Abstract

This paper studies the optimal pricing problem for firms using two types of restrictions. Linear programming techniques will be used to derive the optimal pricing policies under three different scenarios, namely the nested restrictions, mutually exclusive restrictions and the general case. For the first two scenarios, there exists an optimal pricing policy consisting of at most four price levels that sells restricted units at lower prices. For the general case, we show that there exists an optimal pricing policy consisting of at most five price levels. As an application, we will address the airline fare-pricing problem when a membership restriction and a product restriction are used together. We demonstrate how the models developed in this paper can be applied in these realistic situations with additional insights that help firms pursuing simple pricing structures and capitalizing additional revenue opportunities. Examples, such as, volume discount for tour operators and pricing strategies for special clienteles, are used to support three different ways of using the membership restriction.

Subject classifications: Inventory/production: perishable items, pricing policy; Marketing: pricing, segmentation; Programming: applications.
Pricing Non-Storable Perishable Items by Using Two Types of Restrictions

Li (2001, 2002) develops an optimal pricing model for a firm that uses a restriction as a mean of segmenting the market demand. There are two main results: (a) by properly setting the level of the highest restricted price and rationing the sales at lower prices, the monopolist needs to charge at most three prices to maximize its revenue, and (b) rationing the sales at lower prices for restricted units is an optimal practice. His model provides a simple analytical framework that can lead to a better understanding on the use of restrictions in many service industries, such as airlines, hotels, cruise lines, car rentals and theatres, which share the following common characteristics: (i) the product is perishable and not storable for consumers, that is, it must be consumed in a given, normally short, time period; (ii) unsold units have no salvage value to the firm; and (iii) computerized reservation systems have become more powerful and sophisticated to accommodate real-time booking so that yield/revenue management tools can be applied in real-time as well. These industries also have a common dilemma: the prices must be made public in advance, usually with a long lead-time.

In their excellent survey on revenue management, McGill and van Ryzin (1999) commented, “it is now common for airline practitioners to view pricing as part of the revenue management process” (p.243). This view, even though pervasive (for example, Cross (1997)), may have shortcutted the “true” pricing decision problem: how to determine the fare structure that is precedent to the allocation decisions? The existing dynamic pricing models in yield management literature do not question the optimality of prices per se and normally start with arbitrarily many prices. The real role of the yield management techniques is to assist firms in better facilitating the allocations of a fixed capacity among different consumer segments so that the limited capacity can be utilized in the most profitable manner under a pre-specified set of prices. In essence, pricing has not become an intrinsic part of modern yield management techniques yet. Therefore, the pure pricing problem that presets the price levels, which subsequently allow firms to apply yield management tools to further micro-manage the actual allocations, should be addressed in its own context and then become fully integrated into the yield management system. This paper is primarily motivated to address this concern and in the meantime to extend an early model of Li (2001).

It is well known that the main motivation of using restrictions is to segment the market so that the firm can further exploit revenue potentials in different consumer groups. Traditional economic theory of
third degree price discrimination, which is meant to take advantage of market segmentation, rarely provides practical solutions on how to set the correct prices when the segments are not perfectly sealed, which becomes a serious technical problem for firms to price discriminate. Li (2001, 2002) investigates this problem from a different perspective that is in the spirit of mechanism design. Restrictions such as those commonly used by airlines or hotels, are usually artificial in nature in the sense that the costs of imposing or disposing the restriction to the firm are negligible, even though costs of a purchase restriction to consumers are usually positive. So treating a restriction that channels consumers into different segments as a mechanism is a natural approach from pure economics standpoint. At the firm level, the key issues in mechanism design are the desirability and effectiveness.

On one hand, relevant literatures from economics on mechanism design, for example, Gale and Holmes (1992, 1993) and Dana (1998, 1999), primarily focus on the desirability of a mechanism, such as “advance purchase”; but they provide no practical guide on the effectiveness at the firm level. On the other hand, relevant literatures from yield management, for example, Gallego and van Ryzin (1994, 1997) and You (1999), had not integrated the impact of purchase restrictions even though purchase restrictions do affect the market and hence have a direct impact on the allocation decisions. This paper intends to fill this gap by extending the one-restriction model of Li (2001) to two restrictions. The case of two restrictions warrants an additional study for two main reasons: (a) the extension involves additional complications, hence is non-trivial and (b) two restrictions will give a firm more flexibility in fare structure design and certainly provide more opportunities for the firm to tap the market potentials.

The rest of the paper is organized as follows. Section 1 starts with an extension of the one-restriction model of Li (2001) to the case that there are two nested restrictions, which is a quite common practice for airlines, for example, offering the discount fares with both advance booking requirement and Saturday-night stayover requirement and the discount fares just with Saturday-night stayover requirement. It will be shown that there exist optimal policies that can be characterized by a linear programming problem, which implies that the firm needs to offer at most four price levels to maximize its revenue. Section 2 will extend the basic one-restriction model to the case of two types of restrictions that are mutually exclusive. It will also be shown that in this case the firm needs to offer no more than four different price levels with three different types of product. In Section 3, we will use the results developed in Sections 1 and 2 to study the pricing problem when using two general types of restrictions, which leads to four different types of product. We will show that the firm can limit itself to the type of pricing policies that consist of at most five different
price levels. Then in Section 4, we will present an application to the pricing problem when an airline uses a product restriction in the presence of a membership condition. In this section, we will discuss three common forms of the membership discount: (i) cheaper restricted fares only; (ii) cheaper unrestricted fares only; and (iii) cheaper restricted fares and cheaper unrestricted fares. Additional examples will be used to demonstrate how to fit each of these cases into a model developed early. Finally, the last section is the conclusion.

1. Pricing Problem by Using Two Nested Restrictions

Consider that a monopoly firm has a limited quantity of certain perishable items, such as airline seats and hotel rooms. Let $D(p)$ be the market demand function for the product at price $p$. Without further market information, the firm must treat all consumers as identical. And we can interpret $D(p)$ as the number of consumers in the market who are willing to buy exactly one unit of the product at price $p$. With this interpretation, we assume that the demand function $D(p)$ is a decreasing function of the price. It is well known that a well-designed restriction can effectively segment the original market. As demonstrated in Theorem 1 of Li (2001), except for trivial cases, the use of the restrictions will always improve a firm's revenue and lead to better utilization of limited capacity.

