Chapter 1

Exercise 1.1 The payoff $C$ is that of a put option with strike price $K = 3$.

Exercise 1.2 Each of the two possible scenarios yields one equation:

\[
\begin{align*}
5\alpha + \beta &= 0 \\
2\alpha + \beta &= 6,
\end{align*}
\]

with solution

\[
\begin{align*}
\alpha &= -2 \\
\beta &= +10.
\end{align*}
\]

The hedging strategy at $t = 0$ is to shortsell $-\alpha = +2$ units of the asset $S$ priced $S_0 = 4$, and to put $\beta = $10 in savings. The price $V_0 = \alpha S_0 + \beta$ of the initial portfolio at time $t = 0$ is

\[
V_0 = \alpha S_0 + \beta = -2 \times 4 + 10 = 2,
\]

which yields the price of the claim at time $t = 0$. In order to hedge then option, one should:

i) At time $t = 0$,

a. Charge the $2 option price.

b. Shortsell $-\alpha = +2$ units of the stock priced $S_0 = 4$, which yields $8$.

c. Put $\beta = $8 + $2 = $10 in savings.

ii) At time $t = 1$,

a. If $S_1 = $5, spend $10 from savings to buy back $-\alpha = 2$ stocks.

b. If $S_1 = $2, spend $4 from savings to buy back $-\alpha = 2$ stocks, and deliver a $10 -$4 = $6 payoff.

Pricing the option by the expected value $E^*[C]$ yields the equality

\[
$2 = E^*[C]
\]
$$= 0 \times \mathbb{P}^*(C = 0) + 6 \times \mathbb{P}^*(C = 6)$$
$$= 0 \times \mathbb{P}^*(S_1 = 2) + 6 \times \mathbb{P}^*(S_1 = 5)$$
$$= 6 \times q^*,$$

hence the risk-neutral probability measure $\mathbb{P}^*$ is given by

$$p^* = \mathbb{P}^*(S_1 = 5) = \frac{2}{3} \quad \text{and} \quad q^* = \mathbb{P}^*(S_1 = 2) = \frac{1}{3}.$$  

Exercise 1.3

a) i) Does this model allow for arbitrage? Yes ✓ No |

ii) If this model allows for arbitrage opportunities, how can they be realized?

By shortselling | By borrowing in savings ✓ N.A. |

b) i) Does this model allow for arbitrage? Yes | No ✓

ii) If this model allows for arbitrage opportunities, how can they be realized?

By shortselling | By borrowing in savings | N.A. ✓

c) i) Does this model allow for arbitrage? Yes ✓ No |

ii) If this model allows for arbitrage opportunities, how can they be realized?

By shortselling ✓ By borrowing in savings | N.A. |

Exercise 1.4

a) We need to search for possible risk-neutral probability measure(s) $\mathbb{P}^*$ such that $\mathbb{E}^*[S^{(1)}] = (1 + r)\pi^{(1)}$. Letting

$$\begin{cases}
p = \mathbb{P}^*(S^{(1)} = \pi^{(1)}(1 + a)) = \mathbb{P}^* \left( \frac{S^{(1)} - \pi^{(1)}}{\pi^{(1)}} = a \right), \\
\theta = \mathbb{P}^*(S^{(1)} = \pi^{(1)}(1 + b)) = \mathbb{P}^* \left( \frac{S^{(1)} - \pi^{(1)}}{\pi^{(1)}} = b \right), \\
q = \mathbb{P}^*(S^{(1)} = \pi^{(1)}(1 + c)) = \mathbb{P}^* \left( \frac{S^{(1)} - \pi^{(1)}}{\pi^{(1)}} = c \right),
\end{cases}$$

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We have
\[
\begin{align*}
\begin{cases}
p\pi^{(1)}(1+a) + \theta\pi^{(1)}(1+b) + q\pi^{(1)}(1+c) = (1+r)\pi^{(1)} \\
p + \theta + q = 1,
\end{cases}
\end{align*}
\]
from which we obtain
\[
\begin{align*}
\begin{cases}
pa + \theta b + qc = r, \\
p + \theta + q = 1.
\end{cases}
\Rightarrow
\begin{cases}
p = \frac{(1-\theta)c + \theta b - r}{c - a} \in (0, 1), \\
q = \frac{r - (1-\theta)a - \theta b}{c - a} \in (0, 1),
\end{cases}
\end{align*}
\]
for any \(\theta \in (0, 1)\) such that
\[(1-\theta)a - \theta b < r < (1-\theta)c + \theta b,
\]
or \((1-\theta)a < r < (1-\theta)c\) in case \(b = 0\). Therefore there exists an infinity of risk-neutral probability measures depending on the values of \(\theta \in (0, 1)\), and the market is without arbitrage but not complete.

b) Hedging a claim with possible payoff values \(P_a, P_b, P_c\) would require to solve
\[
\begin{align*}
\begin{cases}
\alpha \pi^{(1)}(1+a) + \beta \pi^{(0)}(1+r) = P_a \\
\alpha \pi^{(1)}(1+b) + \beta \pi^{(0)}(1+r) = P_b \\
\alpha \pi^{(1)}(1+c) + \beta \pi^{(0)}(1+r) = P_c,
\end{cases}
\end{align*}
\]
for \(\alpha\) and \(\beta\), which is generally not possible due to the existence of three conditions with only two unknowns.

Exercise 1.5

a) The risk-neutral condition \(\mathbb{E}^*[R_1] = 0\) reads
\[
b\mathbb{P}^*(R_1 = b) + 0 \times \mathbb{P}^*(R_1 = 0) + (-b) \times (R_1 = -b) = bp^* - bq^* = 0,
\]
hence
\[
p^* = q^* = \frac{1-\theta^*}{2},
\]
since \(p^* + q^* + \theta^* = 1\).

b) We have
\[
\text{Var}^* \left[ \frac{S_1^{(1)} - S_0^{(1)}}{S_0^{(1)}} \right] = \mathbb{E}^*[R_1^2] - (\mathbb{E}^*[R_1])^2
\]
\[= \mathbb{E}^*[R_1^2]\]
\[= b^2 \mathbb{P}^*(R_1 = b) + 0^2 \times \mathbb{P}^*(R_1 = 0) + (-b)^2 \times (R_1 = -b)\]
\[= b^2 (p^* + q^*)\]
\[= b^2 (1 - \theta^*)\]
\[= \sigma^2,\]

hence
\[\theta^* = 1 - \frac{\sigma^2}{2b^2}\]

and hence
\[p^* = q^* = \frac{1 - \theta^*}{2} = \frac{\sigma^2}{2b^2}.\]

**Exercise 1.6**

a) The possible values of \(R\) are \(a\) and \(b\).

b) We have
\[
\mathbb{E}^*[R] = a \mathbb{P}^*(R = a) + b \mathbb{P}^*(R = b)
= \frac{a}{b - a} + \frac{b - r}{b - a}
= r.
\]

c) By Theorem 1.5, there do not exist arbitrage opportunities in this market since from Question (b) there exists a risk-neutral probability measure \(\mathbb{P}^*\) whenever \(a < r < b\).

d) The risk-neutral probability measure is unique hence the market model is complete by Theorem 1.11.

e) Taking
\[\eta = \frac{\alpha (1 + b) - \beta (1 + a)}{\pi_1 (b - a)}\quad \text{and} \quad \xi = \frac{\beta - \alpha}{S_0 (b - a)},\]
we check that
\[
\begin{aligned}
\eta \pi_1 + \xi S_0 (1 + a) &= \alpha \quad \text{if } R = a, \\
\eta \pi_1 + \xi S_0 (1 + b) &= \beta \quad \text{if } R = b,
\end{aligned}
\]
which shows that
\[\eta \pi_1 + \xi S_1 = C\]
in both cases \(R = a\) and \(R = b\).

f) We have
\[
\pi_0 (C) = \eta \pi_0 + \xi S_0
= \frac{\alpha (1 + b) - \beta (1 + a)}{(1 + r)(b - a)} + \frac{\beta - \alpha}{b - a}
\]
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\[
\begin{align*}
\frac{\alpha(1+b) - \beta(1+a) - (1+r)(\alpha - \beta)}{(1+r)(b-a)} & = \frac{\alpha b - \beta a - r(\alpha - \beta)}{(1+r)(b-a)}. \\
\end{align*}
\]

(A.1)

g) We have

\[
\mathbb{E}^*[C] = \alpha P^*(R = a) + \beta P^*(R = b) = \frac{b-r}{b-a} + \frac{r-a}{b-a}.
\]

(A.2)

h) Comparing (A.1) and (A.2) above we do obtain

\[
\pi_0(C) = \frac{1}{1+r} \mathbb{E}^*[C]
\]

i) The initial value \(\pi_0(C)\) of the portfolio is interpreted as the arbitrage price of the option contract and it equals the expected value of the discounted payoff.

j) We have

\[
C = (K - S_1)^+ = (11 - S_1)^+ = \begin{cases} 
11 - S_1 & \text{if } K > S_1, \\
0 & \text{if } K \leq S_1.
\end{cases}
\]

k) We have \(S_0 = 1, a = 8, b = 11, \alpha = 2, \beta = 0\), hence

\[
\xi = \frac{\beta - \alpha}{S_0(b-a)} = \frac{0 - 2}{11 - 8} = \frac{2}{3},
\]

and

\[
\eta = \frac{\alpha(1+b) - \beta(1+a)}{\pi_1(b-a)} = \frac{22}{3 \times 1.05}.
\]

l) The arbitrage price \(\pi_0(C)\) of the contingent claim \(C\) is

\[
\pi_0(C) = \eta \pi_0 + \xi S_0 = 6.952.
\]

Exercise 1.7 Let \(a := -(152 - 180)/180 = 7/45\) and \(b := (203 - 180)/180 = 23/180\) denote the potential market returns, with \(r = 0.03\). From the strike price \(K\) and the risk-neutral probabilities

\[
p_r^* = \frac{r-a}{b-a} = 0.6549 \quad \text{and} \quad q_r^* = \frac{b-r}{b-a} = 0.3451,
\]

the price of the option at the beginning of the year is given from Proposition 1.13 as the discounted expected value

\[
\mathbb{E}^*[C] = \frac{1}{1+r} \mathbb{E}^*[C]
\]

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\[
\frac{1}{1+r} \mathbb{E}^*[K - S_1^+] = \frac{1}{1+r} \left( p_r^* (K - 203)^+ + q_r^* (K - 152)^+ \right).
\]

Equating this price with the intrinsic value \((K - 180)^+\) of the put option yields the equation

\[
(K - 180)^+ = \frac{1}{1+r} \left( p_r^* (K - 203)^+ + q_r^* (K - 152)^+ \right)
\]

which requires \(K > 180\) (the case \(K \leq 152\) is not considered because both the option price and option payoff vanish in this case). Hence we consider the equation

\[
K - 180 = \frac{1}{1+r} \left( p_r^* (K - 203)^+ + q_r^* (K - 152)^+ \right),
\]

with the following cases.

i) If \(K \in [180, 203]\) we get

\[
(1 + r)(K - 180) = q_r^*(K - 152),
\]

hence

\[
K = \frac{(1 + r)180 - q_r^* 152}{1 + r - q_r^*} = \frac{(1 + r)180 - q_r^* 152}{p_r^* + r} = 194.11.
\]

ii) If \(K \geq 203\) we find

\[
K = \frac{180(1 + r) - 203p_r^* - 152q_r^*}{r} < 203,
\]

which is out of range and leads to a contradiction.

We note that the above formula

\[
K = \frac{(1 + r)180 - q_r^* 152}{p_r^* + r} = \frac{28b - 180a + r(180(b - a) + 152)}{(b + 1 - a)r - a}
\]

yields a decreasing function \(K(r)\) of \(r\) in the interval \([0, 100\%]\), although the function is not monotone over \(\mathbb{R}_+\).
Chapter 2

Exercise 2.1 Let \( m := \$2,550 \) denote the amount invested each year. By (2.1), the value of the plan after \( N = 10 \) years becomes

\[
m \sum_{k=1}^{N} (1 + r)^k = m(1 + r) \frac{(1 + r)^N - 1}{r},
\]

which in turns becomes

\[
(1 + r)^N m \sum_{k=1}^{N} (1 + r)^k = m(1 + r)^{N+1} \frac{(1 + r)^N - 1}{r},
\]

after \( N \) additional years without further contributions to the plan. Equating

\[
A = m(1 + r)^{N+1} \frac{(1 + r)^N - 1}{r}
\]

shows that

\[
\frac{(1 + r)^{2N+1} - (1 + r)^{N+1}}{r} = \frac{A}{m},
\]

or

\[
\frac{(1 + r)^{21} - (1 + r)^{11}}{r} = \frac{30835}{2550} \approx 12.09215,
\]

showing that \( r \approx 1.23\% \) according to Figure S.2.
Exercise 2.2 Let \( m := \$3,581 \) denote the amount invested each year. After multiplying (2.1) by \((1 + r)^N\) in order to account for the compounded interests from year 11 until year 20, we get the equality

\[
A = m(1 + r)^{N+1} \frac{(1 + r)^N - 1}{r}
\]

shows that

\[
(1 + r)^{21} - (1 + r)^{11} = r \frac{50862}{3581} \simeq 14.2033r,
\]

showing that \( r \simeq 2.28\% \) according to Figure S.3.

Exercise 2.3 We check that for any \( \mathcal{P}^* \) of the form \( \mathcal{P}^*(R_t = -1) := p^* \), \( \mathcal{P}^*(R_t = 0) := 1 - 2p^* \), \( \mathcal{P}^*(R_t = 1) := p^* \), we have

\[
\mathbb{E}^*[S_1] = S_0(2p^* + 1 - 2p^*) = S_0,
\]

and similarly

\[
\mathbb{E}^*[S_2 \mid S_1] = S_1(2p^* + (1 - 2p^*)) = S_1,
\]
hence the probability measure $\mathbb{P}^*$ is risk-neutral.

