NANYANG RESEARCH PROGRAMME

MAE03 - INVESTIGATION OF HYPERTRANSCENDENTAL FUNCTIONS AND TRANSCENDENTAL NUMBERS RELATED TO THE RIEMANN ZETA FUNCTION AND THE GAMMA FUNCTION

Shen Hanzhang
Dunman High School
Zhu Yuxin
Victoria Junior College

Associate Professor Shu Jian Jun
School of Mechanical and Aerospace Engineering
Nanyang Technological University

Abstract - To investigate the Riemann zeta function $\zeta(s)$, a crucial function in analytic number theory, properties of the Gamma function $\Gamma(s)$ are of great importance since $\zeta(s)$ can be expressed in terms of $\Gamma(s)$. One of such properties, that $\Gamma(s)$ (as well as some other special functions) is transcendentally transcendental, is the focus of this paper. The hypertranscendence of $\Gamma(s)$ is proven in Hölder's Theorem using field theory and proof by contradiction. Identifying the inherent limitation of this proving method as it is based on properties of function only specific to $\Gamma(s)$, this paper aims to show a more generalised method of proving transcendentally transcendental functions largely based on 2 introduced theorems, and therefore, constructively provide a relatively more complete list of transcendentally transcendental functions and transcendental numbers than existing research work. These transcendental numbers are obtained with algebraic manipulations of the functional equations of special functions, all of which are related to $\zeta(s)$ and $\Gamma(s)$. From there, we have made several extensions which can be explored in the future.

Keywords: abstract algebra, complex analysis, transcendentally transcendental functions, transcendental numbers, Riemann zeta function, Gamma function

1 INTRODUCTION

1.1 A transcendentally transcendental function, or hypertranscendental function, is a transcendental analytic function that does not satisfy any algebraic differential equations with algebraic initial conditions. In other words, functions that are solutions of differential equations of the form

$$F(x,\ y,\ y',\ \dots\ ,\ y^{(n)})\ =\ 0,$$

where F is a polynomial with constant coefficients, are algebraically transcendental or differentially algebraic, and those transcendental functions that are not solutions to the equations of the form above are transcendentally transcendental functions [1].

In order to investigate the prime number theorem, Bernhard Riemann extensively studied the Zeta function [2], which was first raised by the Swiss mathematician Leonhard Euler.

1.2 Riemann zeta function is defined as

$$\zeta(s)\ =\ \sum_{n=1}^{\infty}\frac{1}{n^s}\ =\ \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{x^{s-1}}{e^x-1}dx.$$

As can be seen from the definition above, the Riemann zeta function is closely related to another special mathematical function, the Gamma function. Hölder's Theorem [3] states that the Gamma function does not satisfy any algebraic differential equation, i.e., it is transcendentally transcendental. This was proposed by the German mathematician Otto Hölder in 1887.

1.3 Reasons for undertaking the study: Current studies in this field are limited to a few special mathematical functions, such as the Gamma function. A relatively complete list of transcendentally transcendental functions has yet to be validated. Furthermore, the methods of proof are constrained by the properties of specific functions. Few methods are applicable to many special mathematical functions.

1.4 A transcendental number is a number that is not algebraic—that is, not the root of a non-zero polynomial of finite degree with rational coefficients, with the best known transcendental numbers being π and e [4]. The concept of transcendental number was first raised by Leibniz in 1682, defined by Euler in the 18th century, and proved the existence of by Joseph Liouville in 1844. Later developments in transcendental number theory include 2 most notable results: (i)Lindemann–Weierstrass theorem [5]:

If $\alpha_1, \ldots, \alpha_n$ are algebraic numbers that are linearly independent over the rational numbers $\mathbb{Q}$, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$; and

(ii)Gelfond–Schneider theorem [6]:

If a and b are algebraic numbers with $a \neq 0, 1$, and b being irrational, then any value of a^b is a transcendental number.

The following graph (Figure 1) helps to classify the terms mentioned above, such as algebraic numbers, transcendental numbers, rational numbers, and irrational numbers. It also visualises the relations between the terms.

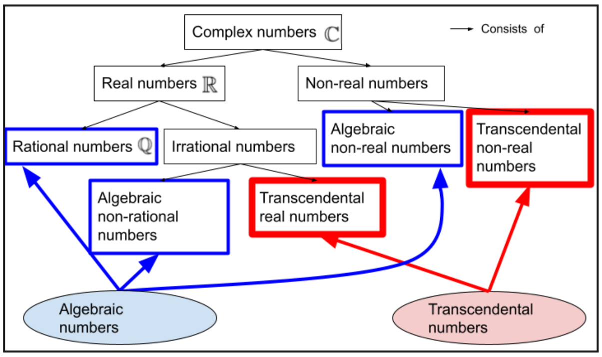

Figure 1: Classification of complex numbers based on whether they are transcendental or algebraic

These 2 theorems are then extended by Baker's theorem [7], and all of these are further generalised by Schanuel's conjecture [8].

1.5 Inherent limitations: Despite our best efforts in completing an in-depth study, our study is only a theoretical one of transcendentally transcendental functions and transcendental numbers, excluding their real-life applications.

1.6 General notations:

$\in$ denotes set membership

$\setminus$ denotes set difference

$\forall$ denotes universal quantification

$\circ$ denotes composition of functions

D denotes the differential operator satisfying: $D(x + y) = Dx + Dy, D(xy) = yDx + xDy$

$\mathbb{C}$ denotes the set of complex numbers

$\mathbb{Q}$ denotes the set of rational numbers

$\mathbb{Z}$ denotes the set of integers

$\mathbb{N}$ denotes the set of natural numbers

A denotes the set of algebraic numbers in Theorem 4.2.1

T denotes the set of transcendental numbers in Theorem 4.2.1

K denotes the differential field K: $K = (K, +, ., D, 0, 1)$

L denotes the differential field L: $L \in K$

1.7 Outline:

1. Introduction
2. Objective and Methodology
3. Literature Review
    I. Hölder's Theorem
    II. Other Transcendentally Transcendental Functions
    III. Transcendental Numbers
4. Results and Discussion
    I. Transcendentally Transcendental Functions
    II. Transcendental Numbers
5. Conclusion

2 OBJECTIVE AND METHODOLOGY

This research project aims to provide a relatively more complete list of transcendentally transcendental functions and transcendental numbers than existing research work, using a generalised proving method which is replicable.

