Generic Characterization of Equilibrium Speed-Flow Relationships

Michael Z. F. Li
Nanyang Business School
Nanyang Technological University
Nanyang Avenue, Singapore 639798
Tel: (65) 790-4659 & Fax: (65) 791-3697
e-mail: zfli@ntu.edu.sg

(2/5/2003 3:08 PM)

Abstract

By introducing the speed-elasticity of flow, this paper provides a generic characterization for equilibrium speed-flow curves that satisfy the fundamental identity of the traffic flow. With this new characterization, we have integrated several well-known speed-flow models, including the classic Greenshield-type and Underwood-type models, and the most recent specifications for highways from Highway Capacity Manual (HCM) 2000. Using the generic characterization, two new classes of two-parameter speed-flow models are generated: the car-following models that extend the Greenshields-type model and an extension of HCM models that possess the backward-bending property. A unifying characterization for multi-regime speed-flow models is also given. The paper concludes with a discussion that connects the generic characterization of a speed-flow curve to the theory of congestion pricing by establishing a simple relationship between the toll estimation and the speed-elasticity function.

Keywords: Transportation, traffic models: speed-flow relationships, Highway Capacity Manual (HCM). Economics: congestion pricing;
The specification of equilibrium speed-flow relationships under stationary states has been an important component of highway capacity planning ever since the first edition of Highway Capacity Manual (HCM) in 1965. The origin of the speed-flow studies can be at least traced back to Greenshields (1935), followed by significant advances in 1950s and 1960s, including Wardrop (1952), Wardrop and Charlesworth (1954), Lighthill and Whitham (1955), Richards (1956), Gazis et al (1959), Greenberg (1959), Haight (1960, 1963), Edie (1961), Underwood (1961), Drew (1965, 1968), Dick (1966), Drake et al (1967), Pikes (1967), Thomson (1967), and Wardrop (1968), among others. In these early years, there were two basic streams of research on traffic flow theory, namely the mathematical approach and the empirical (behavioral) approach.

Except for various editions of HCM, a full specification of a speed-flow curve has become increasingly rare since late 1970s. There may be two possible explanations on this reluctance. First, there have been observations of actual traffic on some highways indicating that the speed-flow relationships can be discontinuous – normally having two disjoint sections. The subsequent multi-regime speed-flow models were primarily motivated to address this problem. The other concern is that there may be several speed-flow curves for the same road corresponding to different capacities, which imply that a unique specification of a traditional speed-flow curve is no longer appropriate. As a result, many empirical studies just report the graphic relationship between observed speeds and flows without providing a statistically fitted model, for instance, Duncan (1976, 1979), Duncan et al (1980), Hall et al (1986), Williams et al (1987), Hall and Hall (1990), Chin and May (1991), Banks (1991a, 1991b), and Polus et al (1991). Refer to May (1990) and Hall et al (1992) for excellent reviews for the development during this period. Recent developments include Newell (1993), Del Castillo and Benitez (1995), Cassidy (1998), Cassidy and Bertini (1999), and Cassidy and Mauch (2001).
There is another stream of research on the nonequilibrium traffic flow theory that basically focuses on higher-order continuum traffic models, which is also known as the Payne-Whitham (PW) theory (Payne 1971; and Whitham 1974). Recent works in this direction include Daganzo (1995), Zhang (1998, 1999, 2000), and Li and Zhang (2001).

The purpose of this paper is to examine the specification of an equilibrium speed-flow-density relationship, which corresponds to the so-called LWR theory (named after Lighthill and Whitham (1955), and Richards (1956)). Section 1 discusses several well-known classical speed-flow relationships and provides re-specifications by using a common set of parameters that are always well-defined. Section 2 reviews the work of Del Castillo and Benitez (1995), which is, to our knowledge, the only paper in the literature that focuses on a general specification of speed-flow curves. Unfortunately, Del Castillo and Benitez’s characterization excludes most of the classical models in the literature; hence a more generic characterization is needed, which is the main motivation of this paper. In Section 3, starting with the fundamental identity of traffic flow and introducing the speed-elasticity function of flow as the instrument function, we derive a generic characterization that captures all well-behaved speed-flow models. Also in this section, two new classes of two-parameter speed-flow models are proposed: the car-following models as an extension to the Greenshields-type model and another two-parameter Greenshields-type model that is motivated by the HCM speed-flow models. A unified characterization for multi-regime speed-flow models is also provided in this section. Section 4 makes the key connection between the speed-elasticity function of flow and the congestion toll estimation, which indirectly validates the regularity conditions on the speed-elasticity function. And Section 5 is the conclusion.
1. CLASSICAL SPEED-FLOW CURVES

In establishing an equilibrium speed-flow relationship, the basic principle is that during stationary states, there is a fundamental identity among the three key traffic characteristics:

\[ q = V \times K, \]  

(1)

where \( q \) is the traffic flow usually measured by the passenger-car-equivalent (PCE) per hour per lane, \( V \) is the speed, and \( K \) is the density measured by PCEs per km per lane. Since the final speed-flow curve is to represent \( q \) as a function of speed, it is natural to express the density \( K \) as a function of speed \( V \), which leads to the so-called speed-density function \( K = K(V) \). Table 1 summarizes some well-known classical speed-flow relationships that are consistent with the fundamental identity (1).