**Exercise 2.4**

a) In order to check for arbitrage opportunities we look for a risk-neutral probability measure $\mathbb{P}^*$ which should satisfy

$$\mathbb{E}^* \left[ S_{k+1}^{(1)} \mid \mathcal{F}_k \right] = (1 + r)S_k^{(1)}, \quad k = 0, 1, \ldots, N - 1.$$ 

Rewriting $\mathbb{E}^* \left[ S_{k+1}^{(1)} \mid \mathcal{F}_k \right]$ as

$$\mathbb{E}^* \left[ S_{k+1}^{(1)} \mid \mathcal{F}_k \right] = (1 - b)S_k^{(1)}P^*(R_{k+1} = a \mid \mathcal{F}_k) + S_k^{(1)}P^*(R_{k+1} = 0 \mid \mathcal{F}_k) + (1 + b)S_k^{(1)}P^*(R_{k+1} = b),$$

$k = 0, 1, \ldots, N - 1$, it follows that any risk-neutral probability measure $\mathbb{P}^*$ should satisfy the equations

$$\begin{cases} 
(1 + b)S_k^{(1)}P^*(R_{k+1} = b) + S_k^{(1)}P^*(R_{k+1} = 0) + (1 - b)S_k^{(1)}P^*(R_{k+1} = a) = S_k^{(1)} \\
P^*(R_{k+1} = b) + P^*(R_{k+1} = 0) + P^*(R_{k+1} = -b) = 1, 
\end{cases}$$

$k = 0, 1, \ldots, N - 1$, i.e.

$$\begin{cases} 
 bP^*(R_k = b) - bP^*(R_k = -b) = 0, \\
P^*(R_k = b) + P^*(R_k = -b) = 1 - P^*(R_k = 0), 
\end{cases}$$

$k = 1, 2, \ldots, N$, with solution

$$P^*(R_k = b) = P^*(R_k = -b) = \frac{1 - \theta^*}{2},$$

$k = 1, 2, \ldots, N$.

b) We have

$$\text{Var}^* \left[ \frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}} \mid \mathcal{F}_k \right] = \mathbb{E}^* \left[ \left( \frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}} \right)^2 \mid \mathcal{F}_k \right] - \left( \mathbb{E}^* \left[ \frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}} \mid \mathcal{F}_k \right] \right)^2.$$
\[
\begin{align*}
&= \mathbb{E}^*[\left(\frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}}\right)^2 | \mathcal{F}_k] \\
&= b^2 \mathbb{P}_\sigma^*(R_{k+1} = -b | \mathcal{F}_k) + b^2 \mathbb{P}_\sigma^*(R_{k+1} = b | \mathcal{F}_k) \\
&= b^2 \left(1 - \mathbb{P}_\sigma^*(R_{k+1} = 0)\right) + b^2 \left(1 - \mathbb{P}_\sigma^*(R_{k+1} = 0)\right) \\
&= b^2 (1 - \theta) \\
&= \sigma^2,
\end{align*}
\]

\(k = 0, 1, \ldots, N - 1\), hence

\[
\mathbb{P}_\sigma^*(R_k = 0) = \theta = 1 - \frac{\sigma^2}{b^2},
\]

and therefore

\[
\mathbb{P}_\sigma^*(R_k = b) = \mathbb{P}_\sigma^*(R_k = -b) = \frac{1 - \mathbb{P}_\sigma^*(R_k = 0)}{2} = \frac{\sigma^2}{2b^2},
\]

\(k = 0, 1, \ldots, N - 1\), under the condition \(0 < \sigma^2 < b^2\).

Exercise 2.5

a) The possible values of \(R_t\) are \(a\) and \(b\).

b) We have

\[
\mathbb{E}^*[R_{t+1} | \mathcal{F}_t] = a \mathbb{P}^*(R_{t+1} = a | \mathcal{F}_t) + b \mathbb{P}^*(R_{t+1} = b | \mathcal{F}_t)
\]

\[
= a \frac{b - r}{b - a} + b \frac{r - a}{b - a} = r.
\]

c) Letting \(p^* = (r - a)/(b - a)\) and \(q^* = (b - r)/(b - a)\) we have

\[
\mathbb{E}^*[S_{t+k} | \mathcal{F}_t] = \sum_{i=0}^{k} \binom{k}{i} (p^*)^i (q^*)^{k-i} \left(\frac{1 + b}{i} + (1 + a)^{k-i}\right) S_t
\]

\[
= S_t \sum_{i=0}^{k} \binom{k}{i} \left(p^* (1 + b)\right)^i \left(q^* (1 + a)\right)^{k-i}
\]

\[
= S_t \left(\frac{r - a}{b - a} (1 + b) + \frac{b - r}{b - a} (1 + a)\right)^k
\]

\[
= (1 + r)^k S_t.
\]

Assuming that the formula holds for \(k = 1\), its extension to \(k \geq 2\) can also be proved recursively from the “tower property” (18.38) of conditional
expectations, as follows:

\[
\mathbb{E}^*[S_{t+k} | \mathcal{F}_t] = \mathbb{E}^*[\mathbb{E}^*[S_{t+k} | \mathcal{F}_{t+k-1} | \mathcal{F}_t]]
\]

\[
= (1 + r) \mathbb{E}^*[S_{t+k-1} | \mathcal{F}_t]
\]

\[
= (1 + r) \mathbb{E}^*[\mathbb{E}^*[S_{t+k-1} | \mathcal{F}_{t+k-2} | \mathcal{F}_t]]
\]

\[
= (1 + r)^2 \mathbb{E}^*[S_{t+k-2} | \mathcal{F}_t]
\]

\[
= (1 + r)^2 \mathbb{E}^*[\mathbb{E}^*[S_{t+k-2} | \mathcal{F}_{t+k-3} | \mathcal{F}_t]]
\]

\[
= (1 + r)^3 \mathbb{E}^*[S_{t+k-3} | \mathcal{F}_t]
\]

\[
= \ldots
\]

\[
= (1 + r)^{k-2} \mathbb{E}^*[S_{t+2} | \mathcal{F}_t]
\]

\[
= (1 + r)^{k-2} \mathbb{E}^*[\mathbb{E}^*[S_{t+2} | \mathcal{F}_{t+1} | \mathcal{F}_t]]
\]

\[
= (1 + r)^{k-1} \mathbb{E}^*[S_{t+1} | \mathcal{F}_t]
\]

\[
= (1 + r)^k S_t.
\]

**Chapter 3**

Exercise 3.1 After drawing the trinomial tree, at time \( t = 0 \) we find

\[
\pi_0(C) = \frac{1}{(1+r)^2} \mathbb{E}^*[\frac{K - S_2}{(1+r)^2}] = p^* + (1 - 2p^*)p^* + (p^*)^2 = 2p^* - (p^*)^2.
\]

At time \( t = 1 \) we find

\[
\pi_1(C) = \frac{1}{1+r} \mathbb{E}^*[\frac{K - S_2}{(1+r)^2} | S_1] = \begin{cases} 
  p^* & \text{if } S_1 = 2S_0, \\
  p^* & \text{if } S_1 = S_0, \\
  1 & \text{if } S_1 = 0.
\end{cases}
\]

Exercise 3.2 We have \( p^* = (r-a)/(b-a) = 1/2 \) and \( q^* = (b-r)/(b-a) = 1/2 \), and the following underlying asset price tree:
We first price, and then hedge. At time \( t = 1 \), by Theorem 3.5 we have

\[
\pi_1(C) = V_1 = \begin{cases} 
\frac{3p^* + q^*}{1 + r} = \frac{4}{3} & \text{if } S_1 = 2 \\
\frac{p^* + 3q^*}{1 + r} = \frac{4}{3} & \text{if } S_1 = 1,
\end{cases}
\]

and \( V_0 = \frac{4p^* + q^*}{3(1 + r)} = \frac{8}{9} \).

This leads to the following option pricing tree:

Regarding hedging, if \( S_1 = 2 \) the condition \( \bar{\xi}_2 \cdot \bar{S}_2 = 0 \) reads

\[
S_1 = 2 \implies \begin{cases} 
4\xi_2 + \eta_2(1 + r)^2 = 3 \\
2\xi_2 + \eta_2(1 + r)^2 = 1,
\end{cases}
\]

hence \((\xi_2, \eta_2) = (1, -4/9)\). On the other hand, if \( S_1 = 1 \) we have
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\[ S_1 = 1 \implies \begin{cases} 
2\xi_2 + \eta_2(1 + r)^2 = 1 \\
\xi_2 + \eta_2(1 + r)^2 = 3, 
\end{cases} \]

hence \((\xi_2, \eta_2) = (-2, 20/9)\). Finally, at time \(t = 0\) with \(S_0 = 1\) we have

\[ \begin{cases} 
2\xi_1 + \eta_1(1 + r) = \frac{4}{3} \\
\xi_1 + \eta_1(1 + r) = \frac{4}{3}, 
\end{cases} \]

hence \((\xi_1, \eta_1) = (0, 8/9)\). The results can be summarized in the following table: In addition it can be checked that the portfolio strategy \((\xi_k, \eta_k)_{k=1,2}\)

| \(S_0 = 1\) | \(S_1 = 2\) | \(V_1 = 4/3\) | \(S_2 = 4\) |
| \(V_0 = 8/9\) | \(\xi_2 = 1\) | \(\eta_2 = -4/9\) | \(V_2 = 3\) |
| \(\xi_1 = 0\) | \(S_1 = 1\) | \(V_1 = 4/3\) | \(S_2 = 1\) |
| \(\eta_1 = 8/9\) | \(\xi_2 = -2\) | \(\eta_2 = 20/9\) | \(V_2 = 3\) |

Table 19.1: CRR pricing and hedging tree.

is self-financing as we have

\[ \xi_1 S_1 + \eta_1 A_1 = \frac{8}{9} \times \frac{3}{2} \]

\[ = \begin{cases} 
2 - \frac{4}{9} \times \frac{3}{2} \\
-2 + \frac{20}{9} \times \frac{3}{2} 
\end{cases} \]

\[ = \xi_2 S_1 + \eta_2 A_1. \]

Exercise 3.3

a) We have

\[ \mathbb{E}^*[S_{t+1} \mid \mathcal{F}_t] = \mathbb{E}^*[S_{t+1} \mid S_t] \]

\[ = \frac{S_t}{2} \mathbb{P}^*(R_t = -0.5) + S_t \mathbb{P}^*(R_t = 0) + 2S_t \mathbb{P}^*(R_t = 1) \]

\[ = S_t \left( \frac{r^*}{2} + q^* + 2p^* \right) \]

\[ = S_t, \quad t = 0, 1, \]
with \( r = 0 \).

b) We have the following graph:

```
S_0 = 1
  \[ p = \frac{1}{4}, r = \frac{1}{2}, q = \frac{1}{4} \]
  \[ S_1 = 1 \]
  \[ p = \frac{1}{4}, q = \frac{1}{4}, r = \frac{1}{2} \]
  \[ S_1 = 2 \]
  \[ p = \frac{1}{4}, r = \frac{1}{2} \]
  \[ S_2 = 0.5 \]
```

```
S_1 = 2
  \[ p = \frac{1}{4}, r = \frac{1}{2}, q = \frac{1}{4} \]
  \[ S_1 = 1 \]
  \[ p = \frac{1}{4}, q = \frac{1}{4}, r = \frac{1}{2} \]
  \[ S_1 = 0.5 \]
  \[ p = \frac{1}{4}, r = \frac{1}{2} \]
  \[ S_2 = 0.25 \]
```

c) The down-and-out barrier call option is priced at time \( t = 0 \) as

\[
V_0 = \mathbb{E}^*[C] = 2.5 \times (p^*)^2 + 0.5 \times p^* q^* = \frac{3}{16}.
\]

At time \( t = 1 \) we have

\[
V_1 = 2.5 \times p^* + 0.5 \times q^* = 2.5 \times \frac{1}{4} + 0.5 \times \frac{1}{4} = \frac{3}{4}
\]

if \( S_1 = 2 \), and \( V_1 = 0 \) in both cases \( S_1 = 1 \) and \( S_1 = 0.5 \).

d) This market is not complete, and not every contingent claim is attainable, because the risk-neutral probability measure \( \mathbb{P}^* \) is not unique, for example \((r^*, q^*, p^*) = (1/4, 5/8, 1/8)\) and \((r^*, q^*, p^*) = (1/2, 1/4, 1/4)\) are both risk-neutral probability measures.

Exercise 3.4 The CRR model can be described by the following binomial tree.
a) By the formulas

\[
V_1 = \frac{1}{1+r} \mathbb{E}^*[V_2 \mid \mathcal{F}_1] = \frac{1}{1+r} \mathbb{E}^*[V_2 \mid S_1] = \frac{S_0(1+b)^2 - 8}{1+r} p^*(S_2 = S_0(1+b)^2 \mid S_1) = p^* \frac{(S_0(1+b)^2 - 8)}{1+r} \mathbb{1}_{\{S_1 = S_0(1+b)\}},
\]

and

\[
V_0 = \frac{1}{1+r} \mathbb{E}^*[V_1 \mid \mathcal{F}_0] = \frac{1}{1+r} \left( p^* \frac{(S_0(1+b)^2 - 8)}{1+r} \times \mathbb{P}^*(S_1 = S_0(1+b)) + 0 \times \mathbb{P}^*(S_1 = S_0(1+a)) \right) = (p^*)^2 \frac{(S_0(1+b)^2 - 8)}{(1+r)^2},
\]

we find the table

| \(S_0 = 1\) | \(S_1 = 3, V_1 = 1/4\) | \(S_2 = 9\) |
| \(V_0 = 1/16\) | \(V_2 = 1\) | \(S_2 = 3\) |
| \(S_1 = 1, V_1 = 0\) | \(V_2 = 0\) | \(S_2 = 1\) |
| \(V_2 = 0\) | \(V_2 = 0\) |

Table 19.2: CRR pricing tree.