Inspired by the proving method in Hölder's Theorem using proof by contradiction and field theory, we go on to investigate if some special mathematical functions [9] are differentially algebraic, i.e., transcendentally transcendental, or not. These functions include first and foremost, the Riemann zeta function $\zeta(s)$, which is the starting point of our research project. By using certain proven transcendental numbers, some values of transcendentally transcendental functions are proven to be transcendental. Thus, more transcendental numbers are found. From there, we have made several extensions which can be explored in the future.

The objective and methodology, as well as the flow of the paper, can be visualised by Figure 2.

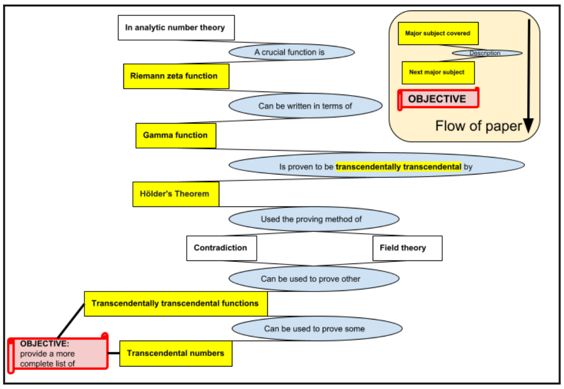

Figure 2: Visualisation of "OBJECTIVE AND METHODOLOGY" and flow of the paper

3 LITERATURE REVIEW

A considerable amount of research regarding transcendentally transcendental functions has been done. Relevant papers include "*Ueber die Eigenschaft der Gammafunction keiner algebraischen Differentialgleichung zu genügen"* [10], *"A Note on Hölder's Theorem Concerning the Gamma Function"* [11], *"A Survey of Transcendentally Transcendental Functions"* [1], and *"Differentially Transcendental Functions"* [12]. These papers have established the basis for transcendentally transcendental functions and have proposed some of such functions. However, only a limited number of functions, such as the Gamma function, have been identified as transcendentally transcendental functions, and we believe that there are many more. Therefore, based on previous research work, this research project aims to extend the list of transcendentally transcendental functions.

3.I HÖLDER'S THEOREM

Although a few mathematicians have proved Hölder's Theorem, their methods are not easily replicable to other special mathematical functions.

Lee A. Rubel used field theory and proof by contradiction to show that the Gamma function is not differentially algebraic in his paper "*A Survey of Transcendentally Transcendental Functions"* [1].

On the other hand, Steven B. Bank and Robert P. Kaufman provided a proof of Hölder's Theorem by Ostrowski in their paper "*A Note on Hölder's Theorem Concerning the Gamma Function"* [11]. The proof supposed the existence of some polynomial P which has a solution in terms of the Gamma function with the minimality assumption. Thereafter, another polynomial Q is defined as a transformation of P, followed by the use of mathematical induction to prove that P does not satisfy the minimality assumption, and hence, the Gamma function is proven not to satisfy any algebraic differential equation via proof by contradiction.

Both proofs are valid. However, it can be sometimes very difficult to similarly apply these two methods to other special functions to prove their transcendental transcendence, as these 2 methods of proof have used properties that are specific to the Gamma function. In order to apply these methods to other functions, each function and its own properties need to be examined thoroughly, which could be extremely difficult sometimes. Therefore, these two methods are not suitable for our study of other transcendentally transcendental functions here.

3.II OTHER TRANSCENDENTALLY TRANSCENDENTAL FUNCTIONS

Rubel investigated a few transcendentally transcendental functions [1], including the Gamma function $\Gamma(s)$. Rubel started by distinguishing differentially algebraic functions from transcendentally transcendental functions, and then proved some transcendentally transcendental functions. Other than Ostrowski's proof mentioned above, Rubel's paper only provides very few transcendentally transcendental functions and power series, hence it is unable to provide a relatively complete list of transcendentally transcendental functions.

In "*Differentially Transcendental Functions"* [12], Žarko Mijajlović and Branko Malešević provided a method to systematically prove some transcendentally transcendental functions using concepts of differential fields, followed by applying this method to a few special functions such as Barnes G-function and Kurepa's function. This method of proof is easily replicable when applied to many other special functions as it is essentially using differential fields. However, similar to Rubel's paper, the paper failed to provide a detailed list of transcendentally transcendental functions. Despite that, this very method of proving transcendentally transcendental functions is of great value to our study.

As mentioned above, although some functions are proven to be transcendentally transcendental in the papers, almost all the papers failed to provide a relatively complete list of such functions, which will be the main focus of this paper. For this very purpose, some useful methods in the reviewed literature are applied in our study.

3.III TRANSCENDENTAL NUMBERS

In "*Transcendence of Periods: The State of the Art"* [13], Michel Waldschmidt proved that

$\Gamma(z + r)$ is a transcendental number and is algebraically independent of π for any integer z and each of the fractions $r = \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}$.

In "*Introduction to Modern Number Theory*" [14], Yuri Ivanovic Manin and Alexei A. Panchishkin proved that (i)π, e^{π}, and $\Gamma(\frac{1}{4})$ are algebraically independent over $\mathbb{Q}$, and that (ii)π, $e^{\pi\sqrt{3}}$, and $\Gamma(\frac{1}{3})$ are algebraically independent over $\mathbb{Q}$. Gisbert Wüstholz [25] proved that some values of the Beta function $B(x, y)$ are transcendental using functional equations of the Beta function as well as Abelian integrals.