--- Insert Table 1 here ---

Greenshields-type model is originated from Greenshields (1935) with \( \beta = 1 \). Drew (1965) extended the specification by proposing \( \beta = (n+1)/2 \). The general specification is due Pipes (1967) and Munjal and Pipes (1971). The origin of Underwood-type model is from Underwood (1961) with \( \delta = 1 \). Drake et al. (1967) extended the specification to \( \delta = 2 \), which is also known as “Bell-Shaped Curve Model” as it corresponds to a speed-flow specification associated with the normal distribution curve. Refer to Dick (1966) and Gerlough and Huber (1975) for additional discussions on these classical models.

It is evident from Table 1 that in order to specify a speed-density relationship, we only need to know one of \( K_j \) and \( K_0 \) and one of \( V_f \) and \( V_0 \). Since the maximum flow \( q_0 \) is commonly interpreted as the road capacity, it always exists empirically and theoretically. On the other hand, there is no guarantee on the existence of jam density \( (K_j, \text{ associated with zero speed}) \) and/or free-flow speed \( (V_f, \text{ associated with zero density}) \). For example, Greenberg model does not have a
finite free-flow speed; while Underwood-type model does not possess a jam density. With these concerns in mind, we express the representations in Table 1 in terms of $V_0$ and $q_0$ that are always well defined. Table 2 summarizes these re-specifications.

-- Insert Table 2 here --

It is easy to check that Greenberg model is in fact the limiting case of the Greenshields-type model when $\beta \to 0$. In all three cases, the speed-flow function $q = q(V)$ obtains its maximum value when the speed $V$ takes the value $V_0$, that is, $V_0$ is the solution of $dq/dV = 0$. This is an important theoretical condition that is associated with the concept of the capacity and the backward-bending property of a speed-flow curve. But ironically, this property has been more or less abandoned in many applications in part due to the emergence of the evidence showing that there is “a vertical drop of the speed” after the traffic volume reaches a certain level (for a review, refer to Hall et al, 1992). A common explanation for the vertical drop of speed is the queue formation and its subsequent discharge. Consequently, the observed speeds drop quickly at a narrow range of flow levels that are still far below the design capacity (about 15 to 25 percent below the capacity).

For example, the speed-flow curves for highways in HCM 2000 do not possess the backward bending property. First, it can be shown that these relevant speed-flow curves in HCM 2000 can be expressed as follows (the details are left to the readers):

$$q = q_0 \left[ \frac{V_f - V}{V_f - V_0} \right]^{1/\beta}, \text{ for } V \in [V_0, V_f],$$  \hspace{1cm} (2)

with $\beta = 1$ corresponding to the two-lane highway, $\beta = 1.31$ to the multi-lane highways, and the traffic volume is adjusted by the amount of free-flow volume. It is evident that the relationship between $q_0$ and $V_0$ under (2) is not consistent with the maximization of the flow because the flow
function is strictly decreasing in $V$. So the underlying speed-flow relationship no longer possesses the backward-bending property. Because of this deficiency, (2) cannot be used as a “full specification” of the speed-flow relationship; instead, it is appropriate only for a specified range of speeds.

Before closing this section, let us highlight a shortcoming of these classical speed-flow models in Table 2. Note that since the three parameters $q_0$, $V_0$ and $V_f$ can fully characterize both Greenshields-type and Underwood-type models, the corresponding curvature parameter $\beta$ or $\delta$ in each model is in fact exogenous. This limitation of course affects the application flexibility of these models, which probably explains why there have been very few fully calibrated specifications of these models since 1970s. Through a generic characterization, we hope to expend the specification domain for the speed-flow curves while, in the meantime, resurrecting the classical speed-flow models.

2. ON DEL CASTILLO-BENITEZ’S GENERAL SPECIFICATION

To our knowledge, Del Castillo and Benitez (1995) is the only paper in the literature that attempts to provide a general characterization for speed-flow relationships. Hence, their work deserves a separate review. First, their primary purpose is to characterize potential functional forms for density-speed curves. Second, the parameterization in their characterization is based on the following three parameters: $K_j$ – the jam density, $V_f$ – the free-flow speed, and a new parameter $C_j$ – the kinetic wave speed at the jam density, which is defined as the value of $dq/dK$ at $K = K_j$. By introducing the instrument variable $\rho = K/K_j$ and defining

$$\lambda = \frac{|C_j|}{V_f} \left( \frac{1}{\rho} - 1 \right),$$
they propose the following general functional form for the density-speed curves:

\[ V = V_f [1 - f(\lambda)], \]

where \( f(\lambda) \) is the generating function with the following six properties (conditions): (a) \( f(\lambda) \to 0 \) as \( \lambda \to \infty \); (b) \( f(0) = 1 \); (c) \( \lambda^2 f'(\lambda) \to 0 \) as \( \lambda \to \infty \); (d) \( f''(\lambda) > 0 \) for \( \lambda > 0 \); (e) \( f'(0) = -1 \); and (f) \( 0 < f(\lambda) < 1 \) for all \( \lambda > 0 \).