Now suppose that a firm considers using two types of restrictions in its pricing decisions. Let $\gamma_t(p)$ be the percentage of consumers who cannot purchase the product if type-$t$ restriction is attached to the product at price $p$, where $t = 1, 2$. Therefore the demand for type-$t$ restricted product at price $p$ is given by $D_t^r(p) = (1 - \gamma_t(p))D(p)$, for $t = 1, 2$. Following Wilson (1988) and Li (2001), we will assume that the demand functions $D(p), D_1^r(p)$ and $D_2^r(p)$ are all step functions defined on the price set $\{p_1, \ldots, p_n\}$, that is,

$$D(p) = \begin{cases} D_1, & \text{if } p \leq p_1, \\ D_i, & \text{if } p \in (p_{i-1}, p_i], \text{for } i = 2, \ldots, n, \\ 0, & \text{if } p > p_n; \end{cases}$$

and for $t = 1, 2$,

$$D_t^r(p) = \begin{cases} D_t^{r*}, & \text{if } p \leq p_1, \\ D_t^{r*}, & \text{if } p \in (p_{i-1}, p_i], \text{for } i = 2, \ldots, n, \\ 0, & \text{if } p > p_n; \end{cases}$$

where $\infty > D_1 > D_2 > \ldots > D_n > 0$ and $\infty > D_1^{r*} > D_2^{r*} > \ldots > D_n^{r*} > 0$. For $i = 1, \ldots, n$, and $t = 1, 2$, we define $\gamma_t = 1 - D_t^{r*}/D_t$. Then it is easy to see that $\gamma(p)$ is also step function defined over $\gamma_1, \ldots, \gamma_n$. 


In this section, an extended model is presented to the case of two types of restrictions that are nested. Let us first introduce the following definition:

**Definition 1**: We say type-2 restriction is nested into type-1 restriction if there exists another function \( \alpha(p) \) such that \( 1 - \gamma_2(p) = (1 - \gamma_1(p))(1 - \alpha(p)) \).

From the definition, it follows that if type-2 restriction is nested into type-1 restriction, then \( D^2(p) = (1 - \alpha(p))D^1(p) \), it implies that those who do not mind the restriction 2 is a subset of those who do not mind the restriction 1. Hence, it is clear that the restriction 2 is more restrictive than the restriction 1. Denote \( \alpha_i = \alpha(p_i) \) for \( i = 1, \ldots, n \). Throughout this section, we will assume the following regularity condition to ensure the effectiveness of the restrictions (refer Li (2001)), that is, both \( \gamma_i \) and \( \alpha_i \) are increasing in \( i \), which immediately imply that \( \gamma_i \) is also increasing in \( i \). When a firm decides to impose restrictions, its pricing policy must address the following issues: (i) what prices will be offered, (ii) which prices are restricted, and (iii) how many units of the product available at each of these prices offered.

Before we move on, we need briefly review key findings from Li (2001, 2002). Li (2001) shows that with one restriction, if the restricted units are sold before the unrestricted units and the units are offered sequentially in the increasing order of the offered prices, then the firm’s optimal pricing problem can be transformed into a linear programming problem. To unlock the potential limitations as a result of this special requirement of sequential availability, Li (2002) establishes two new results – general optimality theorem and the simultaneous availability proposition. The general optimality theorem affirms that the firm should never consider any pricing policy that involves the sales of some restricted units after the unrestricted units. And the simultaneous availability proposition demonstrates that all optimal policies derived from Li (2001) are sustainable even when all prices are offered simultaneously instead of sequentially. Based on these results, with two nested restrictions, the firm can limit its attention to policies with the following properties without loss of generality: (i) type-2 restricted units are sold first, then type-1 restricted, and finally the unrestricted; (ii) if type-2 restricted units and type-1 restricted units are sold at the same price level, then type-2 restricted units are sold first; (iii) if type-1 restricted units and unrestricted units are sold at the same price level, then type-1 restricted units are sold first; and (iv) the product is sold at prices in the order of price levels \( p_1, \ldots, p_n \). In summary, we can limit our attention to the following type of pricing policies:
\((p, q_{r.t}); i = 1, \ldots, m_i; (p, q_{l.i}); j = m_i, \ldots, m_l; (p, q_k); k = m_l, \ldots, n)\),

where \(p_m\) is the highest type-1 restricted price, \(q_{r.i}\) is the quantity available at the type-1 restricted price \(p_r\), \(q_k\) is the quantity available at the unrestricted price \(p_k\), and \(1 \leq m_2 \leq m_1 \leq n\).

To specify the feasibility set of pricing policies for any given pair \((m_2, m_1)\), we can use the same technique developed in Li (2001). First of all, type-2 restricted units are sold according to \{(\(p_1, q_{r.2,2}^1\)) \ldots (\(p_{m_2}, q_{r.2,m_2}^2\))\} with demands given by \(D_r^2\) for \(i = 1, \ldots, m_2\). Therefore the feasibility condition for the allocation plan of the type-2 restricted units is given by

\[
\beta_2 = \sum_{i=1}^{m_2} \frac{q_{r.2,i}}{D_r^2} \leq 1.
\]  

(1)

which is originated from Wilson (1988) and further elaborated in Li (2001). According to Lemma 3 of Li (2001), after the sales of the type-2 restricted product, the residual demand for type-1 restricted product is given by \(d_{m_2,j} = D_j - \beta_2 D_r^2\) for \(j = m_2, \ldots, n\). Consequently, the feasibility condition for the allocation plan \{(\(p_{m_2}, q_{r.1,m_2}^1\)) \ldots (\(p_{m_1}, q_{r.1,m_1}^1\))\} for the type-1 restricted units is

\[
\beta_1 = \sum_{j=m_2}^{m_1} \frac{q_{r.1,j}}{D_{m_2,j}} = \sum_{j=m_2}^{m_1} \frac{q_{r.1,j}}{D_j} - \beta_2 D_r^2 \leq 1.
\]  

(2)

Finally, let us consider selling unrestricted units according to the plan \{(\(p_k, q_k\)): \(k = m_1, \ldots, n\)\}. It is easy to check that the residual demand for the unrestricted product after the sales of the product with restrictions is given by \(d_{m_1,k} = \alpha_k D_k + (1 - \beta_2) q_{r.1,j}^1 D_{m_1,j} = D_k - \beta_1 D_r^1 - (1 - \beta_2) D_r^2\) for \(k \geq m_1\). Hence, the feasibility condition for the remaining unrestricted units according to the \{(\(p_{m_1}, q_{r.1,m_1}^1\)) \ldots (\(p_n, q_n\))\} is given by:

\[
\beta_0 = \sum_{k=m_1}^{n} \frac{q_k}{d_{m_1,k}} \leq \sum_{k=m_1}^{n} \frac{D_k - \beta_1 D_r^1 - (1 - \beta_2) D_r^2}{D_k} \leq 1.
\]  

(3)

Summarizing above discussions, we obtain the following formulation of the firm’s pricing problem when using two nested restrictions:

\[
\text{max} \sum_{i=1}^{m_2} p_i q_{r.2,i} + \sum_{j=m_1}^{m_2} p_j q_{r.1,j} + \sum_{k=m_1}^{n} p_k q_k
\]

subject to constraints (1), (2) and (3)

\[
\sum_{i=1}^{m_2} q_{r.2,i} + \sum_{j=m_1}^{m_2} q_{r.1,j} + \sum_{k=m_1}^{n} q_k \leq q
\]

\[
q_{r.2,i} \geq 0, q_{r.1,j} \geq 0, q_k \geq 0, \forall i, j, k,
\]
where $q$, throughout this paper, is the fixed capacity. For ease of reference, we will call the above model as the N-model where $N$ stands for “nested”. Let $R(m_2, m_1)$ be the derived maximum revenue from the N-model. Note that if $\beta_2 = \beta_1 = 1$, then the constraints (1), (2) and (3) all become linear. The following theorem proves that there is a linear programming (l.p.) characterization for the optimal revenue $\max_{1\leq m_1, \ldots, 1\leq m_n} R(m_2, m_1)$ and we can push further by making $\beta_0$ to be one as well.