These theorems are of great importance to our study of transcendental numbers, as these theorems and functional equations of the Gamma function are used in our study to prove transcendental values of other differentially transcendental functions.

4 RESULTS AND DISCUSSION

4.I TRANSCENDENTALLY TRANSCENDENTAL FUNCTIONS

Theorem 4.1.1 For every $n \in \mathbb{N}$**,** there is no non-zero polynomial $P \in \mathbb{C}[X; Y_0, Y_1, ..., Y_n]$ such that

$\forall s \in \mathbb{C}\backslash\mathbb{Z}_{\leq 0}$, $P(s;\ \Gamma(s),\ \Gamma'(s),\ ...,\ \Gamma(s)^{(n)}) = 0$,

where Γ is the Gamma function.

This theorem was proven by Otto Hölder in 1887, called Hölder's Theorem [3][10][11][15].

Theorem 4.1.2 Let differential field $K = (K, +, ., D, 0, 1)$. For the rest of the paper, let field $L \in K$.

Let $a(s)$ be a complex transcendentally transcendental function over $\mathbb{C}$, $f(s, u_0, u_1, ..., u_m, y_1, ..., y_n)$ a rational expression over L, and assume that $e_1, ..., e_m$ are entire functions which are differentially algebraic over $\mathbb{C}$, $e_i' \neq 0$, $1 \leq i \leq m$. If b is meromorphic and $f(s, b, b \circ e_1, ..., b \circ e_m, Db, ..., D^n b) \equiv a(s)$, where D is the differential operator satisfying:

$D(x + y) = Dx + Dy, D(xy) = yDx + xDy$, then b is transcendentally transcendental over $\mathbb{C}$.

This theorem was proven by Žarko Mijajlović and Branko Malešević in their paper "*Differentially Transcendental Functions*" [12].

Theorem 4.1.1 and Theorem 4.1.2 will be used to prove that Riemann zeta function $\zeta(s)$, and some other special functions are also transcendentally transcendental.

Proposition 4.1.3 The Riemann zeta function $\zeta(s)$ is transcendentally transcendental.

Proof: Since the Riemann zeta function [2], $\zeta(s)$, satisfies the equation

$$\zeta(s) = 2^s \pi^{s-1} sin(\frac{\pi s}{2})\Gamma(1 - s)\zeta(1 - s)$$

therefore, we have

$$\frac{\zeta(s)}{\zeta(1-s)} = 2^s \pi^{s-1} sin(\frac{\pi s}{2})\Gamma(1 - s).$$

Let's suppose for the sake of contradiction that $\zeta(s)$ is differentially algebraic on $\mathbb{C}$, meaning that $\zeta(s)$ is a solution to some differential equation of the form $F(x, y, y', ..., y^{(n)}) = 0$ and $\zeta(s) \in L$. As a result, $\frac{\zeta(s)}{\zeta(1-s)} \in L$. Since $2^s \pi^{s-1} sin(\frac{\pi s}{2})$ is an elementary function, it is also differentially algebraic on $\mathbb{C}$ and belongs to L. Therefore, $\Gamma(1 - s)$ is also differentially algebraic on $\mathbb{C}$. However, by Theorem 4.1.1, the Gamma function is transcendentally transcendental and this yields a contradiction. Since we have a contradiction, $\zeta(s)$ is not differentially algebraic on $\mathbb{C}$. Therefore, the Riemann zeta function must be transcendentally transcendental.

Q.E.D.

Proposition 4.1.4 The Hurwitz zeta function is a meromorphic function.

Proof: The Hurwitz zeta function [16], $\zeta(s, k)$ is defined by

$$\zeta(s, k) = \sum_{n=0}^{\infty} \frac{1}{(n+k)^s}$$

The Riemann zeta function is a special case when $k = 1$. Since Riemann zeta function is meromorphic with a pole at $s = 1$, the Hurwitz zeta function, therefore, is meromorphic with a pole when $k = 1$ and $s = 1$.

Q.E.D.

Proposition 4.1.5 The Hurwitz zeta function $\zeta(s, k)$ is transcendentally transcendental when k is a rational number.

Proof: The Hurwitz zeta function [16] satisfies the functional equation

$$\zeta(1 - s, \frac{m}{n}) = \frac{2\Gamma(s)}{(2\pi n)^s} \sum_{k=1}^{n} \left[cos(\frac{\pi s}{2} - \frac{2\pi k m}{n})\zeta(s, \frac{k}{n})\right].$$

Suppose for the sake of contradiction that the Hurwitz zeta function is differentially algebraic over $\mathbb{C}$, which means that $\zeta(1 - s, \frac{m}{n})$ and $\zeta(s, \frac{k}{n})$ are differentially algebraic over $\mathbb{C}$ and belongs to L. Since $(2\pi n)^s$ and $cos(\frac{\pi s}{2} - \frac{2\pi k m}{n})$ are

elementary functions, they are also differentially algebraic over $\mathbb{C}$ and belongs to L. This implies that $\Gamma(s)$ is differentially algebraic over $\mathbb{C}$ and belongs to L. However, from Theorem 4.1.1, the Gamma function is transcendentally transcendental. This yields a contradiction, hence the Hurwitz zeta function must be transcendentally transcendental.
Q.E.D.

Proposition 4.1.6 The multivariate gamma function is transcendentally transcendental.