Under conditions (a) to (f), Del Castillo and Benitez (1995) further demonstrate that the only possible specification for density-speed curves has the following functional form:

\[ V = V_f \left\{ 1 - \exp \left[ \frac{|C_j|}{V_f} \left( 1 - \frac{K_j}{K} \right) \right] \right\}, \]

which is associated with the generating function \( f(\lambda) = \exp(-\lambda) \). Alternatively, the underlying speed-density function is given by:

\[ K = K_j \left[ 1 - \frac{V_f}{|C_j|} \ln \left( 1 - \frac{V_f}{V_f} \right) \right]^{-1}. \]

By showing that the speed \( V_0 \) at the maximum flow level \( q_0 \) satisfies the following equation:

\[ 1 - \frac{V_f}{|C_j|} \ln \left( 1 - \frac{V_0}{V_f} \right) = \frac{V_f}{|C_j|} \frac{V_0}{V_f - V_0}, \]

we obtain the following relationship between \( K_j \) and \( K_0 \equiv q_0/K_0 \):

\[ \frac{K_j}{K_0} = \frac{V_f}{|C_j|} \frac{V_0}{V_f - V_0}. \]

Hence the speed-density function can be expressed as:

\[ K = K_0 \frac{1}{1 - \ln w}, \quad \beta = \frac{V_0}{V_f - V_0}, \quad w = \left( \frac{V_f - V}{V_f - V_0} \right)^{1/\beta}, \]

which implies that the underlying speed-flow relationship is given by,
\[ q = q_0 \left( \frac{V}{V_0} \right) \frac{1}{1 - \ln w}. \]  

(7)

Note that the identities (5) and (6) together indicate that the role of the three parameters \((K_j, V_f, C_j)\) can be replaced by another three parameters \((K_0, V_f, V_0)\) used in this paper. By comparing (7) with the Greenshields-type specification (Table 2), we do see a pattern:

\[ q = VK_0 L(w), \text{ with } w = \left( \frac{V_f - V}{V_f - V_0} \right)^{1/\beta}. \]  

(8)

where \(L(w)\) is a strictly increasing function defined on \([0, 1]\) such that \(L(0) = 0\) and \(L(1) = 1\). For Greenshields-type models, \(L(w) = w\); and for Del Castillo-Benitez model, \(L(w) = 1/[1-\ln(w)]\).

Besides this connection, the Del Castillo-Benitez model is quite different from any of the classical models.

Note that by using the jam density, the characterization (3) automatically excludes any Underwood-type model because \(K_j\) is not well defined. Interested readers can check that none of Greenshields-type models can satisfy all six conditions and Greenberg model fails to meet some conditions as well. Hence none of the classical models is consistent with (3) if the above six conditions are imposed. While fully realizing that the Del Castillo-Benitez model (4) can be derived both analytically and behaviourally from the six conditions, the fact that the derived model does not include any of the classic models should indicate that the requirements as dictated by these six conditions may be too restrictive. Therefore, these requirements as part of a general theory need to be further examined.
3. A GENERIC CHARACTERIZATION OF SPEED-FLOW CURVES

In this section, we aim to develop a generic characterization for the equilibrium speed-density relationships. The main idea is to create a differential equation directly derived from the fundamental identity (1). The characterization is generic because neither prior functional forms nor some rigid initial conditions are required. Therefore, our approach is different from De Castillo and Benitez (1995).

For any given speed-density function, \( K = K(V) \), the speed elasticity of flow, denoted by \( \varepsilon_V^q \), is given by

\[
\varepsilon_V^q = \frac{dq}{q/V} = \frac{q'(V)}{K(V)} = 1 + \frac{dK}{K/V} = 1 + \varepsilon_V^K,
\]

where \( \varepsilon_V^K \) is the speed elasticity of density. From (9), we have the following “differential equation”:

\[
\frac{d \ln q}{dV} = \frac{\varepsilon_V^q}{V} = \frac{(1 + \varepsilon_V^K)}{dV} = \frac{d \ln V}{V} + \frac{\varepsilon_V^K}{V},
\]

which leads to the following differential equation that involves the speed-density function only:

\[
\frac{d \ln K}{dV} = \frac{d(\ln q - \ln V)}{dV} = \frac{\varepsilon_V^K}{V} = \frac{\varepsilon_V^q - 1}{V}.
\]

It is clear that, as long as a speed-density curve is differentiable, both (10) and (11) hold. Furthermore, because of (10), any solution to (11) must satisfy the fundamental identity (1) as well. Therefore the differential equation (11) can be used to characterize any speed-density function.

