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Abstract

An economy with incomplete financial markets, as described by Cass (1989), typically has inflation volatility driven by sunspots. The purpose of this paper is to investigate how the introduction of inflation-indexed bonds to the “Cass” economy influences a monetary market, an indexed bond market, and welfare. The introduction of indexed bonds is considered a sunspot-stabilizing policy. However, this introduction unrealistically causes the complete shutdown of monetary markets. This problem can be avoided in this paper as incorporating proportional transaction costs in the indexed bond market. Specifically, I show that the monetary market can never shut down even with a very high level of inflation volatility if the indexed bonds have transaction costs. In contrast, the indexed bond market can be inactive with a high value of transaction costs or low levels of inflation volatility. This paper shows that the introduction of indexed bonds does not necessarily induce the economy to be Pareto improving. However, by allowing lump-sum tax-transfer plans that are implemented in period-0 money, the market with indexed bonds can be Pareto superior to the market without them. The conclusions derived from a single-good economy can be applied to a multi-good economy if all agents have an identical homothetic utility function.

Keywords: Inflation-indexed bonds, Sunspots, Inflation volatility, Transaction costs, Consumer price index (CPI)

1 INTRODUCTION

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Sunspots (extrinsic uncertainty) provide explanations of excess volatility for both price level and allocations.¹ Cass and Shell's (1983) seminal paper investigates sunspot equilibria in a economy with restricted asset market participation. Cass (1989) further explains that incomplete financial markets can also allow a continuum of sunspot equilibria.² This paper introduces a sunspot-economy where both money and inflation-indexed bond markets are available. The model of the economy is exactly the same as the Cass GEI model with the addition of the indexed bonds.

Several theoretical studies have attempted to understand the role of indexed bonds in an economy with inflation volatility, but the explanation as to the source of inflation volatility varies. For example, Magill and Quinzii (1997) assume that inflation volatility is from monetary shocks, while Geanakoplos (2005) assumes that it is from intrinsic endowment shocks. This paper assumes that inflation volatility comes from extrinsic shocks (sunspots).

In the literature on sunspots, the introduction of indexed bonds is considered a sunspot-stabilizing policy. For example, Mas-Colell (1992) and Goenka and Préchac (2006) have separately shown that the introduction of real securities or inflation-indexed bonds make the economy immune to sunspots.³ In these models, however, the introduction of real assets unrealistically causes a complete shutdown of nominal financial markets. In reality, indexed bonds are used as minor supplements for money, not as the main financial instrument in monetary markets.⁴ This problem can be resolved, as shown in this paper, by introducing a proportional transaction cost for intermediating indexed bonds.⁵

¹See Cass and Shell (1983) and Shell (1987, 2008).

²Kang (2013) also shows that for each level of price volatility (inflation volatility), there is a unique regular sunspot-economy as in Cass's (1989) GEI model.

³Mas-Colell (1992) indicates that financial markets can be immune to sunspots by introducing as many real securities as the number of goods in each state. In addition, Goenka and Préchac (2006) show that the introduction of inflation-indexed bonds completely eliminates sunspot effects in incomplete markets.

⁴When riskless securities (indexed bonds) are available, there is no incentive for consumers to take any unnecessary risks by trading in nominal assets. However, with intrinsic uncertainty both riskless and nominal bonds can co-exist. (See Neumeyer, 1998.)

⁵Cozzi, Goenka and Shell (2013) have a similar idea that commodity taxes have intermediation costs while money taxes do not. Specifically, an iceberg cost is incurred when the government uses a commodity tax system, but no cost is involved when it uses a money tax system. Nevertheless, money taxes can be less attractive if the price-levels are unstable due to the effects of sunspots. Their paper investigates the comparative statics of whether the majority of consumers prefers commodity taxes versus money taxes when the economy is affected by sunspot signals. Their model is constructed based on Bhattacharya, Guzman and Shell (1998).

In this paper, I show that the introduction of indexed bonds can never cause the monetary market to shut down if the transaction cost of indexed bonds is not zero. In contrast, the indexed bond market can be completely inactive if the transaction costs are high enough or the inflation volatility level is low enough. I also show that as inflation volatility increases, the range of transaction costs for the inactive indexed bond market also increases.

There is a possibility not only for governments but also financial entrepreneurs to issue these bonds. Because an economy with inflation volatility always has an arbitrage in the values of risk-free securities, financial entrepreneurs can also open the market if they have the technology to access indexed bonds at a small enough cost. When the economy has a higher level of inflation volatility, there will be a larger gap between asset-buyers' and asset-sellers' values of risk-free assets and, therefore, the indexed bond market will be more profitable for financial entrepreneurs.

It is counter-intuitive that the introduction of indexed bonds actually can decrease the welfare of some consumers. This is due to the substitution effects: the introduction of a new asset causes the demand for money to decrease and consequently devalues money. This devaluation can have a negative impact on the asset sellers' (borrowers') utility as they sell money (risky asset) at a low price. This situation could be a matter of concern for governments that need to gain consensus to adopt a new policy, but it may not apply to profit-seeking financial entrepreneurs. Because the introduction of indexed bonds does not necessarily make the economy Pareto improving, this paper considers a compensation test based on lump-sum tax-transfer plans that are denominated in date-0 money. Specifically, it demonstrates that there are balanced lump-sum tax-transfer plans that would allow the market with indexed bonds to be Pareto superior to one without indexed bonds.

This paper discusses both single-good and multi-good economies. In a single-good economy, the payoff for indexed bonds is straightforward. However, in a multi-good economy, the relative prices and the fixed commodity bundle from Consumer Price Index (CPI) generally do not agree across both the agents and the states. This problem can be resolved and the ideal pay-offs of the inflation-indexed bonds can be defined if all agents' preferences are identically homothetic.

The outline of this paper is as follows. First, in Section 2, I introduce the combined market of a single-good economy in which both money and indexed bonds are available. The existence of a regular economy and the activeness of the monetary market are shown in Section 3. The impact of inflation volatility on the activeness of the indexed bond market is investigated in Section 4. Section 5 discusses the welfare implications of the introduction of

the indexed bonds to the market. Section 6 introduces a multi-good model where the agents have identical homothetic preferences at both dates 0 and 1. Section 7 is a brief conclusion. Some numerical examples are introduced in the appendix.

2 SINGLE-GOOD ECONOMY

There are two periods, today and tomorrow, labelled as subscripts $t = 0, 1$. At date 1, there are two states, $s = \alpha, \beta$ having positive probabilities $0 < \pi_\alpha < 1$, and $\pi_\beta = 1 - \pi_\alpha$, respectively. There are two consumers, labelled as superscripts $i \in I = \{1, 2\}$. Consumer i 's consumption allocation is $x^i = (x_0^i, x_\alpha^i, x_\beta^i) \in X = \mathbb{R}_{++}^3$ corresponding to prices $p = (p_0, p_\alpha, p_\beta) \gg 0$. His endowment is $e^i = (e_0^i, e_\alpha^i, e_\beta^i) \in X$, where $e_\alpha^i = e_\beta^i = e_1^i$. Denote by $\mathcal{U}$ the space of $\mathcal{C}^2$ utility functions on $\mathbb{R}_{++}^2$ which are twice differentiable, strictly increasing, strictly concave, having the closure of indifference curves contained in $\mathbb{R}_{++}^2$ and satisfying the von Neumann-Morgenstern expected utility hypothesis. Denote by $\mathcal{A}$ the set of characteristics $(v^i, e^i) \in \mathcal{U} \times X$. Consumer i 's preferences are

$$u^i(x^i) = u^i(x_0^i, x_\alpha^i, x_\beta^i) = \pi_\alpha v^i(x_0^i, x_\alpha^i) + \pi_\beta v^i(x_0^i, x_\beta^i)$$

Throughout this paper, I assume that there is an incentive to trade, i.e., for consumer $i = 1$ and $i' = 2$:

$$\frac{\partial v^i(e_0^i, e_1^i)}{\partial x_0} / \frac{\partial v^i(e_0^i, e_1^i)}{\partial x_1} > \frac{\partial v^{i'}(e_0^{i'}, e_1^{i'})}{\partial x_0} / \frac{\partial v^{i'}(e_0^{i'}, e_1^{i'})}{\partial x_1}$$

This condition implies that (1) the initial endowment is not Pareto efficient and (2) consumer 1 would be a lender (asset buyer) while consumer 2 would be a borrower (asset seller) in a financial market. Let's define $I = \{B, S\} = \{1, 2\}$ where B and S represent the asset buyer and the seller, respectively.