Proof: The multivariate gamma function [17] satisfies the equation

$$\Gamma_p(s) = \pi^{\frac{p(p-1)}{4}} \prod_{j=1}^{p} \Gamma(s + \frac{1-j}{2}).$$

Suppose for the sake of contradiction that $\Gamma_p(s)$ is differentially algebraic on $\mathbb{C}$, and $\Gamma_p(s) \in L$. Since $\pi^{\frac{p(p-1)}{4}}$ is an elementary function, it is also differentially algebraic on $\mathbb{C}$, and $\pi^{\frac{p(p-1)}{4}} \in L$. This implies that $\prod_{j=1}^{p} \Gamma(s + \frac{1-j}{2})$ is also differentially algebraic on $\mathbb{C}$. However, by Theorem 4.1.1, Hölder proved that the Gamma function does not satisfy any algebraic differential equation, yielding a contradiction. Therefore, the multivariate gamma function, $\Gamma_p(s)$, is transcendentally transcendental.
Q.E.D.

Proposition 4.1.7 The Dirichlet L-function is transcendentally transcendental.

Proof: The Dirichlet L-function [18] satisfies the equation

$$L(s,\chi) = \varepsilon(\chi)2^s\pi^{s-1}q^{\frac{1}{2}-s}sin(\frac{\pi}{2}(s + a))\Gamma(1 - s)L(1 - s,\bar{\chi})$$

where $|\varepsilon(\chi)| = 1$. Since $\Gamma(1 - s)$ is transcendentally transcendental by Theorem 4.1.1 and the Dirichlet L-function, $L(s,\chi)$, is meromorphic with a pole at $s = 1$, $L(s,\chi)$ is transcendentally transcendental by Theorem 4.1.2.
Q.E.D.

Proposition 4.1.8 The Dirichlet eta function is transcendentally transcendental.

Proof: The Dirichlet eta function [19], $\eta(s)$ satisfies the equation

$$\eta(s) = (1 - 2^{1-s})\zeta(s),$$

where $\zeta(s)$ is the Riemann zeta function. Suppose for the sake of contradiction that $\eta(s)$ is differentially algebraic over $\mathbb{C}$ and $\eta(s) \in L$. Since $(1 - 2^{1-s})$ is an elementary function, obviously it is differentially algebraic over $\mathbb{C}$. This suggests that $\zeta(s)$ is also differentially algebraic over $\mathbb{C}$. However, by Theorem 4.1.3, the Riemann zeta function is transcendentally transcendental, yielding a contradiction. Therefore, the Dirichlet eta function must be transcendentally transcendental.
Q.E.D.

Proposition 4.1.9 The Beta function is transcendentally transcendental.

Proof: The Beta function [20], $B(x,y)$, satisfies the functional equation

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where $\Gamma(x)$ is the Gamma function.
Suppose for the sake of contradiction that $B(x,y)$ is differentially algebraic, and $B(x,y) \in L$. This implies that $\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is differentially algebraic and belongs to L. However, by Theorem 4.1.1, the Gamma function is transcendentally transcendental. This yields a contradiction and the Beta function must be transcendentally transcendental.
Q.E.D.

Proposition 4.1.10 Gauss's Pi function is transcendentally transcendental.

Proof: Since Gauss's Pi function [21] and Euler's Gamma function are essentially the same:

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx$$

$$\Pi(s) = \Gamma(s + 1) = \int_0^\infty x^s e^{-x}dx,$$

It naturally follows from Theorem 4.1.1 that Gauss's Pi function is also transcendentally transcendental.
Q.E.D.

4.II TRANSCENDENTAL NUMBERS

Theorem 4.2.1 Let α be an algebraic number, and β a transcendental number, i.e. $\alpha \in A$, $\beta \in T$, then:

(1) $\alpha\cdot\beta \in A$ if $\alpha = 0$. (since 0 is algebraic)

$\alpha\cdot\beta \in T$ if $\alpha \neq 0$.

Proof: If $\alpha\cdot\beta$ is algebraic, then $\alpha^{-1}\alpha\beta = \beta$ will be algebraic, yielding a contradiction. Hence $\alpha\cdot\beta$ must be transcendental.
Q.E.D.

(2) $\beta^n \in T$, $\forall n \in \mathbb{N}$.

Proof: Suppose that $\beta^n \in A$, i.e. there is a nontrivial polynomial $P(x) \in \mathbb{Z}[x]$ and

$P(\beta^n) = 0$. Replacing every x^m in $P(x)$ with $x^{m \cdot n}$, where $m \in \mathbb{N}$, another non-trivial polynomial $Q(x)$ is obtained. It follows that $Q(\beta) = P(\beta^n) = 0$. Therefore, β is algebraic, yielding a contradiction. Hence, if $\beta \in T$, then $\beta^n \in T$.
Q.E.D.

(3) $\pi^k \in T$, where $k \in \mathbb{Q}$, $k \neq 0$.

Proof: Suppose some $A = \pi^k = \pi^{\frac{a}{b}}$ is algebraic, where a, b are non-zero integers. Then $A^{\frac{b}{a}}$ would be algebraic, but $A^{\frac{b}{a}} = \pi^{\frac{a}{b} \cdot \frac{b}{a}} = \pi$, which is transcendental. This yields a contradiction. Hence, π^k must be transcendental.
Q.E.D.

(4) $(\alpha + \beta) \in T$.

Proof: Suppose that $\alpha + \beta = a$, where $a \in A$. Then $a - \alpha = \beta$ would be transcendental. However, the difference between 2 algebraic numbers a and α must be algebraic, yielding a contradiction. Hence, $(\alpha + \beta) \in T$.
Q.E.D.

Proposition 4.2.2 $\Gamma(\frac{1}{2} + z)$ is transcendental $\forall z \in \mathbb{Z}$.

Proof: $\Gamma(\frac{1}{2} + 0) = \sqrt{\pi}$, which is transcendental [22].

