINANCE FOR RISK-AVERSE AND RISK-SEEKING INVESTORS WITH APPLICATION IN RISK MANAGEMENT

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Abstract

This paper develops new financial theory to link the third order stochastic dominance for risk-averse and risk-seeking investors and provide illustration of application in risk management. We present some interesting new properties of third order stochastic dominance (TSD) for risk-averse and risk-seeking investors. We show that the means of the assets being compared should be included in the definition of TSD for both investor types. We also derive the conditions on the variance order of two assets with equal means for both investor types and extend the second order SD (SSD) reversal result of Levy and Levy (2002) to TSD. We apply our results to analyze the investment behaviors on traditional stocks and internet stocks for both risk averters and risk seekers.

Keywords: Third order stochastic dominance, expected-utility maximization, risk aversion, risk seeking, investment behaviors.

JEL Classification: C00, G11

1 Introduction

Third-order stochastic dominance (TSD) is becoming an important area of research in finance. For example, Post, *et al.* (2015) developed and implemented linear formulations of convex stochastic dominance relations based on decreasing absolute risk aversion (DARA) for discrete and polyhedral choice sets in which DARA is related to the TSD. In addition, Post and Kopa (2013) developed and implemented linear formulations of general N^{th} order stochastic dominance criteria for discrete probability distributions. Post and Kopa (2017) developed an optimization method for constructing investment portfolios that dominate a given benchmark portfolio in terms of TSD. Although risk averse investor behavior is the conventional assumption in most financial research, risk seeking behavior has long been recognized as an important element that should also be considered. For example, Friedman and Savage (1948) observed investors' risk-seeking behavior to buy both insurance and lottery tickets. To circumvent this problem, Markowitz (1952) suggested including convex functions in both the positive and the negative domains. Williams (1966) found evidence to get a translation of outcomes that produces a dramatic shift from risk aversion to risk seeking. Tobin (1958) proposed the mean-variance rules for risk seekers and Kahneman and Tversky (1979) observed risk-seeking behavior in the negative domain. In this paper we include risk-seeking behavior in our study of TSD. More specifically, we analyze TSD in the context of both risk-averse and risk-seeking investors and present some interesting new properties of TSD for both types of investor. Before we discuss our contribution to the literature, we first discuss the literature.

In the von Neuman and Morgenstern theory of expected utility, the utility functions for risk averters are concave and increasing while the utility functions for risk seekers are convex and increasing. In this context stochastic dominance (SD) theory has generated a rich and growing academic literature. There have been numerous developments in the theory and application of SD. Most of them relate to second order SD (SSD) and studies of TSD are relatively rare. Here we list some of the studies on TSD. For example, Whitmore (1970) first introduced the concept of TSD. Porter, *et al.* (1973) examined the factors responsible for the time-consuming nature of SD tests up to the third order and used their results to develop efficient algorithms for conducting SD tests. Bawa (1975) proved that the optimal rule for comparing uncertain investments with equal means is the TSD rule. Fishburn and Vickson (1978) showed that TSD and DARA stochastic dominance are equivalent concepts when the means of the random alternatives are equal to one another. Bawa, *et al.* (1979) developed an algorithm to obtain the second and third order SD admissible sets by using the empirical distribution function for each stock as a surrogate for the true but unknown distribution. Eeckhoudt and Kimball (1992) made the stronger assumption that the distribution of background risk conditional on a given level of insurable loss deteriorates in the sense of TSD as the amount of insurable loss increases. In addition, Aboudi and Thon (1994) developed an efficient technique for determining TSD. Ekern (1980) showed that G has more n^{th} degree risk than F if and only if every n^{th} degree risk averter prefers F to G.

SD theory for risk seekers was developed by Hammond (1974), Meyer (1977), Stoyan and Daley (1983), Levy (2015), and many others. There have also been some studies on TSD that relate to both risk averters and risk seekers. For instance, Wong and Li (1999) extended the convex SD theory for risk averters originally developed by Fishburn (1974) to the first three orders for both risk averters and risk seekers. Bawa, *et al.* (1985) proposed exact linear programming algorithms and implemented for assigning return distributions into the first- and second-order stochastic dominance optimal sets. They also defined a superconvex stochastic dominance approach to work on the third-order stochastic dominance. Li and Wong (1999) extended the SD theory and diversification for risk averters developed by Hadar and Russell (1971) and others by including the TSD and developing the theory to examine the preferences for risk seekers. Wong (2007) further extended the SD theory of the first three orders to compare both return and loss. Recently, Niu, *et al.* (2017) developed the relationship between higher-order) stochastic dominance with Kappa Ratios.

There are also some applications of TSD theory that link it to other theories. For example, Gotoh and Konno (2000) showed that many efficient portfolios obtained by the mean-lower semi-skewness model are also efficient in the sense of TSD. In a study of 24 country stock market indices from 1989-2001, Fong, *et al.* (2005) showed SSD and TSD of winner portfolios over loser portfolios. By considering SSD and TSD, Gasbarro, *et al.* (2007) showed how switching across funds could increase investor utility. Zagst and Kraus (2011) derived parameter conditions implying the SSD and TSD of the Constant Proportion Portfolio Insurance strategy. TSD has also been promoted as a normative criterion to refine the partial ordering over income distributions (Davies and Hoy, 1994). In addition, Le Breton and Peluso (2009) introduced the concepts of TSD by using the Lorenz characterization of the second-degree stochastic order. Ng (2000) constructed two examples in TSD. Thorlund-Petersen (2001) developed the necessary and sufficient conditions for TSD.

This paper presents and studies some interesting new properties of SD for risk-averse and risk-seeking investors. We refer to SD for risk averse investors as SD and follow Levy (2015) to call the SD for risk seeking investors as risk-seeking SD (RSD). We first discuss the basic property of SD and RSD linking the SD and RSD of the first three orders to expected-utility maximization. We then we show that the means of the assets being compared should be included in the definition of TSD for both risk-averse and risk-seeking investors, thereby providing a solution to the controversy in the literature on whether or not the means of the assets being compared should be included in the definition of TSD for SD and RSD investors.

We extend the second order SD (SSD) reversal result of Levy and Levy (2002) for two assets that have the same mean to TSD and show that the dominance relationship can be in the same direction as well as being reversed. Inspired by Levy (2015), [1] we explore the relationship between the variances and the integrals of two assets and obtain the conditions on the order of the variances of two assets for TSD and TRSD under the condition of equal means. In addition, Rothschield and Stiglitz (1970) proved that if two

[1]Levy (2015) found that for any two assets with equal means, the one having smaller variance is a necessary condition for TSD.

assets have the same mean, then there could still be some risk averters that prefer the asset with the bigger variance. In this paper, we construct an example to show that this is possible. We then develop the property in which the dominance relationship can be in the same direction as well as being reversed when the means of the assets are different. All the properties developed in this paper are illustrated with examples.

Another contribution in this paper is that besides comparing the dominance of the integrals of two different distributions, we show that the dominance of the means for the distributions should also be checked to draw inference of SD for third order risk averters and risk seekers. Including "checking the dominance of the means" in the procedure of testing for third order SD is very important because it can change the results completely. For example, in our paper, before we include the "means test" in the testing procedure, our conclusion is *"the third-order risk averters prefer investing in the S&P 500 index to the Nasdaq 100 index in the second sub-period"*. However, after we include "the means test", our conclusion is *"the third-order risk averters are indifferent between the S&P 500 and Nasdaq 100 indices in the second sub-period"*.

Finally, we apply the results developed in the paper by comparing the preferences for traditional and internet stocks for the third-order risk averters and risk seekers. From our analysis, we conclude that the markets are efficient and there is no arbitrage opportunity between the S&P 500 and Nasdaq 100 indices in the entire period and in any sub-period, including any bull run, bear market, the dotcom bubble, and the recent financial crisis. In general, the third-order risk averters prefer investing in the S&P 500 index to the

Nasdaq 100 index while the third-order risk seekers prefer investing in the Nasdaq 100 index to the S&P 500 index over the entire period as well as many of the sub-periods to maximize their expected utilities (but not their expected wealth). Interestingly, we find that both the third-order risk averters and risk seekers are indifferent between the S&P 500 and Nasdaq 100 indices in the bear market during the recent global financial crisis, However, in the bull run during the dotcom bubble and in the bull run after the recent global financial crisis, the third-order risk averters are indifferent between the S&P 500 and Nasdaq 100 indices while the third-order risk seekers prefer the Nasdaq 100 index to the S&P 500 index.

In the conclusion we address the question of whether risk seekers actually exist and discuss how the theory developed in this paper can be used to explain some well-known financial anomalies such as momentum profits and when efficient mean-variance portfolios recommended by Markowitz (1952b) are preferred to the equally weighted portfolio suggested by Frankfurter *et al.* (1971), De Miguel *et al.* (2009), and others. We also suggest how it can be used to test other outstanding arguments in the literature.

The rest of the paper is organized as follows. In the next section we present definitions and notations. Section 3 is devoted to developing the theorems and properties for SD and RSD. Section 4 provides examples for SD and RSD that illustrate all the properties that have been developed. In Section 5, we illustrate the theory developed in this paper by comparing the investment behaviors of both third-order risk averters and risk seekers in traditional and internet stocks. Section 6 concludes and discusses.

2 Definitions and Notations

We define the "*cumulative distribution function*" (CDF) of the measure μ on the support $\Omega = [a, b] \subset \overline{\mathbb{R}}$ as $F_1(x) \equiv F(x) \equiv \mu[a, x]$ with $a < b$ and $\mu(\Omega) = 1$. We also define the *reversed CDF* $F_1^R(x) \equiv \mu[x, b]$ for all $x \in \Omega$. In addition, we assume that F_1 and F_1^R are measurable with $F_1(a) = 0$ and $F_1^R(b) = 0$. For random variables X and Y with CDFs F and G and probability density functions f and g, respectively, we define the mean of X, $\mu_F = \mu_X$, and the mean of Y, $\mu_G = \mu_Y$, to be

$$\mu_F = \mu_X = E(X) = \int_a^b t \, d\, F(t)$$

and define the following to be used throughout this paper:

$$H_j(x) = \int_a^x H_{j-1}(y)\, dy \quad , \quad H_j^R(x) = \int_x^b H_{j-1}^R(y)\, dy \quad j = 1, 2, 3; \tag{2.1}$$

where $H_0(x) = H_0^R(x) = h(x)$ with $H = F$ or G and $h = f$ or g.

We call H_i the i^{th}-order integrals and we note that the definition of H_i figures in the development of SD theory for risk averters (see, for example, Quirk and Saposnik 1962). We call H_i^R the i^{th}-order reversed integral and we note that the definition of H_i^R figures in the development of SD theory for risk seekers (see, for example, Hammond, 1974). Levy (2015) called the SD theory for risk seekers risk seeking SD theory and in this paper we follow Levy to call it risk-seeking SD and denote it by RSD. Risk averters typically have a preference for assets with a lower probability of loss while risk seekers have a preference for assets with a higher probability of gain. When choosing between two assets F or G, risk averters will compare their corresponding i^{th} order SD integrals F_i and G_i and choose

F if F_i is smaller, since it has a lower probability of loss. In the same vein, risk seekers will compare their corresponding i^{th} order RSD integrals F_i^R and G_i^R and choose F if F_i^R is larger since it has a higher probability of gain. This paper studies the properties of SD and RSD in detail with an emphasis on third order SD. We now turn to the definitions of the first-, second-, and third-order SDs as applied to risk averters and RSDs as applied to risk seekers.

Definition 2.1 *Let F and G be the CDFs of X and Y, X (F) is at least as large as Y (G) in the sense of:*

1. *FSD (SSD), denoted by $X \succeq_1 Y$ or $F \succeq_1 G$ ($X \succeq_2 Y$ or $F \succeq_2 G$), if and only if $F_1(x) \leq G_1(x)$ ($F_2(x) \leq G_2(x)$) for each x in $[a, b]$, and*

2. *TSD, denoted by $X \succeq_3 Y$ or $F \succeq_3 G$, if and only if $F_3(x) \leq G_3(x)$ for each x in $[a, b]$ and $\mu_X \geq \mu_Y$,*

where FSD, SSD, and TSD stand for first-, second-, and third-order stochastic dominance for risk averters, respectively.

