SC1004 Freshmen Exemption Test Information

Syllabus

This course aims to support you to learn mathematical concepts related to linear algebra and complex numbers. You will develop set of mathematical skills for applications in computer science and engineering, e.g., machine learning, computer graphics, data science etc.

1. Introduction to vectors and matrices	
2. Linear Equations	System of linear equations; Row reduction and echelon forms; Vector equations; The matrix equation; Solutions sets of linear equations; Linear independence; The matrix of a linear transformation
3. Matrix Algebra	Matrix operations; Inverse of a matrix; Matrix LU factorization
4. Determinants	Properties; volume and linear transformations
5. Vector Spaces	Vector spaces and subspaces; Null spaces, column spaces and linear transformations; linearly independent sets and bases; Dimension of a vector space; Rank; Change of Basis
6. Applications of topics 1-4	
7. Orthogonality	Inner product, length and orthogonality; Orthogonal sets; Orthogonal projections; The Gram-Schmidt process; QR factorization
8. Least Squares	Minimization Problems, Projection Matrix Least-squares solution
9. Complex Numbers	Arithmetic Operations, Geometric representation, Euler's formula, De Moivre's Theorem, Examining Discrete Fourier Transform as a change of basis-complex sequences, complex inner product, orthogonal decomposition, Unitary Matrixes
10. Eigen Values and Singular Values	Eigen Decomposition: Diagonalizing a matrix Singular value decomposition and Pseudo-Inverse

Reference Book

Linear Algebra and its applications, David C. Lay, Steven R. Lay and Judi J. McDonald, Pearson, 6th edition, 2021.

Sample Questions

1. (a) Find the coefficients a, b and c in the following linear system of equations such that the solution of the system is $x = 1, y = -1, z = 2$.

$$\begin{aligned} ax + by - 3z &= -3 \\ -2x - by + cz &= -1 \\ ax + 3y - cz &= -3 \end{aligned}$$

The working should clearly show the steps towards a reduced row echelon form obtained by elementary row operations.

(6 marks)

(b) Consider the vectors $\mathbf{a} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$. Do these vectors span a line, a plane or R^3? Justify.

(4 marks)

(c) A linear system $A\mathbf{x} = \mathbf{b}$ can be transformed through elementary row operations to $R\mathbf{x} = \mathbf{d}$, where R is the reduced row echelon form of A. Suppose the complete solution of $A\mathbf{x} = \mathbf{b}$ is given by

$$\mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}, \quad c_1 \text{ and } c_2 \text{ are scalars.}$$

Find the 3×3 reduced row echelon form R and the vector **d.**

(6 marks)

(d) Let $T: R^2 \to R^3$ be a linear transformation such that $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$. Consider the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Writing $\mathbf{v}$ as a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, find $T(\mathbf{v})$.

(5 marks)

(e) If Q is a matrix such that $Q^TQ = I$, find $|Q|$.

(4 marks)

2. (a) For the matrix $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$, find the value of k if $A^2 = kA - 2I$ and verify its correctness.

(6 marks)

(b) Find the dimension of the null space $\mathbf{N}(B^TB)$ and rank of B^TB for the matrix

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

(10 marks)

(c) Let $\mathbf{x_1}$ and $\mathbf{x_2}$ be vectors in R^n and $S = Span\{\mathbf{x_1}, \mathbf{x_2}\}$. Verify that the conditions for S to be a subspace of R^n are satisfied. Show your working clearly.

(6 marks)

(d) State whether the following statements are True or False:

(i) The column vectors of every 3×5 matrix A are linearly dependent.

(ii) If $T: R^n \to R^n$ is such that $T(\mathbf{x}) = \mathbf{0}$ for every vector $\mathbf{x}$ in R^n, then the matrix A representing the transformation T is the $n \times n$ zero matrix.

(iii) Let A be an $n \times n$ matrix. The linear system $A\mathbf{x} = 4\mathbf{x}$ has a unique solution if and only if $A - 4I$ is an invertible matrix.

(3 marks)

3. Given the following matrix,

$$A = \begin{bmatrix} -2/\sqrt{2} & -3/\sqrt{2} & 0 \\ -2/\sqrt{2} & +3/\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(a) (i) Show that the columns of A form an orthogonal set and then normalize these columns to form an orthogonal matrix Q.