Let the speed-density function be written as \( K = K_0 k(z) \), where \( z = V_0/V \) is the instrument variable. Then the speed elasticity of the flow has the following simple form:
where \( \eta \equiv V_0/V_f < 1 \). The first reason why we choose the instrument variable \( z \) defined as \( V_0/V \), instead of any other forms, is that we want the variable to be bounded above by 1, sharing a similar reason as in Del Castillo and Benitez (1995) when they define their instrument variable \( \rho = K/K_f \). The other reason is that the variable \( z \) is always well defined regardless of circumstance, which eliminates ratio \( V/V_f \) as a valid choice since \( V_f \) can be infinite for the Greenberg model. Consequently, our characterization will not exclude any model because of the default choice of certain parameters. For ease of reference, the function \( e(z) \) is named as the speed-elasticity function of flow.

Now note that from (12), we will have the following simple differential equation regarding the speed-density function,

\[
\frac{d \ln k(z)}{dz} = \frac{e(z) - 1}{z}.
\]

Therefore, to characterize the speed-density function, it is necessary and sufficient to specify the speed-elasticity function \( e(z) \).

Table 3 summarizes the speed-elasticity functions for Greenshields-type and Underwood-type models, where the Greenberg model is dropped because it is the limiting case of Greenshields-type model.

--- Insert Table 3 here ---

We now present the following generic characterization for the speed-density curves, which consequently leads to the specification of speed-flow relationships.

**Proposition 1.** Suppose that a speed-elasticity function \( e(z) \) satisfies the following regularity conditions:
(a) $e(z)$ is a continuous function defined on an interval $(\eta, 1]$;

(b) $e(z) < 0$ for all $z \in (\eta, 1)$;

(c) $e(z)$ is strictly increasing in $z \in (\eta, 1)$;

(d) $e(z) \to -\infty$ as $z \downarrow \eta$.

Then for a given speed range $V \in [V_0, V_f]$ such that $\eta = V_0/V_f$ and a maximum flow $q_0$ such that $q_0 = K_0\times V_0$, the corresponding speed-density curve is given by:

$$K(V) = K_0 \exp \left\{ \int_{V_0/V}^{1} \frac{e(z)-1}{z} \, dz \right\}, \quad V_0 \leq V \leq V_f. \quad (14)$$

**Proof.** From (11), (12) and (13), we know that

$$d \ln K = [e(V_0/V) - 1] \frac{dV}{V}.$$

Therefore, since $K(V_0) = K_0$, it follows that

$$\ln K = \int_{V_0/V}^{1} [e(V_0/V) - 1] \frac{dV}{V} + \ln K_0$$

$$= \int_{V_0/V}^{1} (e(z) - 1)(-\frac{dz}{z}) + \ln K_0 = \int_{V_0/V}^{1} \frac{e(z)-1}{z} \, dz + \ln K_0$$

which immediately leads to (14). ∎

Among the four conditions in the proposition, we only need to elaborate the last two. First, it follows from (12) that $e(z)$ is strictly increasing in $z \in (\eta, 1)$ if and only if $e''_V$ is strictly decreasing in $V \in (V_0, V_f)$. On the other hand, from (9), we know that

$$\frac{d\epsilon''_V}{dV} = \frac{q''(V)K(V) - q'(V)K'(V)}{[K(V)]^2},$$

which implies that as long as the speed-flow curve $q = q(V)$ is concave in $V$, that is, $q''(V) < 0$, condition (c) holds. Since our focus here is to investigate these speed-flow curves with the
backward-bending property that have an interior solution for the maximum flow problem, the concavity condition of \( q(V) \) is commonly satisfied. As for condition (d), it holds naturally because the density becomes when the speed approaches the free-flow speed \( V_f \).

To illustrate that Proposition 1 is expending the specification domain of speed-flow curves, we present the following result that leads to a two-parameter Greenshield-type speed-density relationships that are associated with the car-following models.

**Proposition 2.** For any positive parameters \( \alpha \) and \( \beta \), there exists a two-parameter Greenshields-type speed-density relationship given by:

\[
K = K_0 \left( 1 + \frac{\beta}{\alpha} - \frac{\beta}{\alpha} \left( \frac{V}{V_0} \right)^{\alpha/\beta} \right)
\]

which corresponds to the following speed-elasticity function:

\[
e(z) = \frac{z^\alpha - 1}{z^\alpha - \eta^\alpha}, \quad \eta \leq z \leq 1; \quad \eta = \left( \frac{\beta}{\alpha + \beta} \right)^{1/\alpha} = \frac{V_0}{V_f}.
\]

**Proof.** Since it follows easily from Proposition 1, the details are left to the reader. □

Note that under (15), the corresponding jam density is given by

\[
K_j = K_0 \left( \frac{\alpha + \beta}{\alpha} \right)^{1/\beta},
\]

which, together with (16), leads to the following speed-density function:

\[
K = K_j \left[ 1 - \left( \frac{V}{V_f} \right)^{\alpha/\beta} \right].
\]