Since $\forall n \in \mathbb{N}_0$,

$$\Gamma(\frac{1}{2} + n) = \frac{(2n-1)!!}{2^n}\sqrt{\pi} = \frac{(2n)!}{4^n n!}\sqrt{\pi}$$

$$\Gamma(\frac{1}{2} - n) = \frac{(-2)^n}{(2n-1)!!}\sqrt{\pi} = \frac{(-4)^n n!}{(2n)!}\sqrt{\pi},$$

it follows that the product of $\sqrt{\pi}$ and $\frac{(2n)!}{4^n n!}$ (algebraic), and the product of $\sqrt{\pi}$ and $\frac{(-4)^n n!}{(2n)!}$ (algebraic) are both transcendental (Theorem 4.2.1). Since $\{0\} \cup \mathbb{N} \cup \{(-1) \times \mathbb{N}\} = \mathbb{Z}$, it can be generalised that $\Gamma(\frac{1}{2} + z)$ is transcendental $\forall z \in \mathbb{Z}$.
Q.E.D.

Theorem 4.2.3 $sin(k\pi)$ and $cos(k\pi)$ are always algebraic, where $k \in \mathbb{Q}$.

Proof: Let $k = \frac{a}{b}$, where a, b are integers. We know that $e^{i\frac{a\pi}{b}} = cos(\frac{a\pi}{b}) + isin(\frac{a\pi}{b})$, where $cos(\frac{a\pi}{b})$ and $sin(\frac{a\pi}{b})$ are purely real (Euler's formula). Since $e^{i\frac{a\pi}{b}}$ is a root of the polynomial $x^{2b} - 1$ (root of unity), $e^{i\frac{a\pi}{b}}$ is algebraic (by definition). We also know that i is algebraic.

Since $e^{ik\pi}$ and $e^{-ik\pi}$ are both algebraic as they are solutions of $x^{\frac{1}{k}} + 1 = 0$, and since we know $sin(k\pi) = \frac{e^{ik\pi} - e^{-ik\pi}}{2i}$, $cos(k\pi) = \frac{e^{ik\pi} + e^{-ik\pi}}{2}$, by Theorem 4.2.1, we can confirm that $sin(k\pi)$ and $cos(k\pi)$ are always algebraic.
Q.E.D.

Theorem 4.2.4 The numbers π and $\Gamma(\frac{1}{3})$ are algebraically independent over $\mathbb{Q}$ [14].

Theorem 4.2.5 The numbers π and $\Gamma(\frac{1}{4})$ are algebraically independent over $\mathbb{Q}$ [14].

Consider the functional equation [23]: $\frac{\zeta(s)}{\zeta(1-s)} = 2^s \pi^{s-1} sin(\frac{\pi s}{2})\Gamma(1 - s)$.
When $s \in \mathbb{Q}$, $s \neq 1$, π^{s-1} is transcendental (Theorem 4.2.1(3)). The right-hand side of the functional equation is the product of an algebraic number $2^s sin(\frac{\pi s}{2})$ and another number $\pi^{s-1}\Gamma(1 - s)$ whose transcendence is to be determined. In fact, $\frac{\zeta(s)}{\zeta(1-s)}$ will be transcendental if $\pi^{s-1}\Gamma(1 - s)$ is transcendental, and $\frac{\zeta(s)}{\zeta(1-s)}$ will be algebraic if $\pi^{s-1}\Gamma(1 - s)$ is algebraic. Thus, we have the following conjecture:

Conjecture 4.2.6 If π and $\Gamma(s)$ are algebraically independent, then $\pi^k\Gamma(s)$ is transcendental, where $k \in \mathbb{Q}$.

Proposition 4.2.7 The number $\frac{\zeta(\frac{2}{3})}{\zeta(\frac{1}{3})}$ is transcendental.

Proof: Since $\zeta(s)$ satisfies the functional equation

$$\frac{\zeta(s)}{\zeta(1-s)} = 2^s \pi^{s-1} sin(\frac{\pi s}{2})\Gamma(1 - s).$$

When $s = \frac{2}{3}$,

$$\frac{\zeta(\frac{2}{3})}{\zeta(\frac{1}{3})} = 2^{\frac{2}{3}} \pi^{-\frac{1}{3}} sin(\frac{\pi}{3})\Gamma(\frac{1}{3}).$$

Simplifying,

$$\frac{\zeta(\frac{2}{3})}{\zeta(\frac{1}{3})} = 2^{-\frac{1}{3}} \sqrt{3} \pi^{-\frac{1}{3}} \Gamma(\frac{1}{3}).$$

From Theorem 4.2.4 and Conjecture 4.2.6, $\pi^{-\frac{1}{3}}\Gamma(\frac{1}{3})$ is transcendental. Since $2^{-\frac{1}{3}}\sqrt{3}$ is algebraic, $2^{-\frac{1}{3}}\sqrt{3}\pi^{-\frac{1}{3}}\Gamma(\frac{1}{3})$ is transcendental. We can then deduce that $\frac{\zeta(\frac{2}{3})}{\zeta(\frac{1}{3})}$ is transcendental.
Q.E.D.

Proposition 4.2.8 The number $\frac{\zeta(\frac{3}{4})}{\zeta(\frac{1}{4})}$ is transcendental.

Proof: Since $\zeta(s)$ satisfies the functional equation

$$\frac{\zeta(s)}{\zeta(1-s)} = 2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s).$$

When $s = \frac{3}{4}$,

$$\frac{\zeta(\frac{3}{4})}{\zeta(\frac{1}{4})} = 2^{\frac{3}{4}}\pi^{-\frac{1}{4}}sin(\frac{3\pi}{8})\Gamma(\frac{1}{4}).$$

Simplifying,

$$\frac{\zeta(\frac{3}{4})}{\zeta(\frac{1}{4})} = 2^{-\frac{1}{4}}\pi^{-\frac{1}{4}}\left(\sqrt{2+\sqrt{2}}\right)\Gamma(\frac{1}{4}).$$

From Theorem 4.2.5 and Conjecture 4.2.6, $\pi^{-\frac{1}{4}}\Gamma(\frac{1}{4})$ is transcendental. Since $2^{-\frac{1}{4}}\left(\sqrt{2+\sqrt{2}}\right)$ is algebraic, $2^{-\frac{1}{4}}\pi^{-\frac{1}{4}}\left(\sqrt{2+\sqrt{2}}\right)\Gamma(\frac{1}{4})$ is transcendental. We can then deduce that $\frac{\zeta(\frac{3}{4})}{\zeta(\frac{1}{4})}$ is transcendental.
Q.E.D.