In a monetary market, there is only one financial instrument (money). m^i denotes consumer i 's money holdings. An economic fundamental $\mathcal{E}$ is simply a list $(u^i, e^i) \in \mathcal{A}$, $i \in I = \{B, S\}$. Denote the space of economic fundamentals by $\mathcal{M}$ where $\mathcal{M} = \prod_{i \in I} \mathcal{A}$. Equilibrium in a *pure monetary market* is defined as follows: There are some positive spot prices $(p_0, p_\alpha, p_\beta) \gg 0$ and associated money holdings m such that each household is optimized, (x^i, m^i) is the solution to the maximization problem, denoted as

$$\max \quad u^i(x_0^i, x_\alpha^i, x_\beta^i)$$

$$\text{subject to } \begin{cases} p_0 x_0^i + m^i \leq p_0 e_0^i \\ p_\alpha x_\alpha^i \leq p_\alpha e_1^i + m^i \\ p_\beta x_\beta^i \leq p_\beta e_1^i + m^i \end{cases} \quad \text{and } x^i \in X$$

each market clears,

$$\sum_i x_0^i = \sum_i e_0^i, \quad \sum_i x_\alpha^i = \sum_i x_\beta^i = \sum_i e_1^i$$

$$\text{and } \sum_i m^i = 0.$$

I now introduce inflation-indexed bonds with a proportional transaction cost θ . The amount of this transaction cost is proportional to the trading price. For example, if $\theta = 1\%$ and the selling price of “one” unit of the asset is \$10, the amount of transaction cost for trading “one” unit would be \$0.01 ($=\$10 \times 1\%$). The cost is incurred at date 0, implying that the intermediaries exchange the transaction fees with the period-0 good. Because of the existence of θ , the selling price of the bond is not the same as the purchase price. Therefore, the market clearing condition for the first period is dependent on the amount of the indexed bonds traded in the market.⁶ I label a monetary market with indexed bonds as a *combined market* (CM). The equilibrium in CM is defined as

Given the prices $(p_0, p_\alpha, p_\beta, p)$, $(m^i, n^i, x_0^i, x_\alpha^i, x_\beta^i)_{h=1}^H$ solve the following maximization problem (P-CM):

$$\max \quad u^i(x_0^i, x_\alpha^i, x_\beta^i)$$

subject to

$$p_0 x_0^i + m^i + p(1 + \theta) \max(n^i, 0) + p \min(n^i, 0) \leq p_0 e_0^i \quad (1)$$

$$p_\alpha x_\alpha^i \leq p_\alpha e_1^i + p_\alpha n^i + m^i \quad (2)$$

$$p_\beta x_\beta^i \leq p_\beta e_1^i + p_\beta n^i + m^i \quad (3)$$

Market clear conditions are

$$\sum_i m^i = 0, \sum_i n^i = 0$$

$$\sum_i x_0^i + \frac{p}{p_0} \theta \sum_i \max(n^i, 0) = \sum_i e_0^i$$

⁶In a similar way, Foley (1970) introduced endowment costs in the operation of commodity markets.

$$\text{and } \sum_i x_\alpha^i = \sum_i x_\beta^i = \sum_i x_1^i.$$

Here, $p(1 + \theta)$ and p represent the purchase and selling prices of the indexed bonds, respectively. The nominal payoffs of the indexed bonds are p_α and p_β in state α and β , respectively. For a single-good economy, the government can always design the nominal payoffs of those bonds, guaranteeing the same purchasing power across states. However, defining the payoffs to buy the same commodity bundle across states is not clear in a multi-good economy, as discussed in Section 6.

Finally, the budget constraint set in a combined market is still convex, which is necessary for the existence of single-valued demand functions. In the next section, I investigate the existence and regularity of the equilibria in a combined market.

3 EQUILIBRIUM

Kang (2013) defined price-level volatility (equivalently, inflation volatility) based on the relative standard deviation (RSD) of price-level.⁷ As in Kang (2013), inflation volatility in this paper is defined as

$$\sigma^R = \frac{\sigma(\tilde{p}_1/p_0)}{E(\tilde{p}_1/p_0)} \quad (4)$$

where $\sigma(\tilde{p}_1/p_0)$ and $E(\tilde{p}_1/p_0)$ are the standard deviation and the expected value of the level of inflation, respectively.^{8,9} In this section, I show that a regular economy can be defined in a combined market as fixing the inflation volatility level σ^R . In this model, the volatility σ^R is given exogenously, which implies that the occurrence of non-fundamental volatility does not vanish nor diminish with the introduction of real securities.

Using a method similar to Kang (2013), and for the convenience of mathematics and better intuition, I define the price ratio $\mathcal{P}$:

$$\mathcal{P} = \frac{p_\beta}{p_\alpha} \geq 1.$$

The following relationship shows that the price ratio $\mathcal{P}$ has one-to-one rela-

⁷Kang (2013) showed that there exists a generic set $\mathcal{M}^* \subset \mathcal{M}$ such that any economy in a pure monetary market (MM), specified as a pair $(\mathcal{E}, \sigma^R)_M \in \mathcal{M}^* \times \mathbb{R}_+$ is regular and determinate.

⁸With this two-state model, $\tilde{p}_1$ is defined as $\tilde{p}_1 = \{p_\alpha, p_\beta; \pi_\alpha, \pi_\beta\}$

⁹Price-level volatility and inflation volatility based on the relative standard deviation (SRD) have exactly the same value in a single-good economy.

tionship with inflation volatility σ^R :

$$\sigma^R = \frac{\sigma(\tilde{p}_1/p_0)}{E(\tilde{p}_1/p_0)} = \frac{\sqrt{\pi_\alpha \pi_\beta} |1 - \mathcal{P}|}{\pi_\alpha + \pi_\beta \mathcal{P}}.$$

Given price ratio $\mathcal{P}$, it is possible to interpret nominal security (money) as a security with a real return (R_α, R_β) satisfying that $\mathcal{P} = \frac{R_\alpha}{R_\beta}$. Then, the equivalent economy with two types of real securities¹⁰ can be defined (Lemma 1). Finally, I show that the economy is regular and determinate (Proposition 1).

First, the equivalent maximization problem based on two types of real securities is defined as (E-CM)

$$\max_{z_0^i, n^i, B^i} u^i(e_0^i + z_0^i, e_1^i + n^i + R_\alpha B^i, e_1^i + n^i + R_\beta B^i) \quad (5)$$

$$\text{subject to } q_0 z_0^i + q B^i + p(1 + \theta) \max(n^i, 0) + p \min(n^i, 0) \leq 0 \quad (6)$$

$$\text{where } \frac{R_\beta}{R_\alpha} = \frac{1}{\mathcal{P}}.$$

Next, the following lemma shows that the equivalent maximization problem (E-CM) with the two real securities is equivalent to the original maximization problem (P-CM) with money and indexed bonds.

Lemma 1 *Given $\mathcal{P} = p_\beta/p_\alpha$, P-CM is equivalent to E-CM where*

$$\begin{aligned} x_0^i &= e_0^i + z_0^i, & x_\alpha^i &= e_1^i + n^i + R_\alpha B^i, & x_\beta^i &= e_1^i + n^i + R_\beta B^i \\ m^i &= q B^i, & p_0 &= q_0 (= 1), & p_\alpha &= \frac{q}{R_\alpha}, & p_\beta &= \frac{q}{R_\beta} \end{aligned}$$

Proof. (i) (P-CM $\Rightarrow$ E-CM) P-CM is defined with the given parameters $(p_0, p_\alpha, p_\beta, p, r, \mathcal{P})$ while the given parameters are $(q_0, q, R_\alpha, R_\beta, p, \mathcal{P})$ in E-CM. Defining p_0 and m^i as

$$p_0 \equiv q_0 \quad \text{and} \quad m^i \equiv q B^i,$$

it is clear that the budget constraint (eqn 6) in E-CM can be derived from that in P-CM. Next, defining p_α and p_β as

$$p_\alpha \equiv \frac{q}{R_\alpha}, \quad p_\beta \equiv \frac{q}{R_\beta},$$

¹⁰One is the equivalent real security to money and the other is the inflation-indexed bonds. In this paper, I call them a risky asset and a risk-free asset, respectively.

x_α^i and x_β^i can be expressed as

$$x_\alpha^i \leq e_1^i + n^i + R_\alpha B^i \text{ and } x_\beta^i \leq e_1^i + n^i + R_\beta B^i.$$

Because utility functions are strictly increasing, $\leq$ can be replaced with $=$. Finally, by the equation $x_0^i = e_0^i + z_0^i$, the utility function (eqn 5) in E-CM is the same as the one in P-CM.

(ii) (E-CM $\Rightarrow$ P-CM) Defining q_0 and B^i as

$$q_0 \equiv p_0, \quad B^i \equiv \frac{m^i}{q}$$

the date-0 budget constraint (eqn 1) in P-CM is equivalent to that (eqn 6) in E-CM. Let's define x_α^i and x_β^i as

$$x_\alpha^i \equiv e_1^i + n^i + R_\alpha B^i \text{ and } x_\beta^i \equiv e_1^i + n^i + R_\beta B^i.$$

then, where

$$R_\alpha = \frac{q}{p_\alpha}, \quad R_\beta = \frac{q}{p_\beta},$$

x_α^i and x_β^i can be expressed as

$$\begin{aligned} p_\alpha x_\alpha^i &\leq p_\alpha e_1^i + p_\alpha n^i + m^i \\ p_\beta x_\beta^i &\leq p_\beta e_1^i + p_\beta n^i + m^i \end{aligned}$$

which are the same as eqns (2) and (3), respectively. Finally, because $x_0^i \equiv e_0^i + z_0^i$, the utility function in P-CM is equivalent to that in E-CM. ■

In a competitive equilibrium, the mean of the real return (R_α, R_β) does not affect the equilibrium allocations because the value of q is adjusted according to the value of the mean. For the convenience of proofs and computation, I assume that the mean value of the real return is fixed as one, i.e., $\pi_\alpha R_\alpha + \pi_\beta R_\beta = 1$. Throughout this paper, all of the analyses and proofs are based on the equivalent problem (E-CM).

The main difference between this model and a conventional general equilibrium model is that this model has a transaction cost and, therefore, the budget constraint is not differentiable at $n^i = 0$, i.e., when the indexed bonds markets are inactive. Where $n^i = 0$, the equilibrium price p of indexed bonds is not locally unique. However, even in this case, all the equilibrium allocations and all the prices except the price of indexed bonds are locally unique. The proof of Proposition 1 is based the equilibrium property that if a financial market is inactive, the financial market has no real impact on the

economy.