**Theorem 1:** Let $\tilde{R}(m_2, m_1)$ be the optimal objective value of the following l.p. (named as the tight N-model):

$$\max \sum_{i=1}^{m_2} p_i q_{r_2,i} + \sum_{j=1}^{m_1} p_j q_{r_1,j} + \sum_{k=1}^{m_n} p_k q_k$$

subject to

$\beta_2 = 1; \beta_1 = 1; \beta_0 = 1$

$$\sum_{i=1}^{m_2} q_{r_2,i} + \sum_{j=1}^{m_1} q_{r_1,j} + \sum_{k=1}^{m_n} q_k \leq q$$

$q_{r_2,i} \geq 0, q_{r_1,j} \geq 0, q_k \geq 0, \forall i, j, k$.

Then $\max_{1\leq m_1, \ldots, 1\leq m_n} R(m_2, m_1) = \max_{1\leq m_1, \ldots, 1\leq m_n} \tilde{R}(m_2, m_1)$. Consequently, there exists an optimal pricing policy that consists of at most four different prices.

**Proof:** Please refer to the Appendix.

2. Pricing Problem by Using Two Mutually Exclusive Restrictions

This section discusses another case of two types of restrictions. Before moving on, let us introduce the following definition of mutually exclusiveness of restrictions.

**Definition 3:** Let $\gamma(p)$ be the percentage of consumers who cannot purchase the product if both restrictions are attached to the product. Then we say that these two restrictions are *mutually exclusive* if $\gamma(p) = 1$, i.e., nobody will be able to accommodate both restrictions.

The mutual exclusiveness here means that those who can accommodate the type-1 restriction cannot accommodate the type-2 restriction, and vice versa. Or intuitively speaking, if two types of restrictions are mutually exclusive, then they are targeting two distinct and disjoint consumer groups. This seems to be very restrictive; but it is still possible to have mutually exclusive restrictions. For example, the airline employees enjoy special (deep) discount if they want confirmed seats. If the internal travellers...
decided to free ride, they will be bumped if the flight is full. Therefore they are just filling up otherwise empty seats. Also, most airlines will charge the food costs to these “free-riders”; so they have no impact on revenue derived from the flight. The other restriction is that a traveller must come from a strategic partner airline. Nowadays, it is quite common that a major airline will make strategic alliances with several other carriers, either domestic feeders or international partners. In this context, the two “restrictions” are indeed mutually exclusive. Using the same notation as in Section 1, define $\gamma_t = 1 - \frac{D^{1t}_i}{D_t}$. We will assume the usual regularity condition on the effectiveness of the restrictions, that is, $\gamma_t$ is increasing in $i$ for $t = 1, 2$.

Based on the general optimality theorem of Li (2002), which indicates that no unrestricted units should be sold at a price level that is strictly lower than a restricted price, if the two restrictions are mutually exclusive, we can limit, without loss of generality, our discussion to policies such that both restrictions share the same highest price, i.e., policies of the form $\{(p_i, q_{r1,i}, q_{r2,i}); i = 1, \ldots, m; (p_k, q_k); k = m, \ldots, n\}$, where $p_m$ is the common highest restricted price for both restrictions, $q_{r1,i}$ is the quantity available at price $p_i$ with the restriction-$t$, $q_k$ is the quantity available at the unrestricted price $p_k$ after the sales of the product with restrictions, and $1 \leq m \leq n$. Consequently, the firm’s pricing problem can be formulated as the following mathematical programming problem:

$$\max \sum_{i=1}^{m} p_i (q_{r2,i} + q_{r1,i}) + \sum_{k=m}^{n} p_k q_k$$
subject to

$$\eta_2 = \sum_{i=1}^{m} \frac{q_{r2,i}}{D^{12}_i} \leq 1$$

$$\eta_1 = \sum_{i=1}^{m} \frac{q_{r1,i}}{D^{12}_i} \leq 1$$

$$\eta_0 = \sum_{k=m}^{n} \frac{q_k}{D_k - \eta_1 D^{11}_k - \eta_2 D^{12}_k} \leq 1$$

$$\sum_{i=1}^{m} (q_{r2,i} + q_{r1,i}) + \sum_{k=m}^{n} q_k \leq q$$

$q_{r1,i} \geq 0, q_{r2,i} \geq 0, q_k \geq 0, \forall i, k$.

We will call above formulation as the ME-model, where “ME” stands for “mutually exclusive”. Again let $R(m)$ be the optimal objective value of the above ME-model for a given value of $m$.

The following theorem demonstrates that there also exists a characterization in linear programming for $\max_{1 \leq m \leq n} R(m)$.

**Theorem 2:** Let $\tilde{R}(m)$ be the optimal objective value of the following l.p. (named as the tight ME-model):
max $\sum_{i=1}^{m} p_i (q_{r2,i} + q_{r1,i}) + \sum_{k=m}^{n} p_k q_k$

subject to

$\eta_2 = 1; \eta_1 = 1; \eta_0 = 1;$

$\sum_{i=1}^{m} (q_{r2,i} + q_{r1,i}) + \sum_{k=m}^{n} q_k \leq q$

$q_{r1,i} \geq 0, q_{r2,i} \geq 0, q_k \geq 0, \forall i, k.$

Then $\max_{\{1,\ldots,n\}} R(m) = \max_{\{1,\ldots,n\}} \tilde{R}(m).$ Consequently, there exists an optimal pricing policy that consists of at most four different prices.

**Proof:** To prove the theorem, it suffices to show that for any $m$, there exists another $\tilde{m}$ such that

$R(m) \leq \tilde{R}(\tilde{m}). \quad (7)$

To begin with, let $\{(p_i, q_{r1,i}, q_{r2,i}): i = 1, \ldots, m; (p_k, q_k): k = m, \ldots, n\}$ be an optimal solution to the ME-model with respect to $m$. If both constraints (4) and (5) are binding at this solution, then (7) holds for $\tilde{m} = m$.