Definition 2.2 *Let F and G be the CDFs of X and Y, X (F) is at least as large as Y (G) in the sense of:*

1. *FRSD (SRSD), denoted by $X \succeq_1^R Y$ or $F \succeq_1^R G$ ($X \succeq_2^R Y$ or $F \succeq_2^R G$), if $F_1^R(x) \geq G_1^R(x)$ ($F_2^R(x) \geq G_2^R(x)$) $\forall\, x \in [a, b]$, and*

2. *TRSD, denoted by $X \succeq_3^R Y$ or $F \succeq_3^R G$, if $F_3^R(x) \geq G_3^R(x)$ $\forall\, x \in [a, b]$ and $\mu_X \geq \mu_Y$,*

where FRSD, SRSD, and TRSD stand for first-, second-, and third-order risk-seeking stochastic dominance, respectively.

We note that one could define strict SD and RSD in Definitions 2.1 and 2.2 by adding the condition that there is a nonempty subinterval $I \subset [a, b]$ such that for any $x \in I$ the inequalities in Definitions 2.1 and 2.2 are strict. Without loss of generality, we will discuss the "weak" form of SD and skip the discussion of the "strict" form of SD in our paper.

Stochastic dominance is considered as especially useful for ranking investment prospects in conditions of uncertainty because ranking assets is equivalent to maximizing the expected utility preferences of decision makers with different types of utility functions. The following definition specifies the relevant types of utility functions:

Definition 2.3 *Sets of utility functions, U_n and U_n^R ($n = 1, 2, 3$), for risk averters and risk seekers are:*

$$U_n = \{u : (-1)^i u^{(i)} \leq 0\ ,\ i = 1, \cdots, n\} \quad \textit{and} \quad U_n^R = \{u : u^{(i)} \geq 0\ ,\ i = 1, \cdots, n\}, \tag{2.2}$$

respective, where $u^{(i)}$ is the i^{th} derivative of the utility function u.

We note that in Definition 2.3, $U_1 = U_1^R$. It is straightforward to extend the theory to include non-differentiable utilities defined in Definition 2.3 to be non-differentiable (Wong and Ma, 2008). One may extend the theory to any order $n \geq 1$, see, for example, Guo and Wong (2016). Since we are only interested in the third order in this paper, we stop at $n = 3$. Keeping in mind that investors in U_n are risk averse while investors in U_n^R are risk seeking, it is well known that a negative second derivative for the utility function infers that investors are risk averse and a positive third derivative for the utility function is a necessary, but not sufficient condition for decreasing absolute risk aversion (DARA).

In this case, the more wealthy the investor is, the less, on average, he/she is willing to pay for the insurance against a given risk. That is, $\frac{\partial r(\omega)}{\partial \omega} < 0$. This property is called decreasing absolute risk aversion (DARA). Let U_d be the set of all DARA utility functions. It is well-known (Levy, 2015) that $\frac{\partial r(\omega)}{\partial \omega} < 0$ implies $u^{(3)} > 0$ but U_3 is wider than U_d. Thus, $U_d \subset U_3$. Menezes, *et al.* (1980) showed that one CDF is an increase in downside risk from another if and only if the latter is preferred to the former by all decision makers with utility functions possessing a positive third derivative. Utility functions in U_3 have non-negative third derivatives, which implies the empirically attractive feature of DARA. This property is also called prudence (Menezes, *et al.* , 1980). Post and Levy (2005) suggested that a third-order polynomial utility function implies that only the first three central moments of the return distribution (mean, variance, and skewness) are relevant to investors. On the other hand, Post and Versijp (2007) suggested that TSD efficiency applies if and only if a portfolio is optimal for some nonsatiable, risk-averse, and skewness-loving investor. Fong, *et al.* (2008) commented that TSD adds to risk aversion with the assumption of a preference for skewness. Crainich, *et al.* (2013) provided more detailed information on the shapes of utility functions and their properties for risk averters and risk seekers.

Levy (2015) pointed out that the prizes of a lottery game are generally positively skewed because of the small probability of winning a very large prize and the value of an uninsured house is negatively skewed because of the small probability of a heavy loss due to a fire or burglary. He argued that people insure their homes because they dislike negative skewness. On the other hand, people buy a lottery ticket because they like bigger variance and positive skewness because with a lottery ticket both variance and skewness increase. He also pointed out that stock returns are generally positively skewed because a stock price can drop to zero but the stock price is unbounded from above. He showed that investors with $u^{(3)} > 0$ dislike negative skewness and like positive skewness and commented that the behaviors of people buying home insurance, participating in lotteries, and buying stock conform with the hypothesis that investors dislike negative skewness and like positive skewness which, in turn, provides support for the hypothesis that $u^{(3)} > 0$. We also note that the empirical studies by Arditti (1967) and others suggested Levy's argument that investors with $u^{(3)} > 0$ buy stocks. Levy (2015) also commented that positive skewness plays a central role in TSD. However, it does not tell the whole story. He constructed an example in which there is no FSD and no SSD and the two distributions are symmetrical but yet there is TSD between these two distributions.

According to the von Neumann-Morgenstern (1944) consistency properties, between two prospects F and G, investors will prefer F to G, or prefer X to Y if $\Delta Eu \equiv E\big[u(X)\big] - E\big[u(Y)\big] \geq 0$, where $E\big[u(X)\big] \equiv \int_a^b u(x)dF(x)$ and $E\big[u(Y)\big] \equiv \int_a^b u(x)dG(x)$.

3 The Theory

We first state the following basic result linking the SD and RSD of the first three orders to expected-utility maximization for risk-averse and risk-seeking investors :

Theorem 3.1 *Let u be a utility function and F and G be CDFs of X and Y. Then, for $j = 1$, 2, and 3,*

1. *$X \succeq_j Y$ if and only if $E\big[u(X)\big] \geq E\big[u(Y)\big]$ $\forall u \in U_j$, and*
2. *$X \succeq_j^R Y$ if and only if $E\big[u(X)\big] \geq E\big[u(Y)\big]$ $\forall u \in U_j^R$.*

Several studies have obtained results similar to those in the above proposition for orders 1 and 2. For instance, Hadar and Russell (1971) and Bawa (1975) established the FSD and SSD results for continuous utility functions and continuous probability density functions. For general distribution functions Hanoch and Levy (1969) proved first and second order SD. Rothschild and Stiglitz (1970, 1971) suggested a condition equivalent to the SSD results for the special case of cumulative distributions with equal means. Second order SD for risk averters and risk seekers was discussed by Meyer (1977), Stoyan (1983), and Levy (2015).

The result in Theorem 3.1 that is still controversial is the result of order 3 because for order 3 of SD, some (Whitmore, 1970; Bawa, 1975; Levy, 2015) suggest that both conditions (i) $F_3(x) \leq G_3(x)$ for each x in $[a, b]$ and (ii) $\mu_X \geq \mu_Y$ as stated in Definition 2.1 are necessary while some suggest that condition (ii) is redundant. For example, Schmid (2005) proved that (i) implies (ii) and thus he suggested that condition (ii) is not necessary.

One could draw similar arguments for RSD. In this paper, we confirm that the condition $\mu_X \geq \mu_Y$ in Definitions 2.1 and 2.2 is necessary in order to obtain the result of order 3 in Theorem 3.1. Without this condition, the assertions of Theorem 3.1 do not hold for the case $j = 3$. We will construct examples in our illustration section to show that $\mu_f \geq \mu_g$ is not related to $F_3(x) \leq G_3(x)$. One could easily modify our example to construct another example to show that $\mu_f \geq \mu_g$ is not related to $F_3^R(x) \geq G_3^R(x)$. We note that there are some studies related to Theorem 3.1. For example, Jean (1975) expressed the common moments in terms of successive integrals of a probability density function to allow a systematic comparison of the two methods, and Ekern (1980) shows that G has more n^{th} degree risk than F if and only if every n^{th} degree risk averter prefers F to G. However, as far as we know, no paper has done what we draw the conclusion for condition (ii) in detail. Thus, our result in Theorem 3.1 is still new. We provide the proof of Theorem 3.1 in an appendix.

We are now ready to discuss some other relationships between the third orders of SD and RSD. Before we do so, we first state the well-known hierarchy property in the following theorem:

Theorem 3.2 *For any pair of random variables X and Y, for $i = 1$ and 2, we have:*

1. *if $X \succeq_i Y$, then $X \succeq_{i+1} Y$; and*

2. *if $X \succeq_i^R Y$, then $X \succeq_{i+1}^R Y$.*

Theorem 3.2 shows that a hierarchy exists in both SD and RSD relationships and that the higher orders of SD and RSD can be inferred by the lower orders of SD and RSD but not vice versa. This suggests that practitioners should report the SD and RSD results to the lowest order in empirical analyses. We state the well-known hierarchy property in our paper because it is useful in the proof of other theorems developed in our paper. Levy and Levy (2002) showed that if F and G are of the same mean, then F dominates G in SRSD while G dominates F in SSD. We extend their result to include SD and RSD to the third order SD as stated in the following theorem:

Theorem 3.3 *For any pair of random variables X and Y, if F and G have the same mean, which is finite, and if either $X \succeq_2 Y$ or $Y \succeq_2^R X$, then we have*

$$X \succeq_3 Y \quad \textit{and} \quad Y \succeq_3^R X\,. \tag{3.3}$$

The proof of Theorem 3.3 is straightforward. Levy and Levy (2002) showed that $X \succeq_2 Y$ if and only if $Y \succeq_2^R X$ when $\mu_X = \mu_Y$. The result of Theorem 3.3 could then be obtained by applying Theorem 3.2.

From Theorem 3.3, we find that the dominance relationships of X and Y are reversed for SD and RSD. One may wonder whether the relationships of SD and RSD are always of different directions? The answer is NO. We develop a theorem to show this possibility as follows:

Theorem 3.4 *For any random variables X and Y, if either $X \succeq_1 Y$ or $X \succeq_1^R Y$, then we have*

$$X \succeq_3 Y \quad \textit{and} \quad X \succeq_3^R Y\,. \tag{3.4}$$

The proof of Theorem 3.4 could be obtained by applying Lemma 3 in Li and Wong (1999) and Theorem 3.2 in this paper. One might argue that the third orders SD and RSD in both Theorems 3.3 and 3.4 are trivial. We get the third orders SD and RSD because the first or second order SD and RSD relationships exist.

Levy (2015) found that for any two assets X and Y with means $\mu_x = \mu_y$, he claimed that $\sigma_x^2 < \sigma_y^2$ is a necessary condition for TSD. Inspired by Levy's idea, we explore the relationship between the variances and the integrals of two assets and obtain the following theorem to state the relationship between the difference of the variances, the difference of the third-order integrals, and the difference of the third-order reversed integrals for two different assets under the condition of equal means:

Theorem 3.5 *Given $\mu_F = \mu_G$, we have*

$$G_3(b) - F_3(b) = G_3^R(a) - F_3^R(a) = \frac{1}{2}\left(\sigma_G^2 - \sigma_F^2\right) . \tag{3.5}$$

The proof of Theorem 3.5 is in the appendix. From Theorem 3.5, we can obtain the following corollary to show the necessary and sufficient conditions on the order of the variances of two assets for both TSD and TRSD under the condition of equal means:

Corollary 3.1 *Given $\mu_F = \mu_G$, we have*

1. *$F_3(b) \leq G_3(b)$ if and only if $\sigma_F^2 \leq \sigma_G^2$, and*

2. *$F_3^R(a) \geq G_3^R(a)$ if and only if $\sigma_F^2 \geq \sigma_G^2$.*

From Corollary 3.1, we obtain the following corollary to show the necessary and sufficient conditions on the order of the variances of two assets for TSD and TRSD under the condition of equal means:

Corollary 3.2 *Given $\mu_F = \mu_G$, we have*

1. *if $F \succeq_3 G$, then $\sigma_F^2 \le \sigma_G^2$, and*

2. *if $F \succeq_3^R G$, then $\sigma_F^2 \ge \sigma_G^2$.*

We note that Levy's claim that for any two assets X and Y with means $\mu_x = \mu_y$, $\sigma_x^2 < \sigma_y^2$ is a necessary condition for TSD is Part 1 of Corollary 3.2. The converse of Corollary 3.2 is not true. We illustrate that the converse of Corollary 3.2 is not true by using Example 4.7 as shown in next section.