Therefore, the underlying two-parameter Greenshields-type speed-flow model is given by

\[
q = VK_j \left[ 1 - \left( \frac{V}{V_f} \right)^{\alpha/\beta} \right],
\]
which, according to Del Castillo and Benitez (1995), is exactly the same as the family of the car-
following models. Alternatively, following a similar specification for the Greenshields-type
model as in Table 2, we have the following representation:

\[ q = q_o \left( \frac{V}{V_o} \right) \left( \frac{V_f^\alpha - V^\alpha}{V_f^\alpha - V_0^\alpha} \right)^{1/\beta} \quad \text{for } V \in [V_0, V_f]. \]

Even though there seems to have little connection between the Greenshields-type model and
the Underwood-type model, there is a surprisingly simple linkage between the two models if we
focus on the speed-elasticity function. The following result captures all the models discussed so
far.

**Proposition 3.** For any function \( g(z) \) defined on \([0, 1] \) with following properties:

(a) \( g(z) \) is continuous on \([0, 1] \) such that \( g(1) \leq 1 \);

(b) \( g(z) \) is positive and strictly increasing in \((0, 1) \).

Then for any \( \eta \in (0, 1) \), the following specification

\[ e(z) = \frac{\ln g(z)}{\ln g(\eta) - \ln g(\eta)} \quad \text{for } z \in [\eta, 1]. \]

is a valid speed-elasticity function. In particular,

(i) when \( g(z) = \exp(z - 1) \), it leads to the Greenshields-type model;

(ii) when \( g(z) = z \), it leads to the Underwood-type model;

(iii) when \( g(z) = \exp(z^\alpha - 1) \), it leads to the car-following model.

**Proof.** It is easy to verify that the function \( e(z) \) given above satisfies all the regularity conditions
in Proposition 1. The details are left to the reader. □

Note that the exponential function \( e^{z-1} \) has the following Taylor expansion:

\[ e^{z-1} = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}. \]
Therefore, for the Underwood-type model, \( g(z) = z \) is a result of the first-order Taylor approximation to \( g(z) = e^{z-1} \) at \( z = 1 \), which is associated with the Greenshields-type model.

To conclude the section, let us consider an application to the multi-regime speed-flow models. The literature on multi-regime speed-flow specification was mainly motivated by the dissatisfaction of mis-fitting by a traditional model that has a single curvature parameter. After observing that a Greenberg model fits better for high density cases and an Underwood model fits the data better for relatively lower density cases, Edie (1961) proposes a two-regime specification of the speed-flow model by using a composite of a Greenberg model and an Underwood model, which causes discontinuity at the joint point of the two regimes. Because it is difficult to explain the “transition” from one regime to another, the multi-regime speed-flow models of this type have not gained any popularity since its inception. The following result aims to provide a consistent characterization of the multi-regime speed-flow models.

**Proposition 4 (Generic Multi-regime Speed-Flow Models).** For \( i = 1, \ldots, k \), let \( e_i(z) \) be a speed-elasticity function of flow satisfying all regularity conditions in Proposition 1 with a common initial conditions on \( V_0 \) and \( V_f \). Consider any set of positive constants \( \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \) such that \( \sum_{i=1}^{k} \alpha_i = 1 \). Then a new speed-elasticity function given by:

\[
e(z) = \sum_{i=1}^{k} \alpha_i e_i(z)
\]

corresponds to the following speed-density function:

\[
K(V) = \prod_{i=1}^{k} (K_i(V))^{\alpha_i},
\]

where \( K_i(V) \) is the underlying speed-density function associated with \( e_i(z) \).

**Proof.** It follows immediately from Proposition 1 and the details are left to the reader.
If we want to integrate a Greenshields-type model with an Underwood-type model, then by Proposition 4, it is sufficient to introduce a flexible parameter $\alpha$ and use the following speed-density relationship:

$$K(V) = [K_{\text{Greenshields}}(V)]^\alpha [K_{\text{Underwood}}(V)]^{1-\alpha} \quad (0 \leq \alpha \leq 1). \quad (17)$$

The value of $\alpha$ can be estimated based on the empirical data. It is evident that the specification (17) captures the two-regime model of Edie (1961) painlessly with $\alpha$ defined as follows: for some $V_1 \in (V_0, V_f)$,

$$\alpha = \begin{cases} 
1 & \text{if } V_1 < V \leq V_f; \\
0 & \text{otherwise}. 
\end{cases}$$

The subsequent empirical issue is to find the best possible value of $V_1$ that “joins” the two speed-flow models. It is straightforward to check that the corresponding speed-density relationship is usually discontinuous at $V_1$.