By Theorem 4 of Li (2001), we know that if either one of the constraints (4) and (5) is not binding, then exactly one of $q_{r1}, \ldots, q_n$ can be strictly positive. Let us call it $q_j$. According to Theorem 3 of Li (2001), the constraint (6) for the unrestricted units then must be binding so that $q_j = D_j - \eta_1 D_j^{t1} - \eta_2 D_j^{t2}$. Now take $\tilde{m} = j$ and define a new policy as follows: for $t = 1, 2$,

$$\tilde{q}_{r2,t} = \begin{cases} q_{r2,i} & \text{if } 1 \leq i \leq m \\ 0 & \text{if } m+1 \leq i \leq j-1 \\ (1-\eta_t)D_j^{t} & \text{if } i = j, \end{cases}$$

and $\tilde{q}_j = D_j - D_j^{t1} - D_j^{t2}$ if $j = j$ and $= 0$ if $j \neq j$. It is easy to check that this new policy is a feasible solution to the tight ME-model associated with $\tilde{m}$ and the corresponding revenue satisfies:

$$\tilde{R}(\tilde{m}) \geq \sum_{i=1}^{m} p_i (\tilde{q}_{r1,i} + \tilde{q}_{r2,i}) + \sum_{k=m}^{n} p_k \tilde{q}_k = R(m).$$

Therefore (7) is true. Hence the theorem is proved as required.

### 3. Pricing Problem by Using Two General Restrictions

Recall that $\gamma(p)$ is the percentage of consumers who can not accommodate the type-$t$ restriction; and $\gamma(p)$ is the percentage of consumers who cannot purchase the product if both restrictions are attached to the product.

In this section, we will consider the general case where $\gamma(p) < 1$ and $\gamma(p) > \gamma(p)$ for $t = 1$ and 2.
By the general optimality theorem of Li (2002), we will limit our discussion to pricing policies such that: (a) the firm first sells some units with two restrictions attached; (b) the firm then offers type-1 and type-2 restricted units at certain orders; and (c) the firm finally offers the rest of units without any restrictions. To facilitate our discussion, we will use the following notation:

- Define $\gamma_t = 1 - D_t^n / D_t$, as before.
- We will name the bundle of type-1 and type-2 restrictions as the type-3 restriction.
- Let $D_t^{ij}(p) = (1 - \gamma_t(p))D(p)$ and denote $D_t^{ij} = D_t^{ij}(p_i)$ for $i = 1, \ldots, n$.
- Define $1 - \alpha_{ti} = D_t^{ij} / D_t^n$, for $i = 1, \ldots, n$ and $t = 1, 2$, which measures the percentage of consumers in the market for type-$t$ restricted units who can accommodate both restrictions;
- Let $D_{ti}$ be the demand for any restricted unit at $p_i$. Then it is clear that $D_{ti} = D_i^{1i} + D_i^{2i} - D_i^{3i}$.
- Define $1 - \gamma_{ti} = D_t^{ij} / D_t'$, which measures the percentage of consumers in the market for any restricted unit who cannot accommodate the type-$t$ restriction, for $t = 1, 2, 3$.

Now, it is easy to verify the following two identities: for $t = 1, 2$,

\begin{align*}
(1 - \gamma_t)(1 - \alpha_{ti}) &= 1 - \gamma_t, \quad (8) \\
(1 - \gamma_{ti})(1 - \alpha_{ti}) &= 1 - \gamma_{ti}. \quad (9)
\end{align*}

The regularity conditions for the effectiveness of restrictions are given as follows:

- (a) $\gamma_t$ and $\alpha_t$ are increasing in $i$ for $t = 1, 2$;
- (b) $\gamma_t - \gamma_t$ is decreasing in $I$ for $t = 1, 2$.

There is no need to justify the condition (a) as it is the usual condition. To see why condition (b) is also natural, note that $\gamma_t - \gamma_t = (D_t^n - D_t^{3i}) / D_t$. So condition (b) basically says that after excluding those who can accommodate both restrictions, the proportional market size of the type-$t$ consumers to the overall market is declining as the price increases, which is exactly the basic effectiveness condition for a restriction.

Before we move on, let us briefly discuss the implications of the two identities (8) and (9). Not surprisingly, according to Definition 1, both of them underline some sort of nested structure of restrictions. In particular, identity (8) implies that within the whole market, characterized by $D_t$, the type-3 restriction is nested into type-$t$ restriction for $t = 1, 2$. And identity (9) says that within the market for restricted units, characterized by $D_t'$, the type-3 restriction is nested into type-$t$ restriction as well, for $t = 1, 2$. 
The following lemma ensures that there are two valid nested structures within the market for restricted units, which will allow us to apply Theorem 1 for the restricted market. It plays a key role in the main result of this section. Note that from (8), we know that $\gamma_i$ is increasing in $i$.

**Lemma 1**: For $t = 1, 2$, $\gamma'_t$ is increasing in $i$.

**Proof**. Let us prove the result for $t = 1$. By definition, it follows that

$$\gamma'_1 = \frac{D^2_1 - D^3_1}{D^1_1 + D^2_1 - D^3_1} = \frac{(D^2_1 / D_1) - (D^3_1 / D_1)}{(D^1_1 / D_1) + (D^2_1 / D_1) - (D^3_1 / D_1)}$$

which is increasing in $i$ if and only if $(1 - \gamma_1) / (\gamma_1 - \gamma_2)$ is decreasing, which follows from the fact that $\gamma_i$ is increasing in $i$ and the regularity condition (b) associated with $t = 2$. Similarly, we can prove the result for $t = 2$.

We now prove the main result of this section.

**Theorem 3**: Suppose that the firm uses two restrictions that satisfy the regularity conditions (a) and (b), then the firm's pricing problem can be formulated as the following linear programming problem:

$$\max \sum_{i=1}^{m_1} p_i q_{r,i,j} + \sum_{j=1}^{m} p_j (q_{r_1,j} + q_{r_2,j}) + \sum_{k=1}^{n} p_k q_k$$

subject to

$$\delta_j \equiv \sum_{i=1}^{m_1} \frac{q_{r,i,j}}{D^3_j} = 1$$

$$\sum_{j=1}^{m} \frac{q_{r_1,j}}{D^3_j} = 1$$

$$\sum_{j=1}^{m} \frac{q_{r_2,j}}{D^3_j} = 1$$

$$\sum_{k=1}^{n} \frac{q_k}{D^3_k} = 1$$

$$\sum_{i=1}^{m_1} q_{r,i,j} + \sum_{j=1}^{m} (q_{r_1,j} + q_{r_2,j}) + \sum_{k=1}^{n} q_k \leq q$$

$q_{r,i,j} \geq 0, q_{r_1,j} \geq 0, q_{r_2,j} \geq 0, q_k \geq 0, \forall i, j, k$.

where $1 \leq m_1 \leq m \leq n$. Consequently, the firm has an optimal policy that consists of at most five prices.