Note that we could also directly compute \(V_0\) from
\[ V_0 = \frac{1}{(1+r)^2} \mathbb{E}^*[V_2 \mid \mathcal{F}_0]. \]

b) When \( S_1 = S_0(1+b) \), the equation \( \xi_2 S_2 + \eta_2 A_2 = V_2 \) reads

\[
\begin{cases}
\xi_2 S_0(1+b)^2 + \eta_2 A_0(1+r)^2 = S_0(1+b)^2 - 8 \\
\xi_2 S_0(1+b)(1+a) + \eta_2 A_0(1+r)^2 = 0,
\end{cases}
\]

which yields

\[
\xi_2 = \frac{S_0(1+b)^2 - 8}{S_0(b-a)(1+b)} \quad \text{and} \quad \eta_2 = -\frac{(S_0(1+b)^2 - 8)(1+a)}{(b-a)A_0(1+r)^2}.
\] (A.3)

On the other hand, when \( S_1 = S_0(1+a) \) the equation \( \xi_2 S_2 + \eta_2 A_2 = V_2 \) reads

\[
\begin{cases}
\xi_2 S_0(1+a)^2 + \eta_2 A_0(1+r)^2 = 0 \\
\xi_2 S_0(1+b)^2 + \eta_2 A_0(1+r)^2 = 0,
\end{cases}
\]

which has the unique solution \((\xi_2, \eta_2) = (0,0)\). Next, the equation \( \xi_1 S_1 + \eta_1 A_1 = V_1 \) reads

\[
\begin{cases}
\xi_1 S_0(1+b) + \eta_1 A_0(1+r) = \frac{p^*(S_0(1+b)^2 - 8)}{1+r}, \\
\xi_1 S_0(1+a) + \eta_1 A_0(1+r) = 0
\end{cases}
\]

which yields

\[
\xi_1 = p^* \frac{S_0(1+b)^2 - 8}{S_0(b-a)(1+b)} \quad \text{and} \quad \eta_1 = -p^* \frac{(1+a)(S_0(1+b)^2 - 8)}{(b-a)A_0(1+r)^2}.
\] (A.4)

This can be summarized in the following table:

| \( S_0 = 1 \) | \( S_1 = 3, V_1 = 1/4 \) | \( S_2 = 9 \) |
| \( V_0 = 1/16 \) | \( \xi_2 = 1/6, \eta_2 = -1/8 \) | \( V_2 = 1 \) |
| \( \xi_1 = 1/8 \) | \( S_1 = 1, V_1 = 0 \) | \( S_2 = 3 \) |
| \( \eta_1 = -1/16 \) | \( \xi_2 = 0, \eta_2 = 0 \) | \( V_2 = 0 \) |

Table 19.3: CRR pricing and hedging tree.

When \( S_1 = S_0(1+a) \) at time \( t = 1 \) the option price is \( V_1 = 0 \) and the hedging strategy is to cut all positions: \( \xi_2 = \eta_2 = 0 \). On the other hand, if
\[ S_1 = S_0(1 + b) \] then there is a chance of being in the money at maturity and we need to increase our position in the underlying from \( \xi_1 = 1/8 \) to \( \xi_2 = 1/6 \).

Note that the self-financing condition

\[ \xi_1 S_1 + \eta_1 A_1 = \xi_2 S_1 + \eta_2 A_1, \]  

(A.5)

is verified. For example when \( S_1 = S_0(1 + a) \) we have

\[
\frac{1}{8} \times S_1 - \frac{1}{16} A_1 = 0 \times S_1 + 0 \times A_1 = 0, 
\]

while when \( S_1 = S_0(1 + b) \) we find

\[
\frac{1}{8} \times S_1 - \frac{1}{16} A_1 = \frac{1}{6} \times S_1 - \frac{1}{8} \times A_1 = \frac{1}{4}. 
\]

On the other hand, we can also use the self-financing condition (A.5) to recover (A.4) by rewriting the system of equations as

\[
\begin{align*}
\xi_1 S_0(1 + b) + \eta_1 A_0(1 + r) &= \xi_2 S_0(1 + b) + \eta_2 A_0(1 + r) \\
\xi_1 S_0(1 + a) + \eta_1 A_0(1 + r) &= 0,
\end{align*}
\]

with \((\xi_2, \eta_2)\) given by (A.3), which recovers

\[
V_1 = \xi_1 S_1 + \eta_1 A_1 = \begin{cases} 
\frac{3}{8} - \frac{2}{16} = \frac{1}{4} & \text{if } S_1 = 3, \\
\frac{1}{8} - \frac{2}{16} = 0 & \text{if } S_1 = 1.
\end{cases}
\]

Exercise 3.5

a) Taking \( q^* = 1 - p^* = 1/4 \), we find the binary tree
b) We find the binary tree

\[
\begin{array}{c}
S_0 = 1 \\
S_0 = 1 \text{ and } V_0 = 1/64 \\
0.5 = 1 + a \\
2.5 = 1 + b \\
6.25 = (1 + b)^2 \\
1.25 = (1 + a)(1 + b) \\
0.25 = (1 + a)(1 + b) \\
\end{array}
\]

\[
\begin{array}{c}
p^* \\
p^* \\
q^* \\
p^* \\
p^* \\
q^* \\
q^* \\
q^* \\
\end{array}
\]

\[
\begin{array}{c}
S_0 = 1 \\
S_1 = 2.5 \text{ and } V_1 = 0 \\
S_1 = 0.5 \text{ and } V_1 = 1/8 \\
S_2 = 6.25 \text{ and } V_2 = 0 \\
S_2 = 1.25 \text{ and } V_2 = 0 \\
S_2 = 0.25 \text{ and } V_2 = 1 \\
\end{array}
\]

and the table

<table>
<thead>
<tr>
<th>( S_0 = 1 )</th>
<th>( S_1 = 2.5, V_1 = 0 )</th>
<th>( S_2 = 6.25 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_0 = 1/64 )</td>
<td>( S_1 = 0.5, V_1 = 1/8 )</td>
<td>( S_2 = 1.25 )</td>
</tr>
<tr>
<td>( V_2 = 0 )</td>
<td>( S_2 = 0.25 )</td>
<td>( V_2 = 1 )</td>
</tr>
</tbody>
</table>

Table 19.4: CRR pricing tree.

c) Here we compute the hedging strategy from the option prices. When \( S_1 = S_0(1 + b) \) we clearly have \( \xi_2 = \eta_2 = 0 \). When \( S_1 = S_0(1 + a) \), the equation

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\[ \xi_2 S_2 + \eta_2 A_2 = V_2 \] reads
\[
\begin{align*}
\xi_2 S_0 (1 + a)^2 + \eta_2 (1 + r)^2 &= S_0 (K - (1 + a)(1 + b)) \\
\xi_2 S_0 (1 + b)(1 + a) + \eta_2 (1 + r)^2 &= 0
\end{align*}
\]
hence
\[
\xi_2 = -\frac{(K - S_0 (1 + a)(1 + b))}{S_0 (b - a)(1 + a)} \quad \text{and} \quad \eta_2 = \frac{(K - S_0 (1 + a)(1 + b))(1 + b)}{S_0 (b - a)(1 + r)^2}.
\]

Next, at time \( t = 1 \) the equation \( \xi_1 S_1 + \eta_1 A_1 = V_1 \) reads
\[
\begin{align*}
\xi_1 S_0 (1 + a) + \eta_1 (1 + r) &= S_0 q^* (K - (1 + a)(1 + b)) \\
\xi_1 S_0 (1 + b) + \eta_1 (1 + r) &= 0
\end{align*}
\]
which yields
\[
\xi_1 = -\frac{q^* (K - S_0 (1 + a)(1 + b))}{S_0 (b - a)(1 + r)} \quad \text{and} \quad \eta_1 = \frac{q^* (K - S_0 (1 + a)(1 + b))(1 + b)}{S_0 (b - a)(1 + r)^2}.
\]

This can be summarized in the following table:

| \( S_0 = 1 \) | \( V_0 = 1/64 \) | \( \xi_1 = -1/16 \) | \( \eta_1 = 5/64 \) | \( \xi_2 = 0 \) | \( \eta_2 = 0 \) | \( S_1 = 2.5 \) | \( V_1 = 0 \) | \( S_2 = 6.25 \) | \( V_2 = 0 \) | \( S_2 = 1.25 \) | \( V_2 = 0 \) | \( S_2 = 0.25 \) | \( V_2 = 1 \) |

Table 19.5: CRR pricing and hedging tree.

If \( S_1 = S_0 (1 + a) \) then there is a chance of being in the money at maturity and we need short sell by decreasing \( \xi_1 \) from \( \xi_1 = -1/16 \) to \( \xi_2 = -1 \). Note that the self-financing condition
\[ \xi_1 S_1 + \eta_1 A_1 = \xi_2 S_1 + \eta_2 A_1, \]
is satisfied.

Exercise 3.6

a) Taking the risk-free rate \( r \) equal to zero, the binary call option can be priced as
\[
E^*[C] = E^*[\1_{[K,\infty)}(S_N)] = P^*(S_N \geq K) =: p^*
\]
under the risk-neutral probability measure $\mathbb{P}^*$.
b) Investing $p^*$ by purchasing one binary call option yields a potential net return of
\[
\begin{align*}
\frac{1-p^*}{p^*} &= \frac{1}{p^*} - 1 \text{ if } S_N \geq K, \\
\frac{0-p^*}{p^*} &= -100\% \text{ if } S_N < K.
\end{align*}
\]
c) The corresponding expected return is
\[
p^* \times \left( \frac{1}{p^*} - 1 \right) + (1-p^*) \times (-1) = 0.
\]
d) The corresponding expected return is
\[
p^* \times 0.86 + (1-p^*) \times (-1) = p^* \times 1.86 - 1,
\]
which will be \textit{negative} if
\[
p^* < \frac{1}{1.86} \simeq 0.538.
\]
That means, the expected gain can be negative even if
\[
0.538 > p^* = \mathbb{P}^*(S_N \geq K) > 0.5.
\]
Similarly, the expected gain
\[
(1-p^*) \times 0.86 + p^* \times (-1) = 0.86 - p^* \times 1.86,
\]
on binary put options will be \textit{negative} if $1-p^* > 1/1.86$, i.e. if
\[
p^* > \frac{0.86}{1.86} \simeq 0.462.
\]
That means, the expected gain can be negative even if $1 - 0.462 > \mathbb{P}^*(S_N < K) > 0.5$. In conclusion, both call and put average gains will be negative if $p^* \in (0.462, 0.538)$.

Note that the average of call and put gains will still be negative, as
\[
\frac{p^* \times 1.86 - 1}{2} + \frac{0.86 - p^* \times 1.86}{2} = \frac{0.86 - 1}{2} < 0.
\]

Exercise 3.7 A put spread collar option requires its holder to sell an asset at the price $f(x)$ when its market price is $x$. 

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a) The payoff function graph of the put spread collar option is given in Figure S.5.

b) The payoff function can be written as

\[-(K_1 - x)^+ + (K_2 - x)^+ - (x - K_3)^+\]
\[= -(80 - x)^+ + (90 - x)^+ - (x - 110)^+,\]

see also http://optioncreator.com/stp7xy2. Hence this collar payoff can be realized by

- holding (purchasing) one put option with strike price \(K_2\),
- issuing (or shorting) one put option with strike price \(K_1\), and
- issuing (or shorting) one call option with strike price \(K_3\).
Exercise 3.8 We have
\[
\mathbb{E}^* \left[ \frac{\phi \left( S_1 + \cdots + S_N \right)}{N} \right] \leq \mathbb{E}^* \left[ \frac{\phi(S_1) + \cdots + \phi(S_N)}{N} \right]
\]
\[
= \frac{\mathbb{E}^*[\phi(S_1)] + \cdots + \mathbb{E}^*[\phi(S_N)]}{N}
\]
\[
= \mathbb{E}^*[\phi(\mathbb{E}^*[S_N | F_1]) + \cdots + \mathbb{E}^*[\phi(S_N) | F_N]]
\]
\[
\leq \mathbb{E}^*[\mathbb{E}^*[\phi(S_N) | F_1]] + \cdots + \mathbb{E}^*[\mathbb{E}^*[\phi(S_N) | F_N]]
\]
\[
= \mathbb{E}^*[\phi(S_N)] + \cdots + \mathbb{E}^*[\phi(S_N)]
\]
\[
= \mathbb{E}^*[\phi(S_N)].
\]

since \( \phi \) is convex,

because \((S_n)_{n \in \mathbb{N}}\) is a martingale,

by Jensen’s inequality,

by the tower property,

Exercise 3.9

a) The condition \( V_N = C \) reads
\[
\begin{align*}
\eta_N \pi_N + \xi_N (1 + a) S_{N-1} &= (1 + a) S_{N-1} - K \\
\eta_N \pi_N + \xi_N (1 + b) S_{N-1} &= (1 + b) S_{N-1} - K,
\end{align*}
\]
from which we deduce the (static) hedging strategy \( \xi_N = 1 \) and \( \eta_N = -K(1 + r)^{-N} / \pi_0 \).