To conclude this section, let us briefly discuss the speed-flow curves in HCM 2000 specified by (2) and then propose another extension of two-parameter Greenshields-type model. First, it can be shown that the speed-elasticity function of flow for the speed-flow curves given by (2) is given by:

$$e'_q = -\frac{V}{\beta(V_f-V)}.$$

which leads to the following speed-elasticity function:

$$e(z) = -\frac{\eta}{\beta(z-\eta)}, \quad \eta \leq z \leq 1 \quad (\text{with } \eta \equiv \frac{V_0}{V_f}).$$

that satisfies all regularity conditions in Proposition 1. But there is a key difference between the HCM model and the classic models. Note that from Table 3, $e(z) = 0$ when $z = 1$ (corresponding to $V = V_0$) for both classic models, which, by (9), is equivalent to requiring $dq/dV = 0$ at $V_0$. 

14
Therefore, the boundary condition that $e(1) = 0$ implies that the underlying speed-flow curve is backward bending. For HCM models, since $e(1) < 0$, these speed-flow curves for highways are not backward bending as pointed out earlier.

Finally, by further examining the HCM 2000 model (2) and the specification of Greenshields-type (refer to Table 2), we can see that there are alternative specifications in between the two forms. A natural extension is the following two-parameter specification:

$$q = q_0 \left( \frac{V}{V_0} \right)^\alpha \left( \frac{V_f - V}{V_f - V_0} \right)^{1/\beta}.$$  \hfill (18)

Note that

$$\frac{dq}{dV} = q_0 \left( \frac{V}{V_0} \right)^{\alpha-1} \left( \frac{V_f - V}{V_f - V_0} \right)^{(1/\beta)-1} \frac{\alpha \beta V_f - (1 + \alpha \beta)V}{\beta V_0 (V_f - V_0)},$$  \hfill (19)

which implies that in order for the speed-flow curve (18) to have the backward bending property, we must have $\alpha \beta > 0$. Consequently, both $\alpha$ and $\beta$ must be positive since $\frac{dq}{dV} < 0$ when $V \in [V_0, V_f]$. Therefore, letting $\frac{dq}{dV} = 0$, we obtain the speed $V_0$ at the maximum flow level $q_0$, given by,

$$V_0 = \frac{\alpha \beta}{1 + \alpha \beta} V_f.$$  \hfill (20)

Now, from (19), we can derive the speed elasticity of flow as follows:

$$\varepsilon^q_V = \frac{1 + \alpha \beta}{\beta} \times \frac{V_0 - V}{V_f - V},$$

which, together with (20), leads to the following speed-elasticity function of flow,

$$e(z) = \alpha \frac{z - 1}{z - \eta}, \ (\alpha > 0)\ \hfill (21)$$
where \( z = V_0/V \) and \( \eta = V_0/V_f \) as usual. It is clear that (21) is an extension of the Greenshields-type model (with \( \alpha = 1 \)).

Even though the specification (18) is somehow new to the literature, there is a similar form known as BPR-type model (Singh, 1999), which corresponds to a negative value of \( \alpha = -1/\beta \). Because the BPR-type models are intended to discuss the speeds behavior when the volume-to-capacity ratio \( (q/q_0) \) becomes greater than one, it is expected that the underlying speed-flow curves does not possess the backward bending property, which, by (19), explains why the value of \( \alpha \) must be negative. Therefore, the BPR-type models are not the standard speed-flow curves that we intend to characterize in this paper.

4. SPEED-ELASTICITY FUNCTION AND CONGESTION TOLL ESTIMATION

It is well-known that the theory of congestion pricing is critically related to the speed-flow relationships, as originally illustrated by Walters (1961) and most recently by Li (2002). Therefore, it is worthwhile to investigate the connection between the congestion pricing and the generic characterization of speed-flow curves in the previous section.

Let \( q \) be the number of vehicles on a given section of a road following an equilibrium speed-flow curve given by \( q = q(V) \). For the congestion pricing purpose, we can limit \( V \in [V_0, V_f] \), namely, the upper section of the speed-flow curve that is strictly decreasing in \( V \). Therefore, there is an one-to-one correspondence between \( q \) and \( V \) when \( V \in [V_0, V_f] \) so we can write \( V \) as a function \( q \), i.e., \( V = V(q) \). Let \( D \) be the distance to travel in kilometers; and for convenience, we take \( D = 1 \). Let \( c \) be the (average) generalized travel cost (say, in $) per hour (vehicle operating costs plus time costs), assumed to be constant. Then the average cost to a traveler is given by
$AC(q) = c/V \text{ (per km)}$ when there are $q$ vehicles on the road since $1/V$ is the travel time for one kilometre of distance. Hence the total cost with $q$ vehicles on the road is given by $TC(q) = (qc)/V$ and consequently the marginal (social) cost is given by:

$$MC(q) = \frac{dTC(q)}{dq} = \frac{Vc - qc(dV/dq)}{V^2} = AC(q) - \frac{qc}{V^2} \frac{dV}{dq}.$$  \hfill (22)