**Proof**: The key is to take care of the market for restricted product. For this, suppose that at an optimal solution, $\bar{q}$ is the total allocation for unrestricted units and $p_{m_1}$ is the lowest positive unrestricted price level in the optimal solution. So the remaining capacity $q - \bar{q}$ will be allocated for the restricted units. Based on
Lemma 1 and the regularity condition (9), we know that type-3 restricted is nested into type-1 restriction and type-2 restriction. Therefore, we can apply Theorem 1 for the restricted market twice and use the one that generates more revenue. Regardless of which case, the most important feature is that the market for type-3 restricted units will be fully exhausted at the highest positive type-3 restricted price, say, $p_m$, that is, $\delta_i = 1$. Once type-3 consumers are fully satisfied, in the residual market, type-1 restriction and type-2 restriction now become mutually exclusive. Therefore, we can apply Theorem 2 and will immediately lead the formulation in the theorem. To complete this part, we still need to validate the regularity conditions required by Theorem 2, that is, $(D_j^{\alpha} - D_j^{\gamma_3})/(D_j - D_j^{\gamma_3})$ is decreasing in $j$ for $j \geq m_1$. Note that

\[
\frac{D_j^{\alpha} - D_j^{\gamma_3}}{D_j - D_j^{\gamma_3}} = \frac{(D_j^{\alpha} / D_j) - (D_j^{\gamma_3} / D_j)}{1 - (D_j^{\gamma_3} / D_j)} = \frac{(1 - \gamma_j) - (1 - \gamma_j)}{\gamma_j} = \gamma_j - \gamma_j
\]

which is indeed decreasing since $\gamma_j$ is increasing and $(\gamma_j - \gamma_j)$ is decreasing. Therefore, Theorem 2 is applicable. This proves the theorem as required.

4. Airline Pricing Problem by Using Membership-Restriction and Product-Restriction

In this section, we will discuss an application to the pricing problem when an airline uses a membership restriction and a product restriction at the same time. As it is well known, airlines have some special fares designated to some special clienteles, such as, tour operators, corporate clients, government clients, clients associated with a partner airline, and internal travellers. These clients can usually purchase a ticket at a price lower than what the general public pays. But on the other hand, the availability of the seats for these clienteles needs to be controlled and a good yield management system must be capable of handling the presence of these special clienteles. For ease of presentation, we will use the following convention on terminologies: (i) fares and tickets are the same as prices and units; (b) restricted fares and unrestricted fares are understood as fares that are available to the general public, which means that all consumers are allowed to purchase these fares; and (c) these fares that are specifically available to members are called either restricted membership fares or unrestricted membership fares.

Note that there is an asymmetry between members and non-members on the access to fares offered. A member can purchase a ticket either at a membership fare or at a public fare whichever is less. But non-members can only purchase the public fares. Therefore, membership is another restriction in addition to the product restriction. As a result, the airline needs to deal with two restrictions and can offer up to four
different types of fares: restricted membership fares, restricted fares, unrestricted membership fares and unrestricted fares. Because of the membership agreement, we know: (a) restricted membership fares, if offered, should not be higher than the planned restricted fares and (b) unrestricted membership fares, if offered, should not be higher than the planned unrestricted fares. In this section, we will consider three common arrangements in term of membership privileges: (i) cheaper restricted fares only; (ii) cheaper unrestricted fares only; and (iii) cheaper restricted fares and cheaper unrestricted fares.

The purpose of this section is to use the results developed in the previous sections to illustrate how they can be applied in this context and what are the managerial implications we can learn. We will briefly discuss several different situations based on different membership privileges. As a result of this analysis, we hope to get a better understanding about the operating environment for each of these situations.

Let \( D(p) \) be the total market demand function at (unrestricted) price \( p \), among which \( D^M(m) \) of them have the membership. Denote \( D^N(p) \) to be the demand from the non-member group, so \( D^N(p) = D(p) - D^M(p) \). Let \( \gamma_d(p) \) be the percentage of consumers in the market who are non-members at price \( p \), i.e., \( 1 - \gamma_d(p) = \frac{D^M(p)}{D(p)} \). Let \( \gamma_{dM}(p) \) be the percentage of consumers in the total demand market who cannot purchase the product at the restricted membership fare \( p \), that is, \( 1 - \gamma_{dM}(p) = \frac{D^{Mr}(p)}{D(p)} \), where \( D^{Mr}(p) \) is the demand for the product at the restricted membership fare \( p \). And \( \gamma(p) \) is the percentage of consumers in the total demand market who cannot accommodate the product restriction at price \( p \), that is, \( 1 - \gamma(p) = \frac{D'(p)}{D(p)} \), where \( D'(p) \) is the demand for product at price \( p \) with product-restriction only. Further assume that \( D(p) \) and \( D^M(p) \) are step functions defined on the set of fares \( \{p_1, ..., p_n\} \). As usual, for \( i = 1, ..., n \), denote \( \gamma_{Mi} = \gamma_d(p_i), \gamma_{Mr.i} = \gamma_{dMr}(p_i), \gamma_{M} = \gamma_{dM}(p_i), \text{and } \gamma_j = \gamma(p_j). \) And let

\[
D_i = D(p_i), D'_i = D'(p_i), D^M = D^M(p_i), D^{Mr} = D^{Mr}(p_i), i = 1, ..., n.
\]

Throughout this section, we will assume that \( \gamma_{Mr,i} \) and \( \gamma_j \) are increasing in \( j \). The following subsections will discuss the three proposed cases in details.

4.1. Cheaper Restricted Membership Fares Only

Let us first analyze the case where an airline only offers the members lower restricted fares. As a consequence of this, as a member, the traveller has three types of fares to choose from: restricted membership fares, (public) restricted fares and (public) unrestricted fares. On the other hand, any non-member can only purchase the public restricted fares and the public unrestricted fares.
Now if we call the bundle of membership restriction and product restriction as the type-3 restriction and the product restriction as the type-1 restriction, then it is clear that type-3 restriction is nested into type-1 restriction. Let $\alpha_{Mr,i}$ be the percentage of consumers in the demand market at restricted price $p$ who are non-members. And we have $1 - \alpha_{Mr,i} = \frac{D_i^{Mr}}{D_i^{r}}$, for $i \geq 1$. Then it is clear that $1 - \gamma_{Mr,i} = (1 - \gamma_{r,i})(1 - \alpha_{Mr,i})$, which indicates that type-3 restriction is indeed nested into type-1 restriction. Based on this setting, Theorem 1 is applicable in this context, which implies that the airline needs to offer at most four fares.

To understand the managerial implications of the derived optimal fare structure, we need to investigate the implications of the model assumptions, i.e., the regularity conditions. Clearly the assumption that $\gamma_{r,i}$ is increasing is a basic requirement for an effective product restriction. For the assumption that $\alpha_{Mr,i}$ is increasing, it simply means that as the price increases, among those who can purchase restricted product, the percentage of the members is decreasing. As demonstrated in Theorem 1, this assumption plays a key role in the process of obtaining simple pricing structure, or a simple fare structure in our context here. Basically, it allows us to focus on the type of policies with the property that the demand for restricted membership fares will be exhausted right after the sales of these restricted fares targeted to members. Offering lower restricted fares to special clienteles is a very common approach for airlines to deal with membership issues. We now present two examples that are consistent with this particular case.