b) We have
\[
\begin{align*}
\eta_{N-1} \pi_{N-1} + \xi_{N-1} (1 + a) S_{N-2} &= \eta_N \pi_{N-1} + \xi_N (1 + a) S_{N-2} \\
\eta_{N-1} \pi_{N-1} + \xi_{N-1} (1 + b) S_{N-2} &= \eta_N \pi_{N-1} + \xi_N (1 + b) S_{N-2},
\end{align*}
\]
which yields \( \xi_{N-1} = \xi_N = 1 \) and \( \eta_{N-1} = \eta_N = -K(1 + r)^{-N} / \pi_0 \). Similarly, solving the self-financing condition
\[
\begin{align*}
\eta_t \pi_t + \xi_t (1 + a) S_{t-1} &= \eta_{t+1} \pi_t + \xi_{t+1} (1 + a) S_{t-1} \\
\eta_t \pi_t + \xi_t (1 + b) S_{t-1} &= \eta_{t+1} \pi_t + \xi_{t+1} (1 + b) S_{t-1}
\end{align*}
\]
at time \( t \) yields
\[
\xi_t = 1 \quad \text{and} \quad \eta_t = -(1 + r)^{-N} \frac{K}{\pi_0}, \quad t = 1, 2, \ldots, N.
\]

c) We have
\[
\pi_t(C) = V_t = \eta_t \pi_t + \xi_t S_t = S_t - K(1 + r)^{-N} \frac{\pi_t}{\pi_0}
\]
We have
\[ S_t - K(1 + r)^{-(N-t)}. \]

d) For all \( t = 0, 1, \ldots, N \) we have
\[
(1 + r)^{-(N-t)} \mathbb{E}^*[C \mid \mathcal{F}_t] = (1 + r)^{-(N-t)} \mathbb{E}^*[S_N - K \mid \mathcal{F}_t],
\]
\[
= (1 + r)^{-(N-t)} \mathbb{E}^*[S_N \mid \mathcal{F}_t] - (1 + r)^{-(N-t)} \mathbb{E}^*[K \mid \mathcal{F}_t]
\]
\[
= (1 + r)^{-(N-t)} (1 + r)^{N-t} S_t - K(1 + r)^{-(N-t)}
\]
\[
= S_t - K(1 + r)^{-(N-t)}
\]
\[
= V_t = \pi_t(C).
\]

For a future contract expiring at time \( N \) we take \( K = S_0(1 + r)^N \) and the contract is usually quoted at time \( t \) using the forward price
\[
(1 + r)^{N-t}(S_t - K(1 + r)^{N-t}) = (1 + r)^{N-t} S_t - K = (1 + r)^{N-t} S_t - S_0(1 + r)^N,
\]
or simply using \((1 + r)^{N-t} S_t\). Future contracts are “marked to market” at each time step \( t = 1, 2, \ldots, N \) via a positive or negative cash flow exchange \((1 + r)^{N-t} S_t - (1 + r)^{N-t+1} S_{t-1}\) from the seller to the buyer, ensuring that the absolute difference \(|(1 + r)^{N-t} S_t - K|\) has been credited to the buyer’s account if it is positive, or to the seller’s account if it is negative.

### Exercise 3.10

a) We write
\[
V_N = \begin{cases} 
\xi_N S_{N-1} (1 + 1/2) + \eta_N = (S_{N-1} (1 + 1/2))^2 \\
\xi_N S_{N-1} (1 - 1/2) + \eta_N = (S_{N-1} (1 - 1/2))^2 
\end{cases}
\]
which yields
\[
\begin{cases} 
\xi_N = 2 S_{N-1} \\
\eta_N = -3 (S_{N-1})^2 / 4.
\end{cases}
\]

b) We have
\[
\mathbb{E}^*[S_N^2 \mid \mathcal{F}_{N-1}] = p^*(S_{N-1})^2 (1 + 1/2)^2 + (1 - p^*) (S_{N-1})^2 (1 - 1/2)^2
\]
\[
= \frac{1}{2} (S_{N-1})^2 ((1 + 1/2)^2 + (1 - 1/2)^2)
\]
\[
= 5 (S_{N-1})^2 / 4.
\]

c) We have
\[
\xi_{N-1} S_{N-1} + \eta_{N-1} A_0 = \begin{cases} 
\xi_{N-1} S_{N-2} (1 + 1/2) + \eta_{N-1} \\
\xi_{N-1} S_{N-2} (1 - 1/2) + \eta_{N-1}
\end{cases}
\]
$$= V_{N-1}$$  
$$= 5(S_{N-1})^2/4$$  
$$= \begin{cases} 
5(S_{N-2}(1+1/2))^2/4 \\
5(S_{N-2}(1-1/2))^2/4, 
\end{cases}$$

hence

$$\begin{cases} 
\xi_{N-1} = 5S_{N-2}/2 \\
\eta_{N-1} = -15(S_{N-2})^2/16. 
\end{cases}$$

d) We have

$$\xi_{N-1}S_{N-1} + \eta_{N-1}A_0 = 5S_{N-2}S_{N-1}/2 - 15(S_{N-2})^2/16$$  
$$= \begin{cases} 
5(S_{N-2})^2(1+1/2)/2 - 15(S_{N-2})^2/16 \\
5(S_{N-2})^2(1-1/2)/2 - 15(S_{N-2})^2/16 \\
15(S_{N-2})^2/4 - 15(S_{N-2})^2/16 \\
5(S_{N-2})^2 - 15(S_{N-2})^2/16 \\
45(S_{N-2})^2/16 \\
5(S_{N-2})^2/16, 
\end{cases}$$

and on the other hand,

$$\xi_N S_{N-1} + \eta_N A_0 = 2(S_{N-1})^2 - 3(S_{N-1})^2/4$$  
$$= \begin{cases} 
2(S_{N-2})^2(1+1/2)^2 - 3(S_{N-2})^2(1+1/2)^2/4 \\
2(S_{N-2})^2(1-1/2)^2 - 3(S_{N-2})^2(1-1/2)^2/4 \\
45(S_{N-2})^2/16 \\
5(S_{N-2})^2/16. 
\end{cases}$$

Remark: We could also determine \((\xi_{N-1}, \eta_{N-1})\) as in Proposition 3.11, from \((\xi_N, \eta_N)\) and the self-financing condition

$$\xi_{N-1}S_{N-1} + \eta_{N-1}A_0 = \xi_N S_{N-1} + \eta_N A_{N-1},$$

as

$$\xi_{N-1}S_{N-1} + \eta_{N-1}A_0 = \begin{cases} 
\xi_{N-1}S_{N-2}(1+1/2) + \eta_{N-1} \\
\xi_{N-1}S_{N-2}(1-1/2) + \eta_{N-1}. 
\end{cases}$$
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\[ = \xi_N S_{N-1} + \eta_N A_0 \]
\[ = 2(S_{N-1})^2 - 3(S_{N-1})^2/4 \]
\[ = \begin{cases} 
2(S_{N-2})^2(1 + 1/2)^2 - 3(S_{N-2})^2(1 + 1/2)^2/4 \\
2(S_{N-2})^2(1 - 1/2)^2 - 3(S_{N-2})^2(1 - 1/2)^2/4, 
\end{cases} \]

which recovers \( \xi_{N-1} = 5S_{N-2}/2 \) and \( \eta_{N-1} = -15(S_{N-2})^2/16. \)

Exercise 3.11

a) By Theorem 2.15 this model admits a unique risk-neutral probability measure \( \mathbb{P}^* \) because \( a < r < b \), and from (2.11) we have

\[ \mathbb{P}^*(R_t = a) = \frac{b - r}{b - a} = \frac{0.07 - 0.05}{0.07 - 0.02}, \]

and

\[ \mathbb{P}(R_t = b) = \frac{r - a}{b - a} = \frac{0.05 - 0.02}{0.07 - 0.02}, \]

\( t = 1, 2, \ldots, N. \)

b) There are no arbitrage opportunities in this model, due to the existence of a risk-neutral probability measure.

c) This market model is complete because the risk-neutral probability measure is unique.

d) We have

\[ C = (S_N)^2, \]

hence

\[ \tilde{C} = \frac{(S_N)^2}{(1 + r)^N} = h(X_N), \]

with

\[ h(x) = x^2(1 + r)^N. \]

Now we have

\[ \tilde{V}_t = \tilde{v}(t, X_t), \]

where the function \( v(t, x) \) is given from Proposition 3.8 by

\[ \tilde{v}(t, x) = \sum_{k=0}^{N-t} \frac{(N-t)!}{k!(N-t-k)!} \]
\[ \times (p^*)^k (q^*)^{N-t-k} \]
\[ \times (1 + b)^k \]
\[ \times (1 + a)^{N-t-k} \]
\[ \times x \]
\[ \times \left( \frac{1 + b}{1 + r} \right)^k \]
\[ \times \left( \frac{1 + a}{1 + r} \right)^{N-t-k} \]
\[ = x^2(1 + r)^N \sum_{k=0}^{N-t} \frac{(N-t)!}{k!(N-t-k)!}. \]
e) We have

\[
\begin{align*}
\xi^1_t &= \frac{v \left( t, \frac{1+b}{1+r} X_{t-1} \right) - v \left( t, \frac{1+a}{1+r} X_{t-1} \right)}{X_{t-1}(b-a)/(1+r)} \\
&= X_{t-1} \frac{\left( \frac{1+b}{1+r} \right)^2 - \left( \frac{1+a}{1+r} \right)^2}{(b-a)/(1+r)} \frac{(1+r(a+b+2) - ab)^{N-t}}{(1+r)^{N-2t}} \\
&= S_{t-1}(a+b+2) \frac{(1+r(a+b+2) - ab)^{N-t}}{(1+r)^{N-t}}, \quad t = 1, 2, \ldots, N,
\end{align*}
\]

representing the quantity of the risky asset to be present in the portfolio at time \( t \). On the other hand we have

\[
\begin{align*}
\xi^0_t &= \frac{V_t - \xi^1_t X_t}{X^0_t} \\
&= \frac{V_t - \xi^1_t X_t}{\pi_0} \\
&= \frac{X_t(1+r(a+b+2) - ab)^{N-t} X_t - X_{t-1}(a+b+2)/(1+r)}{\pi_0 (1+r)^{N-2t}} \\
&= S_t(1+r(a+b+2) - ab)^{N-t} \frac{S_t - S_{t-1}(a+b+2)}{\pi_0 (1+r)^N}
\end{align*}
\]
\[ -(S_{t-1})^2(1 + r(a + b + 2) - ab)^{N-t} \frac{(1 + a)(1 + b)}{\pi_0(1 + r)^N}, \]

\( t = 1, 2, \ldots, N. \)

d) Let us check that the portfolio is self-financing. We have

\[
\dot{\xi}_{t+1} \cdot S_t = \xi_{t+1}^0 S_t^0 + \xi_{t+1}^1 S_t^1 \\
= -(S_t)^2(1 + r(a + b + 2) - ab)^{N-t} \frac{(1 + a)(1 + b)}{\pi_0(1 + r)^N} S_t^0 \\
+ (S_t)^2(a + b + 2) \frac{(1 + r(a + b + 2) - ab)^{N-t-1}}{(1 + r)^{N-t}} \\
= (S_t)^2 \frac{(1 + r(a + b + 2) - ab)^{N-t-1}}{(1 + r)^{N-t}} \\
\times ((a + b + 2)(1 + r) - (1 + a)(1 + b)) \\
= \frac{1}{(1 + r)^{N-3t}} (X_t)^2 (1 + r(a + b + 2) - ab)^{N-t} \\
= (1 + r)^t V_t \\
= \dot{\xi}_t \cdot S_t, \quad t = 1, 2, \ldots, N.
\]

Exercise 3.12

a) We have

\[ V_t = \xi_t S_t + \eta_t \pi_t \]
\[ = \xi_t (1 + R_t) S_{t-1} + \eta_t (1 + r) \pi_{t-1}. \]

d) We have

\[
\mathbb{E}^* [R_t | \mathcal{F}_{t-1}] = a \mathbb{P}^* (R_t = a | \mathcal{F}_{t-1}) + b \mathbb{P}^* (R_t = b | \mathcal{F}_{t-1}) \\
= \frac{b - r}{b - a} + \frac{r - a}{b - a} \\
= \frac{b r - a}{b - a} - a \frac{r}{b - a} \\
= r.
\]

c) By the result of Question (a) we have

\[
\mathbb{E}^* [V_t | \mathcal{F}_{t-1}] = \mathbb{E}^* [\xi_t (1 + R_t) S_{t-1} | \mathcal{F}_{t-1}] + \mathbb{E}^* [\eta_t (1 + r) \pi_{t-1} | \mathcal{F}_{t-1}] \\
= \xi_t S_{t-1} \mathbb{E}^* [1 + R_t | \mathcal{F}_{t-1}] + (1 + r) \mathbb{E}^* [\eta_t \pi_{t-1} | \mathcal{F}_{t-1}] \\
= (1 + r) \xi_t S_{t-1} + (1 + r) \eta_t \pi_{t-1} \\
= (1 + r) \xi_{t-1} S_{t-1} + (1 + r) \eta_{t-1} \pi_{t-1} \\
= (1 + r) V_{t-1},
\]

where we used the self-financing condition.
d) We have

\[ V_{t-1} = \frac{1}{1 + r} \mathbb{E}^*[V_t \mid \mathcal{F}_{t-1}] \]

\[ = \frac{3}{1 + r} \mathbb{P}^*(R_t = a \mid \mathcal{F}_{t-1}) + \frac{8}{1 + r} \mathbb{P}^*(R_t = b \mid \mathcal{F}_{t-1}) \]

\[ = \frac{1}{1 + 0.15} \left( \frac{0.25 - 0.15}{0.25 - 0.05} + \frac{8(0.15 - 0.05)}{0.25 - 0.05} \right) \]

\[ = \frac{1}{1.15} \left( \frac{3}{2} + \frac{8}{2} \right) \]

\[ = 4.78. \]

Problem 3.13

a) In order to check for arbitrage opportunities we look for a risk-neutral probability measure \( \mathbb{P}^* \) which should satisfy

\[ \mathbb{E}^* \left[ S^{(1)}_{k+1} \mid \mathcal{F}_k \right] = (1 + r)S^{(1)}_k, \quad k = 0, 1, \ldots, N - 1. \]