Corollary 3.2 gives us an impression that for any two distributions F and G, if $\mu_F = \mu_G$, then if we want to have $F \succeq_3 G$, then we must have the condition that $\sigma_F^2 \le \sigma_G^2$. Is this true? Rothschield and Stiglitz (1970) showed that this is not true by showing that "if F and G have the same mean, F may have a lower variance and yet G will be preferred to F by some risk averse individuals." We report this result in the following property:

Property 3.1 *F and G are two distributions with means μ_F and μ_G and variances σ_F^2 and σ_G^2 and there exists a concave utility $u \in U_2$ that $E[u(G)] > E[u(F)]$ if $\sigma_G^2 > \sigma_F^2$.*

One could easily show that this does not contradict Corollary 3.2 or any of the theorems/corollaries we developed in our paper. In this paper we will construct Example 4.8 in next section to show that Property 3.1 holds.

One might wonder whether there is any non-trivial third order SD and RSD relationship. Or, more specifically, one might ask: it is possible that there are X and Y such that they do not possess first and second order SD and RSD but there exist third order SD and RSD and there is a relationship between their third order SD and RSD. Our answer is YES and we derive one as follows:

Theorem 3.6 *If F and G satisfy $\mu_F = \mu_G$ and $\sigma_G^2 = \sigma_F^2$,*

then

$$F \succeq_3 G \quad \textit{if and only if} \quad F \succeq_3^R G\,.$$

The proof of Theorem 3.6 is in the appendix.

So far, the theory we developed in this paper from Theorem 3.3 to Theorem 3.6 on the relationship of SR and RSD is assumed that $\mu_F = \mu_G$. One may wonder is there any relationship for SR and RSD in which we do not have to assume $\mu_F = \mu_G$? Our answer is YES and we now develop the following theorems and corollary to show that it is possible to relax the condition of equal mean:

Theorem 3.7 *If F and G satisfy $\sigma_F = \sigma_G$ and $\mu_F - \mu_G \geq 0$, then we have*

$$F \succeq_3 G \Longrightarrow F \succeq_3^R G\,.$$

Theorem 3.8 *If F and G satisfy $\sigma_F = \sigma_G$ and $\mu_F - \mu_G \geq 0$, then we have*

$$F \succeq_3^R G \Longrightarrow F \succeq_3 G\,.$$

Corollary 3.3 *If F and G satisfy $\sigma_F = \sigma_G$ and $\mu_F - \mu_G \geq 0$, then we have*

$$F \succeq_3^R G \Longleftrightarrow F \succeq_3 G \, .$$

The proofs of Theorems 3.7 and 3.8 is in the appendix. Theorems 3.7 and 3.8 and Corollary 3.3 show that with $\mu_F > \mu_G$, it is possible that there is no FSD or FRSD between F and G, F dominates G in the sense of the third-order SD and RSD from the same direction. We illustrate in Example 4.9 to show that this is possible.

4 Illustration

As mentioned above, some papers suggest that the condition $\mu_X \geq \mu_Y$ stated in Definition 2.1 is not necessary to obtain the result in Theorem 3.1. For example, Schmid (2005) proved that $F_3(x) \leq G_3(x)$ implies $\mu_X \geq \mu_Y$ and thus he suggested condition $\mu_X \geq \mu_Y$ is not necessary. In this paper, we confirm that the condition $\mu_X \geq \mu_Y$ in both Definitions 2.1 and 2.2 is necessary in order to obtain the result of order 3 in Theorem 3.1. Without this condition, the assertions of Theorem 3.1 do not hold for the case $j = 3$. In this section, we first construct the following example to illustrate that $\mu_F \geq \mu_G$ is not related to $F_3(x) \leq G_3(x)$. One could easily modify our example to construct another example to show that $\mu_F \geq \mu_G$ is not related to $F_3^R(x) \geq G_3^R(x)$.

Example 4.1 $\underline{\mu_F \geq \mu_G \text{ is not related to } F_3(x) \leq G_3(x)}$

a. We first construct an example in which $G_3(x) > F_3(x)$ for all x but yet $\mu_G > \mu_F$.

Let $F(x) = x$, the uniform distribution on [0, 1]. Let $G(x)$ be such that

$$G(x) = \begin{cases} \frac{3x}{2} & 0 \le x \le 0.24, \\ 0.24 + \frac{x}{2} & 0.24 \le x \le 0.74, \\ \frac{3x}{2} - 0.5 & 0.74 \le x \le 1. \end{cases}$$

We can see that $\mu_G = 0.505 > 0.5 = \mu_F$ and $G_3(x) - F_3(x) \ge 0$ for all $0 \le x \le 1$.

b. *Next, we construct an example where $G_3(x) > F_3(x)$ for all x but yet $\mu_G < \mu_F$. Again, let $F(x) = x$, the uniform distribution on [0, 1]. Let $G(x)$ be such that*

$$G(x) = \begin{cases} \frac{3x}{2} & 0 \le x \le 0.26, \\ 0.26 + \frac{x}{2} & 0.26 \le x \le 0.76, \\ \frac{3x}{2} - 0.5 & 0.76 \le x \le 1. \end{cases}$$

We can see that $\mu_G = 0.495 < 0.5 = \mu_F$ while $G_3(x) - F_3(x) \ge 0$ for all $0 \le x \le 1$.

From this example, we can conclude that (i) $G_3(x) \ge F_3(x)$ for all x and (ii) $\mu_G < \mu_F$ have no relationship at all.

We note that most, if not all, of the examples constructed in this paper could be used to illustrate Theorem 3.2. To construct an example to illustrate Theorem 3.3, we modify the Production/Operations Management example used by Weeks (1985), Dillinger *et al.* (1992), and Wong (2007) as follows:

Example 4.2 *A production/operations manager make a choice between two investments with profit (x) and their associated probabilities f and g as shown in Table 2. From probabilities f and g, we obtain their SD and RSD integrals H_j and H_j^R ($j = 1, 2$ and 3)*

for $H = F$ and G defined in (2.1) and we define their differentials

$$GF_j = G_j - F_j \quad \text{and} \quad GF_j^R = G_j^R - F_j^R \ . \tag{4.6}$$

Thereafter, we exhibit the results of the SD and RSD integrals and their differentials for the first three orders in Table 2.

In this example, our results show that there is no first order SD or RSD between F and G but we have $F \succeq_j G$ and $G \succeq_j^R F$ for $j = 2$ and 3. Thus, this example illustrates Theorem 3.3 (as well as Theorem 3.2).

To illustrate Theorem 3.4, we construct Example 4.3 with Experiment 1 which is used in Levy and Levy (2002) as follows:

Example 4.3 *The gains one month later and their probabilities for an investor who invests $10,000 either in stock A or in stock B is shown in the following experiment:*

Experiments 1					
Stock A		Stock B			
Gain (×$1000)	Probability	Gain (×$1000)	Probability	Gain (×$1000)	Probability
0.5	0.3	-0.5	0.1	1	0.2
2	0.3	0	0.1	2	0.1
5	0.4	0.5	0.1	5	0.4

We let X and Y be the gain or profit for investing in Stocks A and B with the corresponding probability functions f and g and CDFs F and G, respectively. We depict the SD and RSD integral differentials GF_j and GF_j^R for the gain of investing in Stocks A and B in Table 3 in which GF_j and GF_j^R are defined in (4.6) for $j = 1, 2$ and 3.

From Table 3, we obtain $X \succeq_1 Y$ and $X \succeq_1^R Y$, $X \succeq_2 Y$ and $X \succeq_2^R Y$, as well as $X \succeq_3 Y$ and $X \succeq_3^R Y$. This example illustrates Theorem 3.4 (as well as Theorem 3.2).

In the above examples, we find that we have both SSD, SRSD, TSD, and TRSD for a pair of random variables. Is it possible to have TSD and TRSD but no SSD or SRSD? The answer is YES and this is exactly what Theorem 3.6 tells us. Thus, herewith we construct an example to illustrate Theorem 3.6 as follows:

Example 4.4 *Consider*

$$F(x) = \frac{x+1}{2}, \quad -1 \le x \le 1 \quad \text{and} \quad G(x) = \begin{cases} 0 & -1 \le x \le -3/4, \\ x + \frac{3}{4} & -3/4 \le x \le -1/4, \\ \frac{1}{2} & -1/4 \le x \le 0, \\ x + \frac{1}{2} & 0 \le x \le 1/4, \\ \frac{3}{4} & 1/4 \le x \le 3/4, \\ x & 3/4 \le x \le 1. \end{cases}$$

Both distributions have the same zero mean. One could easily find for this example that we do not have $F \succeq_2 G$ or $G \succeq_2 F$ but we have $G \succeq_3 F$ since the difference $G_3 - F_3$ is nonpositive. In addition, we can find that $F_3(b) = G_3(b) = 2/3$, so the conditions of Theorem 3.6 hold and we expect $G \succeq_3^R F$. Indeed the difference $G_3^R - F_3^R$ is nonnegative which means that $G \succeq_3^R F$ as predicted by Theorem 3.6.

We now ask: is it possible that we have TSD and TRSD but no SSD or SRSD, and the conditions of Theorem 3.6 do not hold? The answer is YES and we construct an example to illustrate this possibility.

Example 4.5 *Consider*

$$F(x) = x, \quad 0 \leq x \leq 1 \quad \text{and} \quad G(x) = \begin{cases} 2x & 0 \leq x \leq 1/6, \\ 1/3 & 1/6 \leq x \leq 5/9, \\ 3x - 4/3 & 5/9 \leq x \leq 2/3, \\ 2x - 2/3 & 2/3 \leq x \leq 5/6, \\ 1 & 5/6 \leq x \leq 1. \end{cases}$$

In this example, $\mu_F = 0.5 \neq 0.48148 = \mu_G$. *Thus, the first condition of Theorem 3.6 does not hold. In addition, one could easily check that we do not have* $F \succeq_2 G$ *or* $G \succeq_2 F$ *but we have* $F \succeq_3 G$ *since the difference* $F_3 - G_3$ *is nonpositive. Notice that* $F_3(b) \neq G_3(b)$ *and* $F_3^R(a) \neq G_3^R(a)$*, so the second condition of Theorem 3.6 does not hold either. However, we have* $F \succeq_3^R G$ *since one could check that the difference* $F_3^R - G_3^R$ *is nonnegative.*

In the above examples, we find that we have both TSD and TRSD for a pair of random variables. Is it possible to have TSD but no TRSD or vice versa? The answer is YES and we construct two examples in Example 4.6 in which in the first example there exist F and G such that $G \succeq_3^R F$ but neither $F \succeq_3 G$ nor $G \succeq_3 F$ holds. In the second example in which there exist F and G such that $F \succeq_3 G$ but neither $F \succeq_3^R G$ nor $G \succeq_3^R F$ holds.