Proposition 1 *There exists a generic set $\mathcal{M}^* \subset \mathcal{M}$ such that any economy in a combined market, specified as a triple $(\mathcal{E}, \sigma^R, \theta)_C \in \mathcal{M}^* \times \mathbb{R}_+^2$,*

- (1) is regular if $n^i \neq 0$.*
- (2) has locally unique equilibrium allocation and prices except p if $n^i = 0$.*
- (3) has locally unique equilibrium allocation and prices except (p_α, p_β) if $m^i = 0$ (equivalently, $B^i = 0$).*

Proof. Case 1: In the equivalent maximization problem (E-CM), the budget sets are convex and the utility function is strictly concave for all $(q_0, R_\alpha, R_\beta, q, \mathcal{P}, p) \gg 0$. Therefore, the excess demand is single valued. (The proof can be done by defining a compact subset of the budget set.)

For the proof of a regular and determinate economy, we need to define the aggregate excess demand and check if it satisfies sufficient conditions for the existence of regular economies. The market aggregate demand function is defined as

$$Z(Q) = \sum_i \begin{pmatrix} z_0^i(Q) + \theta \frac{p}{q_0} \max(n^i(Q), 0) \\ B^i(Q) \\ n^i(Q) \end{pmatrix},$$

where $Q = (q_0, q, p)$.

This market aggregate demand function is not the same as a conventional one because of the additional term $\theta \frac{p}{q_0} \max(n^i(Q), 0)$ which is the amount of first-period goods the intermediaries charge as transaction fees.

It can be easily shown that $Z(Q)$ satisfies the five properties:

- (i) continuous
- (ii) homogeneous of degree zero
- (iii) (Walras' law) $Q \cdot Z(Q) = 0$
- (iv) There is an $s > 0$ such that $Z(Q) > (-s, -s, -s)$ for all P .
- (v) If $Q^n \rightarrow Q$, where $Q \neq 0$ and $p = 0$, then $\sum_i n^i(Q) \rightarrow \infty$.

Therefore, the economy is regular and determinate. (See Debreu 1970.)

Case 2: Because the indexed bond market is inactive, the equilibrium allocations with the indexed bond market is exactly the same as those without the market. The market aggregate demand function without the indexed bond market is defined as

$$Z'(Q) = \sum_i \begin{pmatrix} z_0^i(Q') \\ B^i(Q) \end{pmatrix},$$

where $Q' = (q_0, q)$.

Where $n^i = 0$, $Z'(Q)$ satisfies the five properties necessary for defining a regular economy.

Case 3: This can be proven in the same way as case 2. In this case, there is a continuum of equilibrium price q in the equivalent problem (E-CM). This implies that there is also a continuum of equilibrium price-level (p_α, p_β) by the equations $p_\alpha = \frac{q}{R_\alpha}$ and $p_\beta = \frac{q}{R_\beta}$ in Lemma 1. ■

The proof for Case 2 in Proposition 1 can be directly applied to the proof for defining a regular economy in the pure monetary market. The economy in the pure monetary market is specified as $(\mathcal{E}, \sigma^R)_M \in \mathcal{M}^* \times \mathbb{R}_+$ is regular.

The existence of equilibria does not necessarily imply that either (both) an indexed bond market or (and) a monetary market are active. The main difference from previous articles¹¹ dealing with the combined market of money and indexed bonds in a sunspot economy is that the monetary market in my model is always active even after the introduction of indexed bonds, if the transaction cost of indexed bonds is not zero. The following proposition shows this.

Proposition 2 *The monetary market is always active, i.e., $m^i \neq 0$ for some $i \in I$ if $\theta > 0$.*¹²

Proof. (by contradiction) Let's assume that $m^i = 0$ for $i \in \{B, S\}$, labeling the asset buyer as $i = B$ and the asset seller as $i = S$.

Case 1: Assuming that $n^B = n^S = 0$, money should be active because the initial endowment is not Pareto efficient.

Case 2: Next, we need to check the case where $n^B \neq 0$ and $n^S \neq 0$. For this case, we can compute each individual's value of the risky-asset (money). The value is the ratio between the marginal utility of risk assets and that of the date-0 good. Remembering that the risky asset's return (R_α, R_β) satisfies that $\pi_\alpha R_\alpha + \pi_\beta R_\beta = 1$, the value (price) of the risky asset for the buyer is computed as

$$\begin{aligned} q_R^B &= \frac{\pi_\alpha \frac{\partial v^B(x_0^B, e_1^B + R_\alpha B^B)}{\partial B^B} + \pi_\beta R_\beta \frac{\partial v^B(x_0^B, e_1^B + R_\beta B^B)}{\partial B^B}}{\pi_\alpha \frac{\partial v^B(x_0^B, x_\alpha^B)}{\partial x_0} + \pi_\beta \frac{\partial v^B(x_0^B, x_\beta^B)}{\partial x_0}} \\ &= \frac{\pi_\alpha R_\alpha \frac{\partial v^B(x_0^B, x_\alpha^B)}{\partial x_1} + \pi_\beta R_\beta \frac{\partial v^B(x_0^B, x_\beta^B)}{\partial x_1}}{\pi_\alpha \frac{\partial v^B(x_0^B, x_\alpha^B)}{\partial x_0} + \pi_\beta \frac{\partial v^B(x_0^B, x_\beta^B)}{\partial x_0}} \left(\begin{array}{c} \pi_\alpha R_\alpha + \pi_\beta R_\beta = 1 \\ \mathcal{P} = R_\alpha / R_\beta \end{array} \right) \end{aligned}$$

¹¹See Mas-Colell (1992) and Goenka and Pr  chac (2006).

¹²It is proven that where $\theta = 0$, money is not traded in the market and the sunspot effects will disappear. (See Goenka and Pr  chac 2006.)

Because $m^B = 0$ ($B^B = 0$), $x_\alpha^B = x_\beta^B$. Also, it is true that $\pi_\alpha R_\alpha + \pi_\beta R_\beta = \pi_\alpha + \pi_\beta$. Therefore, we get:

$$q_R^B = \frac{\pi_\alpha \frac{\partial v^B(x_0^B, x_\alpha^B)}{\partial x_1} + \pi_\beta \frac{\partial v^B(x_0^B, x_\beta^B)}{\partial x_1}}{\pi_\alpha \frac{\partial v^B(x_0^B, x_\alpha^B)}{\partial x_0} + \pi_\beta \frac{\partial v^B(x_0^B, x_\beta^B)}{\partial x_0}}. \quad (7)$$

Equation (7) is the same as the asset buyer's value of the indexed bonds. This is also the same as the purchase price of the indexed bond. Therefore, we can get

$$q_R^B = p(1 + \theta).$$

In the same way, we can show that the asset seller's value of a risky-asset is the same as the selling price of the indexed bond:

$$q_R^S = p.$$

Because $q_R^B > q_R^S$, there is a price $q \in (q_R^S, q_R^B)$ in which the two consumers have an incentive to trade with money. This contradicts that $m^i = 0$ for $i = B, S$. ■

Proposition 2 shows that the market for money is still operating even with an extremely high level of inflation volatility if $\theta > 0$. However, without a transaction cost, i.e., $\theta = 0$, the existence of inflation volatility makes the market for money shut down completely, which was proven by Goenka and Préchac (2006).

4 INDEXED BONDS AND INFLATION VOLATILITY

The indexed bond market is not necessarily active if the transaction cost θ is too large. This section shows that there exists a *threshold* transaction cost $\bar{\theta}$ such that only if the transaction cost is lower than $\bar{\theta}$, the indexed bond market is active. To show the existence of the threshold cost, we will re-address the pure monetary market (equivalent to Case 2 in Proposition 2).

Although inflation-indexed bonds are not issued in a monetary market, the values of those securities can be computed at the equilibrium. Let's define p_F^i as the value of a risk-free asset (whose return is "one" in each date) for

consumer $i \in \{B, S\}$:

$$p_F^i = \frac{\pi_\alpha \frac{\partial v^i(x_0^i, x_\alpha^i)}{\partial x_1} + \pi_\beta \frac{\partial v^i(x_0^i, x_\beta^i)}{\partial x_1}}{\pi_\alpha \frac{\partial v^i(x_0^i, x_\alpha^i)}{\partial x_0} + \pi_\beta \frac{\partial v^i(x_0^i, x_\beta^i)}{\partial x_0}} \quad (8)$$

which is the ratio between the marginal utility of risk-free assets and that of the date-0 good. The following lemma shows that there is an arbitrage between two values p_F^B and p_F^S where “B” and “S” represent the asset buyer and the asset seller, respectively.