Rewriting \( \mathbb{E}^* \left[ S^{(1)}_{k+1} \mid \mathcal{F}_k \right] \) as

\[ \mathbb{E}^* \left[ S^{(1)}_{k+1} \mid \mathcal{F}_k \right] = (1 + a)S^{(1)}_k \mathbb{P}^*(R_{k+1} = a \mid \mathcal{F}_k) + S^{(1)}_k \mathbb{P}^*(R_{k+1} = 0 \mid \mathcal{F}_k) \]

\[ + (1 + b)S^{(1)}_k \mathbb{P}^*(R_{k+1} = b \mid \mathcal{F}_k) \]

\[ = (1 + a)S^{(1)}_k \mathbb{P}^*(R_{k+1} = a) + S^{(1)}_k \mathbb{P}^*(R_{k+1} = 0) \]

\[ + (1 + b)S^{(1)}_k \mathbb{P}^*(R_{k+1} = b), \]

\( k = 0, 1, \ldots, N - 1, \) it follows that any risk-neutral probability measure \( \mathbb{P}^* \) should satisfy the equations

\[ \begin{cases} 
(1 + r)S^{(1)}_k = \\
(1 + b)S^{(1)}_k \mathbb{P}^*(R_{k+1} = b) + S^{(1)}_k \mathbb{P}^*(R_{k+1} = 0) + (1 + a)S^{(1)}_k \mathbb{P}^*(R_{k+1} = a), \\
\mathbb{P}^*(R_{k+1} = b) + \mathbb{P}^*(R_{k+1} = 0) + \mathbb{P}^*(R_{k+1} = a) = 1, \\
k = 0, 1, \ldots, N - 1, \ i.e. \end{cases} \]

\[ \begin{cases} 
b \mathbb{P}^*(R_k = b) + a \mathbb{P}^*(R_k = a) = r, \\
\mathbb{P}^*(R_k = b) + \mathbb{P}^*(R_k = a) = 1 - \mathbb{P}^*(R_k = 0), \\
k = 1, 2, \ldots, N, \ \text{with solution} \]
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\[ P^*(R_k = b) = \frac{r - (1 - P^*(R_k = 0))a}{b - a} = \frac{r - (1 - \theta^*)a}{b - a}, \]

and

\[ P^*(R_k = a) = \frac{(1 - P^*(R_k = 0))b - r}{b - a} = \frac{(1 - \theta^*)b - r}{b - a}, \]

\( k = 1, 2, \ldots, N. \) We check that this ternary tree model is without arbitrage if and only if there exists \( \theta^* := P^*(R_k = 0) \in (0, 1) \) such that

\[ (1 - \theta^*)a < r < (1 - \theta^*)b, \quad (A.6) \]

or

\[ 0 < \theta^* < \min \left( \frac{r - a}{-a}, \frac{b - r}{b} \right) = \begin{cases} 1 - \frac{r}{b} & \text{if } r \geq 0, \\ 1 - \frac{r}{a} & \text{if } r \leq 0. \end{cases} \]

Condition (A.6) is necessary in order to have

\[ P^*(R_k = b) > 0 \quad \text{and} \quad P^*(R_k = a) > 0, \]

and it is sufficient because it also implies

\[ P^*(R_k = b) = 1 - \theta^* - P^*(R_k = a) \leq 1 \]

and

\[ P^*(R_k = a) = 1 - \theta^* - P^*(R_k = b) \leq 1. \]

b) We will show that this ternary tree model is without arbitrage if and only if \( a < r < b. \)

(i) Indeed, if the condition \( a < r < b \) is satisfied there always exists \( \theta \in (0, 1) \) such that

\[ a < (1 - \theta)a < r < (1 - \theta)b < b, \]

as can be seen by taking

\[ \theta \in \left( 0, \min \left( \frac{r - a}{-a}, \frac{b - r}{b} \right) \right), \]

hence there exists a risk-neutral probability measure \( P^*_\theta \), and the market model is without arbitrage.

(ii) Conversely, if this ternary tree model is without arbitrage there exists some \( \theta = P^*(R_t = 0) \in (0, 1) \) such that

\[ (1 - \theta)a < r < (1 - \theta)b. \]
c) When \( r \leq a < 0 < b \) the risky asset overperforms the risk-free asset, therefore we can realize arbitrage by borrowing from the risk-free asset to purchase the risky asset. When \( a < 0 < b \leq r \) the risk-free asset overperforms the risky asset, therefore we can realize arbitrage by shortselling the risky asset and save the profit of the short sale on the risk-free asset.

d) Under the absence of arbitrage condition \( a < r < b \), every value of \( \theta \in (0, 1) \) such that

\[
0 < \theta < \min \left( \frac{r-a}{-a}, \frac{b-r}{b} \right)
\]

satisfies

\[
(1-\theta)a < r < (1-\theta)b,
\]

and gives rise to a different risk-neutral probability measure, hence this ternary tree model is not complete.

In particular, every risk-neutral probability measure \( P_\theta^* \) will give rise to a different claim price

\[
\pi_t^\theta(C) = \frac{1}{(1+r)^{N-t}} E_\theta^*[C \mid F_t], \quad t = 0, 1, \ldots, N.
\]

e) We have

\[
\begin{align*}
\text{Var}^* \left[ \frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}} \mid F_k \right] &= E^* \left[ \left( \frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}} \right)^2 \mid F_k \right] - \left( E^* \left[ \frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}} \mid F_k \right] \right)^2 \\
&= E^* \left[ \left( \frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}} \right)^2 \mid F_k \right] - r^2 \\
&= a^2 P_\sigma^*(R_{k+1} = a \mid F_k) + b^2 P_\sigma^*(R_{k+1} = b \mid F_k) - r^2 \\
&= a^2 \left( 1 - P_\sigma^*(R_{k+1} = 0) \right) b - r + b^2 r - (1 - P_\sigma^*(R_{k+1} = 0)) a - r^2 \\
&= ab(\theta - 1) + r(a + b) - r^2 \\
&= \sigma^2,
\end{align*}
\]

\( k = 0, 1, \ldots, N-1 \), hence

\[
P_\sigma^*(R_k = 0) = \theta = 1 + \frac{\sigma^2 + r^2 - r(a + b)}{ab},
\]

and therefore
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\[ P^*_\sigma(R_k = b) = \frac{r - (1 - P^*_\sigma(R_k = 0))a}{b - a} = \frac{\sigma^2 - r(a - r)}{b(b - a)}, \]
and
\[ P^*_\sigma(R_k = a) = \frac{(1 - P^*_\sigma(R_k = 0))b - r}{b - a} = \frac{r(b - r) - \sigma^2}{a(b - a)}, \]

\( k = 1, 2, \ldots, N, \) under the condition

\( \sigma^2 > \max(-r(r - a), r(b - r)), \)

in addition to the condition \( 0 < \theta < 1, \) i.e.

\( r(b - r) + rb < \sigma^2 < (b - r)(r - a). \)

Finally, we find

\[ -r(r - a) < \sigma^2 < (b - r)(r - a), \]

if \( r \in (a, 0], \) and

\[ r(b - r) < \sigma^2 < (b - r)(r - a), \]

if \( r \in [0, b). \)

f) In this case the ternary tree becomes a trinomial recombining tree, and the expression of the risk-neutral probability measure becomes

\[ P^*_\theta(R_k = b) = \frac{r(b + 1) + (1 - \theta)b}{b^2 + 2b}, \]

and

\[ P^*_\theta(R_k = a) = (b + 1)^{(1 - \theta)b - r}, \]

\( k = 1, 2, \ldots, N. \) The market model is without arbitrage if and only if there exists \( \theta := P^*_\theta(R_k = 0) \in (0, 1) \) such that

\[ -(1 - \theta) \frac{b}{b + 1} < r < (1 - \theta)b, \]

or

\[ 0 < \theta < 1 - \frac{r}{b}. \]

g) Using the tower property (18.38) of conditional expectations, we have

\[ f(k, S^{(1)}_k) = \frac{1}{(1 + r)^{N-k}} \mathbb{E}^*[C | \mathcal{F}_k] \]

\[ = \frac{1}{(1 + r)^{N-k}} \mathbb{E}^*[\mathbb{E}^*[C | \mathcal{F}_{k+1}] | \mathcal{F}_k] \]

\[ = \frac{1}{(1 + r)^{N-k}} \mathbb{E}^* [(1 + r)^{N-(k+1)}f(k + 1, S^{(1)}_{k+1}) | \mathcal{F}_k] \]

\[ \text{(Continued)} \]
\[
\begin{align*}
&\frac{1}{1+r} \mathbb{E}^* \left[ f(k+1, S_{k+1}^{(1)}) \mid \mathcal{F}_k \right] \\
= &\frac{1}{1+r} \left( f(k+1, S_k^{(1)}(1+a)) \mathbb{P}_\theta^*(R_k = a) + f(k+1, S_k^{(1)}) \mathbb{P}_\theta^*(R_k = 0) \\
&\quad + f(k+1, S_k^{(1)}(1+b)) \mathbb{P}_\theta^*(R_k = b) \right).
\end{align*}
\]

h) In this case we have \( f(N, x) = (K - x)^+ \).
i) See the attached code. Download and install the Anaconda distribution from https://www.continuum.io/downloads or try it online at https://try.jupyter.org/.
j) Taking \( \theta = 0.5 \) we find the following graph:

![Graph](image.png)

Fig. S.6: Put option pricing.

* Download the modified (trinomial) IPython notebook that can be run here.
† Download the corresponding (binomial) IPython notebook.
Chapter 4

Exercise 4.1

a) We need to check whether the four properties of the definition of Brownian motion are satisfied. Checking Conditions 1-2-3 does not pose any particular problem since the time changes $t \mapsto c + t$, $t \mapsto t/c^2$ and $t \mapsto ct^2$ are deterministic, continuous, and increasing. As for Condition 4, we need to check that the joint distribution of $B_t$ and $B_{t+c}$ is the same as the joint distribution of $B_t$ and $B_{t+c}$ for any $t, c \geq 0$.
tion 4, $B_{c+t} - B_{c+s}$ clearly has a centered Gaussian distribution with variance $c + t - (c - s) = t - s$, and the same property holds for $cB_t/c^2$ since

$$\text{Var}(c(B_t/c^2 - B_{s/c^2})) = c^2\text{Var}(B_t/c^2 - B_{s/c^2}) = c^2(t - s)/c^2 = t - s.$$ 

As a consequence, (i) and (ii) are standard Brownian motions.

Concerning (iii), we note that $c B_t^2$ is a centered Gaussian random variable with variance $c^2 t - s$, hence $(B_t^2/c^2)_{t \in \mathbb{R}^+}$ is not a standard Brownian motion.

Regarding (iv), this process does not have independent increments, hence it cannot be a Brownian motion. For example, by (4.1) we have

$$\mathbb{E}[(B_t + B_{t/2} - (B_s + B_{s/2}))(B_s + B_{s/2})]$$

$$= \mathbb{E}[B_t B_s + B_t B_{s/2} + B_{t/2} B_s + B_{t/2} B_{s/2} - B_s B_t - B_s B_{s/2} - B_{s/2} B_s - B_s B_{s/2}]$$

$$= s + s + s + s - s - s - s - s - s - s = \frac{s}{2},$$

which is not 0, hence the two increments are not independent - otherwise we would have

$$\mathbb{E}[(B_t + B_{t/2} - (B_s + B_{s/2}))(B_s + B_{s/2})]$$

$$= \mathbb{E}[B_t + B_{t/2} - (B_s + B_{s/2})] \mathbb{E}[(B_s + B_{s/2})]$$

$$= 0.$$ 

b) We have

$$\int_{0}^{T} 2dB_t = 2(B_T - B_0) = 2B_T,$$

which has a Gaussian distribution with mean 0 and variance $4T$. On the other hand,

$$\int_{0}^{T} (2 \times 1_{[0,T/2]}(t) + 1_{(T/2,T]}(t))dB_t = 2(B_{T/2} - B_0) + (B_T - B_{T/2})$$

$$= B_T + B_{T/2},$$

which has a Gaussian distribution with mean 0 and variance

$$\text{Var}[B_T + B_{T/2}] = \text{Var}[(B_T - B_{T/2}) + 2B_{T/2}]$$

$$= \text{Var}[B_T - B_{T/2}] + 4 \text{Var}[B_{T/2}]$$

$$= \frac{T}{2} + \frac{4T}{2}.$$
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\[ T = \frac{5T}{2}. \]

Equivalently, using the Itô isometry (4.7), we have

\[
\begin{align*}
\text{Var} \left[ \left( \int_0^T (2 \times \mathbb{1}_{[0,T/2]}(t) + \mathbb{1}_{(T/2,T]}(t)) dB_t \right) \right] &= \mathbb{E} \left[ \left( \int_0^T (2 \times \mathbb{1}_{[0,T/2]}(t) + \mathbb{1}_{(T/2,T]}(t)) dB_t \right) \right] \\
&= \int_0^T (2 \times \mathbb{1}_{[0,T/2]}(t) + \mathbb{1}_{(T/2,T]}(t)) dt \\
&= 4 \int_0^{T/2} dt + \int_T^T dt \\
&= \frac{5T}{2}.
\end{align*}
\]

c) The stochastic integral \( \int_0^{2\pi} \sin(t) dB_t \) has a Gaussian distribution with mean 0 and variance

\[ \int_0^{2\pi} \sin^2(t) dt = \frac{1}{2} \int_0^{2\pi} (1 - \cos(2t)) dt = \pi. \]

d) If \( 0 \leq s \leq t \) we have

\[
\mathbb{E}[B_t B_s] = \mathbb{E}[(B_t - B_s) B_s] + \mathbb{E}[B_s^2] \\
= \mathbb{E}[(B_t - B_s)] \mathbb{E}[B_s] + \mathbb{E}[B_s^2] \\
= 0 + s \\
= s,
\]

and similarly we obtain \( \mathbb{E}[B_t B_s] = t \) when \( 0 \leq t \leq s \), hence in general we have

\[ \mathbb{E}[B_t B_s] = \min(s,t), \quad s, t \in \mathbb{R}_+. \]

e) By the Itô (4.26) we have

\[
\begin{align*}
d(f(t)B_t) &= f(t)dB_t + B_tf(t)dt + df(t) \cdot dB_t \\
&= f(t)dB_t + B_tf'(t)dt + f'(t)dt \cdot dB_t \\
&= f(t)dB_t + B_tf'(t)dt,
\end{align*}
\]

and by integration on both sides we get

\[
\begin{align*}
\int_0^T f(t)dB_t + \int_0^T B_tf'(t)dt &= \int_0^T d(f(t)B_t) \\
&= f(T)B_T - f(0)B_0 \\
&= 0,
\end{align*}
\]
since \( f(T) = 0 \) and \( B_0 = 0 \), hence the conclusion. Note that this result can also be obtained by integration by parts.