Example 4.6 *$F \succeq_3 G$ and $F \succeq_3^R G$ are not related*

a. *We construct an example in which there exist* F *and* G *such that* $G \succeq_3^R F$ *but neither* $F \succeq_3 G$ *nor* $G \succeq_3 F$ *holds.*

Consider

$$F(t) = \begin{cases} 0 & 0 \le t \le 1/10, \\ 2t - 1/5 & 1/10 \le t \le 1/5, \\ 5t/4 - 1/20 & 1/5 \le t \le 3/5, \\ 3t/4 + 1/4 & 3/5 \le t \le 1, \end{cases} \quad and \quad G(t) = t$$

In this example, one could easily check that we do not have $F \succeq_3 G$ or $G \succeq_3 F$ but we have $G \succeq_3^R F$.

b. *Next we construct an example in which there exist F and G such that $F \succeq_3 G$ but neither $F \succeq_3^R G$ nor $G \succeq_3^R F$ holds.*

Consider

$$F(t) = t \quad and \quad G(t) = \begin{cases} 5t/4 & 0 \le t \le 0.4, \\ 3t/4 + 1/5 & 0.4 \le t \le 0.8, \\ 0.8 & 0.8 \le t \le 0.9, \\ 2t - 1 & 0.9 \le t \le 1, \end{cases}$$

One could easily find that $F \succeq_3 G$ but we do not have $F \succeq_3^R G$ or $G \succeq_3^R F$.

In Example 4.6 we show that we (i) do not have $F \succeq_3 G$ or $G \succeq_3 F$ but have $G \succeq_3^R F$ and (ii) do not have $F \succeq_3^R G$ or $G \succeq_3^R F$ but have $F \succeq_3 G$.

We note that Levy's claim that for any two assets X and Y with means $\mu_x = \mu_y$, $\sigma_x^2 < \sigma_y^2$ is a necessary condition for TSD is Part 1 of Corollary 3.2. We also note that the converse of Corollary 3.2 is not true as illustrated by using Example 4.7:

Example 4.7 *Consider*

$$f(x) = 1/2 \text{ in } [-1,1] \quad \text{and} \quad g(x) = \begin{cases} 0 & -1 \le x < -0.9, \\ 1 & -0.9 \le x < -0.4, \\ 0 & -0.4 \le x < 0.4, \\ 1 & 0.4 \le x < 0.9, \\ 0 & 0.9 \le x \le 1. \end{cases}$$

We have (i) $\mu_F = \mu_G = 0$ and (ii) $1/3 = \sigma_F^2 \le \sigma_G^2 = 0.443$, but one could easily check that $F \not\succeq_3 G$ and $F \not\preceq_3 G$.

In this paper we construct the following example to show that Property 3.1 holds:

Example 4.8 *Consider*

$$G(x) = x, \quad 0 \le x \le 1 \quad \text{and} \quad F(x) = \begin{cases} 5x/4 & 0 \le x \le 0.1, \\ 3x/4 + 1/20 & 0.1 \le x \le 0.2, \\ x/2 + 1/10 & 0.2 \le x \le 0.35, \\ 3x/2 - 1/4 & 0.35 \le x \le 0.5, \\ 3x/2 - 1/4 & 0.5 \le x \le 0.65, \\ x/2 + 2/5 & 0.65 \le x \le 0.8, \\ 3x/4 + 1/5 & 0.8 \le x \le 0.9, \\ 5x/4 - 1/4 & 0.9 \le x \le 1. \end{cases}$$

In this example, $\mu_F = \mu_G$ and $\sigma_G > \sigma_F$. Now, we set $u(x) = x - \frac{1}{8}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4$ for $0 \le x \le 1$. Then, one can easily show that u is an increasing concave utility function and $E[u(F)] - E[u(G)] \le 0$.

Theorems 3.7 and 3.8 and Corollary 3.3 show that with $\mu_F > \mu_G$, it is possible that there is no FSD or FRSD between F and G, F dominates G in the sense of the third-order SD and RSD from the same direction. We construct the following example to illustrate this relationship.

Example 4.9 *Let F and G be two assets with probabilities f and g, respectively. One could obtain their SD and RSD integrals F_j, F_j^R, G_j and G_j^R by using (2.1). We also define their differentials as*

$$GF_j = G_j - F_j \quad \textit{and} \quad GF_j^R = G_j^R - F_j^R \tag{4.7}$$

Thereafter, we exhibit the results of the SD and RSD integral differentials for the first three orders in the following: The results from Table 1 show that with $\mu_F > \mu_G$ and $\sigma_F > \sigma_G$, there are $F \succ_j G$ and $F \succ_j^R G$ for j=2 and 3 at the same time.

5 Application

Ofek and Richardson (2003), Scheinkman and Xiong (2003) and Hong, *et al.* (2006) have highlighted the insights to be gained from examining investor preferences for traditional versus internet stocks. There are many studies on this issue with respect to risk averse investors (See Fong, *et al.* (2008) and the references therein for more information). However, as far as we know, there have been no studies that consider both risk averters and risk seekers, especially for third-order risk averters and risk seekers. To bridge the gap in the literature, we use the SD theory developed in this paper to analyze the preferences

Table 1: SD and RSD

x	f	g	GF_1	GF_2	GF_3	GF_1^R	GF_2^R	GF_3^R
1	0.1	0.2	0.1	0	0	0	-0.2	-0.8
2	0.2	0.1	0	0.1	0.05	-0.1	-0.1	-0.65
3	0.25	0.15	-0.1	0.1	0.15	0	-0.1	-0.55
4	0.15	0.25	0	0	0.2	0.1	-0.2	-0.4
5	0.1	0.2	0.1	0	0.2	0	-0.2	-0.2
6	0.1	0.1	0.1	0.1	0.25	-0.1	-0.1	-0.05
7	0.1	0	0	0.2	0.4	-0.1	0	0
mean	3.65	3.45						
variance	3.228	2.648						

of risk averters and risk seekers for traditional versus internet stocks, where the S&P 500 index represents the traditional stocks and the NASDAQ 100 index represents the internet stocks. In some existing papers SD tests only compare the dominance of $\hat{F}_n(x)$ with $\hat{G}_n(x)$ and draw inference for the preference of investors between F and G. We note that this is not good enough. In this paper, we include the test of $\mu_X \geq \mu_Y$, to make the SD statistics properly test the true SD relationship among the prospects being compared.

5.1 Data

To illustrate the theory we developed in this paper, we use the daily returns of the S&P 500 (S&P) and NASDAQ 100 (Nasdaq). Data for the S&P 500 and NASDAQ 100 were obtained from Yahoo Finance. The sample covers the period from January 1, 1986 through

December 31, 2015. Furthermore, for robustness checking, we divide the entire period into six sub-periods. In order to compare the results from Fong, *et al.* (2008), we choose the same sub-periods used in Fong, *et al.* (2008), that is, January 1, 1998 to March 9, 2000 (as our second sub-period) and March 10, 2000 to December 31, 2003 (as our third sub-period). Besides the sub-periods specified by Fong *et al.* (2008), we identify four other sub-periods. The first sub-period runs from January 1, 1986 to December 31, 1997. The fourth sub-period runs from trough to peak of the S&P 500 and the fifth sub-period from peak to trough. The sixth sub-period is from January 1, 2009 to December 31, 2015. In this way we can capture and compare the performance between S&P 500 and NASDAQ 100 for different bull and bear markets. One could classify sub-period 1 to be the bull run before the dotcom bubble, sub-period 2 to be the bull run during the dotcom bubble, sub-period 3 to be the bear market during the dotcom bubble, sub-period 4 to be the bull run after the dotcom bubble, sub-period 5 to be the bear market during the recent global financial crisis, and Sub-period 6 to be the bull run after the recent global financial crisis.

We use R_t, at time t defined as $R_t = \ln(P_t/P_{t-1})$ for the daily returns of S&P 500 and NASDAQ 100. We display the time series plots of the daily S&P500 and NASDAQ 100 in Figure 1. Figure 1 confirms that, basically, the first, third, and fifth sub-periods are bull runs while the second and fourth sub-periods are bear markets.

5.2 Mean Variance Analysis

Before we conduct the SD test, we first apply the MV criterion and display the descriptive statistics of the data in Table 4 for the entire periods as well as all the sub-periods. Let $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ be the returns of S&P 500 and Nasdaq 100 indices with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively, for the (sub-)period being studied. The t statistic is used to test the equality of the means of X_i and Y_i while the F statistic is used to test the ratio of the variances, σ_1^2 and σ_2^2 to be unity, respectively.

Markowitz (1952) and Tobin (1958) introduced a MV rule for risk averters such that for any two returns X and Y with means μ_X and μ_Y and standard deviations σ_X and σ_Y, respectively, X is said to dominate Y by the MV rule for risk averters, (denoted by $X\ MV_{RA}\ Y$) if $\mu_X \geq \mu_Y$ and $\sigma_X \leq \sigma_Y$ in which the inequality holds in at least one of the two. On the other hand, Wong (2007) modified the rule to get the MV rule for risk seekers such that X is said to dominate Y by the MV rule for risk seekers (denoted by $X\ MV_{RS}\ Y$) if $\mu_X \geq \mu_Y$ and $\sigma_X \geq \sigma_Y$ in which the inequality holds in at least one of the two.

From Table 4, the daily mean return of the S&P 500 index is not significantly different from that of the Nasdaq index neither for the entire period nor for sub-periods 3, 4, 5 and 6. It is only significantly different for sub-period 2. In the other hand, the standard deviation of daily returns of the S&P index is significantly lower than that of the Nasdaq index over the entire period as well as over all the sub-periods except subperiod 5 where the standard deviation of daily returns of the S&P 500 index is still lower but not statistically significant. We note that our results of sub-period 2 are consistent with the finding in Fong *et al.* (2008) that, on average, the NASDAQ 100 Index outperforms the the S&P 500 index from 1998 to early March 2000. According to the MV criterion, our MV findings imply that 1) risk averters prefer the S&P 500 to the Nasdaq index for the entire period as well as all the sub-periods except sub-period 2; (2) risk averters are indifferent between the S&P 500 and Nasdaq indices in sub-period 2; and (3) risk seekers prefer the Nasdaq 100 index to the S&P 500 index over the entire period as well as over all the sub-periods.

Table 4 further indicates that in general both the S&P 500 and the Nasdaq 100 indices have significant skewness and kurtosis. Interestingly, the distributions of the returns of both the S&P 500 and the Nasdaq 100 indices are skewed to the left in the bull markets and to the right in the bear markets. These results imply that there is a possibility of greater loss in the bull market even though the mean return is positive and a possibility of greater gain in the bear market even though the mean return is negative. The presence of significant skewness (except the skewness of S&P in the fifth sub-period and Nasdaq 100 in the fourth and fifth sub-periods) and kurtosis further supports the hypothesis of non-normality of the return distributions. The highly significant Jarque-Bera statistic for the entire period as well as for each sub-period also confirms that the return distributions of both the S&P 500 and the Nasdaq 100 index are non-normal. This implies that the conclusions drawn from the MV criterion may be misleading. Thus, we turn to the SD theory developed above.

5.3 SD Analysis

5.3.1 SD Tests for Risk Averters

The tests developed by Davidson and Duclos (DD, 2000), Barrett and Donald (BD, 2003), and Linton, *et al.* (LMW, 2005) are the most commonly used statistics to investigate the preferences of risk averters. Bai, *et al.* (2015) modified the DD test to include risk seekers as well as risk averters. Since the test developed by DD is found[2] to be one of the most powerful statistics to test the significance of stochastic dominance, one of the least conservative in size, and one which is also robust to non-i.i.d. observations, in this paper we apply only the modified DD tests in our analysis.

For $j = 1, 2, 3$; one can test the following hypothesis, $H_0 : \ F_j \equiv G_j$, against three alternatives

$$H_1 \ : \ F \not\equiv_j G \ , \quad H_{1l} \ \ F \succ_j G_j \ , \quad \text{and } H_{1r} \ \ F \prec_j G \ , \tag{5.1}$$

for the preferences of risk averters. The three hypotheses are equivalent to $H_1 : F_j(x) \neq G_j(x)$ for some x and both $H_{1l} : F_j(x) \leq G_j(x), \forall x$ and $H_{1r} : F_j(x) \geq G_j(x), \forall x$ and the inequality is strict for at least one interval of x.