Lemma 2 *If $\sigma^R \neq 0$ in a pure monetary market, then¹³*

$$p_F^S < p_F^B.$$

Proof. q^+ represents the equilibrium price of the risky asset in a pure monetary market.

$$\begin{aligned} q^+ &= \frac{\pi_\alpha \frac{\partial v^B(x_0^B, e_1^B + R_\alpha B^B)}{\partial B^B} + \pi_\beta R_\beta \frac{\partial v^B(x_0^B, e_1^B + R_\beta B^B)}{\partial B^B}}{\pi_\alpha \frac{\partial v^B(x_0^B, x_\alpha^B)}{\partial x_0} + \pi_\beta \frac{\partial v^B(x_0^B, x_\beta^B)}{\partial x_0}} \\ &= \frac{\pi_\alpha R_\alpha \frac{\partial v^i(x_0^i, x_\alpha^i)}{\partial x_1} + \pi_\beta R_\beta \frac{\partial v^i(x_0^i, x_\beta^i)}{\partial x_1}}{\pi_\alpha \frac{\partial v^i(x_0^i, x_\alpha^i)}{\partial x_0} + \pi_\beta \frac{\partial v^i(x_0^i, x_\beta^i)}{\partial x_0}} \quad (9) \\ &\text{where } \begin{pmatrix} \pi_\alpha R_\alpha + \pi_\beta R_\beta = 1 \\ \mathcal{P} = R_\alpha / R_\beta \end{pmatrix} \end{aligned}$$

Let's label the asset buyer ($B^i > 0$) as $i = B$ and the asset seller ($B^i < 0$) as $i = S$. Because the initial endowment allocation is not Pareto efficient, they will trade and therefore $B^B > 0$ and $B^S < 0$. First, we want to show that $p_F^B > q^+$. Because the state β is inflationary $\mathcal{P} > 1$, we get: $R_\alpha > R_\beta$ and $x_\alpha^B > x_\beta^B$. From equations (8) and (9), $\frac{p_F^B}{q^+}$ is

$$\frac{p_F^B}{q^+} = \frac{\pi_\alpha \frac{\partial v^B(x_0^B, x_\alpha^B)}{\partial x_1} + \pi_\beta \frac{\partial v^B(x_0^B, x_\beta^B)}{\partial x_1}}{\pi_\alpha R_\alpha \frac{\partial v^B(x_0^B, x_\alpha^B)}{\partial x_1} + \pi_\beta R_\beta \frac{\partial v^B(x_0^B, x_\beta^B)}{\partial x_1}}. \quad (10)$$

¹³ An economy without inflation volatility ($\sigma^R = 0$) is the same as a certainty economy. Therefore, where $\sigma^R = 0$ there is no arbitrage, i.e., $p_F^S = p_F^B$.

Because v^i is strictly concave, it is true that

$$\frac{\partial v^B(x_0^B, x_\alpha^B)}{\partial x_1} < \frac{\partial v^B(x_0^B, x_\beta^B)}{\partial x_1}. \quad (11)$$

Because $\pi_\alpha R_\alpha + \pi_\beta R_\beta = 1$, $\pi_\alpha + \pi_\beta = 1$ and $\pi_\alpha R_\alpha > \pi_\alpha$, $\pi_\beta R_\beta < \pi_\beta$, from (11), we can get:

$$\pi_\alpha \frac{\partial v^B(x_0^B, x_\alpha^B)}{\partial x_1} + \pi_\beta \frac{\partial v^B(x_0^B, x_\beta^B)}{\partial x_1} > \pi_\alpha R_\alpha \frac{\partial v^B(x_0^B, x_\alpha^B)}{\partial x_1} + \pi_\beta R_\beta \frac{\partial v^B(x_0^B, x_\beta^B)}{\partial x_1}.$$

Therefore, we know that

$$\frac{p_F^B}{q^+} > 1.$$

In the same way, we can prove the following:

$$\frac{p_F^S}{q^+} < 1.$$

From the two inequalities, we know that $p_F^S < p_F^B$. ■

The arbitrage in the values of the risk-free securities between two consumers provides the incentive for them to accept real securities as the financial instrument in a combined market. However, the existence of the arbitrage does not necessarily imply that the indexed bonds are active. The magnitude of the arbitrage should be large enough, compared to the transaction cost, for the indexed bonds to be traded in a combined market. The following propositions and corollary show that the indexed bond markets can be active or inactive depending on the level of inflation volatility and the level of a transaction cost, also as shown in Figure 1.

Proposition 3 *If $\sigma^R \neq 0$, there exists $\bar{\theta} \in \mathbb{R}_{++}$ such that the indexed bond market is active (inactive) if $\theta < \bar{\theta}$ ($\theta \geq \bar{\theta}$), i.e.,*

$$n^i \begin{cases} \neq 0 & \text{if } \theta < \bar{\theta} \\ = 0 & \text{if } \theta \geq \bar{\theta} \end{cases} \quad \text{for } i = B, S.$$

where

$$\bar{\theta} = \frac{p_F^B - p_F^S}{p_F^S}. \quad (12)$$

Proof. (i) For $n^i \neq 0$ where $\theta < \bar{\theta}$.

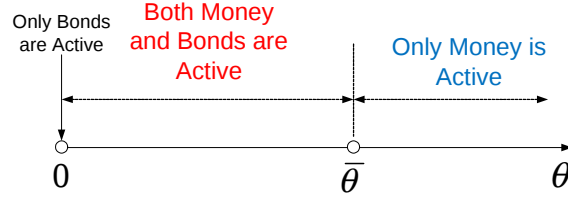


Figure 1: Transaction cost and financial markets

(Proof by contradiction) Let's assume that $n^i = 0$ for $i = B, S$. Then, the equilibrium allocations in a combined market are the same as those in a pure monetary market. If $\theta < \bar{\theta} = (p_F^B - p_F^S) / p_F^S$, there exists $p > 0$ such that $p_F^S < p < p(1 + \theta) < p_F^B$. (This can be proven as follows: Let $p = p_F^S + \varepsilon$ and $\theta = \frac{p_F^B - p_F^S - \delta}{p_F^S} > 0$. Then, for all $\delta \in (0, p_F^B - p_F^S)$, there exists ε which satisfies the inequality $p_F^S \leq p < p(1 + \theta) \leq p_F^B$.) This implies that the two consumers have an incentive to trade with the indexed bonds. This contradicts that $n^i = 0$.

(ii) For $n^i = 0$ where $\theta \geq \bar{\theta}$.

(Proof by contradiction) Let's assume that $n^i \neq 0$ for $i = B, S$. Assuming that $\theta \geq \bar{\theta}$, there does not exist $p > 0$ such that $p_F^S < p < p(1 + \theta) < p_F^B$. This contradicts that $n^i \neq 0$ for $i = B, S$. ■

Next, it will be shown that as the market has higher inflation volatility (more uncertainty about the future price level), $\bar{\theta}$ is also higher. In the model, the decision about whether financial entrepreneurs should enter the market (or whether the government should issue bonds) depends on whether the transaction cost θ is lower than $\bar{\theta}$. The following proposition shows the direct connection between the value of $\bar{\theta}$ and the level of inflation volatility.

Proposition 4 *If v^i is additively separable, i.e., $v^i(x_0^i, x_1^i) = f^i(x_0^i) + g^i(x_1^i)$, $\bar{\theta}$ is strictly increasing in σ^R . (See Appendix for the proof.)*

Proof. (See Appendix for the proof.) ■

The proposition is the extended version of Lemma 1 saying that $\bar{\theta}$ is strictly positive where $\sigma^R > 0$ and equals zero where $\sigma^R = 0$. The proof for the non-separable utility functions remains open. Proposition 4 implies that as the economy has higher inflation volatility, financial entrepreneurs can make higher profits by charging higher intermediation costs.

From Propositions 3 and 4, we derive the following result.

Corollary 1 *If v^i is additively separable, for any $\theta > 0$, there exists a constant $\bar{\sigma}^R \in \mathbb{R}_{++}$ such that only the monetary market is active if $\sigma^R < \bar{\sigma}^R$*

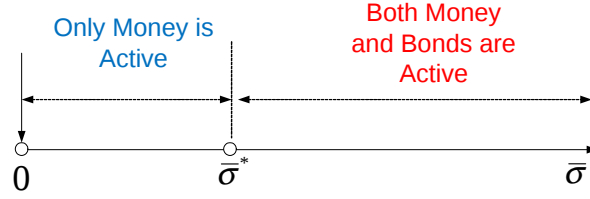


Figure 2: Inflation volatility and financial markets

while both indexed bond and monetary markets are active if $\sigma^R \geq \bar{\sigma}^R$. (See Figure 2.)

Proof. Trivial from Propositions 3 and 4. ■

5 WELFARE

A combined market provides consumers more trading choices than a pure monetary market. Therefore, it can be expected that the equilibrium in the combined market is Pareto superior to that in the pure monetary market. However, it has been shown that financial innovations to incomplete markets do not necessarily induce the economy to be Pareto improving.¹⁴ Financial innovations are known to affect market prices under a general equilibrium setting. These price changes can negatively affect agents' real wealth and, consequently, their utility values. Specifically, the introduction of indexed bonds causes the value of money to decrease due to substitution effects, which has a negative impact on the asset sellers' (borrowers') wealth.

The following example illustrates how the introduction of indexed bonds can make some consumers worse off. There are two consumers who have the same expected utility functions $v^1 = v^2 = \log(x_0) + \log(x_1)$ and whose endowments are $e^1 = (10, 0)$ and $e^2 = (0, 10)$. In this case, as the level of inflation volatility increases, the second consumer's utility value increases. The intuition is as follows. For a high level of inflation volatility, the real return of assets becomes small in an inflationary state. If the inflationary state is realized, the first consumer will end up getting a small amount of the good at date 1 and, consequently, his utility will be particularly low. Therefore, the first consumer's demand for the asset will be higher to insure against the inflation state in a higher level of inflation volatility. The high demand of the first consumer results in a high price (value) of the risky asset

¹⁴See Cass and Citanna (1998), Elul (1995) and Hart (1997) .

(money). Finally, the second consumer, who sells the risky asset (money) to the first one, obtains more income as the asset price goes higher in a pure monetary market.

However, the introduction of new riskless securities, which are substitutes for money, has caused a considerable decrease in money demand. Therefore, the value of money in a combined market is smaller than that in a pure monetary market. A decrease in the money value negatively affects the second consumer's utility level. In this example, even though the asset seller has more trading choices in the combined market, his utility level is actually lower than that in the pure monetary market. (See the Appendix for more details about the example.)

Although some of the consumers may be negatively affected, it is not possible for all of them to be worse off by the introduction of the indexed bonds according to the revealed preference hypothesis. The following proposition shows this.