Exercise 4.2

a) The probability distribution of \( X_n \) is Gaussian with mean zero and variance

\[
\text{Var}[X_n] = \mathbb{E} \left[ \left( \int_0^{2\pi} \sin(nt) dB_t \right)^2 \right] = \int_0^{2\pi} \sin^2(nt) dt = \frac{1}{2} \int_0^{2\pi} \cos(0) dt - \frac{1}{2} \int_0^{2\pi} \cos(2nt) dt = \pi, \quad n \geq 1.
\]

b) The random variables \((X_n)_{n \geq 1}\) have same Gaussian distribution, and they are pairwise independent because

\[
\mathbb{E}[X_n X_m] = \mathbb{E} \left[ \int_0^{2\pi} \sin(nt) dB_t \int_0^{2\pi} \sin(mt) dB_t \right] = \int_0^{2\pi} \sin(nt) \sin(mt) dt = \frac{1}{2} \int_0^{2\pi} \cos((n-m)t) dt - \frac{1}{2} \int_0^{2\pi} \cos((n+m)t) dt = 0, \quad n, m \geq 1, n \neq m,
\]

and the vector \((X_n, X_m)\) is jointly Gaussian. Note that this condition implies independence only when the random variables have a Gaussian distribution.

Exercise 4.3 We have \( X_t = f(B_t) \) with \( f(x) = \sin^2 x \), \( f'(x) = 2 \sin x \cos x = \sin(2x) \), and \( f''(x) = 2 \cos(2x) \), hence

\[
\begin{align*}
    dX_t &= d\sin^2(B_t) \\
    &= df(B_t) \\
    &= f'(B_t) dB_t + \frac{1}{2} f''(B_t) dt \\
    &= \sin(2B_t) dB_t + \cos(2B_t) dt.
\end{align*}
\]

Exercise 4.4 Taking expectations on both sides of (4.36) shows that

\[
\mathbb{E}[S_T] = c + \mathbb{E} \left[ \int_0^T \zeta_{t,T} dB_t \right] = c,
\]
hence
\[
\begin{align*}
c &= \mathbb{E}[S_T] \\
&= \mathbb{E}[S_0 e^{\mu T + \sigma B_T - \sigma^2 T/2}] \\
&= S_0 e^{\mu T - \sigma^2 T/2} \mathbb{E}[e^{\sigma B_T}] \\
&= S_0 e^{\mu T - \sigma^2 T/2 + \sigma^2 T/2} \\
&= S_0 e^{\mu T},
\end{align*}
\]
where we used the moment generating function
\[\mathbb{E}[e^{\sigma B_T}] = e^{\sigma^2 T/2}\]
of the Gaussian random variable \(B_T \sim \mathcal{N}(0, T)\). On the other hand, the
discounted asset price \(X_t := e^{-rt S_t}\) satisfies \(dX_t = \sigma X_t dB_t\), which shows that
\[X_T = X_0 + \sigma \int_0^T X_t dB_t.\]
Multiplying both sides by \(e^{rT}\) shows that
\[S_T = e^{rT} S_0 + \sigma \int_0^T e^{rT} X_t dB_t = e^{rT} S_0 + \sigma \int_0^T e^{r(T-t)} S_t dB_t,
\]
which recovers the relation \(c = S_0 e^{r T}\), and shows that \(\zeta_{t,T} = \sigma e^{r(T-t)} S_t, t \in [0,T]\).

Exercise 4.5 Taking expectation on both sides of (4.37) shows that \(c = 0\).
Next, applying Itô’s formula to the function \(f(x) = x^3\) shows that
\[
(B_T)^3 = f(B_T) = f(B_0) + \int_0^T f'(B_t) dB_t + \frac{1}{2} \int_0^T f''(B_t) dt
\]
\[= 3 \int_0^T B_t^2 dB_t + 3 \int_0^T B_t dt.
\]
By the integration by parts formula (4.11) applied to \(f(t) = t\), we find
\[
\int_0^T B_t dt = TB_T - \int_0^T tdB_t = \int_0^T (T-t) dB_t,
\]
\[
(B_T)^3 = 3 \int_0^T B_t^2 dB_t + 3 \left( TB_T - \int_0^T tdB_t \right)
\]
\[= 3 \int_0^T (T-t + B_t^2) dB_t,
\]
and we find $\zeta_{t,T} = 3(T-t + B_t^2)$, $t \in [0,T]$. This type of stochastic integral decomposition can be used for option hedging, cf. Section 6.5.

Exercise 4.6 Let $f \in L^2([0,T])$. We have

$$E\left[e^{\int_0^T f(s)dB_s} \mid \mathcal{F}_t\right] = e^{\int_0^t f(s)dB_s} E\left[e^{\int_t^T f(s)dB_s} \mid \mathcal{F}_t\right]$$

$$= e^{\int_0^t f(s)dB_s} E\left[e^{\int_0^T f(s)dB_s}\right]$$

$$= \exp\left(\int_0^t f(s)dB_s + \frac{1}{2} \int_t^T |f(s)|^2 ds\right),$$

$0 \leq t \leq T$, where we used the Gaussian moment generating function $\mathbb{E}[e^{X}] = e^{\sigma^2/2}$ for $X \sim \mathcal{N}(0, \sigma^2)$ and the fact that $\int_t^T f(s)dB_s \sim \mathcal{N}\left(0, \int_t^T f^2(s)ds\right)$ by Proposition 4.6.

Exercise 4.7 We have

$$\mathbb{E}\left[\exp\left(\int_0^t f(s)dB_s - \frac{1}{2} \int_0^t f^2(s)ds\right) \mid \mathcal{F}_u\right]$$

$$= \exp\left(-\frac{1}{2} \int_0^t f^2(s)ds\right) \mathbb{E}\left[\exp\left(\int_0^t f(s)dB_s\right) \mid \mathcal{F}_u\right]$$

$$= \exp\left(-\frac{1}{2} \int_0^t f^2(s)ds\right) \mathbb{E}\left[\exp\left(\int_0^u f(s)dB_s + \int_u^t f(s)dB_s\right) \mid \mathcal{F}_u\right]$$

$$= \exp\left(\int_0^u f(s)dB_s - \frac{1}{2} \int_0^u f^2(s)ds\right) \mathbb{E}\left[\exp\left(\int_u^t f(s)dB_s\right) \mid \mathcal{F}_u\right]$$

$$= \exp\left(\int_0^u f(s)dB_s - \frac{1}{2} \int_0^u f^2(s)ds\right) \mathbb{E}\left[\exp\left(\int_0^t f(s)dB_s\right)\right]$$

$$= \exp\left(\int_0^u f(s)dB_s - \frac{1}{2} \int_0^u f^2(s)ds + \frac{1}{2} \int_u^t f^2(s)ds\right)$$

$$= \exp\left(\int_0^u f(s)dB_s - \frac{1}{2} \int_0^u f^2(s)ds\right), \quad 0 \leq u \leq t.$$

Exercise 4.8

a) We have $S_t = f(X_t)$, $t \in \mathbb{R}_+$, where $f(x) = S_0e^x$ and $(X_t)_{t \in \mathbb{R}_+}$ is the Itô process given by

$$X_t := \int_0^t \sigma_s dB_s + \int_0^t u_s ds, \quad t \in \mathbb{R}_+,$$

or in differential form

$$dX_t := \sigma_t dB_t + u_t dt, \quad t \in \mathbb{R}_+.$$

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hence
\[ dS_t = df(X_t) \]
\[ = f'(X_t) dX_t + \frac{1}{2} f''(X_t) (dX_t)^2 \]
\[ = u_t f'(X_t) dt + \sigma_t f'(X_t) dB_t + \frac{1}{2} \sigma_t^2 f''(X_t) dt \]
\[ = S_0 u_t e^{X_t} dt + S_0 \sigma_t e^{X_t} dB_t + \frac{1}{2} S_0 \sigma_t^2 e^{X_t} dt \]
\[ = u_t S_t dt + \sigma_t S_t dB_t + \frac{1}{2} \sigma_t^2 S_t dt. \]

b) The process \((S_t)_{t \in \mathbb{R}_+}\) satisfies the stochastic differential equation
\[ dS_t = \left( u_t + \frac{1}{2} \sigma_t^2 \right) S_t dt + \sigma_t S_t dB_t. \]

Exercise 4.9

a) We have \(\mathbb{E}[S_t] = 1\) because the expected value of the Itô stochastic integral is zero. Regarding the variance, using the Itô isometry (4.7) we have
\[ \text{Var}[S_t] = \sigma^2 \mathbb{E} \left[ \left( \int_0^t e^{\sigma B_s - \sigma^2 s/2} dB_s \right)^2 \right] \]
\[ = \sigma^2 \mathbb{E} \left[ \int_0^t \left( e^{\sigma B_s - \sigma^2 s/2} \right)^2 ds \right] \]
\[ = \sigma^2 \int_0^t \mathbb{E} \left[ e^{2 \sigma B_s - \sigma^2 s} \right] ds \]
\[ = \sigma^2 \int_0^t e^{-\sigma^2 s} \mathbb{E} \left[ e^{2 \sigma B_s} \right] ds \]
\[ = \sigma^2 \int_0^t e^{-\sigma^2 s} e^{2 \sigma^2 s} ds \]
\[ = \sigma^2 \int_0^t e^{\sigma^2 s} ds \]
\[ = e^{\sigma^2 t} - 1. \]

b) Taking \(f(x) = \log x\), we have
\[ d \log(S_t) = df(S_t) \]
\[ = \sigma f'(S_t) dS_t + \frac{1}{2} \sigma^2 f''(S_t) (dS_t)^2 \]
\[ = \sigma f'(S_t) e^{\sigma B_t - \sigma^2 t/2} dB_t + \frac{1}{2} \sigma^2 f''(S_t) e^{2 \sigma B_t - \sigma^2 t} dt \]
c) We check that when \( S_t = e^{\sigma B_t - \sigma^2 t/2}, t \in \mathbb{R}_+ \), we have

\[
\log S_t = \sigma B_t - \frac{\sigma^2 t}{2}, \quad \text{and} \quad d\log S_t = \sigma dB_t - \frac{\sigma^2}{2} dt.
\]

On the other hand, we also find

\[
\sigma dB_t - \frac{\sigma^2}{2} dt = \frac{\sigma}{S_t} e^{\sigma B_t - \sigma^2 t/2} dB_t - \frac{\sigma^2}{2S_t^2} e^{2\sigma B_t - \sigma^2 t} dt,
\]

showing by (A.7) that the equation

\[
d\log S_t = \frac{\sigma}{S_t} e^{\sigma B_t - \sigma^2 t/2} dB_t - \frac{\sigma^2}{2S_t^2} e^{2\sigma B_t - \sigma^2 t} dt
\]

is satisfied. By uniqueness of solutions, we conclude that \( S_t := e^{\sigma B_t - \sigma^2 t/2} \) solves

\[
S_t = 1 + \sigma \int_0^t e^{\sigma B_s - \sigma^2 s/2} dB_s, \quad t \in \mathbb{R}_+.
\]

Exercise 4.10

a) We have \( f(t) = f(0) e^{rt} \) (interest rate compounding) and

\[
S_t = S_0 e^{\sigma B_t - \sigma^2 t/2 + rt}, \quad t \in \mathbb{R}_+,
\]

(geometric Brownian motion).