Suppose that the sample is drawn such that $\{(f_i, g_i), i = 1, \cdots, m, \ f_k, g_l, k = m + 1, \cdots, N_f; l = m + 1, \cdots, N_g\}$ where f_i and g_i are observations drawn from the returns of the S&P and Nasdaq, denoted by Y and Z with distribution functions F and G, respectively. The integrals F_j and G_j for F and G are defined in (2.1). For a grid of pre-selected points $\{x_k, \ k = 1, \cdots, K\}$, we follow Bai, *et al.* (2015) to use the following

[2]See Lean, *et al.* (2008) and the references therein for more information.

j^{th} order modified DD test statistic, $T_j(x)$ for risk averters to test for H_1, H_{1l}, and H_{1r}:

$$T_j(x) = \frac{\hat{F}_j(x) - \hat{G}_j(x)}{\sqrt{\hat{V}_j(x)}}, \tag{5.2}$$

where

$$\begin{aligned}
\hat{H}_j(x) &= \frac{1}{N_h(j-1)!}\sum_{i=1}^{N_h}(x-h_i)_+^{j-1}\,, \\
\hat{V}_j(x) &= \hat{V}_{F_j}(x) + \hat{V}_{G_j}(x) - 2\hat{V}_{FG_j}(x)\,, \\
\hat{V}_{FG_j}(x) &= \frac{1}{N_f N_g((j-1)!)^2}\sum_{i=1}^{m}(x-f_i)_+^{j-1}(x-g_i)_+^{j-1} - \frac{m}{N_f N_g}\hat{F}_j(x)\hat{G}_j(x)\,, \\
\hat{V}_{H_j}(x) &= \frac{1}{N_h}\left[\frac{1}{N_h((j-1)!)^2}\sum_{i=1}^{N_h}(x-h_i)_+^{2(j-1)} - \hat{H}_j(x)^2\right],\ H = F, G;\ h = f, g\,.
\end{aligned}$$

For a pre-designated finite number of values $\{x_k,\, k = 1, \cdots, K\}$, we test the following weaker hypotheses: H_0^K : $F_j(x_k) = G_j(x_k)$ for all x_k; H_1^K : $F_j(x_k) \neq G_j(x_k)$ for some x_k; H_{1l}^K : $F_j(x_k) \leq G_j(x_k)$ for all x_k and $F_j(x_k) < G_j(x_k)$ for some x_k; and H_{1r}^K : $F_j(x_k) \geq G_j(x_k)$ for all x_k and $F_j(x_k) > G_j(x_k)$ for some x_k. Under the null hypothesis H_0^K, the T_j is computed at each grid point and the null hypothesis, H_0^K, is rejected if $\max_{k \leq K} |T_j(x_k)| > M_{\alpha/2}^K$ for the alternative H_1^K; $\min_{k \leq K} T_j(x_k) < -M_\alpha^K$ for the alternative H_{1l}^K; and $\max_{k \leq K} T_j(x_k) > M_\alpha^K$ for the alternative H_{1r}^K. We follow Bai, *et al.* (2011) to get the simulated critical value M_α^K by using a bootstrap approach.

In this paper we only examine whether there is any first and third-order SD between S&P 500 index and Nasdaq 100 index in this paper. We skip reporting the second order SD because our paper is to study the relationship of the third order SD. The first order SD is reported because this information is useful to examine whether the market is efficient and whether there is any arbitrage opportunity. From Table 5, we observe that there is no first-order SD (FSD) between the S&P 500 index and the Nasdaq 100 index because there is 18.00 percent of the first-order modified SD statistic T_1 is significantly negative in negative domain and 22.00 percent of it is significantly positive in positive domain at the 5 per cent bootstrap simulated critical level. Hence, we conclude that the markets are efficient and there is no arbitrage opportunity between the S&P 500 and Nasdaq 100 indices in the entire period. In addition, from Table 5, we observe that 8.00 (43.00) percent of the third-order modified SD statistic T_3 is significantly negative in the negative (positive) domain and none of it is significantly positive at the 5 per cent bootstrap simulated critical level. Hence, we conclude that there is a dominance of S&P 500 index and Nasdaq 100 index in terms of third order SD (TSD) at the 5 per cent significance level, inferring that third-order risk averters prefer investing in the S&P 500 rather than the Nasdaq 100 index. We also apply the testing procedure by using $\max_x |T_j(x)|$. The inference drawn from this approach leads to the same conclusion. We now investigate the preference for risk averters in each of the sub-periods. Table 5 indicates that there are 11.00, 30.00, 32.00, 25.00, 0 and 10.00 percent significantly positive T_1 and 12.00, 25.00, 26.00, 26.00, 0 and 1.00 percent of significantly negative T_1 in the first, second, third, fourth, fifth, and sixth sub-periods at the 5 per cent bootstrap simulated critical level. Thus, there is no FSD between the S&P 500 index and the Nasdaq 100 index at the 5 per cent significance level in any of the sub-periods. This implies that the markets are efficient and there is no arbitrage opportunity between the S&P 500 and the Nasdaq 100 indices in any sub-period.[3]

In addition, from Table 5, we find that 25.00, 44.00, 87.00, 72.00, 0 and 0 per cent of T_3 are significantly negative in the first, second, third, fourth, fifth and sixth sub-periods, respectively, but none of it is significantly positive at the 5 per cent bootstrap simulated critical level for the sub-periods, respectively, implying that the S&P 500 index stochastically dominates the Nasdaq 100 index in the first, second, third, and fourth sub-periods but not in the fifth and sixth sub-periods in the sense of TSD.

5.3.2 SD Tests for Risk Seekers

We now turn to examine the preferences of risk seekers between the S&P 500 index and the Nasdaq 100 index. We follow Bai, *et al.* (2015) and use the modified DD test statistic for risk seekers, which we call the risk-seeking DD test statistic. Let $\{f_i\}$ $(i = 1, 2, \cdots, N_f)$ and $\{g_i\}$ $(i = 1, 2, \cdots, N_g)$ be observations drawn from the returns of the S&P and Nasdaq, respectively. For a grid of pre-selected points $\{x_k,\ k = 1, \cdots, K\}$ and for $j = 1$, 2 and 3, the j^{th} order risk-seeking DD test statistic, $T_j^R(x)$, is:

$$T_j^R(x) = \frac{\hat{F}_j^R(x) - \hat{G}_j^R(x)}{\sqrt{\hat{V}_j^R(x)}}, \tag{5.3}$$

[3]Readers may refer to Guo *et al.* (2017) and the reference therein for more discussion on arbitrage opportunity.

where

$$\begin{aligned}
\hat{H}_j^R(x) &= \frac{1}{N_h(j-1)!}\sum_{i=1}^{N_h}(h_i-x)_+^{j-1}\,,\\
\hat{V}_j^R(x) &= \hat{V}_{F_j}^R(x)+\hat{V}_{G_j}^R(x)-2\hat{V}_{FG_j}^R(x)\,,\\
\hat{V}_{FG_j}^R(x) &= \frac{1}{N_fN_g((j-1)!)^2}\sum_{i=1}^{m}(f_i-x)_+^{j-1}(g_i-x)_+^{j-1}-\frac{m}{N_fN_g}\hat{F}_j^R(x)\hat{G}_j^R(x)\,,\\
\hat{V}_{H_j}^R(x) &= \frac{1}{N_h}\left[\frac{1}{N_h((j-1)!)^2}\sum_{i=1}^{N_h}(h_i-x)_+^{2(j-1)}-\hat{H}_j^R(x)^2\right],\ H=F,G;\ h=f,g\,;
\end{aligned}$$

in which the integrals ${F_j}^R$ and ${G_j}^R$ are defined in (2.1). For $k=1,\cdots,K$, the following hypotheses are tested for risk seekers: $H_0^R: F_j^R(x_k)=G_j^R(x_k)$ for all x_k; $H_1^R: F_j^R(x_k)\neq G_j^R(x_k)$ for some x_k; $H_{1l}^R: F_j^R(x_k)\geq G_j^R(x_k)$ for all x_k, and $F_j^R(x_k)>G_j^R(x_k)$ for some x_k; and $H_{2r}^R: F_j^R(x_k)\leq G_j^R(x_k)$ for all x_k and $F_j^R(x_k)<G_j^R(x_k)$ for some x_k.

To implement the risk-seeking DD test, T_j^R, one can test the following hypotheses at each grid point being computed: $H_0^R: F_j^R\equiv G_j^R$, against three alternatives

$$H_1^R\quad F\not\equiv_j^R G,\quad H_{1l}^R\ :\ F\succ_j^R G_j,\quad \text{and } H_{1r}^R\ \ F\prec_j^R G\,. \tag{5.4}$$

The three hypotheses are equivalent to $H_1^R: F_j^R(x)\neq G_j^R(x)$, for some x and H_{1l}^R : $F_j^R(x)\geq G_j^R(x)$,$\forall x$ and $H_{1r}^R: F_j^R(x)\leq G_j^R(x)$, $\forall x$ and the inequality is strict for at least one interval of x.

To test the hypotheses (5.4), we reject the null hypothesis H_0^R if $\max_{a<x<b}|T_j^R(x)|>M_{\alpha/2}^R$ for the alternative H_1^R; $\max_{a<x<b}T_j^R(x)>M_\alpha^R$ for the alternative H_{1l}^R; and $\min_{a<x<b}T_j^R(x)<-M_\alpha^R$ for the alternative H_{1r}^R. We follow Bai, *et al.* (2011) to get the simulated critical value M_α^K by using a bootstrap approach.

Since the conclusion drawn from testing the hypotheses in (5.4) for risk seekers is the same as that drawn from testing the hypotheses in (5.1) for risk averters for $j = 1$, we skip reporting our analysis for testing the hypotheses in (5.4) for $j = 1$ and only discuss our analysis for testing the hypotheses in (5.4) for risk seekers for $j = 3$. We first examine the preference of third-order risk seekers between S&P 500 and Nasdaq 100 indices in the entire period. To do so, we employ the third-order RSD statistic, T_3^R, for risk seekers as stated in (5.3) to analyze the preferences of the third-order risk seekers between S&P 500 index and Nasdaq 100 index and report the result in Table 5. From the table, we find that 14.00 (57.00) percent of T_3^R is significantly negative at the positive (negative) domain, and no portion of T_3^R is significantly positive at the 5 percent significant level, implying that Nasdaq 100 index TRSD dominates S&P 500 index in the entire period.

We now examine the preferences of third-order risk seekers between the S&P 500 and the Nasdaq 100 indices in all the sub-periods. From Table 5, we find that 64.00, 86.00, 56.00, 75.00, 0 and 59.00 percent of T_3^R are significantly negative and none of it is significantly positive at the 5 per cent bootstrap simulated critical level for the first, second, third, fourth, fifth, and sixth sub-periods, respectively. This implies that the Nasdaq 100 index TRSD dominates the S&P 500 index at the 5 per cent significant level in all the sub-periods except the fifth subperiod.

Overall, the results from the SD tests for both risk averters and risk seekers imply that the markets are efficient, and there is no arbitrage opportunity between the S&P 500 and the Nasdaq 100 indices neither in the entire period nor in any sub-period, including any *bull run, bear market, the dotcom bubble, and the recent financial crisis. Nevertheless, the third-order risk averters prefer investing in the S&P 500 index to the Nasdaq 100 index while the third-order risk seekers prefer investing in the Nasdaq 100 index to the S&P 500 index in the entire period as well as in the first, second, third, and fourth sub-periods. However, both the third-order risk averters and risk seekers are indifferent between the S&P 500 and Nasdaq 100 indices in the fifth sub-period. Last, the third-order risk averters are indifferent between the S&P 500 and Nasdaq 100 indices but the third-order risk seekers prefer investing in the Nasdaq 100 index to the S&P 500 index in the sixth sub-period.*

5.3.3 Completed SD Tests for Risk Averters and Risk Seekers

The conclusion highlighted in italic in Section 5.3.2 is drawn by conducting the j^{th} order modified DD test statistic, $T_j(x)$, in (5.2) and the risk-seeking DD test statistic, $T_j^R(x)$, in (5.3). We note that this is the approach recommended by Bai, *et al.* (2015) and others and has been used in several studies. For example, Qiao, *et al.* (2012) apply this approach and conclude that spot dominates futures for risk averters while futures dominates spot for risk seekers in the sense of the second and third order SD. We now analyze whether this approach is correct.