Proposition 5 (i) *For any economic fundamental $\mathcal{E} \subset \mathcal{M}$, there exists at least one consumer who is better off.* (ii) *However, all consumers can be better off for some economic fundamentals in $\mathcal{M}$.*

Proof. (i) When the value of money, which is considered q in the equivalent problem, is not the same in both markets, there must be at least one consumer who will benefit from the change of the value. (The consumer should be the asset buyer (seller) if the value decreases (increases) in a combined market.) Because the consumer has more choices in the combined market, according to the revealed preference argument, the consumer must be better off.

(ii) If variation in the value of money is small enough or zero, according to the revealed preference argument, all consumers can be better off. ■

The introduction of indexed bonds may be a sunspot-stabilizing policy in the sense that it makes the equilibrium allocations less volatile.^{15,16} However, Proposition 5 shows that the introduction of indexed bonds does not necessarily make all agents better off. This means that the government can

¹⁵More accurately, the relative standard deviation (RSD) of the excess demand $\{x_\alpha - e_1, x_\beta - e_1; \pi_\alpha, \pi_\beta\}$ is lowered with the introduction of the indexed bonds.

¹⁶However, it is different from conventional sunspot-stabilizing policies because it cannot completely eliminate the consumption volatility unless the transaction cost is zero. Several studies have suggested stabilizing policies to completely eliminate the effects of sunspots on incomplete markets. Three dominant policies have been shown: 1) the introduction of new types of nominal securities. (see Cass and Shell 1983 [Proposition 3] and Balasko 1983 [Theorem 1]); 2) the introduction of as many real securities as the number of goods in each state (see Mas-Colell 1992); and 3) the introduction of options. (see Antinolfi and Keister 1998 and Kajii 1997).

fail to gain consensus in adopting the financial innovation—the introduction of indexed bonds. However, Proposition 6 shows that if balanced lump-sum tax plans are allowed along with financial innovations, consumer consensus on the policy can be achieved. Although there is no clear Pareto ranking between a combined market and a pure monetary market, it can be shown that a combined market is Kaldor-Hicks superior to a pure monetary market by considering a compensation test based on balanced lump-sum tax-transfer plans. The space of the balanced lump-sum tax plans is defined as

$$T = \{(\tau^1, \dots, \tau^I) \in \mathbb{R}^I \mid \sum_{i \in I} \tau^i = 0\}.$$

The tax-transfer plan τ is applied to either the combined economy or the pure monetary market. Then, the budget constraints at date 0 are modified to be

$$s.t \quad \begin{cases} p_0 x_0^i + m^i + p(1 + \theta) \max(n^i, 0) + p \min(n^i, 0) \leq p_0 e_0^i - \tau^i \\ p_\alpha x_\alpha^i \leq p_\alpha e_1^i + p_\alpha n^i + m^i \\ p_\beta x_\beta^i \leq p_\beta e_1^i + p_\beta n^i + m^i \end{cases} \quad (\text{P-CM})$$

The budget constraints in the second period are invariant. Assuming that $p^0 = 1$, we can interpret that the taxes and transfers are denominated by the date-0 commodity.

Definition 1 *Market "A" is Kaldor-Hicks superior to market "B", i.e.,*

$$\text{market } A \succeq_{KH} \text{market } B,$$

if there exist(s) $\tau \in T$ such that market A with the tax-transfer plan τ is Pareto superior to market B.

In the above definition, the tax-transfer plans are applied to only the first market (market A) but not to the second market (market B).

The following proposition shows that a combined market is “superior” to a pure monetary market based on the compensation principle.

Proposition 6 *For any $\mathcal{E} \in \mathcal{M}$, $\sigma^R > 0$ and $\theta > 0$,*

$$(\mathcal{E}, \sigma^R, \theta)_C \succeq_{KH} (\mathcal{E}, \sigma^R)_M.$$

Proof. See the Appendix. ■

Propositions 5 and 6 show that there is no clear Pareto ranking between pure monetary and combined markets, but there does exist Kaldor-Hicks

welfare ranking between the two markets. In the same way, it is possible to compare two types of combined markets with different levels of transaction costs. Surprisingly, there are some consumers (among asset sellers) who prefer an economy with higher transaction costs because higher transaction costs make the value of money higher, which is preferred by asset sellers (borrowers). (See the Appendix for a detailed numerical example.) In the same sense as Proposition 6, it can be shown that a combined market with lower transaction costs is Kaldor-Hicks superior to a market with higher costs, i.e.,

$$(\mathcal{E}, \sigma^R, \theta_1)_C \succeq_{KH} (\mathcal{E}, \sigma^R, \theta_2)_C \quad \text{if } \theta_1 < \theta_2.$$

I do not put the results as a proposition because this is considered somewhat trivial.

6 MULTI-GOOD ECONOMY

The object of this section is to construct a more realistic model by extending the number of commodities in each spot from one to $L > 1$. To clearly distinguish the notations from the single-good case, I use the symbol “ $\rightarrow$ ” in all vectors in a multi-good economy. Consumer i 's consumption allocation is $\vec{x}^i = (\vec{x}_0^i, \vec{x}_\alpha^i, \vec{x}_\beta^i) \in X = \mathbb{R}_{++}^{3L}$ where

$$\vec{x}_s^i = (x_{s1}^i, \dots, x_{sl}^i, \dots, x_{sL}^i) \quad s = 0, \alpha, \beta.$$

Corresponding prices are $\vec{p} = (\vec{p}_0, \vec{p}_\alpha, \vec{p}_\beta) \in \mathbb{R}_{++}^{3L} \gg 0$ where

$$\vec{p}_s = (p_{s1}, \dots, p_{sl}, \dots, p_{sL}) \quad s = 0, \alpha, \beta.$$

Consumer i 's endowment is $\vec{e}^i = (\vec{e}_0^i, \vec{e}_\alpha^i, \vec{e}_\beta^i) \in X$ where $\vec{e}_\alpha^i = \vec{e}_\beta^i = \vec{e}_1^i$ and

$$\vec{e}_t^i = (e_{t1}^i, \dots, e_{tl}^i, \dots, e_{tL}^i) \quad t = 0, 1$$

Then, the budget constants in the multi-good economy are:

$$\vec{p}_0 \cdot \vec{x}_0^i + m^i + p(1 + \theta) \max(n^i, 0) + p \min(n^i, 0) \leq \vec{p}_0 \cdot \vec{e}_0^i$$

$$\vec{p}_\alpha \cdot \vec{x}_\alpha^i = \vec{p}_\alpha \cdot \vec{e}_1^i + \vec{p}_\alpha \cdot \vec{w}_1 n^i + m^i \quad (13)$$

$$\vec{p}_\beta \cdot \vec{x}_\beta^i = \vec{p}_\beta \cdot \vec{e}_1^i + \vec{p}_\beta \cdot \vec{w}_1 n^i + m^i \quad (14)$$

where $\vec{w}_1$ is a fixed commodity bundle from the consumer price index (CPI).

The payoffs of indexed bonds are based on nominal returns $\vec{p}_\alpha \cdot \vec{w}_1$ and

$\vec{p}_\beta \cdot \vec{w}_1$ in the states α and β , respectively. The question is whether all consumers will buy the same fixed commodity bundle $\vec{w}_1$ across the states with nominal returns $\vec{p}_s \cdot \vec{w}_1$, $s = \alpha, \beta$.¹⁷ In a single good-economy, the answer is always “yes”, because consumers have no other choice except to purchase the single good. However, in a multi-good economy, it is not trivial to define the payoffs of indexed bonds, which guarantee the same purchasing power across the states without any further restrictions on preferences.

To deal with this issue, it is necessary to define the Consumer Price Index (CPI) in this paper. The payoffs of the indexed bonds are based on the CPI.¹⁸ CPI consists of two factors: the fixed commodity bundle and the price index. These factors should be estimated in advance to design the optimal indexed bonds. The fixed commodity bundle is defined as

$$\vec{w}_t = (w_{t1}, w_{t2}, \dots, w_{tl}, \dots, w_{tL}) \quad \text{where } t = 0, 1$$

where w_{tl} represents the quantity of good l at date t .

The second factor in CPI is the (relative) price index.¹⁹ It can be defined as

$$\vec{p}_t = (p_{t1}, p_{t2}, \dots, p_{tl}, \dots, p_{tL}) \quad \text{where } t = 0, 1.$$

where p_{tl} represents the price of good l at date t .

To design the payoffs guaranteeing the same purchasing power across agents and states, the following conditions should be satisfied

Condition 1: There exists a common price index at date 1 across states.

Condition 2: There exists a common fixed commodity bundle at date 1 for all agents.

If Condition 1 is violated, i.e., there exists no common relative prices across states, a consumer would not buy the same fixed commodity bundle $\vec{w}_1$ across the states with nominal returns $\vec{p}_s \cdot \vec{w}_1$, $s = \alpha, \beta$. Condition 2 represents that the fixed commodity bundle $\vec{w}_1$ should be common for *all* consumers.²⁰

To get a consistent price index and commodity bundle satisfying Conditions 1 and 2, additional restrictions on utility functions and endowments

¹⁷This is the same question as whether the indexed bonds have the exact same effects on the economy as real securities.

¹⁸For example, Treasury Inflation Protected Securities (TIPS) are U.S. inflation-indexed bonds where coupons and principal are adjusted according to the evolution of the consumer price index.

¹⁹The relative price index is used to know the relative prices among commodities in each state, but it has no information about price levels. The price level in each state is defined later in this section. See equation (15).