Noticing the trend of Proposition 4.2.7 and Proposition 4.2.8, we now make a generalisation.

Proposition 4.2.9 $\frac{\zeta(s)}{\zeta(1-s)}$ is transcendental, where $s = z - r$, $z \in \mathbb{Z}$, $r \in \left\{\frac{1}{3}, \frac{1}{4}\right\}$.

Proof: $\zeta(s)$ satisfies the functional equation $\frac{\zeta(s)}{\zeta(1-s)} = 2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s)$.

(1) Firstly, we need to ensure that $sin(\frac{\pi s}{2})$ is non-zero, otherwise $\frac{\zeta(s)}{\zeta(1-s)}$ would be 0 which is algebraic. Since $s = z - r$, $z \in \mathbb{Z}$, $r \in \left\{\frac{1}{3}, \frac{1}{4}\right\}$, $sin(\frac{\pi s}{2}) \neq 0$. Moreover, from Theorem 4.2.3, $sin(\frac{\pi s}{2})$ is algebraic.
(2) Secondly, since $s = z - r$, $1 - s = 1 - z + r$. Therefore, $\Gamma(1-s) = \Gamma(1-z+r) = \Gamma(m+r)$, where $m \in \mathbb{Z}$. By properties of the Gamma function, $\Gamma(s+1) = s\Gamma(s)$. It can be deduced that
$\Gamma(1-s)$
$= \Gamma(m+r)$
$= (r+m-1)\dots(r+1)(r)\Gamma(r)$

When $r \in \left\{\frac{1}{3}, \frac{1}{4}\right\}$, by Theorem 4.2.4, Theorem 4.2.5, and Conjecture 4.2.6, $\pi^{s-1}\Gamma(r)$ is transcendental.
Since $(r+m-1)\dots(r+1)(r)$ is algebraic, $(r+m-1)\dots(r+1)(r)\Gamma(r)\pi^{s-1}$ is transcendental, i.e., $\pi^{s-1}\Gamma(1-s)$ is transcendental. Then the product of a non-zero algebraic number $2^s sin(\frac{\pi s}{2})$ and a transcendental number $\pi^{s-1}\Gamma(1-s)$ is transcendental. Hence, $\frac{\zeta(s)}{\zeta(1-s)}$ is transcendental.
Q.E.D.

Now suppose that Conjecture 4.2.6 is false. Even though we know that both π^{s-1}, $s \neq 1$ (Theorem 4.2.1(3)) and $\Gamma(1-s)$ (Hölder's theorem) are transcendental, we cannot determine the transcendence of their product unless we know sufficient properties about s. In other words, we have to prove that Conjecture 4.2.6 is true for Proposition 4.2.7 and Proposition 4.2.8 to stand true. This leads to a question for future research:

Question 4.2.10 What is the necessary condition for a product of 2 transcendental numbers to be transcendental?

Attempted answer: Notice that a simple example of $t \cdot \frac{1}{t} = 1$, where t is a transcendental number, shows that the product of 2 transcendental numbers needs not be always transcendental. Although it is difficult to arrive at a conclusive answer at this point, we can know more about the properties of the product of transcendental numbers from the following lemma:

Lemma 4.2.11 If α and β are two transcendental numbers, then at least one of $\alpha + \beta$ or $\alpha \cdot \beta$ is transcendental. Alternatively, α and β are both algebraic if and only if $\alpha + \beta$ and $\alpha \cdot \beta$ are both algebraic.

Proof: Suppose there exist α and β that are both transcendental numbers with $\alpha + \beta = a$ and $\alpha \cdot \beta = b$, where a, b are algebraic. Simplifying these 2 equations, we obtain a polynomial equation with algebraic coefficients, $\alpha^2 + b\alpha - a = 0$. By the Fundamental Theorem of Algebra, a polynomial with algebraic coefficients always gives algebraic roots. Hence α will be an algebraic root, contradicting the assumption that α is transcendental. Thus, for transcendental numbers α and β, $\alpha + \beta$ and $\alpha \cdot \beta$ cannot be algebraic simultaneously. In other words, with the condition $\alpha + \beta$ and $\alpha \cdot \beta$ are both algebraic, α and β must be algebraic.
Q.E.D.

Concluding from Theorem 4.2.1 and Lemma 4.2.11, the following graph (Figure 3) provides a visual representation of some useful generalisations about calculations that involve: algebraic numbers only, transcendental numbers only, and both of them.

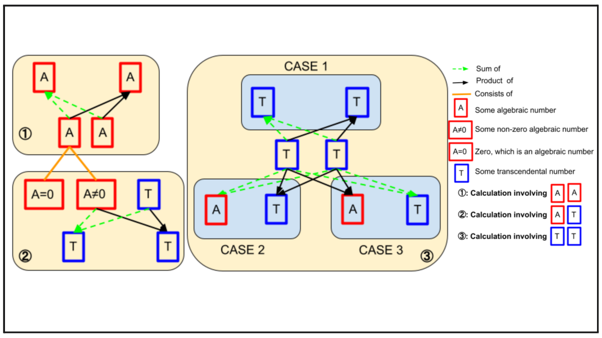

Figure 3: Classification of the results of sum and product of: ①2 algebraic numbers, ②1 algebraic and 1 transcendental number, and ③2 transcendental numbers