We note that this is correct for $j = 1$ and 2 but not correct for $j = 3$. To see why this is not correct for $j = 3$, one can refer to Part 2 of Definition 2.1 that $X \succ_3 Y$ or $F \succ_3 G$ if and only if (i) $H_0^1 : F_3(x) \leq G_3(x)$ for each x in $[a, b]$, $F_3(x) < G_3(x)$ for at least one interval of x in $[a, b]$, and (ii) $H_0^2 : \mu_X \geq \mu_Y$. Similarly, Part 2 of Definition 2.2

tells us that $X \succ_3^R Y$ or $F \succ_3^R G$ if and only if (i*) $H_0^{1*}: F_3^R(x) \geq G_3^R(x)\ \forall\ x \in [a,b]$, $F_3^R(x) > G_3^R(x)$ for at least one interval of x in $[a,b]$, and (ii) $H_0^2 : \mu_X \geq \mu_Y$.

The conclusion highlighted in italic in Section 5.3.2 is obtained by testing H_0^1 for risk averters and H_0^{1*} for risk seekers but the hypothesis that $H_0^2\ \mu_X \geq \mu_Y$ has not been tested. Thus, in order to complete the SD tests, we must test the hypothesis H_0^2 whether $\mu_X \geq \mu_Y$. To do so, we need to consider the joint Type I error rate to test for both H_0^1 and H_0^2 or to test for both H_0^{1*} and H_0^2, that is, the probability that a randomly chosen sample (of the given size, satisfying the appropriate model assumptions) will give a Type I error for at least one of the hypothesis tests performed. One easy approach to fix the problem is to use the Bonferroni correction method (Bonferroni, 1935).[4] The Bonferroni correction method is based on the idea that if an experimenter is testing m hypotheses, then one way of maintaining the family-wise error rate is to test each individual hypothesis at a statistical significance level of $1/m$ times what it would be if only one hypothesis were tested. So, if the desired significance level for the whole family of tests should be (at most) α, then the Bonferroni correction would test each individual hypothesis at a significance level of α/m. In our case, a trial is testing two hypotheses with a desired $\alpha = 0.1$, then the Bonferroni correction would test each individual hypothesis at $\alpha = 0.1/2 = 0.05$.

The conclusion drawn in Section 5.3.1 for risk averters is obtained by testing H_0^1 for risk averters. In order to complete the SD test for risk averters, we have to test for the hypothesis that $H_0^2\ \mu_X \geq \mu_Y$. The result for testing $H_0^2\ \mu_X \geq \mu_Y$ has already been

[4]Besides using the Bonferroni correction method, there are some other multiple comparison approaches that can be used. Readers may refer to Miller (1981) for more information.

reported in Table 4. From the table, we find that the daily mean return of S&P 500 index is not significant different from that of the Nasdaq 100 index for the entire period as well as for any sub-period except Sub-period 2 from Jan 1, 1998 to Mar 9, 2000 – a period of the bull market for the dotcom bubble in which the daily mean return of S&P 500 index is significantly smaller than that of the Nasdaq 100 index at $\alpha = 0.05$.

Thus, at $\alpha = 0.10$, all the conclusion drawn in Section 5.3.1 for risk averters is correct except that in Subperiod 2 in which we have to change the original conclusion:

"the third-order risk averters prefer investing in the S&P 500 index to the Nasdaq 100 index in the second sub-period"

to

"the third-order risk averters are indifferent between the S&P 500 and Nasdaq 100 indices in the second sub-period".

On the other hand, the conclusion drawn in Section 5.3.2 for risk seekers is obtained by testing H_0^{1*} for risk seekers but the hypothesis $H_0^2\ \mu_X \geq \mu_Y$ has not been tested. Thus, in order to complete the SD test for risk seekers, we have to test for the hypothesis that $H_0^2\ \mu_X \geq \mu_Y$. The result for testing $H_0^2\ \mu_X \geq \mu_Y$ has already been reported in Table 4. From the table, we conclude that the daily mean return of the Nasdaq 100 index is either the same or bigger than that of the S&P 500 index. Thus, at $\alpha = 0.10$, all the conclusion drawn in Section 5.3.2 for risk seekers holds.

Based on this, we correct the conclusion for both risk averters and risk seekers as follows:

Overall, the result from the SD tests for both risk averters and risk seekers implies that the markets are efficient, and there is no arbitrage opportunity between the S&P 500 and Nasdaq 100 indices in the entire period and in any sub-period, including any bull run, bear market, the dot-com bubble, and the recent financial crisis. Nevertheless, the third-order risk averters prefer investing in the S&P 500 index to the Nasdaq 100 index while the third order risk seekers prefer investing in the Nasdaq 100 index to the S&P 500 index in the entire period as well as in the first, third, and fourth sub-periods. However, both the third order risk averters and risk seekers are indifferent between the S&P 500 and Nasdaq 100 indices in the fifth sub-period. Finally, the third-order risk averters are indifferent between the S&P 500 and the Nasdaq 100 indices but the third-order risk seekers prefer investing in the Nasdaq 100 index to the S&P 500 index in the second and sixth sub-periods.

6 Concluding Remarks

This paper develops and studies several interesting new properties of TSD for risk-averse and risk-seeking investors. We show that the means of the assets being compared should be included in the definition of TSD for both risk-averse and risk-seeking investors. We extend the second order SD (SSD) reversal result of Levy and Levy (2002) for two assets that have the same mean to TSD and show that the dominance relationship can be in the same direction as well as being reversed. We derive the conditions on the order of the variances of two assets for TSD and TRSD under the condition of equal means. We provide examples to illustrate all the properties developed in this paper and show they can be applied in an empirical comparison of the S&P 500 and Nasdaq 100 indices.

Another contribution in this paper is that besides comparing the dominance of the integrals of two different distributions, we show that the dominance of the means for the distributions should also be checked to draw inference of SD for third order risk averters and risk seekers. We illustrate this idea by comparing the preferences of the S&P 500 and the Nasdaq 100 indices for the third-order risk averters and risk seekers. We find that, in general, the third-order risk averters prefer investing in the S&P 500 index to the Nasdaq 100 index while the third-order risk seekers prefer investing in the Nasdaq 100 index to the S&P 500 index in the entire period as well as many of the sub-periods. Interestingly, however, these preferences can vary, depending on economic conditions. For example, both the third-order risk averters and risk seekers are indifferent between the S&P 500 and Nasdaq 100 indices in the bear market during the recent global financial crisis. However, the third-order risk averters are indifferent between the S&P 500 and Nasdaq 100 indices but the third-order risk seekers prefer investing in the Nasdaq 100 index to the S&P 500 index in the bull run during the dotcom bubble and in the bull run after the recent global financial crisis.

We note that global risk aversion has been criticized for not describing how investors actually behave. For example, examining the relative attractiveness of various forms of investments, Friedman and Savage (1948) noticed that the strictly concave functions may not be able to explain why investors buy insurance or lottery tickets. Hartley and Farrell (2002) and others proposed using global convex utility functions, the functions for risk seekers, to indicate risk-seeking behavior. Markowitz (1952) addressed Friedman and Savage's concern and proposed a utility function that has convex and concave regions in both the positive and the negative domains. Williams (1966) reported data whereby a translation of outcomes produces a dramatic shift from risk aversion to risk seeking, while Fishburn and Kochenberger (1979) documented the prevalence of risk seeking in choices between negative prospects. Post and Levy (2005) concluded that investors are risk averse in bear markets and risk seeking in bull markets.

In reality, investor utility functions could be very complicated. They could be concave in some regions and convex in others or even a combination of different concave and convex functions. In this paper, it is not our intention to prove that the utility functions of investors are concave, convex, sometimes concave and sometimes convex, etc. In this paper we have developed a set of results for pure concave and convex utility functions. However, the results we developed could be useful for research using utility functions that are concave in some regions and convex in other regions, or are combinations of different concave and convex functions. In addition, if investors with concave or convex utility functions do exist in the market, then the results we developed in this paper can be applied directly. In fact, there are some studies that find evidence to support the argument that both risk averters and risk seekers could exist in the markets. For example, examining the Taiwan spot and futures markets, Qiao, *et al.* (2012) found that the second- and third-order risk averters prefer investing in spot to futures while the second- and third-order risk seekers prefer investing in futures to spot. Qiao, *et al.* (2013) further examined the issue and noticed that the conclusion drawn in Qiao, *et al.* (2012) is only true for the emerging markets, not for the mature markets in which there is no dominance between spot and futures. Lean, *et al.* (2015) also revealed that risk-averse investors prefer the oil spot, whereas risk seekers prefer to invest in oil futures for the entire period as well as for the sub-period before the 2008 Global Financial Crisis (GFC) and the sub-period during and after the GFC. In addition, some academics, for example, Clark, *et al.* (2016) find evidence in the Taiwan stock and futures markets for investors with S-shaped and reverse S-shaped utility functions. It turns out that their findings support the existence of risk averters and risk seekers in the markets. We note that their findings do not imply that there are risk seekers in the markets. Their findings can only imply that if there are risk seekers in the market, they will prefer futures to spot. In practice, there are many investors that prefer futures to spot. Thus, this shows that among those who buy futures, some could be risk seekers.

We also note that the SD theory developed in our paper could be used to explain some financial anomalies. For example, Jegadeesh and Titman (1993) documented a financial anomaly on momentum profit in stock markets that extreme movements in stock prices will be followed by subsequent price movements in the same direction. In other words, former winners continue to win and former losers continue to lose. If investors know that past winners continue to win and past losers continue to lose, they would buy winners and sell losers, thereby driving up the price of winners relative to losers until the market price of winners relative to losers is high enough to make the momentum profit disappear.

However, after many years and many studies momentum is still an empirical reality.[5] We note that SD theory could well explain this financial anomaly. For example, Fong, *et al.* (2005) found that risk averters will prefer investing in winners to losers while Sriboonchita, *et al.* (2009) concluded that risk seekers prefer investing in losers to winners. This finding could explain why the momentum profit could persist after discovery. If risk averters and risk seekers do both exist in the markets, risk averters prefer to invest in winners while risk seekers prefer to invest in losers. Thus, both risk averters and risk seekers would get what they want in the market and will not drive up the price of winners or drive down the price of losers, thereby allowing the persistence of momentum profits after discovery of the anomaly. Besides financial anomalies, the theory developed in our paper could also be used to examine and compare different investment strategies, such as optimization versus stock picking, or market timing versus buy and hold, etc.