²⁰Simply, the two conditions ensure that there is a common income expansion path both across states and among all consumers.

are required. One advantage of our model is that the endowments across states are identical and, consequently, there are no relative-price fluctuations from endowment shocks. Therefore, we do not need any more restrictions on endowments. However, we still need an additional assumption on utility functions:

Assumption 1 *Utility functions u^i , $i \in I$ are weakly separable across states and identically homothetic within states. That is, there exist functions v^i for all $i \in I$ and f such that²¹*

$$v^i : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad i \in I$$

which are smooth, strictly increasing and strictly concave, and functions

$$f : \mathbb{R}_+^L \rightarrow \mathbb{R}$$

which are smooth, strictly increasing, strictly concave, having the closure of indifference curves contained in $\mathbb{R}_{++}^L$ and homogeneous of degree one. Then, u^i is defined as

$$u^i(\vec{x}_0^i, \vec{x}_\alpha^i, \vec{x}_\beta^i) = \pi_\alpha v^i(f(\vec{x}_0^i), f(\vec{x}_\alpha^i)) + \pi_\beta v^i(f(\vec{x}_0^i), f(\vec{x}_\beta^i)).$$

The following lemma shows that a common commodity bundle and price index exist at both date 0 and 1.

Lemma 3 *If Assumption 1 holds, the price index and the fixed commodity bundle can be expressed as*

$$\vec{p}_t = c_t^{pi} \nabla f(e_t) \quad \text{and} \quad \vec{w}_t = c_t^w \vec{e}_t \quad t = 0, 1$$

where

$$\vec{e}_t = (e_{t1}, \dots, e_{tL}) = \sum_i \vec{e}_t^i.$$

and

$$c_t^{pi}, c_t^w > 0$$

Proof. Because (1) all the agents' spot-utility functions $f(\cdot)$ are identical, and (2) the aggregate endowment in each spot is deterministic, there exists a representative agent at each spot with a utility function $f(\cdot)$ at both date 0 and 1 and the representative agent's endowment is $\vec{e}_t (= \sum_i \vec{e}_t^i)$ at date

²¹The two theoretical articles for the inflation-indexed bonds from Geankoplos (2005) and Magill and Quinzii (1997) used almost the same utility format. Both articles assumed a homothetic embedded function $f(x^i)$ at date 1 and utility v^i based on utility f .

t . Then, $\vec{p}i_t$ and $\nabla f(\vec{e}_t)$ are collinear by first-order conditions of the representative agents. Because $f(\cdot)$ is homothetic, $\vec{x}_t^i$ and $\vec{e}_t$ are collinear for all $i \in I$. Therefore, the fixed commodity bundle can be defined as $\vec{w}_t = c_t^w \vec{e}_t$. ■

$\vec{p}i_t$ and $\vec{w}_t$ are estimated by the indexed-bond issuer to design the payoffs of the indexed bonds. c_t^{pi} and c_t^w are positive constant, and their values do not affect the equilibrium allocation because the price of the indexed bond, “ p ” is automatically adjusted according to c_t^{pi} and c_t^w . Therefore, the issuer can assign any values to c_t^{pi} and c_t^w . For the convenience of notation, I normalize $\vec{w}_t$ as $(1, w_{t2}, \dots, w_{tL})$ and $\vec{p}i_t$ as satisfying that $\vec{p}i_t \cdot \vec{w}_t = 1$.

Remark 1 *Geanakoplos 2005 also assumed that $f(x_s^i)$ is a homogeneous degree of one. In his model, there exists intrinsic uncertainty on endowments and, therefore, the aggregate endowment across states is not identical. In that case, it is not possible to derive any common CPI price and commodity bundle, even with the homothetic utility functions. However, he pointed out that the payoffs that guarantee the same utility in terms of the embedded utility $f(x_s^i)$ still can be achievable. This is due to the homogeneity of degree one of $f(x_s^i)$. In my model, the payoffs of those bonds guarantee both the same CPI commodity bundle and the same utility because there is no intrinsic uncertainty.*

Lemma 3 shows that common relative prices $\vec{p}i_1$ exist across the states. Therefore, the price in each state can be expressed as

$$\vec{p}_\alpha = p_\alpha \vec{p}i_1 \quad \text{and} \quad \vec{p}_\beta = p_\beta \vec{p}i_1, \quad (15)$$

where p_α and p_β are scalars, which are determined endogenously. p_α and p_β is interpreted as the price levels while $\vec{p}i_1$ represents the relative prices among commodities. In the same way, the prices in date 0 can be expressed as $\vec{p}_0 = p_0 \vec{p}i_0$. Then, consumer i 's budget constraints can be modified as:

$$\begin{aligned} p_0 \vec{p}i_0 \cdot \vec{x}_0^i + m^i + p(1 + \theta) \max(n^i, 0) + p \min(n^i, 0) &\leq p_0 \vec{p}i_0 \cdot \vec{e}_0^i \\ p_\alpha \vec{p}i_1 \cdot \vec{x}_\alpha^i &= p_\alpha \vec{p}i_1 \cdot (\vec{e}_1^i + n^i \vec{w}_1) + m^i \\ p_\beta \vec{p}i_1 \cdot \vec{x}_\beta^i &= p_\beta \vec{p}i_1 \cdot (\vec{e}_1^i + n^i \vec{w}_1) + m^i \end{aligned} \quad (16)$$

where

$$\vec{p} = (\vec{p}_0, \vec{p}_\alpha, \vec{p}_\beta) = (p_0 \vec{p}i_0, p_\alpha \vec{p}i_1, p_\beta \vec{p}i_1).$$

In this section, p_0, p_α and p_β are scalars, which are determined endogenously, but they do not represent the prices of any specific good. The return

of the indexed bonds can be interpreted as real or nominal returns. $\vec{w}_1$ is the consumption bundle vector of the real return of the bonds while $p_\alpha \vec{p}_1 \cdot \vec{w}_1$ and $p_\beta \vec{p}_1 \cdot \vec{w}_1$ are the nominal returns, which have the same purchasing power to have the consumption bundle $\vec{w}_1$ across the states. Whether the returns are delivered in terms of commodities or money, it does not affect the equilibrium allocations nor prices. The values of the two scalars p_α and p_β depend on price-level volatility; however, the price index ratio $\vec{p}_1$ and the commodity bundle $\vec{w}_1$ do not. Therefore, the government can perfectly estimate the weighted factor $\vec{w}_1$ and the price index $\vec{p}_1$ even without any information about price volatility σ^R .

Next, we need to define price-level volatility and inflation volatility in a multi-good economy. Since the relative prices are the same across states, the price-level volatility can be computed with the same formula as that in the single-good economy:

$$\begin{aligned}\sigma^R &= \frac{\sigma(\tilde{p}_{1l})}{E(\tilde{p}_{1l})} = \frac{\sigma(\tilde{p}_1)}{E(\tilde{p}_1)} \quad \text{for all } l \in L \\ &= \frac{\sqrt{\pi_\alpha \pi_\beta} |p_\alpha - p_\beta|}{\pi_\alpha p_\alpha + \pi_\beta p_\beta} = \frac{\sqrt{\pi_\alpha \pi_\beta} |1 - \mathcal{P}|}{\pi_\alpha + \pi_\beta \mathcal{P}}\end{aligned}$$

where $\tilde{p}_1 = \{p_\alpha, p_\beta; \pi_\alpha, \pi_\beta\}$ is a random variable and $\mathcal{P} = \frac{p_\beta}{p_\alpha} \geq 1$.

At a fixed value of price-level volatility σ^R , the indeterminacy is eliminated even in a multi-good economy.²² Next, I show that a multi-good economy can be mathematically equivalent to that of a single-good economy if Assumption 1 holds. With the fixed commodity bundle and the price index, the utility maximization problem can be re-defined in terms of only the first good ($l = 1$) in each spot. Then, the maximization problem of the multi-good economy is equivalent to that of the single-good economy. In this case, the utility function can be expressed with only the date-0 good ($l = 1$):

$$v^i(f(x_0^i), f(x_s^i)) = v^i(x_{01}^i f(\vec{w}_0), x_{s1}^i f(\vec{w}_1)),$$

since f is the homogeneous degree of one and $x_0^i = x_{01}^i \vec{w}_0$ and $x_s^i = x_{s1}^i \vec{w}_1$ for $s = \alpha, \beta$.

Also, with the relation $\vec{p}_t \cdot \vec{w}_t = 1$, the budget constraints in (16) can

²²Cass (1992) indicated that in an incomplete financial market of a conventional two-period model, sunspots can generate $D = S - J$ nominal and real indeterminacy with S sunspot states and $0 < J < S$ distinct types of bonds. Therefore, the set of the equilibria, without specifying the level of inflation volatility, has one degree of indeterminacy. In my model, as fixing the level of inflation volatility, one degree of indeterminacy can be eliminated.

be replaced with

$$\begin{aligned}
p_0 x_{01}^i + m^i + p(1 + \theta) \max(n^i, 0) + p \min(n^i, 0) &\leq p_0 e_{01}^i \\
p_\alpha x_{\alpha 1}^i &= p_\alpha \left(\vec{p}_{i1} \cdot \vec{e}_1 + n^i \right) + m^i \\
p_\beta x_{\beta 1}^i &= p_\beta \left(\vec{p}_{i1} \cdot \vec{e}_1 + n^i \right) + m^i.
\end{aligned} \tag{17}$$

Then, the maximization problem is exactly the same as that of the single-good economy with a modification in endowment $\left(\vec{p}_{i0} \cdot \vec{e}_0^i, \vec{p}_{i1} \cdot \vec{e}_1^i \right)$ and utility $v^i(x_{01}^i f(\vec{w}_0), x_{s1}^i f(\vec{w}_1))$.