(6 marks)

(ii) Let $y = [1 \;\; 2 \;\; 3]^T$. Two columns of Q (from Q3a(i)) are used to form U to approximate y by $Ux = \hat{y}$, where $\hat{y}$ is the least squares approximation to y. Determine which 2 columns of Q should be selected. Provide your reasons and/or workings.

(5 marks)

(iii) Using your chosen Q, calculate the least squares solution x, $\hat{y}$, and the norm of the residual error.

(6 marks)

(b) Given a 3×2 matrix U with orthonormal columns spanning subspace W, comment onUU^T matrix's properties in terms of rank, orthogonality, dimension, type of matrix, and space that it spans.

(8 marks)

4. (a) Perform *QR* decomposition on the given *A* matrix via the Gram-Schmidt process to generate Q with property $Q^TQ = I$, where *I*'s dimension is 2x2 and upper triangular matrix R, where

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

(6 marks)

(b) Given

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$$

(i) Find A's eigen values and vectors.

(6 marks)

(ii) Evaluate $A^{10}x$ for $x = [1 \; 1]^T$.

(4 marks)

(c) With respect to eigen decomposition, complete the following sentences for square matrices A and B:

(i) A is defective if there is an eigen value λ whose geometric multiplicity is ____________

(2 marks)

(ii) A and B are similar if ________________

(2 marks)

(d) Answer True/False for the followings with respect to a square matrix A.

(i) If A is invertible, it will always be diagonalizable.

(ii) All diagonalizable matrices are invertible.

(iii) If A has unique eigenvalues, it is always diagonalizable.

(iv) The null space of $(A - \lambda I)$ is spanned by A's eigenvectors.

(v) If λ is the eigen value of A, then λ^{-1} is an eigen value of A^{-1}.

(5 marks)

5. If **a** and **b** are non-zero and non-collinear vectors such that $\|\mathbf{a}+\mathbf{b}\| < \|\mathbf{a}\|$, then the angle θ between **a** and **b** is

A. $0°$
B. $90°$
C. $0° < \theta < 90°$
D. $90° < \theta < 180°$
E. Cannot be determined

6. Two matrices X and Y are said to be *similar* if $P^{-1}XP = Y$ for some invertible matrix P. Suppose A and B are $n \times n$ invertible matrices, where $n > 1$ and I is the $n \times n$ Identity matrix. If A and B are similar matrices, which of the following statements must be true?

I. $A - 2I$ and $B - 2I$ are similar matrices.
II. A^{-1} and B^{-1} are similar matrices.

A. I only
B. II only
C. Neither I nor II
D. Both I and II
E. Either I or II depending on n

7. Evaluate $z = \left[\frac{1e^{\frac{j\pi}{4}} - \frac{1}{2}(\sqrt{2} - j\sqrt{2})}{1+j}\right]^2$

A. -1,
B. $\sqrt{j}$,
C. j
D. $\sqrt{2} + j\sqrt{2}$
E. None of the above

8. What are the DFT coefficients $X[k]$ for $x(n) = 3e^{\frac{-j2\pi}{8}n} + 1\cos\left(\frac{\pi}{4}n\right)$ for $N = 8, n = 0..(N-1)$.

A. $X[k] = N[0,3.5,0,0,0,0,0,0.5]^T$
B. $X[k] = N[0,0.5,0,0,0,0,0,3.5]^T$
C. $X[k] = 1/N[0,3,0.5,0,0,0,0,0,0.5]^T$
D. $X[k] = N[0,3,0.5,0,0,0,0,0.5]^T$
E. None of the above

9. List the elements of column $k = 6$ of the DFT Synthesis matrix W^H for $N = 8$. Note, index of columns starts from $k = 0$.

A. $w = e^{j\frac{2\pi}{N}}$, $[w^0, w^1, w^2, \dots, w^{N-1}]^T$
B. $w = e^{-j\frac{2\pi}{N}}$, $[w^0, w^1, w^2, \dots, w^{N-1}]^T$
C. $w = e^{-j\frac{2\pi}{N}}$, $\left[w^0, w^6, w^{12}, \dots, w^{(N-1)6}\right]^T$
D. $w = e^{+j\frac{2\pi}{N}}$, $\left[w^0, w^6, w^{12}, \dots, w^{(N-1)6}\right]^T$
E. None of the above