**Example 1: Volume Discounts for Tour Operators Through Wholesale**

Consider vacation operators that sell tour packages or vacation packages with many predetermined destinations. One of key ingredients in these packages is the airfare. Some operators announce that they will get much cheaper fares than the general fares, which implies that the vacation operators must have reached an agreement with a particular airline. From an airline point of view, these vacation operators create some substantial revenue opportunities since: (i) a successful operator usually offers different packages of predictable sizes all year round, which are planned many months advance; and (ii) Almost all vacationers who arrange their vacations through operators will not fly at an unrestricted fare, which is usually much higher than a regular fare. Therefore, for airlines that are willing to take that extra effort to negotiate with vacation operators, additional revenues can be generated. But the airline must address the following issues: (a) given the various requirements (e.g., dates and group sizes) by the vacation operators, the airline needs to tell them which flights are available; (b) if there are many flights that satisfy the time constraint and capacity
constraint, then the airline can either give its own best choice on the basis of most revenue or negotiate with the vacation operator on a common choice; and (c) on the other hand, if the airline cannot find an appropriate flight, then it is possible for the airline to suggest an alternative flight.

All these issues involve careful decisions on pricing and inventory control. The model in Section 1 will be helpful on this regard. First, since the airline only deals with the vacation operator with a clearly defined capacity requirement, our model at least gives a bottom-line value for the total revenue for this particular part of the capacity if these seats are offered to the operator. It will give the airline a base for any negotiation with the operator. As far as we know, there is no existing work that can directly address how airlines price the discount seats to tour operators. This example illustrates how the model developed in Section 1 can be applied to address this difficult pricing (and allocation) decision.

Example 2: Corporate Retreat Programs (Incentive Travel)

Many corporations have annual retreat programs to reward their high performance employees. Every year, the management will select a group of high-performing employees whose contributions and send them to a special place for a couple of days of retreat. Typical examples include companies from insurance industry, lodging industry and real estate industry. The main characteristics for these types of programs are: (i) the company will take care of all expenses and make all the arrangement; (ii) these programs are usually arranged for weekends and the airfare, if the program decides to use air travel, is a major part of the total expense, which is tax deductible; and (iii) some senior executives in the company usually need to show up at the program and these executives are in fact captives of the unrestricted fare or the first class fare since they simply cannot make an early commitment.

Many large corporations use charter services for their retreat programs because it is flexible and convenient. But major airlines can capture a large market share of this segment too. Airline managers who understand the impact of these programs on their capacity and revenue should be able to exploit this market segment. We feel that this situation fits the model in Section 1 very well.

In order for an airline to take the full advantage of these revenue opportunities, the airline needs to have a consistent approach since each of these opportunities represents a market niche, especially for national and regional airlines. An airline must have a systematic management tool that is useful to specifically explore this potential source of revenues. The computer reservation system needs to be enhanced
so that it is capable of providing real time recommendations to these managers on the price to be offered and
the targeted size of block booking.

4.2. Cheaper Unrestricted Membership Fares Only

We now discuss the case where an airline considers only offering lower unrestricted membership fares. There are many situations that lead to this type of practice. For example, travellers who are the airline's employees can fly wherever and whenever they want for free. Most of airlines also have an internal policy that will give an employee a guaranteed seat and all the flexibility if the employee pays a small fraction of the unrestricted fare. This is of course a result of its internal corporate policy. But nevertheless, it has the direct revenue impact. Another example may involve a large consulting corporation, which constantly has its consultants on different assignments all the time. These travellers require the maximum flexibility. As a result of this, most of them must pay the substantially higher unrestricted fares. On the other hand, many companies have cut their travel budget to cope with the high cost of air travel. From an airline's point of view, these travellers are part of the frequent flyers. And increasing these travellers' frequency will create substantial revenue impact for the airline. Technically speaking, an airline may consider negotiating a special travel deal with such a corporation under certain provisions, such as a minimum commitment on the total number of trips. The general feature for this type of membership demand structure is that there is no demand for restricted fares from members. Since the fares targeted to members are not available to non-members, we will two mutually exclusive restrictions Therefore, we can directly use the ME-model developed in Section 2 to find the optimal fare prices, which says that the airline needs to offer at most four fares.

From Theorem 2, we know that in the optimal solution, there is no guarantee that the (unrestricted) membership fare is smaller than the restricted (public) fare. On the other hand, as an internal policy, many airlines allow their employees to buy unrestricted fares at a price that is normally lower the regular discounted fares. With this additional constraint, the airline’s pricing has the following form:

\[
\{(p_i, q_{M,i}) : i = 1, \ldots, m; (p_j, q_{r,j}) : j = m_1, \ldots, m; (p_k, q_k) : k = m_1, \ldots, n\},
\]

where \(q_{M,i}\) is the allocation to members at the (unrestricted) price level \(p_i\) for \(1 \leq i \leq m\); \(q_{r,j}\) is the allocation to the general public at the restricted price \(p_j\) for \(m_1 \leq j \leq m\); and \(q_k\) is the allocation to the general public at
the unrestricted price \( p_k \) for \( m \leq k \leq n \). Consequently, the airline’s pricing problem can be formulated into the following mathematical problem.

\[
\max \sum_{m=1}^{m} p_{m,j} + \sum_{j=m}^{n} p_{j,r,j} + \sum_{i=1}^{s} p_{i} q_{i} \\
\text{subjectto} \\
\mu_2 \equiv \sum_{m=1}^{m} \frac{q_{m,j}}{D_{m}} \leq 1; \quad \mu_1 \equiv \sum_{j=m}^{n} \frac{q_{r,j}}{D_{j}} \leq 1 \\
\sum_{i=1}^{s} q_{i} \leq 1 \\
\sum_{m=1}^{m} q_{m,j} + \sum_{j=m}^{n} q_{r,j} + \sum_{i=1}^{s} q_{i} \leq q \\
q_{m,j} \geq 0, q_{r,j} \geq 0, q_{i} \geq 0, \forall i, j, k.
\]

In the above formulation, it is easy to make \( \mu_1 = 1 \) through a normal regularity condition. In order to obtain a simple optimal fare structure, the key is to make \( \mu_2 \) to be one. But the problem is that there are no natural conditions that generically guarantee \( \mu_2 \) to be 1. Of course, the requirement that \( \mu_2 = 1 \), in most cases, is a corporate policy since the airline will give any employee a guaranteed unrestricted ticket if the employee pays a small fraction of the public unrestricted fare. This is indeed plausible for the above formulation per se, which leads to a simple optimal fare structure because it reduces to a linear programming problem. But this practice can be a bad news to the airline since it may cause a revenue loss. For these flights with low demand, the above formulation will most likely lead an optimal policy that is consistent with an optimal solution derived directly from ME-model. But for some high demand flights or flights at busy travelling seasons such as Christmas time, the commitment that \( \mu_2 = 1 \) may prove to be costly. Therefore, the above analysis calls for close monitoring on these flights and some modification may be necessary. One possible modification is to satisfy a fixed percentage of these who request a booking on the flight. There are several benefits for this provision. First, it will still reduce above formulation to a linear programming since \( \mu_2 \) is a fixed constant and \( \mu_1 \) can always be taken as one. Second, it is not hard to implement it because it involves its own employees. For example, it can be implemented through a lottery draw by a computer with a fixed success rate among these employees who wish to book the same flight. Another possible modification is to ask internal travellers to pay for an unrestricted ticket at a restricted price. The best possible modification is to implement an optimal fare structure derived from ME-model, which may force some internal travellers to pay for an unrestricted ticket at a price that is higher that the restricted fares. In summary, the main point here is that strictly committing \( \mu_2 = 1 \) may be costly.
4.3. Cheaper Restricted Membership Fares and Cheaper Unrestricted Membership Fares