Finally, we need one more clarification about market clearing conditions in date 0. In a single-good economy, it is clear that the issuer uses the revenue from transaction fees to purchase a single good since there is no alternative substitute except the good. However, in a multi-good economy, we need to specify the intermediaries' utility function to determine the corresponding quantity of commodity choices at date 0 from the nominal quantity of the issuer's revenue. It would be reasonable to assume that the intermediaries' utility function at date 0 is the same as $f(\cdot)$. Then, the intermediaries' aggregate budget constraint in the spot market at date 0 is

$$\vec{p}_0 \cdot \vec{x}_0^I = p\theta \sum_i \max(n^i, 0). \tag{18}$$

where $\vec{x}_0^I$ represents intermediates's demand for date-0 commodities.

According to the assumption that the intermediaries' utility function at date 0 is the same the consumer's spot-utility function $f(\cdot)$, $\vec{x}_0^I$ is collinear with $\vec{w}_0$ by Lemma 3. Assuming that $\vec{x}_0 = s\vec{w}_0$, we get the value of s from eqn. (18):

$$\vec{p}_0 \cdot s\vec{w}_0 = p\theta \sum_i \max(n^i, 0) \Rightarrow s = \frac{p\theta \sum_i \max(n^i, 0)}{\vec{p}_0 \cdot \vec{w}_0}.$$

Therefore, the intermediates's demand function $\vec{x}_0^I$ can be derived as

$$\begin{aligned}
\vec{x}_0^I &= \frac{p\theta \sum_i \max(n^i, 0)}{\vec{p}_0 \cdot \vec{w}_0} \vec{w}_0 = \frac{p\theta \sum_i \max(n^i, 0)}{p_0 \vec{p}_{i0} \cdot \vec{w}_0} \vec{w}_0 \\
&= \frac{p}{p_0} \theta \sum_i \max(n^i, 0) \vec{w}_0.
\end{aligned} \tag{19}$$

From eqn (19), we know that $\vec{x}_0^I$ is homogeneous degree one in prices, which is a necessary property in proving the existence and regularity of an economy.

Finally, the market clear conditions can be expressed as

$$\begin{aligned}\sum_i m^i &= 0, \sum_i n^i = 0 \\ \sum_i x_{01}^i + \frac{p}{p_0} \theta \sum_i \max(n^i, 0) &= \sum_i e_{0i}^i\end{aligned}$$

which are equivalent to those in a single-good economy.²³

Therefore, all of the results including Propositions 1-6 can be directly applied for a multi-good case.

Proposition 7 *If Assumption 1 holds in a multi-good economy, the following are true:*

- (i) *A regular economy given price-level volatility σ^R is defined. (Proposition 1)*
- (ii) *Money is always traded if $\theta > 0$. (Proposition 2)*
- (iii) *There exists a $\bar{\theta}$ such that the indexed bonds are (not) traded if $\theta < \bar{\theta}$ ($\theta > \bar{\theta}$). (Proposition 3)*
- (iv) *$\bar{\theta}$ strictly increases in price-level volatility σ^R if v^i is additively separable. (Proposition 4)*
- (v) *(+) There exists at least one consumer who is better off. (++) All consumers can be better off for some economic fundamentals. (Proposition 5)*
- (vi) *A combined market is “superior” to a (pure) monetary market based on the compensation principal (Proposition 6).*

7 CONCLUDING REMARKS

This paper shows that inflation-indexed bonds can be more actively traded in an economy with higher inflation volatility, which is exogenously given. In the same way, we can show that the uncertainty of a nominal interest rate can provide room for inflation-indexed bonds. This idea is basically the same as in Magill and Quinzii (1997). Using a two-period general equilibrium model, they show that the Central Bank’s imperfect process of controlling the money supply can cause the economy to prefer indexed bonds. While

²³The market clearing condition for period-0 good can be derived from the following equations:

$$\begin{aligned}\sum_i \vec{x}_0^i + \frac{p}{p_0} \theta \sum_i \max(n^i, 0) \vec{w}_0 &= \sum_i \vec{e}_0^i, \\ \vec{x}_0^i = x_{01}^i \vec{w}_0 \quad \text{and} \quad \sum_i \vec{e}_0^i &= (\sum_i e_{01}^i) \vec{w}_0\end{aligned}$$

inflation volatility is based on market psychology, volatility in the nominal interest rate is caused by an inconsistent monetary policy.

Recently, a significant number of such inflation-indexed bonds have been issued in many countries. These bonds are taking up a more important position as widely accepted financial instruments. The findings of the study can provide a better understanding of these bonds from a theoretical point of view.

Appendix

A PROOF OF PROPOSITION 4

For the proof of Proposition 6, we need the following lemma.

Lemma 4 x_α^B is increasing but x_β^B is decreasing in $\mathcal{P} > 1$. (Subscript "B" represents the asset buyer.)

Proof. There are four cases:

Case 1 : both x_α^i and x_β^i increase in $\mathcal{P} > 1$.

Case 2 : x_α^i increases but x_β^i decreases in $\mathcal{P} > 1$. (*)

Case 3 : x_α^i decreases but x_β^i increases in $\mathcal{P} > 1$.

Case 4 : both x_α^i and x_β^i decrease in $\mathcal{P} > 1$.

First, Case 3 is ruled out since x_β^i/x_α^i must increase in $\mathcal{P} > 1$. Then, we need to show that Case 1 and 4 are not true.

Assuming that Case 1 is true; For $1 < \mathcal{P} < \mathcal{P}'$, it is true that $x_\alpha^B(\mathcal{P}) < x_\alpha^B(\mathcal{P}')$ and $x_\beta^B(\mathcal{P}) < x_\beta^B(\mathcal{P}')$. By market clearing conditions, we get: $x_\alpha^S(\mathcal{P}) > x_\alpha^S(\mathcal{P}')$ and $x_\beta^S(\mathcal{P}) > x_\beta^S(\mathcal{P}')$.

Choosing $q_0 = 1$, the price of a risky asset is given by

$$\begin{aligned} q^+ &= \frac{\pi_\alpha R_\alpha g^{S'}(x_\alpha^S) + \pi_\beta R_\beta g^{S'}(x_\beta^S)}{f^{S'}(x_0^S)} \\ &= \frac{\pi_\alpha R_\alpha g^{B'}(x_\alpha^B) + \pi_\beta R_\beta g^{B'}(x_\beta^B)}{f^{B'}(x_0^B)}. \end{aligned} \quad (20)$$

where $g^{S'}(x_\alpha^S) = \partial g^{S'}(x_\alpha^S) / \partial x_\alpha^S$ and $g^{B'}(x_\alpha^B) = \partial g^{B'}(x_\alpha^B) / \partial x_\alpha^B$.

By $x_\alpha^B(\mathcal{P}) < x_\alpha^B(\mathcal{P}')$, $x_\beta^B(\mathcal{P}) < x_\beta^B(\mathcal{P}')$ and strictly concavity of g_h , $\pi_\alpha R_\alpha g^{S'}(x_\alpha^S) + \pi_\beta R_\beta g^{S'}(x_\beta^S)$ decreases in $\mathcal{P}$. In the same way, $\pi_\alpha R_\alpha g^{B'}(x_\alpha^B) + \pi_\beta R_\beta g^{B'}(x_\beta^B)$ increases in $\mathcal{P}$. Therefore, by eqn (20), the market clearing condition, $x_0^S +$

$x_0^B = e_0^S + e_0^B$ and strict concavity of f^i , the following inequalities must be satisfied

$$x_0^B(\mathcal{P}) < x_0^B(\mathcal{P}') \text{ and } x_0^S(\mathcal{P}) > x_0^S(\mathcal{P}').$$

The asset buyer's budget constraints are

$$x_0^B(\mathcal{P}) + q^+(\mathcal{P})B^B(\mathcal{P}) = 0 \quad \text{and}$$

$$x_0^B(\mathcal{P}') + q(\mathcal{P}')B^B(\mathcal{P}') = 0$$

Since $x_0^B(\mathcal{P}) < x_0^B(\mathcal{P}')$ and $B^B(\mathcal{P}) < B^B(\mathcal{P}')$,

$$q^+(\mathcal{P}) > q^+(\mathcal{P}') \quad (21)$$

The asset seller's budget constraints are

$$x_0^S(\mathcal{P}) + q^+(\mathcal{P})B^S(\mathcal{P}) = 0 \quad \text{and}$$

$$x_0^S(\mathcal{P}') + q^+(\mathcal{P}')B^S(\mathcal{P}') = 0$$

Since $x_0^S(\mathcal{P}) > x_0^S(\mathcal{P}')$ and $B^S(\mathcal{P}) > B^S(\mathcal{P}')$,

$$q^+(\mathcal{P}) < q^+(\mathcal{P}') \quad (22)$$

Inequalities (21) and (22) contradict each other.