For example, there is a question of whether the efficient portfolio theory developed by Markowitz (1952b) is correct or the idea from De Miguel, *et al.* (2009) on equally weighted portfolios is better. Following the Markowitz (1952b) theory, the portfolios on the efficient frontier should outperform the equally weighted portfolio. However, De Miguel *et al.* (2009) suggest that the naive equally weighted portfolio would outperform efficient portfolios. Actually, the paper De Miguel, *et al.* (2009) is not the first paper pointing the

[5]It is a well established empirical fact going back to the Victorian age on UK data (see: Chabot, Ghysels, and Jagannathan, 2009), over two centuries on US equity data (see: Geczy and Samonov, 2013) and many years of out-of-sample testing in at least 40 other countries (see: Asness, Moskowitz and Pedersen, 2013).

problem. Other previous papers have also discussed it. For example, Frankfurter *et al.* (1971) find that the portfolio selected according to the Markowitz MV criterion is likely to be less effective than an equally weighted portfolio. To answer this question, recently Hoang, *et al.* (2015) examine whether the ideas from both Markowitz (1952b) and De Miguel, *et al.* (2009) can be applied to portfolio selection on Chinese stock and gold markets. They find that risk averters prefer efficient portfolios while risk-seekers prefer an equally weighted portfolio in their study. In other words, to the question of whether the theory developed by Markowitz (1952b) or the idea from Frankfurter *et al.* (1971), De Miguel, *et al.* (2009), and others is correct, the findings from Hoang, *et al.* (2015) suggest that both theories could be correct in the way that the former fits more to risk averters while the latter is more suitable to risk seekers.

We note that the theory of risk averters and risk seekers could be used to address some important issues in economics and finance, especially since the statistics to test SD for risk averters and risk seekers are available (see, for example, Bai, *et al.* (2015)). For example, the different order of the SD tests could be used to compare the different order of income distributions from poor to rich, while the different order of the RSD test could be used to compare the different order of income distributions from rich to poor. These measurements provide specific information about the nature of income disparity between rich and poor for different countries and policy makers could use the information provided by the SD and TSD tests to develop policies to reduce the income disparity. This is another important practical contribution of SD and RSD tests. Last, we note that the theory and test developed in our paper could be easily extended to any higher order, not only for the third order. We also note that our model is useful for academics, practitioners, and decision makers in their studies of different issues, for example, portfolio selection (Guo, *et al.,* 2018a) and risk analysis (Guo, *et al.,* 2018b).

Appendices

Proof of Theorem 3.1.

We only prove the necessary condition for Part 1 of Theorem 3.1. One could easily modify the proof from Huang and Litzenberger (1988) to prove the sufficient condition. In addition, we only prove Part 1 of Theorem 3.1. One could easily modify our proof to obtain that for Part 2 of Theorem 3.1.

$$\begin{aligned}
\Delta Eu &\equiv u(F) - u(G) \equiv \int_a^b u(x)dF(x) - \int_a^b u(x)dG(x) \\
&= [F(x) - G(x)]u(x)|_a^b - \int_a^b [F(x) - G(x)]u^{(1)}(x)\,dx \\
&= \int_a^b [G(x) - F(x)]u^{(1)}(x)\,dx = \int_a^b [G_1(x) - F_1(x)]u^{(1)}(x)\,dx \qquad \text{(A.1)} \\
&= [G_2(x) - F_2(x)]u^{(1)}(x)\Big|_a^b - \int_a^b [G_2(x) - F_2(x)]u^{(2)}(x)\,dx \\
&= A_1 + \int_a^b [F_2(x) - G_2(x)]u^{(2)}(x)\,dx \qquad \text{(A.2)} \\
&= A_1 + \int_a^b u^{(2)}(x)\,d\,[F_3(x) - G_3(x)] \\
&= A_1 + [F_3(x) - G_3(x)]u^{(2)}(x)\Big|_a^b - \int_a^b [F_3(x) - G_3(x)]u^{(3)}(x)\,dx \\
&= A_1 + A_2 + \int_a^b [G_3(x) - F_3(x)]u^{(3)}(x)\,dx \qquad \text{(A.3)}
\end{aligned}$$

where

$$A_1 = [G_2(b) - F_2(b)]u^{(1)}(b) \quad \text{and} \quad A_2 = [F_3(b) - G_3(b)]u^{(2)}(b)\,. \qquad \text{(A.4)}$$

If $F \succeq_1 G$ then $F_1(x) \leq G_1(x)$ for all x. If $u \in U_1$ then $u^{(1)} \geq 0$. Hence, from (A.1), we have $\Delta Eu = u(F) - u(G) \geq 0$ if $F \succeq_1 G$ and $u \in U_1$.

If $F \succeq_2 G$, then $F_2(x) \leq G_2(x)$ for all x. If $u \in U_2$ then $u^{(1)} \geq 0$ and $u^{(2)}(x) \leq 0$ for x. From (A.4), $A_1 \geq 0$, and hence from (A.2), $\Delta Eu = u(F) - u(G) \geq 0$.

If $F \succeq_3 G$, then $F_3(x) \leq G_3(x)$ for all x and $\mu_X \geq \mu_Y$. If $u \in U_3$ then $u^{(1)} \geq 0$, $u^{(2)}(x) \leq 0$ and $u^{(3)} \geq 0$. From (A.4), we have $A_2 \geq 0$. As $\mu_X \geq \mu_Y$ implies $\int_a^b F(t)dt \leq \int_a^b G(t)dt$, which is $F_2(b) \leq G_2(b)$, then we have $A_1 \geq 0$. Hence, from (A.3), $\Delta Eu = u(F) - u(G) \geq 0$.

□

Proof of Theorem 3.5.

First, we note that

$$\begin{aligned}
\mu_F = \mu_G &\iff \int_a^b t dF(t) = \int_a^b t dG(t) \\
&\iff tF(t)|_a^b - \int_a^b F(t)dt = tG(t)|_a^b - \int_a^b G(t)dt \\
&\iff b - \int_a^b F(t)dt = b - \int_a^b G(t)dt \\
&\iff \int_a^b F(t)dt = \int_a^b G(t)dt. \qquad \text{(A.5)}
\end{aligned}$$

Defining

$$D_i(t) = G_i(t) - F_i(t) = \int_a^t G_{i-1}(z) - F_{i-1}(z)dz \ , \quad \text{for } i = 1, 2, 3$$

we get

$$\begin{aligned}
G_3(b) - F_3(b) &= \int_a^b D_2(z)dz = \int_a^b \int_a^z D_1(y)dydz \\
&= z \int_a^z D_1(y)dy \bigg|_a^b - \int_a^b zd\Big(\int_a^z D_1(y)dy\Big) \\
&= b \int_a^b D_1(y)dy - \int_a^b zD_1(z)dz \ . \qquad \text{(A.6)}
\end{aligned}$$

By using (A.5), we have

$$\mu_F = \mu_G \iff \int_a^b G(y) - F(y)dy = 0 = \int_a^b D_1(y)dy \ . \qquad \text{(A.7)}$$

Also,

$$\begin{aligned}\sigma_F^2 &= \int_a^b (x-\mu_F)^2 dF(x) \\ &= (x-\mu_F)^2 F(x)\Big|_a^b - \int_a^b 2(x-\mu_F)F(x)dx \\ &= (b-\mu_F)^2 - 2\int_a^b xF(x)dx + 2\mu_F \int_a^b F(x)dx\end{aligned}$$

and similarly,

$$\sigma_G^2 = (b-\mu_G)^2 - 2\int_a^b xG(x)dx + 2\mu_G \int_a^b G(x)dx \ .$$

by using (A.5) again,

$$\sigma_F^2 - \sigma_G^2 = 2\int_a^b x\big[G(x)-F(x)\big]dx = 2\int_a^b xD_1(x)\ dx \ . \tag{A.8}$$

Thereafter, plugging (A.7) and (A.8) into (A.6), we get

$$\begin{aligned}G_3(b) - F_3(b) &= b\int_a^b D_1(y)dy - \int_a^b zD_1(z)dz \\ &= -\int_a^b zD_1(z)dz = \frac{1}{2}\big(\sigma_G^2 - \sigma_F^2\big) \ .\end{aligned}$$

The result of

$$G_3^R(a) - F_3^R(a) = \frac{1}{2}\big(\sigma_G^2 - \sigma_F^2\big)$$

can be obtained similarly. Thus, the assertion Theorem 3.5 holds. □

Proof of Theorem 3.6.

Let's suppose $F_3(b) = G_3(b)$ first. Then for all $a \le x \le b$, using (A.5), we have

$$\begin{aligned}
& F_3(x) - G_3(x) - [G_3^R(x) - F_3^R(x)] \\
= & \int_a^x \int_a^y [F(t) - G(t)]dtdy - \int_x^b \int_y^b [F(t) - G(t)]dtdy \\
= & \int_a^x \int_a^y [F(t) - G(t)]dtdy + \int_x^b \int_a^y [F(t) - G(t)]dtdy \\
= & \int_a^b \int_a^y [F(t) - G(t)]dtdy = \int_a^b \int_a^y [F_1(t) - G_1(t)]dtdy \\
= & \int_a^b [F_2(y) - G_2(y)]dy = F_3(b) - G_3(b) = 0,
\end{aligned}$$

then using Theorem 3.5, we get

$$G_3(b) - F_3(b) \;=\; \frac{1}{2}\left(\sigma_G^2 - \sigma_F^2\right) .$$

where the last equality follows from the assumption.

Similarly, if $F_3^R(a) = G_3^R(a)$ holds, then using (A.5) again, we have, for all $a \le x \le b$,

$$\begin{aligned}
& G_3^R(x) - F_3^R(x) - [F_3(x) - G_3(x)] \\
= & \int_x^b \int_y^b [F(t) - G(t)]dtdy - \int_a^x \int_a^y [F(t) - G(t)]dtdy \\
= & \int_x^b \int_y^b [F(t) - G(t)]dtdy + \int_a^x \int_y^b [F(t) - G(t)]dtdy \\
= & \int_a^b \int_y^b [F(t) - G(t)]dtdy = \int_a^b \int_y^b [G_1^R(t) - F_1^R(t)]dtdy \\
= & \int_a^b [G_2^R(y) - F_2^R(y)]dy = G_3^R(a) - F_3^R(a) = 0,
\end{aligned}$$

then using Theorem 3.5

$$G_3^R(a) - F_3^R(a) = \frac{1}{2}\left(\sigma_G^2 - \sigma_F^2\right)$$

where the last equality is established from the assumption. $\square$

Proof of Theorem 3.7.

$[F_3(x) - G_3(x)] - [G_3^R(x) - F_3^R(x)]$

$= \int_a^x \int_a^y [F(t) - G(t)]dtdy - \int_x^b \int_y^b [F(t) - G(t)]dtdy$

$= \int_a^x \int_a^y [F(t) - G(t)]dtdy - \int_x^b \int_a^b [F(t) - G(t)]dtdy + \int_x^b \int_a^y [F(t) - G(t)]dtdy$

$= \int_a^b \int_a^y [F(t) - G(t)]dtdy - \int_x^b \int_a^b [F(t) - G(t)]dtdy$

$= [F_3(b) - G_3(b)] - (b - x)[\mu_G - \mu_F]$

since $\mu_G - \mu_F$

$= \int_a^b tdG(t) - \int_a^b tdF(t)$

$= tG(t)|_a^b - \int_a^b G(t)dt - [tF(t)|_a^b - \int_a^b F(t)dt]$

$= \int_a^b [F(t) - G(t)]dt$

so

$[G_3^R(x) - F_3^R(x)] = [F_3(x) - G_3(x)] - [F_3(b) - G_3(b)] - (b - x)[\mu_F - \mu_G]$

Thus, if $F_3(b) - G_3(b) = 0$ that is $\frac{1}{2}\left(\sigma_F^2 - \sigma_G^2\right) = 0$ by using Theorem 3.5 and $\mu_F - \mu_G \geq 0$, then $F_3(x) - G_3(x) \leq 0$ implies $G_3^R(x) - F_3^R(x) \leq 0$, that is, $F \succeq_3 G$ implies $F \succeq_3^R G$.

$\square$

Proof of Theorem 3.8.