In the last two subsections, we have discussed several cases which indicate that it is sensible for an airline to offer either lower restricted membership fares or lower unrestricted membership fares. But in some situations, with enough information about the demand behaviour from the members, the airline may consider to offer both lower restricted membership fares and lower unrestricted membership fares. This leads to the pricing problem with two general restrictions -- one membership restriction and one product restriction.

It is clear that in our case here, the airline has the flexibility of offering four types of fares: restricted membership fares, (public) restricted fares, (unrestricted) membership fares and (public) unrestricted fares. As argued at the beginning of this section, we know that restricted membership fares, if offered, should not be higher than the public restricted fares; and similarly, the unrestricted membership fares, if offered, also should not be higher than the public unrestricted fares. Clearly, the airline should always consider selling restricted membership fares first since it is most restricted among these four proposed fares. By Theorem 3, we know that if the membership deal between the airline and the members requires that airline must give a guaranteed booking for a restricted ticket from any member who requests for it, then the airline's pricing problem can be formulated as a simple linear programming problem, which shows that the airline needs to offer at most five fare levels with four different types of ticket.

We now give some examples that fit into scenario. First, consider the market that consists of travellers from a partner airline. These travellers must pay for their tickets if they want to fly at a flight from its partner airline. On the other hand, because of their affiliation with an airline and its internal benefit of free flying, these travellers are more price sensitive than the general public for both restricted fares and unrestricted fares. We feel that this situation fits nicely into this scenario.

Another example involves agreements with government clients. It is well known that there is an enormous amount of travelling activities from employees from government or/and government agencies. Some of these travels can be planned in advance; but there is a substantial portion of it that cannot be planned ahead. In this case, an airline can give these clients a break on both restricted fares and unrestricted fares. It is evident that the government will be pleased to let its employees to fly with an unrestricted fare at a price that is between the public restricted fares and public unrestricted fares. It is a special treatment that may induce more travels from the government employees. We believe that the results from Theorem 3 will be helpful for airlines to negotiate acceptable fare prices with the government clients.
4.4. Additional Discussions

So far in this section, we have illustrated how the pricing models for two types of restrictions can be used for airlines when product- and membership- restrictions are used at the same time. Unlike product restrictions, which are fairly difficult to design and implement, membership restrictions are relatively easy to plan and execute. In our view, the use of membership restrictions will further enhance an airline's revenues. On the other hand, addressing all possible membership restrictions at the same time could be counter-productive because it is hard to monitor the interactions among many different member groups. A simple solution to this problem is to develop a heuristic procedure that has the following features: (i) it can model the impact of each individual membership restriction on each flight; (ii) the airline can place a flag on a flight showing which member group(s) can book this flight; and (iii) the computer reservation system should be automated so that ordinary travel agents can authorize bookings from members.

The key ingredient here is to tag the affected flights. On the other hand, to tag a flight several months ahead is not an easy task, but with more advanced forecasting tools, we believe that our pricing models can be helpful in this regard as a decision tool. Most importantly they can be used to effectively complement the modern airline yield management systems.

Finally, the use of membership restriction is not unique to airlines. There are many other industries that use memberships extensively. We feel that as long as the membership is used to obtain special access to perishable products, our discussions in this section will be equally useful.

5. Conclusion and Further Comments

This paper addresses the issue of pricing perishable items by using two types of restrictions. After studying two extreme cases, i.e., nested restrictions and mutually exclusive restrictions, we have also discussed the general case. We have shown that for the two extreme cases, there exist optimal pricing structures that consist of at most four prices. And for the general case, we present two situations where there exist optimal pricing structures that consist of at most five prices under plausible conditions. We further show that all of these optimal pricing structures can be fully characterized by linear programming problems, which make these models tractable in applications.

To further argue the practical relevance and the usefulness of the models developed in this paper, we have demonstrated that our models can be applied in the airline context when a membership restriction and a product restriction are used at the same time. As it is relatively easier to design and control
membership restrictions, they should be and can be used to generate additional revenues. Unfortunately, there is a dilemma here: members normally request additional discounts for their tickets. Therefore pricing and allocation decisions are critical. We illustrate, through several examples, that our models can be helpful in several aspects: (a) it gives an optimal pricing and allocation plan, which can help the airline to tag these flights that are available for bookings from certain members; and (b) it specifies the operating conditions that lead to an effective fare structure by validating the regularity conditions for the corresponding model to be applied.

Additional studies are needed to address the integration issues with yield management systems and to develop operational procedures to execute realistic implementations of the models developed in this paper. Of course, there should be more empirical studies to examine the impact of various restrictions that are in use and to propose more innovative restrictions for airlines to use.

Appendix

Proof of Theorem 1: First, it is evident that

$$\max_{1 \leq m \leq n} R(m_2, m_1) \geq \max_{1 \leq n \leq m \leq n} \tilde{R}(m_2, m_1).$$

Hence it suffices to show that

$$\max_{1 \leq m \leq n} R(m_2, m_1) \leq \max_{1 \leq n \leq m \leq n} \tilde{R}(m_2, m_1) \quad (A.1)$$