With the objective of maximizing the social welfare, the congestion externality will be optimized when the traffic flow is set at a level such that the marginal cost equates the marginal benefit. Consequently, at the optimum traffic flow we must have $MC(q) = D(q) \equiv P$, where $D(q)$ is the (derived) inverse demand function and $P$ is the efficient price borne by the traveler. Since a driver only pays for her private cost $AC(q)$, which, according to (22), is always lower than the efficient price $P$. As a result, the government must raise drivers’ private costs by imposing a congestion toll, which is the difference between the efficient price and the private cost. That is, the congestion toll, denoted by $r$, is given by

$$r = P - AC(q) = MC(q) - AC(q) = -\frac{qc}{V^2} \frac{dV}{dq} = -\frac{c}{V \times \varepsilon_v^q}, \hfill (23)$$

which $\varepsilon_v^q$ is the speed-elasticity of the traffic flow as defined before.

We can make two important observations based on (23). First, in the absence of a demand function, the theoretical congestion toll, which is the difference between the marginal cost and the average cost, has a simple expression in terms of the speed. This is important because in the actual implementation, it is much easier to monitor the speed than the traffic volume. As illustrated in Li (2002), it is operationally feasible to bypass the demand function in the process of congestion pricing implementation and the theoretical toll is optimal only when the intended speed that is used to derive the toll according to (23) coincides with the observed speed after the
implementation of the toll. Secondly, the theoretical toll as in (23) can be easily related to the speed-elasticity function of the flow $e(z)$ introduced in this paper:

$$r(z) = -\frac{c}{V_0} \frac{z}{e(z)} = \frac{c}{V_0} \frac{z}{|e(z)|}, \text{ for } z \in [\eta, 1]. \quad (24)$$

Table 4 summarizes the toll estimation results for the classical speed-flow models derived from (23) and (24) respectively.

--- Insert Table 4 here ---

According to the theory, the theoretical toll must be decreasing as the speed increases, which implies that $r(z)$ should be an increasing function of $z$. Since $e(z)$ is a negative function that is increasing, $|e(z)|$ is a decreasing function of $z$, which implies that $r(z)$ is indeed an increasing function of $z$. Even though this connection does not provide a direct justification on the monotonicity requirement on $e(z)$, it shows the plausibility of such a requirement. Condition (d) in Proposition 1 also becomes intuitively clear as well because, based on (24), it simply says that as the speed approaches the free-flow level (equivalently, $z \downarrow \eta$), the congestion toll should be zero. In addition, since $e(1) = 0$ for any speed-flow curve that is backward bending, it follows from (24) that the congestion toll should be extremely high, which is consistent with the theory.

5. CONCLUSION

The main purpose of this paper is to readdress the classical issue on the specification of an equilibrium speed-flow model in stationary states. Based on the fundamental identity of traffic flow, we have developed a generic characterization that has integrated two general classes of specifications in the literature, namely, Greenshields-type and Underwood-type. The main
instrument is the speed-elasticity function of flow, which becomes part of a differential equation that characterizes all well-behaved speed-density relationship.

Since the specification of a speed-flow model relies on curvature parameters derived from physical characteristics of the road or network, there is very little flexibility in the final empirical specification if a classic model is applied. The results in this paper can be used to explore more general specifications, such as, the two-parameter-based car-following models. Using the generic characterization, we can characterize the multi-regime speed-flow models at ease. We demonstrate that these highway speed-flow models from HCM 2000 fit the characterization as well even though they don’t possess the backward bending property.

Finally, we have shown that the speed-elasticity function is closely related to the congestion toll estimation if the underlying speed-flow model is used as the basis of implementing a congestion pricing scheme. This connection serves two purposes. First, it indirectly justifies the regularity conditions associated with the speed-elasticity function of flow. Second, with this connection, it is easy to compare alternative specifications of a speed-flow relationship when implementing congestion pricing scheme. The first point is meant for internal consistency of the development in the paper; while the second point is meant to show the practical relevance of the paper because of the intrinsic connection between the theory of congestion pricing and the speed-flow relationships (Walters (1961)).

We hope that the results in this paper will stimulate fresh empirical research on speed-flow models that provide better fitting of the actual traffic data under diverse situations. Additional research will be needed in developing robust statistical procedures when dealing with multi-parameter speed-flow models. We believe and certainly hope that additional insight may be
gained by applying the results of the paper in transportation planning, especially in highway capacity planning.

REFERENCES


DICK, A.C. 1966. Speed/flow relationships within an urban area. Traffic Engineering & Control 8, 393-396.