$[G_3^R(x) - F_3^R(x)] - [F_3(x) - G_3(x)]$

$= \int_x^b \int_y^b [F(t) - G(t)]dtdy - \int_a^x \int_a^y [F(t) - G(t)]dtdy$

$= \int_x^b \int_y^b [F(t) - G(t)]dtdy - \int_a^x \int_a^b [F(t) - G(t)]dtdy + \int_a^x \int_y^b [F(t) - G(t)]dtdy$

$= \int_a^b \int_y^b [F(t) - G(t)]dtdy - \int_a^x \int_a^b [F(t) - G(t)]dtdy$

$= [G_3^R(a) - F_3^R(a)] - (x-a)[\mu_G - \mu_F]$

so

$[F_3(x) - G_3(x)] = [G_3^R(x) - F_3^R(x)] - [G_3^R(a) - F_3^R(a)] - (x-a)[\mu_F - \mu_G]$

Thus, if $G_3^R(a) - F_3^R(a) = 0$ that is $\frac{1}{2}\left(\sigma_G^2 - \sigma_F^2\right) = 0$ by using Theorem 3.5 and $\mu_F - \mu_G \geq 0$, then $G_3^R(x) - F_3^R(x) \leq 0$ implies $F_3(x) - G_3(x) \leq 0$, that is, $F \succeq_3^R G$ implies $F \succeq_3 G$.

$\square$

Figure 1: Time Series Plots of the S&P 500 and the NASDAQ 100

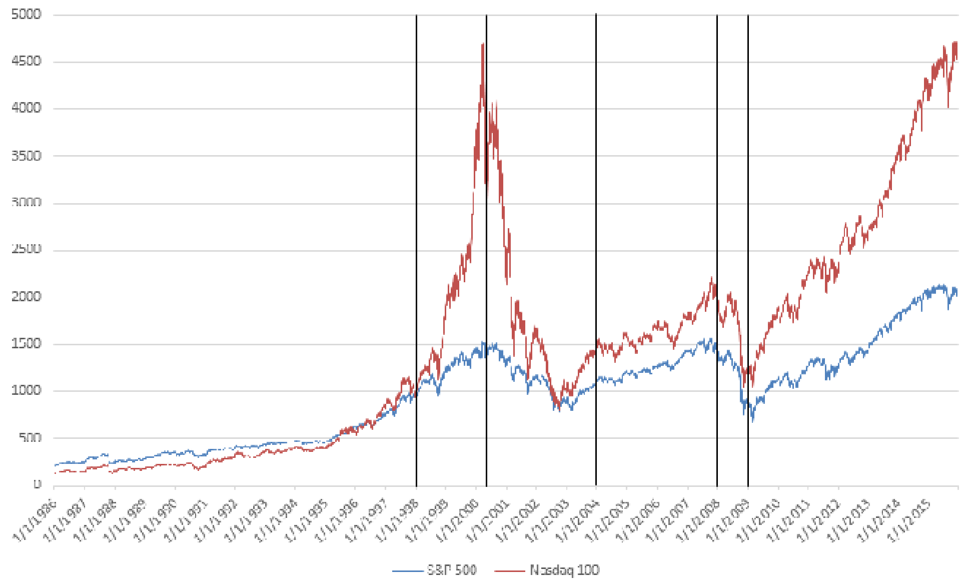

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Table 2: The Profits of two Locations and their SD and RSD Integral Differentials

Profit (in million)	Probability		SD Integral Differentials			RSD Integral Differentials		
x	f	g	GF_1	GF_2	GF_3	GF_1^R	GF_2^R	GF_3^R
1	0.5	0.5	0	0	0	0	0	0.1
2	0	0	0	0	0	0	0	0.1
3	0	0.1	0.1	0	0	0	0	0.1
4	0.25	0.05	-0.1	0.1	0.05	-0.1	0.1	0.05
5	0.25	0.35	0	0	0.1	0.1	0	0

Note: The integral differentials GF_j and GF_j^R are defined in (4.6) for $j = 1, 2$ and 3.

Table 3: The SD and RSD integral differentials for the gain of investing in Stocks A and B.

Profit (in million)	Probability		SD Integral Differentials			RSD Integral Differentials		
x	f	g	GF_1	GF_2	GF_3	GF_1^R	GF_2^R	GF_3^R
-1	0	0	0	0	0	0	-0.35	-0.6625
-0.5	0	0.1	0.1	0	0	0	-0.35	-0.4875
0	0	0.1	0.2	0.05	0.0125	-0.1	-0.3	-0.325
0.5	0.3	0.1	0	0.15	0.0625	-0.2	-0.2	-0.2
1	0	0.2	0.2	0.15	0.1375	0	-0.2	-0.1
2	0.3	0.1	0	0.35	0.3875	-0.2	0	0
5	0.4	0.4	0	0.35	1.4375	0	0	0

Note: The integral differentials GF_j and GF_j^R are defined in (4.6) for $j = 1, 2$ and 3.

Table 4: Descriptive Statistics of S&P 500 Index and Nasdaq 100 index Returns for Entire period and Six Sub-periods

	Mean	Std Dev	Skewness	Kurtosis	J-B	t-test/F-test
Entire period, Jan 1, 1986 to Dec 31, 2015						
S&P 500	0.000300 **	0.011658	-1.2705***	30.488***	240180***	-0.71233
Nasdaq 100	0.000468**	0.017016	-0.10092***	10.431***	17416***	0.46944***
Sub-period 1, Jan 1, 1986 to Dec 31, 1997						
S&P 500	0.000502***	0.010080	-4.2778***	97.900***	1148100***	-0.52836
Nasdaq 100	0.000663***	0.0134377	-0.94909***	15.816***	21225***	0.56265***
Sub-period 2, Jan 1, 1998 to Mar 9, 2000						
S&P 500	0.000667	0.012366	-0.37895***	5.5380***	161.07***	-1.9938**
Nasdaq 100	0.002781***	0.021593	-0.37663***	3.9992***	35.947***	0.32796***
Sub-period 3, Mar 10, 2000 to Dec 31, 2003						
S&P 500	-0.000242	0.013763	0.18523**	4.3630***	79.551***	0.90135
Nasdaq 100	-0.001190	0.0294981	0.35123***	4.9148***	165.87***	0.21767***
Sub-period 4, Jan 1, 2004 to Dec 31, 2007						
S&P 500	0.000276	0.007612	-0.30922***	4.7984***	151.61***	-0.17626
Nasdaq 100	0.000349	0.010572	-0.11797	3.5064***	13.082***	0.51839***
Sub-period 5, Jan 1, 2008 to Dec 31, 2008						
S&P 500	-0.001921	0.025840	-0.033726	6.6754***	142.45***	0.09633
Nasdaq 100	-0.002145	0.026626	0.12476	6.4517***	126.25 ***	0.94186
Sub-period 6, Jan 1, 2009 to Dec 31, 2015						
S&P 500	0.000463*	0.011334	-0.26878***	7.4485***	1474.1***	-0.74355
Nasdaq 100	0.000756***	0.012034	-0.15977***	5.9164***	631.95***	0.88719**

*** significant at 1% level, ** significant at 5% level, * significant at 10%. The last column indicates the result of t-test and F-test. The upper statistic is the result of t-test while the lower indicates the results of F-test. The t-statistic tests the equality of means of S&P 500 and Nasdaq 100 and the F-statistic tests the equality of variances of S&P 500 and Nasdaq 100.

Table 5: Modified first and third order SD and RSD test statistic for Entire period and Six sub-periods

Entire period	FSD		FRSD		TSD		TRSD	
	$T_1 > 0$	$T_1 < 0$	$T_1^R > 0$	$T_1^R < 0$	$T_3 > 0$	$T_3 < 0$	$T_3^R > 0$	$T_3^R < 0$
Total(%)	22.00	18.00	18.00	22.00	0	51.00	0	71.00
+ve Domain(%)	22.00	0	0	22.00	0	43.00	0	14.00
−ve Domain(%)	0	18.00	18.00	0	0	8.00	0	57.00
$\max(\lvert T_j \rvert)$	20.21	19.82	19.82	20.21	1.00	14.91	N/A	16.79
Sub-period 1	**FSD**		**FRSD**		**TSD**		**TRSD**	
	$T_1 > 0$	$T_1 < 0$	$T_1^R > 0$	$T_1^R < 0$	$T_3 > 0$	$T_3 < 0$	$T_3^R > 0$	$T_3^R < 0$
Total(%)	11.00	12.00	12.00	11.00	0	25.00	0	64.00
+ve Domain(%)	11.00	0	0	11.00	0	23.00	0	8.00
−ve Domain(%)	0	12.00	12.00	0	0	2.00	0	56.00
$\max(\lvert T_j \rvert)$	13.26	12.85	12.85	13.26	1.00	5.48	N/A	13.16
Sub-period 2	**FSD**		**FRSD**		**TSD**		**TRSD**	
	$T_1 > 0$	$T_1 < 0$	$T_1^R > 0$	$T_1^R < 0$	$T_3 > 0$	$T_3 < 0$	$T_3^R > 0$	$T_3^R < 0$
Total(%)	30.00	25.00	25.00	30.00	0	44.00	0	86.00
+ve Domain(%)	30.00	0	0	30.00	0	26.00	0	25.00
−ve Domain(%)	0	25.00	25.00	0	0	18.00	0	61.00
$\max(\lvert T_j \rvert)$	10.25	7.49	7.49	10.25	N/A	7.34	N/A	11.71

Sub-period 3	FSD		FRSD		TSD		TRSD	
	$T_1 > 0$	$T_1 < 0$	$T_1^R > 0$	$T_1^R < 0$	$T_3 > 0$	$T_3 < 0$	$T_3^R > 0$	$T_3^R < 0$
Total(%)	32.00	26.00	26.00	32.00	0	87.00	0	56.00
+ve Domain(%)	32.00	0	0	32.00	0	63.00	0	19.00
−ve Domain(%)	0	26.00	26.00	0	0	24.00	0	37.00
$\max(\lvert T_j\rvert)$	12.02	14.40	14.40	12.02	N/A	12.96	N/A	9.16

Sub-period 4	FSD		FRSD		TSD		TRSD	
	$T_1 > 0$	$T_1 < 0$	$T_1^R > 0$	$T_1^R < 0$	$T_3 > 0$	$T_3 < 0$	$T_3^R > 0$	$T_3^R < 0$
Total(%)	25.00	26.00	26.00	25.00	0	72.00	0	75.00
+ve Domain(%)	25.00	0	0	25.00	0	50.00	0	25.00
−ve Domain(%)	0	26.00	26.00	0	0	22.00	0	50.00
$\max(\lvert T_j\rvert)$	8.32	8.19	8.19	8.32	N/A	9.30	N/A	10.93

Sub-period 5	FSD		FRSD		TSD		TRSD	
	$T_1 > 0$	$T_1 < 0$	$T_1^R > 0$	$T_1^R < 0$	$T_3 > 0$	$T_3 < 0$	$T_3^R > 0$	$T_3^R < 0$
Total(%)	0	0	0	0	0	0	0	0
+ve Domain(%)	0	0	0	0	0	0	0	0
−ve Domain(%)	0	0	0	0	0	0	0	0
$\max(\lvert T_j\rvert)$	2.02	2.87	2.87	2.02	0.46	1.00	0.07	1.65

Sub-period 6	FSD		FRSD		TSD		TRSD	
	$T_1 > 0$	$T_1 < 0$	$T_1^R > 0$	$T_1^R < 0$	$T_3 > 0$	$T_3 < 0$	$T_3^R > 0$	$T_3^R < 0$
Total(%)	10.00	1.00	1.00	10.00	0	0	0	59.00
+ve Domain(%)	10.00	0	0	10.00	0	0	0	9.00
−ve Domain(%)	0	1.00	1.00	0	0	0	0	50.00
$\max(\lvert T_j\rvert)$	5.12	3.31	3.31	5.12	1.80	2.50	1.00	6.59

Note: This table summarizes the modified third order SD and RSD test results for risk averters and seekers. The table reports the percentages of modified SD and RSD statistic that are significantly negative or positive at the 5% significance level, based on the critical value generated from a bootstrap method. The test statistic T_3 is defined in (5.2) and T_3^R is defined in (5.3) with $F = S\&P$ and $G =$Nasdaq 100. TSD and TRSD stand for third-order SD and RSD, respectively.