b) Those quantities can be directly computed from the expression of \( S_t \) as a function of the \( \mathcal{N}(0, t) \) random variable \( B_t \). Alternatively, taking expectations in the stochastic differential equations \( dS_t = rS_t dt + \sigma S_t dB_t \) that \( u(t) := \mathbb{E}[S_t] \) satisfies the ordinary differential equation \( u'(t) = ru(t) \) with \( u(0) = S_0 \) and solution \( u(t) = \mathbb{E}[S_t] = S_0 e^{rt} \). On the other hand, taking expectations on both sides of

\[
dS_t^2 = 2S_t dB_t + (dS_t)^2 = 2S_t^2 dt + \sigma^2 S_t^2 dt + 2\sigma S_t dB_t,
\]

or

\[
S_t^2 = S_0^2 + 2r \int_0^t S_u^2 du + \sigma^2 \int_0^t S_u^2 du + 2\sigma \int_0^t S_u dB_u,
\]

we find

\[
v(t) = \mathbb{E}[S_t^2]
\]

\[
= S_0^2 + (2r + \sigma^2) \mathbb{E} \left[ \int_0^t S_u^2 du \right] + 2\sigma \mathbb{E} \left[ \int_0^t S_u dB_u \right]
\]
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\[ S_t^2 = S_0^2 + (2r + \sigma^2) \int_0^t \mathbb{E}_u [S_u^2] \, du \]

\[ = S_0^2 + (2r + \sigma^2) \int_0^t v(u) \, du, \]

hence \( v(t) := \mathbb{E} [S_t^2] \) satisfies the ordinary differential equation

\[ v'(t) = (\sigma^2 + 2r)v(t), \]

with \( v(0) = S_0^2 \) and solution

\[ v(t) = \mathbb{E} [S_t^2] = S_0^2 e^{(\sigma^2 + 2r)t}, \]

hence

\[ \text{Var}[S_t] = \mathbb{E} [S_t^2] - (\mathbb{E} [S_t])^2 \]

\[ = v(t) - u^2(t) \]

\[ = S_0^2 e^{(\sigma^2 + 2r)t} - S_0^2 e^{2rt} \]

\[ = S_0^2 e^{2rt}(e^{\sigma^2 t} - 1), \quad t \in \mathbb{R}_+. \]

c) We have

\[ d \log S_t = \frac{1}{S_t} dS_t - \frac{1}{2S_t^2} (dS_t)^2 = r dt + \sigma dB_t - \frac{\sigma^2}{2} dt, \quad t \in \mathbb{R}_+. \]

d) Using the Itô formula (4.25) in two variables we find

\[ df(S_t, Y_t) = \frac{\partial f}{\partial x} (S_t, Y_t) dS_t + \frac{\partial f}{\partial y} (S_t, Y_t) dY_t \]

\[ + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (S_t, Y_t) (dS_t)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial y^2} (S_t, Y_t) (dY_t)^2 + \frac{\partial^2 f}{\partial x \partial y} (S_t, Y_t) dS_t \cdot dY_t \]

\[ = \frac{\partial f}{\partial x} (S_t, Y_t) (rS_t dt + \sigma S_t dB_t) + \frac{\partial f}{\partial y} (S_t, Y_t) (\mu Y_t dt + \eta Y_t dW_t) \]

\[ + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 f}{\partial x^2} (S_t, Y_t) dt + \frac{\eta^2 Y_t^2}{2} \frac{\partial^2 f}{\partial y^2} (S_t, Y_t) dt + \rho \sigma \eta S_t Y_t \frac{\partial^2 f}{\partial x \partial y} (S_t, Y_t) dt. \]

Exercise 4.11

a) We have

\[ F_t = \xi_t S_t + \eta_t A_t = \beta \frac{F_t}{S_t} S_t - (\beta - 1) \frac{F_t}{A_t} A_t, \quad t \in \mathbb{R}_+, \]

with \( \xi_t = \beta F_t / S_t \) and \( \eta_t = -(\beta - 1) F_t / A_t, t \in \mathbb{R}_+. \)

b) We have

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\[ dF_t = \xi_t dS_t + \eta_t dA_t \]
\[ = \beta \frac{F_t}{S_t} dS_t - (\beta - 1) \frac{F_t}{A_t} dA_t \]
\[ = \beta \frac{F_t}{S_t} dS_t - (\beta - 1) rF_t dt \]
\[ = \beta F_t (rdt + \sigma dB_t) - (\beta - 1) rF_t dt \]
\[ = rF_t dt + \beta \sigma F_t dB_t, \quad t \in \mathbb{R}_+. \]

c) We have
\[ F_t = F_0 e^{\beta \sigma B_t + rt - \beta^2 t^2 / 2} \]
\[ = F_0 \left( e^{\sigma B_t + rt - \beta^2 t^2 / 2} \right)^\beta \]
\[ = F_0 \left( e^{\sigma B_t + rt - \beta^2 t^2 / 2 - (\beta - 1) \sigma^2 t^2 / 2 - (1 - 1/\beta) t^2 / 2} \right)^\beta \]
\[ = F_0 \left( e^{\sigma B_t + rt - \beta^2 t^2 / 2} \right)^\beta e^{-(\beta - 1) rt - \beta (\beta - 1) \sigma^2 t^2 / 2} \]
\[ = \left( S_0 e^{\sigma B_t + rt - \beta^2 t^2 / 2} \right)^\beta e^{-(\beta - 1) rt - \beta (\beta - 1) \sigma^2 t^2 / 2} \]
\[ = S_t^\beta e^{-(\beta - 1) rt - \beta (\beta - 1) \sigma^2 t^2 / 2}, \quad t \in \mathbb{R}_+. \]

Exercise 4.12 We have
\[ E \left[ \exp \left( \beta \int_0^T B_t dB_t \right) \right] = E \left[ \exp \left( \beta (B_T^2 - T) / 2 \right) \right] \]
\[ = e^{-\beta T / 2} E \left[ e^{\beta (B_T)^2 / 2} \right] \]
\[ = \frac{e^{-\beta T / 2}}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{\beta x^2 / 2} e^{-x^2 / (2T)} dx \]
\[ = \frac{e^{-\beta T / 2}}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{(\beta - 1/T)x^2 / 2} dx \]
\[ = \frac{e^{-\beta T / 2}}{\sqrt{1 - \beta T}} \int_{-\infty}^{\infty} \frac{e^{-x^2 / (2(1/T - \beta))}}{\sqrt{2\pi / (1/T - \beta)}} dx \]
\[ = \frac{e^{-\beta T / 2}}{\sqrt{1 - \beta T}}, \]
for all \( \beta < 1/T \), where we applied Relation (18.43) to \( \phi(x) = e^{\beta x^2 / 2} \), knowing that \( B_T \simeq \mathcal{N}(0, T) \).

Exercise 4.13
a) Letting \( Y_t = e^{bt} X_t \), we have

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\[ dY_t = d(e^{bt} X_t) \]
\[ = be^{bt} X_t dt + e^{bt} dX_t \]
\[ = be^{bt} X_t dt + e^{bt} (-bX_t dt + \sigma e^{-bt} dB_t) \]
\[ = \sigma dB_t, \]

hence
\[ Y_t = Y_0 + \int_0^t dY_s = Y_0 + \sigma \int_0^t dB_s = Y_0 + \sigma B_t, \]

and
\[ X_t = e^{-bt} Y_t = e^{-bt} Y_0 + \sigma e^{-bt} B_t = e^{-bt} X_0 + \sigma e^{-bt} B_t. \]

b) Letting \( Y_t = e^{bt} X_t \), we have
\[ dY_t = d(e^{bt} X_t) \]
\[ = be^{bt} X_t dt + e^{bt} dX_t \]
\[ = be^{bt} X_t dt + e^{bt} (-bX_t dt + \sigma e^{-at} dB_t) \]
\[ = \sigma e^{(b-a)t} dB_t, \]

hence we can solve for \( Y_t \) by integrating on both sides as
\[ Y_t = Y_0 + \int_0^t dY_s = Y_0 + \sigma \int_0^t e^{(b-a)s} dB_s, \quad t \in \mathbb{R}_+. \]

This yields the solution
\[ X_t = e^{-bt} Y_t = e^{-bt} X_0 + \sigma e^{-bt} \int_0^t e^{(b-a)s} dB_s, \quad t \in \mathbb{R}_+. \]

Comments:

(i) This type of computation appears anywhere *discounting* by the factor \( e^{-bt} \) is involved.

(ii) The stochastic integral \( \int_0^t e^{(b-a)s} dB_s \) cannot be computed in closed form. It is a centered Gaussian random variable with variance
\[ \int_0^t e^{2(b-a)s} ds = \frac{e^{2(b-a)t} - 1}{2(b-a)} \]
if \( b \neq a \), and variance \( t \) if \( a = b \).

Exercise 4.14 Letting \( X_t := f(t) e^{\sigma B_t - \sigma^2 t/2}, t \in \mathbb{R}_+ \), we have
\[ dX_t = e^{\sigma B_t - \sigma^2 t/2} f'(t) dt + f(t) e^{\sigma B_t - \sigma^2 t/2} \]
\[ = e^{\sigma B_t - \sigma^2 t/2} f'(t) dt + f(t) \sigma e^{\sigma B_t - \sigma^2 t/2} dB_t \]
\[ = \frac{f'(t)}{f(t)} X_t dt + \sigma X_t dB_t \]
\[ h(t)X_t dt + \sigma X_t dB_t, \]

hence
\[ \frac{d}{dt} \log f(t) = \frac{f'(t)}{f(t)} = h(t), \]

which shows that
\[ \log f(t) = \log f(0) + \int_0^t h(s) ds, \]

and
\[ X_t = f(t) e^{\sigma B_t - \sigma^2 t / 2}, \]
\[ = f(0) \exp \left( \int_0^t h(s) ds + \sigma B_t - \frac{\sigma^2}{2} t \right), \]
\[ = X_0 \exp \left( \int_0^t h(s) ds + \sigma B_t - \frac{\sigma^2}{2} t \right), \quad t \in \mathbb{R}_+. \]

Exercise 4.15

a) Note that the stochastic integral
\[ \int_0^T \frac{1}{T - s} dB_s \]

is not defined in \( L^2(\Omega) \) as the function \( s \mapsto \frac{1}{T - s} \) is not in \( L^2([0, T]) \) and by the Itô isometry we have
\[ \mathbb{E} \left[ \left( \int_0^T \frac{1}{T - s} dB_s \right)^2 \right] = \int_0^T \frac{1}{(T - s)^2} ds = \left[ \frac{1}{T - s} \right]_0^\infty = +\infty. \]

By (4.38) we have
\[ d \left( \frac{X_t^T}{T - t} \right) = \frac{dX_t^T}{T - t} + \frac{X_t^T}{(T - t)^2} dt = \frac{\sigma dB_t}{T - t}, \]

hence by integration using the initial condition \( X_0 = 0 \) we have
\[ \frac{X_t^T}{T - t} = \sigma \int_0^t \frac{1}{T - s} dB_s, \quad t \in [0, T). \]

b) We have
\[ \mathbb{E}[X_t^T] = \sigma (T - t) \mathbb{E} \left[ \int_0^t \frac{1}{T - s} dB_s \right] = 0, \quad t \in [0, T). \]

c) By the Itô isometry we have

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\[ \text{Var}[X_t^T] = \sigma^2(T-t)^2 \text{Var} \left[ \int_0^t \frac{1}{T-s} dB_s \right] \]
\[ = \sigma^2(T-t)^2 \mathbb{E} \left[ \left( \int_0^t \frac{1}{T-s} dB_s \right)^2 \right] \]
\[ = \sigma^2(T-t)^2 \int_0^t \frac{1}{(T-s)^2} ds \]
\[ = \sigma^2(T-t)^2 \left( \frac{1}{T-t} - \frac{1}{T} \right) \]
\[ = \sigma^2 \left( 1 - \frac{t}{T} \right), \quad t \in [0,T). \]

d) We have
\[ \lim_{t \to 0} \|X_t^T\|_{L^2(\Omega)} = \lim_{t \to 0} \text{Var}[X_t^T] = 0. \]

Exercise 4.16 Exponential Vasicek model (1). Applying the Itô formula to
\[ X_t = e^{rt} = f(r_t) \text{ with } f(x) = e^x, \]
we have
\[ dX_t = d e^{rt} \]
\[ = e^{rt} dr_t + \frac{1}{2} e^{rt} |dr_t|^2 \]
\[ = e^{rt}((a - br_t)dt + \sigma dB_t) + \frac{1}{2} e^{rt}((a - br_t)dt + \sigma dB_t)^2 \]
\[ = e^{rt}((a - br_t)dt + \sigma dB_t) + \frac{\sigma^2}{2} e^{rt} dt \]
\[ = X_t \left( a + \frac{\sigma^2}{2} - b \log(X_t) \right) dt + \sigma X_t dB_t \]
\[ = X_t(\tilde{a} - \tilde{b} f(X_t)) dt + \sigma g(X_t) dB_t, \]
hence
\[ \tilde{a} = a + \frac{\sigma^2}{2} \quad \text{and} \quad \tilde{b} = b \]
the functions \( f(x) \) and \( g(x) \) are given by \( f(x) = \log x \) and \( g(x) = x \). Note that this stochastic differential equation is that of the exponential Vasicek model.

Exercise 4.17 Exponential Vasicek model (2).

a) We have \( Z_t = e^{-at} Z_0 + \sigma \int_0^t e^{-a(t-s)} dB_s \).
b) We have \( Y_t = e^{-at} Y_0 + \frac{\theta}{a} (1 - e^{-at}) + \sigma \int_0^t e^{-a(t-s)} dB_s \).
c) We have \( dX_t = X_t \left( \theta + \frac{\sigma^2}{2} - a \log X_t \right) dt + \sigma X_t dB_t \).
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d) We have \( r_t = \exp \left( e^{-at} \log r_0 + \frac{\theta}{a} (1 - e^{-at}) + \sigma \int_0^t e^{-a(t-s)} dB_s \right) \), with
\( \eta = \theta + \frac{\sigma^2}{2} \).
e) We have
\[
\mathbb{E}[r_t] = r_0 \exp \left( e^{-at} \log r_0 + \frac{\theta}{a} (1 - e^{-at}) + \frac{\sigma^2}{4a} (1 - e^{-2at}) \right).
\]
f) We have \( \lim_{t \to \infty} \mathbb{E}[r_t] = r_0 \exp \left( \frac{\theta}{a} + \frac{\sigma^2}{4a} \right) \).