We can show that Case 4 is not true in the same way. ■

$\frac{p_F^i}{q^+}$ is

$$\frac{p_F^i}{q^+} = \frac{\pi_\alpha g^{i'}(x_\alpha^i) + \pi_\beta g^{i'}(x_\beta^i)}{\pi_\alpha R_\alpha g^{i'}(x_\alpha^i) + \pi_\beta R_\beta g^{i'}(x_\beta^i)}.$$

We need to show that $\frac{d}{d\mathcal{P}} \left(\frac{p_F^B}{q^+} \right) > 0$ and $\frac{d}{d\mathcal{P}} \left(\frac{p_F^S}{q^+} \right) < 0$. There are four variables $R_\alpha(\mathcal{P}), R_\beta(\mathcal{P}), x_\alpha^i(\mathcal{P})$ and $x_\beta^i(\mathcal{P})$ which should be considered in the total derivative. The total derivative can be divided into two parts:

$$\begin{aligned} \frac{d}{d\mathcal{P}} \left(\frac{p_F^B}{q^+} \right) &= \frac{d}{d\mathcal{P}} \left(\frac{\pi_\alpha g^{B'}(x_\alpha^B) + \pi_\beta g^{B'}(x_\beta^B)}{\pi_\alpha R_\alpha g^{B'}(x_\alpha^B) + \pi_\beta R_\beta g^{B'}(x_\beta^B)} \right)_{x_\alpha^i, x_\beta^i = \text{constant}} \\ &\quad + \frac{d}{d\mathcal{P}} \left(\frac{\pi_\alpha g^{B'}(x_\alpha^B) + \pi_\beta g^{B'}(x_\beta^B)}{\pi_\alpha R_\alpha g^{B'}(x_\alpha^B) + \pi_\beta R_\beta g^{B'}(x_\beta^B)} \right)_{R_\alpha, R_\beta = \text{constant}}. \end{aligned}$$

Since g^i is strictly concave and $x_\alpha^B > x_\beta^B$ for the asset buyer, $g^{B'}(x_\alpha^B) < g^{B'}(x_\beta^B)$. Since $\pi_\alpha R_\alpha(\mathcal{P})$ and $\pi_\beta R_\beta(\mathcal{P})$ increases and decreases in $\mathcal{P}$, re-

spectively, and $\pi_\alpha R_\alpha(\mathcal{P}) + \pi_\beta R_\beta(\mathcal{P})$ is constant in $\mathcal{P}$,

$$\frac{d}{d\mathcal{P}} (\pi_\alpha R_\alpha g^{B'}(x_\alpha^B) + \pi_\beta R_\beta g^{B'}(x_\beta^B))_{R_\alpha, R_\beta = \text{constant}} < 0.$$

Therefore, the first term is strictly positive.

The second term can be expressed as

$$T(G) = \frac{\pi_\alpha + \pi_\beta G}{\pi_\alpha R_\alpha + \pi_\beta R_\beta G} \text{ where } G = \frac{g^{B'}(x_\beta^B)}{g^{B'}(x_\alpha^B)}$$

$T(G)$ is increasing in G since $\pi_\beta > \pi_\beta R_\beta$. $\frac{dT}{d\mathcal{P}}$ is positive since $\frac{dx_\alpha^B}{d\mathcal{P}} > 0$, $\frac{dx_\beta^B}{d\mathcal{P}} < 0$ (by lemma) and strictly concavity of g' . Thus, the second term also increases in $\mathcal{P}$ and $\frac{d}{d\mathcal{P}} \left(\frac{p_F^B}{p_F^S} \right) > 0$.

We can prove that $\frac{d}{d\mathcal{P}} \left(\frac{p_F^S}{q^+} \right) < 0$ in the same way. Finally,

$$\frac{d}{d\mathcal{P}} \left(\frac{p_F^B - p_F^S}{p_F^S} \right) = \frac{d}{d\mathcal{P}} \left(\frac{p_F^B}{p_F^S} \right) = \frac{d}{d\mathcal{P}} \left(\frac{p_F^B}{q^+} \right) / \frac{d}{d\mathcal{P}} \left(\frac{p_F^S}{q^+} \right) > 0.$$

End of Proof.

B PROOF OF PROPOSITION 6

The budget sets in a pure monetary economy is defined as

$$B_+^i(q) = \{(z^i, B^i) \mid z^i + qB^i \leq 0\}.$$

The budget sets in a combined economy is defines as

$$B_*^i(q, p, \tau^i) = \{(z^i, B^i, n^i) \mid z^i + qB^i + p \max(n^i, 0) + p(1 + \theta) \min(n^i, 0) \leq \tau^i\}.$$

We need to show that for any equilibrium of the money market (z_+^i, B_+^i) , there exists a lump-sum transfer plan $\sum_{i \in I} \tau^i = 0$ such that $(z_+^i, B_+^i, 0) \in B_*^i(q, p, \tau^i)$ for all $i \in I$ and any q (because we do not know about the equilibrium price q^* after a tax plan, we need to prove it for any price $q \in \mathbb{R}_{++}$). Then, the allocation $(z_+^i, B_+^i, 0)_{i=1}^I$ is also affordable in the combined market. That means that the combined market is at least weakly Pareto superior to the money market by the revealed preferences hypothesis.

Assuming that the equilibrium price and allocations in the monetary mar-

ket are q_+ and $(z_+^i, B_+^i)_{i=1}^I$, the following equation is satisfied

$$z_+^i + q_+ B_+^i = 0 \text{ for all } i = 1, \dots, I \quad (23)$$

In the combined market, we need to show the existence of τ^i for $i = 1, \dots, I$ such that $\sum_i \tau^i = 0$ and $(z_+^i, B_+^i, 0) \in B_*^i(q, p, \tau^i)$ for any q . Then,

$$z_+^i + q B_+^i = \tau^i \text{ for all } i = 1, \dots, I \quad (24)$$

subtracting eqn (24) with (23), we get

$$(q - q_+) B_+^i = \tau^i$$

That means that if $\tau^i = (q - q_+) B_+^i$, $(z_+^i, B_+^i, 0) \in B_+^i(q, p, \tau^i)$. Also, we can prove that $\sum_i \tau^i = 0$ by market clearing:

$$\sum_i \tau^i = \sum_i (q - q_+) B_+^i = (q - q_+) \sum_i B_+^i = 0.$$

End of Proof.

C EXAMPLE

There are two consumers B and S with endowments of $(10, 0)$ and $(0, 10)$, respectively. Their expected utility functions are the same as $\log(x_0) + \log(x_1)$. State probabilities are $\pi_\alpha = \pi_\beta = 0.5$. Assuming that the inflation level $\mathcal{P} = 2$, the corresponding price-level (inflation) volatility is $\sigma^R = 33.3\%$ by eqn (4). In this case, it is computed that $\bar{\theta} = 23\%$ by eqn (12). Assuming that the transaction cost is 10% ($\theta = 0.1$) in the combined market, the equilibrium outcomes are summarized in the table.

	Monetary Market		Combined Market	
	Buyer	Seller	Buyer	Seller
Utility Level	3.07	3.26	3.19	3.18
$money(m^i)$	0.5	-0.5	0.212	-0.212
$bond(n^i)$	-		0.276	-0.276
Price-level: (p_α, p_β)	(0.820, 1.640)		(0.744, 1.488)	
Bond-price: $(p, p + \theta)$	-		(0.948, 1.048)	
q	1.093		0.992	

The equilibrium price-levels in the monetary market are higher than those in the combined market. Higher price levels imply a higher value of money in

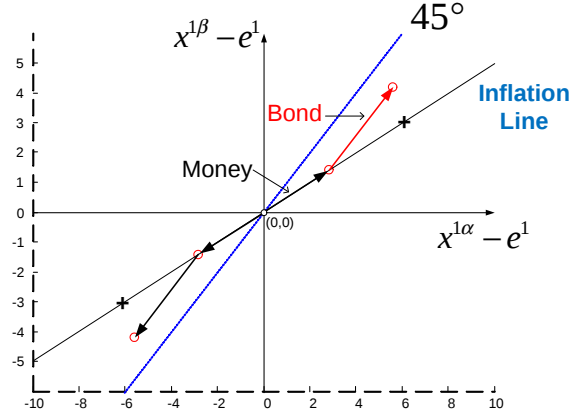


Figure 3:

date 0. As explained in section 5, the higher value of money is from the asset buyer's higher demand for money. In a combined market, the demand for money is lower because of the substitution effects from the introduction of indexed bonds. Consequently, the value of money is not as high as that in a pure monetary market. The change of money value after the introduction of indexed bonds causes the asset seller's utility to decrease from 3.26 to 3.18. (See the table.)

Figure 3 represents the equilibrium allocations in the space of excess demand. The figure should be three dimensional including the excess demand of x_0 but we can imagine the original three-dimensional figure is projected onto a two-dimensional one. The buyer's allocation is located in the northeast quadrant while the seller's is in the southwest quadrant. The allocations are symmetric to (0,0) by market clearing conditions. "+" and "o" represent the equilibrium allocations in pure monetary and combined markets, respectively. From the figure, we know that the equilibrium allocations move closer to the 45 degree line in the combined market although both markets have the same volatility level.

Figure 4 shows how two consumers utility change in transaction costs. Surprisingly, the asset seller's utility is increasing in transaction cost θ if $\theta > 10\%$. Higher transaction costs make the demand for money increase by the substitution effects. Therefore, the price of money goes up and consequently, the asset buyer has more income from selling the money (risky asset) at a higher price. If the transaction cost is higher than 23%, the indexed bond market is inactive. In that case, the equilibrium allocations in a combined economy $(\mathcal{E}, \sigma^R, \theta)_C$ are identical to those in a pure monetary

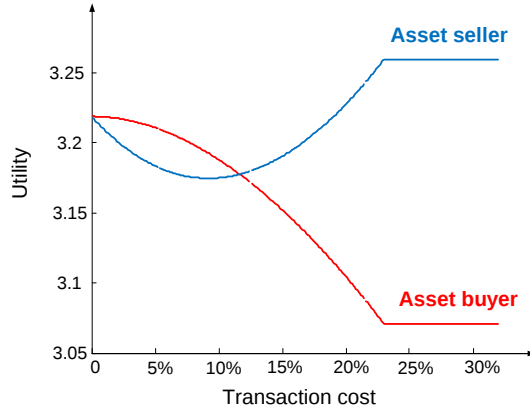


Figure 4:

economy $(\mathcal{E}, \sigma^R)_M$.

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