Example: Now with Figure 3 as a reference, we can review Proof of Proposition 4.2.3 to gain new insights about algebraic and transcendental numbers. Suppose that we know Theorem 4.2.1 and Lemma 4.2.11 but do not know Proposition 4.2.3. To determine whether $cos(k\pi)$ and $sin(k\pi)$ are transcendental or not, we can first examine the product of them: $cos(k\pi)\,sin(k\pi) = \frac{1}{2}sin(2k\pi)$. Then, the sum of them, $cos(k\pi) + sin(k\pi) = \sqrt{2}sin(k\pi + \frac{\pi}{4})$. Notice that now it is only possible for $cos(k\pi)$, $sin(k\pi)$, their product, and their sum to be all transcendental or algebraic (Lemma 4.2.11). Otherwise, if $cos(k\pi)$ and $sin(k\pi)$ are numbers such that one is algebraic and the other is transcendental, $e^{i\frac{a\pi}{b}}$ would be transcendental (Theorem 4.2.1(4)), yielding a contradiction. Then, if we know one of $cos(k\pi)$ and $sin(k\pi)$ to be algebraic or transcendental, the other one is automatically algebraic or transcendental accordingly.

Extensions: In addition to transcendental numbers, *hypertranscendental numbers* can similarly be obtained from hypertranscendental functions. A complex number is said to be hypertranscendental if it is not the value at an algebraic point of a function which is the solution of an algebraic differential equation with coefficients in $\mathbb{Z}[r]$ and with algebraic initial conditions [24], i.e., a function that is transcendentally transcendental.

The term "hypertranscendental number" was introduced by D. D. Morduhai-Boltovskoi in *"Hypertranscendental numbers and hypertranscendental functions" (1949).* In particular, the number e is transcendental but not hypertranscendental, as it can be generated from the solution to the differential equation $y' = y$.

Notice that when functions and numbers are associated with 3 concepts: "algebraic", "transcendental", and "hypertranscendental" respectively, they have important relations with each other. The following graph (Figure 4) makes it more convenient to distinguish these concepts from each other.

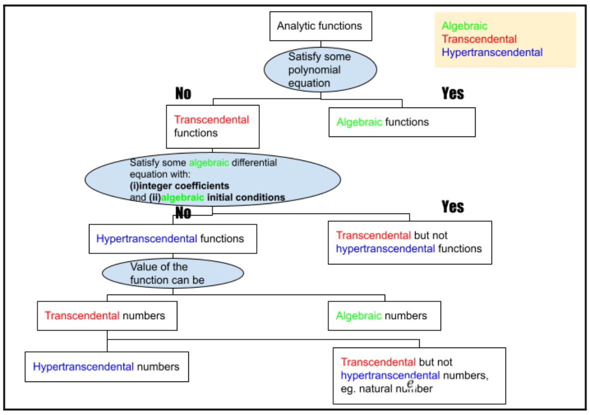

Figure 4: Classification of "algebraic", "transcendental", "hypertranscendental" functions and numbers

5 CONCLUSION

Inspired by Hölder's Theorem, as well as the properties and some specific values of the Riemann zeta function $\zeta(s)$ and the Gamma function $\Gamma(s)$, we have been able to successfully provide a relatively more complete list of transcendentally transcendental functions (Table 1) and transcendental numbers (Table 2) mainly using proof by contradiction and theories related to differential fields. Future researchers can continue to explore the necessary condition(s) for the product of 2 transcendental numbers (e.g. π^k and $\Gamma(s)$) to be transcendental. Thus, with the functional equations of more special functions involving $\Gamma(s)$ or $\zeta(s)$ together with some known values of these functions, more transcendental numbers, or even hypertranscendental numbers, can be obtained.

Moreover, the transcendentally transcendental functions proven here can be useful for the analysis of dynamical systems in physics and engineering [26]. Due to the very nature of transcendental numbers, they can be utilised in cryptography to ensure the security of several systems such as the online banking system.

(Note: for Table 1 and Table 2, generally, $s, x, y \in \mathbb{C}$, $k \in \mathbb{Q}$, $z \in \mathbb{Z}$, χ is a Dirichlet character.)

Table1: List of Transcendentally Transcendental Functions

Index	Transcendentally Transcendental Functions
1	Gamma function $\Gamma(s)$
2	Riemann zeta function $\zeta(s)$
3	Barnes G-function $G(s)$
4	Kurepa's function $K(s)$
5	Hurwitz zeta function $\zeta(s, k)$
6	Multivariate gamma function $\Gamma_p(s)$
7	Dirichlet L-function $L(s, \chi)$
8	Dirichlet eta function $\eta(s)$
9	Beta function $B(x, y)$
10	Gauss's Pi function $\Pi(s)$,

Table2: List of Transcendental Numbers

Index		Transcendental Numbers
1		π^k
2		$e,\ e^{\pi}, e^{\pi\sqrt{3}}$
3		$\Gamma(z + r)$, where $r = \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}$
If $\pi^k\Gamma(s)$ is always transcendental	4	$\frac{\zeta(\frac{2}{3})}{\zeta(\frac{1}{3})}$
If $\pi^k\Gamma(s)$ is always transcendental	5	$\frac{\zeta(\frac{3}{4})}{\zeta(\frac{1}{4})}$
If $\pi^k\Gamma(s)$ is always transcendental	6	$\frac{\zeta(s)}{\zeta(1-s)}, s = z - r,$ $, r \in \left\{\frac{1}{3}, \frac{1}{4}\right\}$

ACKNOWLEDGEMENTS

This paper would not have been possible without the support of the Nanyang Research Programme and Nanyang Technological University. We are especially indebted to Associate Professor Shu Jian Jun, who gave much guidance and has been very supportive of our research. We also appreciate our teacher coordinators for their efforts: Mr. Teo Yik Tee Jerry of Victoria Junior College and Mr. Wee Hock Guan of Dunman High School. Last but not the least, many thanks to the authors' family members and friends for their genuine support.