To prove (A.1), we only need to show that for any pair: $1 \leq m_2 \leq m_1 \leq n$, there exists another pair: $1 \leq \tilde{m}_2 \leq \tilde{m}_1 \leq n$ such that

$$R(m_2, m_1) \leq \tilde{R}(\tilde{m}_2, \tilde{m}_1). \quad (A.2)$$

For any given pair $1 \leq m_2 \leq m_1 \leq n$, let $\{(p_i, q_{i, j}): i = 1, \ldots, m_2; (p_j, q_{j, k}): j = m_2, \ldots, m_1; (p_k, q_k): k = m_2, \ldots, n\}$ be an optimal solution to the N-model. Denote $q_{i,j} \equiv \sum_{i=1}^{m_2} q_{i,j}$, which is the total sale for type-2 restricted units. Note that the residual demand for the type-1 restricted product after the sales of the type-2 restricted product is given by $d^{r_1}_{i,j} = D^{r_1} - \beta^{r_2} D^{r_2}_j$, for $j = m_2, \ldots, n$ and the total residual demand for unrestricted product at price $p_j$ after the sales of the type-2 restricted product is given by $d^{u}_{m_2,j} = D_j - \beta^{u} D^{u}_j$, for $j = m_2, \ldots, n$. Therefore the residual percentage of consumers who can not accommodate the type-1 restriction is given by:
\[
\alpha_{ij} = 1 - \frac{d_{x_{1,j}}^1}{d_{x_{2,j}}} = 1 - \frac{D_j^i - \beta_j D_j^2}{D_j - \beta_j D_j^2} = \frac{\alpha_y}{1 - \beta_j (1 - \alpha_{1,j})(1 - \alpha_{2,j})},
\]

which is increasing since both \( \alpha_y \) and \( \alpha_{ij} \) are increasing by assumption. Therefore, we can apply Theorem 5 of Li (2001) to the residual market for the type-1 restricted product and the unrestricted product. This implies that considering \((q - q_{r2})\) as the new capacity limit, we know that there exists an integer \( \bar{m}_1 : m_2 \leq \bar{m}_1 \leq n \) and a new optimal allocation plan of the form

\[
\{(p_{j}, \tilde{q}_{r1,j}) : j = m_2, ..., \bar{m}_1; (p_k, \tilde{q}_k) : k = \bar{m}_1, ..., n\}
\]
such that:

\[
\sum_{j=m_2}^{\bar{m}_1} \frac{q_{r1,j}}{D_j^i - \beta_j D_j^2} = 1, \quad \text{(A.3)}
\]

\[
\sum_{k=\bar{m}_1}^{n} \frac{\tilde{q}_k}{D_k - D_k^2} = 1, \quad \text{(A.4)}
\]

\[
\sum_{j=m_2}^{\bar{m}_1} \tilde{q}_{r1,j} + \sum_{k=\bar{m}_1}^{n} \tilde{q}_k \leq q - q_{r2}, \quad \text{(A.5)}
\]

\[
\sum_{j=m_2}^{\bar{m}_1} p_j \tilde{q}_{r1,j} + \sum_{k=\bar{m}_1}^{n} p_k \tilde{q}_k \geq \sum_{j=m_2}^{\bar{m}_1} p_j q_{r1,j} + \sum_{k=\bar{m}_1}^{n} p_k q_k. \quad \text{(A.6)}
\]

We should notice that the relationship (A.4) is independent of the value of \( \beta_2 \). In fact, as long as (A.3) holds, that is, the feasibility constraint for the type-1 restricted product is tight, (A.4) is automatically true!

Now let \( \tilde{q}_0 = \sum_{k=\bar{m}_1}^{n} \tilde{q}_k \). Therefore \( \tilde{q}_0 \) is the total number of units allocated for unrestricted units.

Then with \( \tilde{q}_0 \) unrestricted units protected, the firm needs to solve the following sub-problem:

\[
\max \sum_{i=m_2}^{\bar{m}_2} p_i q_{r2,i} + \sum_{j=m_2}^{\bar{m}_1} p_j q_{r1,j}
\]

subject to

\[
\beta_2 \equiv \sum_{i=m_2}^{\bar{m}_2} \frac{q_{r2,i}}{D_j^i} \leq 1; \sum_{j=m_2}^{\bar{m}_1} \frac{q_{r1,j}}{D_j^i - \beta_2 D_j^2} \leq 1
\]

\[
\sum_{i=m_2}^{\bar{m}_2} q_{r2,i} + \sum_{j=m_2}^{\bar{m}_1} q_{r1,j} \leq q - \tilde{q}_0
\]

\[
q_{r2,i} \geq 0, q_{r1,j} \geq 0, \forall i, j.
\]

This again leads to the basic model of Li (2001). Now according to Theorem 5 of Li (2001), there exists another integer \( \bar{m}_2 : 1 \leq \bar{m}_2 \leq \bar{m}_1 \) and another allocation plan given by

\[
\{(p_{j}, \tilde{q}_{r2,j}) : i = 1, ..., \bar{m}_2; (p_{j}, \tilde{q}_{r1,j}) : j = \bar{m}_2, ..., \bar{m}_1\}
\]
such that
\[
\sum_{k=1}^{\tilde{m}_k} \frac{\tilde{q}_{2,k}}{D_k^2} = 1, \quad (A.7)
\]

\[
\sum_{j=1}^{\tilde{m}_j} \frac{\tilde{q}_{1,j}}{D_j^2} = 1, \quad (A.8)
\]

\[
\sum_{i=1}^{\tilde{m}_i} \tilde{q}_{2,i} + \sum_{j=1}^{\tilde{m}_j} \tilde{q}_{1,j} \leq q, \quad (A.9)
\]

\[
\sum_{i=1}^{\tilde{m}_i} p_i \tilde{q}_{2,i} + \sum_{j=1}^{\tilde{m}_j} p_j \tilde{q}_{1,j} \geq \sum_{i=1}^{\tilde{m}_i} p_i \tilde{q}_{2,i} + \sum_{j=1}^{\tilde{m}_j} p_j \tilde{q}_{1,j}. \quad (A.10)
\]

Finally, by (A.4), (A.7), (A.8) and (A.9), we know that the new allocation plan given by

\[
\{(p_i, \tilde{q}_{2,i}) : i = 1, ..., \tilde{m}_2 ; (p_j, \tilde{q}_{1,j}) : j = \tilde{m}_2, ..., \tilde{m}_3 ; (p_k, \tilde{q}_k) : k = \tilde{m}_1, ..., n\}
\]

constitutes a feasible policy for the tight N-model with respect to \(\tilde{m}_2\) and \(\tilde{m}_1\). Furthermore, by (A.6) and (A.10), it follows that

\[
\bar{R}(\tilde{m}_2, \tilde{m}_1) \geq \sum_{i=1}^{\tilde{m}_i} p_i \tilde{q}_{2,i} + \sum_{j=1}^{\tilde{m}_j} p_j \tilde{q}_{1,j} + \sum_{k=1}^{\tilde{m}_k} p_k \tilde{q}_k
\]

\[
\geq \sum_{i=1}^{\tilde{m}_i} p_i \tilde{q}_{2,i} + \sum_{j=1}^{\tilde{m}_j} p_j \tilde{q}_{1,j} + \sum_{k=1}^{\tilde{m}_k} p_k \tilde{q}_k
\]

\[
\geq \sum_{i=1}^{\tilde{m}_i} p_i \tilde{q}_{2,i} + \sum_{j=1}^{\tilde{m}_j} p_j \tilde{q}_{1,j} + \sum_{k=1}^{\tilde{m}_k} p_k \tilde{q}_k = R(m_2, m_1).
\]

So (A.2) is true; and therefore (A.1) is proved. This completes the proof of the theorem.
References


Dana, J.D. 1999. Using Yield Management to Shift Demand When the Peak Time is Unknown. RAND J. of Econ. 30, 456-474.