SINGH, R. 1999. Improved speed-flow relationships: Application to transportation planning models. Presented at the 7th TRB Conference on Application of Transportation Planning Methods, Boston, Massachusetts, March 1999. (Online access link is: [http://www.mtc.ca.gov/datamart/research/boston1.htm](http://www.mtc.ca.gov/datamart/research/boston1.htm))


List of Tables

Table 1  Classical Speed-Density and Speed-Flow Relationships

<table>
<thead>
<tr>
<th>Model</th>
<th>Speed-density function</th>
<th>Speed-flow function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Greenberg (1959)</td>
<td>$K = K_j \exp\left(-\frac{V}{V_0}\right)$</td>
<td>$q = K_j V \exp\left(-\frac{V}{V_0}\right)$</td>
</tr>
<tr>
<td>Greenshields-type</td>
<td>$K = K_j \left(1 - \frac{V}{V_f}\right)^{\frac{1}{\beta}}$ ($\beta &gt; 0$)</td>
<td>$q = K_j V \left(1 - \frac{V}{V_f}\right)^{\frac{1}{\beta}}$</td>
</tr>
<tr>
<td>Underwood-type</td>
<td>$K = K_0 \left(\delta \ln\frac{V}{V_f}\right)^{\frac{1}{\delta}}$ ($\delta &gt; 0$)</td>
<td>$q = K_0 V \left(\delta \ln\frac{V}{V_f}\right)^{\frac{1}{\delta}}$</td>
</tr>
</tbody>
</table>

Notation:
- $K_j$ = the jam density;
- $K_0$ = the corresponding density at the maximum flow capacity $q_0$;
- $V_f$ = the free-flow speed;
- $V_0$ = the corresponding speed at the maximum flow capacity $q_0$;
- $q_0 = V_0 \times K_0$.

Table 2  Re-specifications of Classical Speed-Flow Relationships

<table>
<thead>
<tr>
<th>Model</th>
<th>Parameter relationships</th>
<th>Speed-flow function (for $V_0 \leq V \leq V_f$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Greenberg</td>
<td>$K_0 = K_j / e$</td>
<td>$q = q_0 \left(V \right) \exp\left(1 - \frac{V}{V_0}\right)$</td>
</tr>
<tr>
<td>Greenshields-type</td>
<td>$\beta = \frac{V_0}{V_f - V_0}, \ K_0 = \frac{K_j}{(1 + \beta)^{\frac{1}{\beta}}}$</td>
<td>$q = q_0 \left(V \right) \left(1 + \beta - \frac{V}{V_0}\right)^{\frac{1}{\beta}} = q_0 \left(V \right) \frac{V_f - V}{V_f - V_0}^{\frac{1}{\beta}}$</td>
</tr>
<tr>
<td>Underwood-type</td>
<td>$\delta = \frac{1}{\ln V_f - \ln V_0}$</td>
<td>$q = q_0 \left(V \right) \left(1 + \delta \ln\frac{V_0}{V}\right)^{\frac{1}{\delta}} = q_0 \left(V \right) \frac{\ln V_f - \ln V}{\ln V_f - \ln V_0}^{\frac{1}{\delta}}$</td>
</tr>
</tbody>
</table>

Table 3  Speed-Elasticity Functions for the Classical Models

<table>
<thead>
<tr>
<th>Model</th>
<th>Speed Elasticity of Flow</th>
<th>Speed-Elasticity Function of Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>Greenshields-type</td>
<td>$\varepsilon_f^V = \frac{V_0 - V}{V_0 - (\beta / (1 + \beta))V}$</td>
<td>$e(z) = \frac{z - 1}{z - \eta}$ ($\eta \leq z \leq 1$) $\eta = \frac{\beta}{1 + \beta}$</td>
</tr>
<tr>
<td>Underwood-type</td>
<td>$\varepsilon_f^V = \frac{\delta \ln(V_0 / V)}{1 + \delta \ln(V_0 / V)}$</td>
<td>$e(z) = \frac{\ln z}{\ln z - \ln \eta}$ ($\eta \leq z \leq 1$) $\eta = e^{-1/\delta}$</td>
</tr>
</tbody>
</table>
**Table 4 Congestion Toll Estimations for Classic Speed-Flow Models**

<table>
<thead>
<tr>
<th>Model</th>
<th>Congestion Toll Estimate (23)</th>
<th>Congestion Toll Estimate (24)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Greenshields-</td>
<td>[ r = c \times \frac{V_0 - \frac{\beta}{1+\beta}V}{V - V_0} \quad (V_0 \leq V \leq \frac{1+\beta}{\beta}V_0) ]</td>
<td>[ r(z) = \frac{c}{V_0} \frac{z(z - \eta)}{1 - z}, \quad z = \frac{V_0}{V} ]</td>
</tr>
<tr>
<td>Type</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Underwood-</td>
<td>[ r = \frac{c}{V} \times \frac{\frac{1}{V} \ln(e^{\frac{V}{V_0}}) - \ln V}{\ln V - \ln V_0} \quad (V_0 \leq V \leq e^{\frac{1}{\beta}}V_0) ]</td>
<td>[ r(z) = \frac{c}{V_0} \frac{z(\ln z - \ln \eta)}{\ln 1 - \ln z}, \quad z = \frac{V_0}{V} ]</td>
</tr>
<tr>
<td>Type</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>