REFERENCES

[1] Rubel, L. A. (1989). A Survey of Transcendentally Transcendental Functions. *The American Mathematical Monthly*, *96*(9), 777–788. https://doi.org/10.2307/2324840

[2] Wikipedia contributors. (2021, May 5). *Riemann zeta function*. Wikipedia. Retrieved May 28, 2021, from https://en.wikipedia.org/wiki/Riemann_zeta_function

[3] Wikipedia contributors. (2021, February 13). Hölder's theorem. Wikipedia. Retrieved May 28, 2021, from https://en.wikipedia.org/wiki/H%C3%B6lder%27s_theorem

[4] Wikipedia contributors. (2022, January 5). Transcendental number. Wikipedia. Retrieved January 11, 2022, from https://en.wikipedia.org/wiki/Transcendental_number

[5] Wikipedia contributors. (2021, December 17). Lindemann–Weierstrass theorem. In *Wikipedia, The Free Encyclopedia*. Retrieved January 12, 2022, from https://en.wikipedia.org/w/index.php?title=Lindemann%E2%80%93Weierstrass_theorem&oldid=1060834436

[6] Wikipedia contributors. (2021, October 15). Gelfond–Schneider theorem. In *Wikipedia, The Free Encyclopedia*. Retrieved 11:53, January 12, 2022, from https://en.wikipedia.org/w/index.php?title=Gelfond%E2%80%93Schneider_theorem&oldid=1050110281

[7] Baker, A. (1966), Linear forms in the logarithms of algebraic numbers. Mathematika, 13: 204-216. https://doi.org/10.1112/S0025579300003971

[8] Lang, Serge (1966). Introduction to Transcendental Numbers. Addison–Wesley. pp. 30–31.

[9] Wikipedia contributors. (2020, March 16). List of special functions and eponyms. Wikipedia. Retrieved May 28, 2021, from https://en.wikipedia.org/wiki/List_of_special_functions_and_eponyms

[10] Hölder, O. (1886). Ueber die Eigenschaft der Gammafunction keiner algebraischen Differentialgleichung zu genügen. Mathematische Annalen, 28(1), 1–13. https://doi.org/10.1007/bf02430507
[11] Bank, S. B., & Kaufman, R. P. (1978). A note on Hölder's theorem concerning the gamma function. *Mathematische Annalen*, *232*(2), 115–120. https://doi.org/10.1007/bf01421399
[12] Malešević, B., & Mijajlović, A. (2008). Differentially transcendental functions. *Bulletin of the Belgian Mathematical Society - Simon Stevin*, *15*(2). https://doi.org/10.36045/bbms/1210254818
[13] Waldschmidt, M. (2006). Transcendence of Periods: The State of the Art. Pure and Applied Mathematics Quarterly, 2(2), 435–463. https://doi.org/10.4310/pamq.2006.v2.n2.a3
[14] Manin, & Panchishkin, A. A. (2005). Introduction to Modern Number Theory: Fundamental Problems, Ideas and Theories (2. Aufl., Vol. 49). Springer-Verlag.
[15] Boros, G., & Moll, V. (2004). The Method of Partial Fractions. In *Irresistible Integrals: Symbolics, Analysis and Experiments in the Evaluation of Integrals* (pp. 25-47). Cambridge: Cambridge University Press. doi:10.1017/CBO9780511617041.003
[16] Wikipedia contributors. (2021, December 29). Hurwitz zeta function. Wikipedia. Retrieved December 6, 2021, from https://en.wikipedia.org/wiki/Hurwitz_zeta_function
[17] Wikipedia contributors. (2021, July 9). *Multivariate gamma function*. Wikipedia. Retrieved December 1, 2021, from https://en.wikipedia.org/wiki/Multivariate_gamma_function
[18] Wikipedia contributors. (2021, June 28). Dirichlet L-function. Wikipedia. Retrieved July 8, 2021, from https://en.wikipedia.org/wiki/Dirichlet_L-function
[19] Wikipedia contributors. (2021, November 26). Dirichlet eta function. Wikipedia. Retrieved December 6, 2021, from https://en.wikipedia.org/wiki/Dirichlet_eta_function
[20] Choczewski, Bogdan & Wach-Michalik, Anna. (2004). Euler's Beta function diagonalized and a related functional equation. Opuscula Mathematica. 24. https://www.researchgate.net/publication/265623127_Euler's_Beta_function_diagonalized_and_a_related_functional_equation
[21] Wikipedia contributors. (2022, January 8). Gamma function. In *Wikipedia, The Free Encyclopedia*. Retrieved January 2, 2022, from https://en.wikipedia.org/wiki/Gamma_function
[22] Wikipedia contributors. (2021, November 27). Particular values of the gamma function. Wikipedia. Retrieved January 12, 2022, from https://en.wikipedia.org/wiki/Particular_values_of_the_gamma_function
[23] Murty, M. R. (2008). Transcendental Numbers and Zeta Functions. The Mathematics Student, 77(1–4). https://mast.queensu.ca/~murty/Murty-MathStudent.pdf
[24] Wikipedia contributors. (2013, December 2). Hypertranscendental number. In *Wikipedia, The Free Encyclopedia*. Retrieved January 12, 2022, from https://en.wikipedia.org/w/index.php?title=Hypertranscendental_number&oldid=584281235
[25] Wüstholz, G. (1989). Algebraische Punkte auf Analytischen Untergruppen algebraischer Gruppen. *Annals of Mathematics*, *129*(3), 501–517. https://doi.org/10.2307/1971515
[26] Work on spin-chains by A. Knauf, et. al. (n.d.). Empslocal.Ex.Ac.Uk. Retrieved January 12, 2022, from http://empslocal.ex.ac.uk/people/staff/mrwatkin/zeta/spinchains.htm