Exercise 4.18 Cox-Ingersoll-Ross (CIR) model.

a) We have \( r_t = r_0 + \int_0^t (\alpha - \beta r_s) ds + \sigma \int_0^t \sqrt{r_s} dB_s \).
b) Using the fact that the expectation of the stochastic integral with respect to Brownian motion is zero, we get, taking expectations on both sides of the above integral equation: \( u'(t) = \alpha - \beta u(t) \).
c) Apply Itô's formula to
\[
r_t^2 = f \left( r_0 + \int_0^t (\alpha - \beta r_s) ds + \sigma \int_0^t \sqrt{r_s} dB_s \right),
\]
with \( f(x) = x^2 \), to obtain
\[
d(r_t)^2 = r_t (\sigma^2 + 2\alpha - 2\beta r_t) dt + 2\sigma r_t^3 / 2 dB_t. \quad (A.8)
\]
d) Taking again the expectation on both sides of \((A.8)\), we find
\[
\mathbb{E}[r_t^2] = \mathbb{E}[r_0^2] + \int_0^t (\sigma^2 \mathbb{E}[r_t] + 2\alpha \mathbb{E}[r_t] - 2\beta \mathbb{E}[r_t^2]) dt,
\]
and after differentiation with respect to \( t \) this yields
\[
v' = (\sigma^2 + 2\alpha) u(t) - 2\beta v(t), \quad t \in \mathbb{R}_+.
\]

Exercise 4.19

a) We have
\[
S_t = e^{X_t} \quad = e^{X_0} + \int_0^t u_s e^{X_s} dB_s + \int_0^t v_s e^{X_s} ds + \frac{1}{2} \int_0^t u_s^2 e^{X_s} ds \\
= e^{X_0} + \sigma \int_0^t e^{X_s} dB_s + \nu \int_0^t e^{X_s} ds + \frac{\sigma^2}{2} \int_0^t e^{X_s} ds \\
= S_0 + \sigma \int_0^t S_s dB_s + \nu \int_0^t S_s ds + \frac{\sigma^2}{2} \int_0^t S_s ds.
\]
b) Let $r > 0$. The process $(S_t)_{t \in \mathbb{R}_+}$ satisfies the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dB_t$$

when $r = \nu + \sigma^2/2$.

c) We have

$$\text{Var}[X_t] = \text{Var}[\sigma(B_T - B_t)] = \sigma^2 \text{Var}[B_T - B_t] = \sigma^2 (T - t), \quad t \in [0, T].$$

d) Let the process $(S_t)_{t \in \mathbb{R}_+}$ be defined by $S_t = S_0 e^{\sigma B_t + \nu t}$, $t \in \mathbb{R}_+$. Using the time splitting decomposition

$$S_T = S_t S_T^{S_T} = S_t e^{\sigma(B_T - B_t) + \nu \tau},$$

we have

$$P(S_T > K \mid S_t = x) = P(S_t e^{\sigma(B_T - B_t) + \nu (T-t)} > K \mid S_t = x)$$

$$= P(x e^{\sigma(B_T - B_t) + \nu (T-t)} > K)$$

$$= P(e^{\sigma(B_T - B_t)} > K e^{-\nu(T-t)}/x)$$

$$= P\left(\frac{B_T - B_t}{\sqrt{T-t}} > \frac{1}{\sigma \sqrt{T-t}} \log \left(K e^{-\nu(T-t)}/x\right)\right)$$

$$= 1 - \Phi\left(\frac{\log \left(K e^{-\nu(T-t)}/x\right)}{\sigma \sqrt{T-t}}\right)$$

$$= \Phi\left(-\frac{\log \left(K e^{-\nu(T-t)}/x\right)}{\sigma \sqrt{T-t}}\right)$$

$$= \Phi\left(\frac{\log(x/K) + \nu \tau}{\sigma \sqrt{T-t}}\right),$$

where $\tau = T - t$.

Problem 4.20

a) The Itô formula cannot be applied to the function $f(x) := (x - K)^+$ because it is not (twice) differentiable.

b) The function $x \mapsto f_\varepsilon(x)$ can be plotted as follows with $K = 1$.

We note that $f_\varepsilon$ converges uniformly on $\mathbb{R}$ to the function $x \mapsto (x - K)^+$ as we have

$$0 \leq f_\varepsilon(x) - (x - K)^+ \leq \frac{\varepsilon}{4}, \quad x \in \mathbb{R}. \quad (A.9)$$

c) Applying the Itô formula to the function $f_\varepsilon$ we find

$$f_\varepsilon(B_T) = f_\varepsilon(B_0) + \int_0^T f_\varepsilon'(B_t) dB_t + \frac{1}{2} \int_0^T f_\varepsilon''(B_t) dt$$
Fig. S.7: Graph of the function $x \mapsto f_\varepsilon(x)$.

$$f_\varepsilon(B_0) + \int_0^T f'_\varepsilon(B_t) dB_t + \frac{1}{4\varepsilon} \int_0^T \mathbb{1}_{(K-\varepsilon,K+\varepsilon)}(B_t) dt,$$

and to conclude it suffices to note that

$$\ell \left( \{ t \in [0,T] : K - \varepsilon < B_t < K + \varepsilon \} \right) = \int_0^T \mathbb{1}_{(K-\varepsilon,K+\varepsilon)}(B_t) dt.$$

d) The derivative $f'_\varepsilon(x)$ of $f_\varepsilon(x)$ is given by

$$f'_\varepsilon(x) := \begin{cases} 
1 & \text{if } x > K + \varepsilon, \\
\frac{1}{2\varepsilon}(x - K + \varepsilon) & \text{if } K - \varepsilon < x < K + \varepsilon, \\
0 & \text{if } x < K - \varepsilon.
\end{cases}$$

Fig. S.8: Graph of the derivative $x \mapsto f'_\varepsilon(x)$.

Hence we have

$$\|\mathbb{1}_{[K,\infty)}(\cdot) - f'_\varepsilon(\cdot)\|_{L^2(\mathbb{R}_+)}^2 = \int_0^\infty \left( \mathbb{1}_{[K,\infty)}(x) - f'_\varepsilon(x) \right)^2 dx.$$
e) i) We have

\[
\begin{align*}
    &\mathbb{E} \left[ \int_0^T (\mathbb{1}_{[K,\infty)}(B_t) - f'_\varepsilon(B_t))^2 dt \right] = \int_0^T \mathbb{E} \left[ (\mathbb{1}_{[K,\infty)}(B_t) - f'_\varepsilon(B_t))^2 \right] dt \\
    &\leq \int_0^T \int_{-\infty}^\infty (\mathbb{1}_{[K,\infty)}(x) - f'_\varepsilon(x))^2 e^{-x^2/(2t)} dx \frac{1}{\sqrt{2\pi t}} dt \\
    &\leq \int_0^T \int_{K-\varepsilon}^{K+\varepsilon} (\mathbb{1}_{[K,\infty)}(x) - f'_\varepsilon(x))^2 e^{-(K-\varepsilon)^2/(2t)} dx \frac{1}{\sqrt{2\pi t}} dt \\
    &\leq \| \mathbb{1}_{[K,\infty)}(\cdot) - f'_\varepsilon(\cdot) \|_{L^2(\mathbb{R}_+)} \int_0^T e^{-(K-\varepsilon)^2/(2t)} \frac{1}{\sqrt{2\pi t}} dt \\
    &\leq \left( 2\varepsilon + \frac{2\varepsilon}{3} \right) \int_0^T e^{-(K-\varepsilon)^2/(2t)} \frac{1}{\sqrt{2\pi t}} dt,
\end{align*}
\]

where

\[
\int_0^T e^{-(K-\varepsilon)^2/(2t)} \frac{1}{\sqrt{2\pi t}} dt < \int_0^T e^{-K^2/(8t)} \frac{1}{\sqrt{2\pi t}} dt < \infty,
\]

for \( \varepsilon < K/2 \), hence \( \lim_{\varepsilon \to 0} \mathbb{E} \left[ \int_0^T (\mathbb{1}_{[K,\infty)}(B_t) - f'_\varepsilon(B_t))^2 dt \right] = 0 \), and by the Itô isometry

\[
\mathbb{E} \left[ \left( \int_0^\infty (\mathbb{1}_{[K,\infty)}(B_t) - f_\varepsilon(B_t)) dB_t \right)^2 \right] = \mathbb{E} \left[ \int_0^\infty (\mathbb{1}_{[K,\infty)}(B_t) - f_\varepsilon(B_t))^2 dt \right]
\]

we find that

\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ \left( \int_0^\infty \mathbb{1}_{[K,\infty)}(B_t) dB_t - \int_0^\infty f_\varepsilon(B_t) dB_t \right)^2 \right] = 0,
\]

which shows that \( \int_0^\infty f_\varepsilon(B_t) dB_t \) converges to \( \int_0^\infty \mathbb{1}_{[K,\infty)}(B_t) dB_t \) in \( L^2(\Omega) \) as \( \varepsilon \) tends to zero.

ii) By (A.9) we have

\[
\mathbb{E} \left[ ((B_T - K)^+ - f_\varepsilon(B_T))^2 \right] \leq \frac{\varepsilon}{4},
\]
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hence $f_\varepsilon(B_T)$ converges to $(B_T - K)^+$ in $L^2(\Omega)$.

iii) Similarly, $f_\varepsilon(B_0)$ converges to $(B_0 - K)^+$ for any fixed value of $B_0$.

As a consequence of (ei), (eii) and (eiii) above, the equation (4.44) shows that

$$\frac{1}{2\varepsilon} \ell \left( \{ t \in [0, T] : K - \varepsilon < B_t < K + \varepsilon \} \right)$$

admits a limit in $L^2(\Omega)$ as $\varepsilon$ tends to zero, and this limit is denoted by $L^K_{[0,T]}$. The formula (4.45) is known as the Tanaka formula.

Problem 4.21

a) We have

$$0 \leq \mathbb{E}[(X - \varepsilon)^+]$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_\varepsilon^\infty (x - \varepsilon) e^{-x^2/(2\sigma^2)} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_\varepsilon^\infty x e^{-x^2/(2\sigma^2)} dx - \frac{\varepsilon}{\sqrt{2\pi\sigma^2}} \int_\varepsilon^\infty e^{-x^2/(2\sigma^2)} dx$$

$$= -\frac{\sigma^2}{\sqrt{2\pi\sigma^2}} \left[ e^{-x^2/(2\sigma^2)} \right]_\varepsilon^\infty - \varepsilon \mathbb{P}(X \geq \varepsilon)$$

$$= \frac{\sigma^2}{\sqrt{2\pi\sigma^2}} e^{-\varepsilon^2/(2\sigma^2)} - \varepsilon \mathbb{P}(X \geq \varepsilon),$$

which leads to the conclusion.

b) We have

$$\mathbb{P}(X \in dx \mid X + Y = z) = \frac{\mathbb{P}(X \in dx \text{ and } X + Y \in dz)}{\mathbb{P}(X + Y \in dz)}$$

$$= \frac{\mathbb{P}(X \in dx \text{ and } Y \in (dz) - x)}{\mathbb{P}(X + Y \in dz)}$$

$$= \frac{\sqrt{2\pi(\alpha^2 + \beta^2)}}{2\pi\alpha\beta} e^{-x^2/(2\alpha^2) - (z-x)^2/(2\beta^2)} e^{-z^2/(2(\alpha^2 + \beta^2))} dx$$

$$= \frac{\sqrt{1/\beta^2 + 1/\alpha^2}}{\sqrt{2\pi}} e^{-(x^2(1+\beta^2/\alpha^2)+(x^2+z^2-2xz)(1+\alpha^2/\beta^2)-z^2)/(2(\alpha^2+\beta^2))} dx$$

$$= \frac{\sqrt{1/\beta^2 + 1/\alpha^2}}{\sqrt{2\pi}} e^{-(x^2(2+\beta^2/\alpha^2+\alpha^2/\beta^2)+z^2\alpha^2/\beta^2-2xz(1+\alpha^2/\beta^2))/(2(\alpha^2+\beta^2))} dx$$

$$= \frac{\sqrt{1/\beta^2 + 1/\alpha^2}}{\sqrt{2\pi}} e^{-(x(\beta/\alpha+\alpha/\beta)-z\alpha/\beta^2)/(2(\alpha^2+\beta^2))} dx$$

$$= \frac{\sqrt{1/\beta^2 + 1/\alpha^2}}{\sqrt{2\pi}} e^{-(x((\alpha^2+\beta^2)/(\alpha\beta))-z\alpha/\beta^2)/(2(\alpha^2+\beta^2))} dx$$

$$= \frac{\sqrt{1/\beta^2 + 1/\alpha^2}}{\sqrt{2\pi}} e^{-(x-z\alpha^2/(\alpha^2+\beta^2))^2/(2/(1/\alpha^2+1/\beta^2))} dx.$$
c) Given that \( B_u = x \) we decompose
\[
B_v = (B_v - B_{(u+v)/2}) + (B_{(u+v)/2} - B_u) + x,
\]
and apply the result of Question (b) by taking
\[
X = B_{(u+v)/2} - B_u \quad \text{and} \quad Y = B_v - B_{(u+v)/2},
\]
i.e.
\[
\alpha^2 = \beta^2 = \frac{v-u}{2} \quad \text{and} \quad z = y - x,
\]
which shows that the distribution of \( B_{(u+v)/2} = x + X \) given that \( B_u = x \)
and \( B_v = y \) is Gaussian \( \mathcal{N} \left( \frac{x+y}{2}, \frac{v-u}{4} \right) \) with mean
\[
x + \frac{\alpha^2 z}{\alpha^2 + \beta^2} = x + \frac{y-x}{2} = \frac{x+y}{2}
\]
and variance \( \frac{\alpha^2 \beta^2}{\alpha^2 + \beta^2} = \frac{v-u}{4} \).

d) Four linear interpolations are displayed in Figure S.9.

Fig. S.9: Samples of linear interpolations.

e) Clearly, the statement is true for \( n = 0 \) because \( Z_1^{(0)} \) and \( B_1 \) have the
same \( \mathcal{N}(0, 1) \) distribution. Next, assuming that it holds at the rank \( n \), we
note that the terms appearing in the sequence
\[
Z^{(n+1)} = (0, Z_{1/2^{n+1}}, Z_{2/2^{n+1}}, Z_{3/2^{n+1}}, Z_{4/2^{n+1}}, \ldots, Z_{1/2^{n+1}}).
\]
can be written for any \( k = 0, 1, \ldots, 2^n - 1 \) as
\[
\begin{pmatrix}
\cdots, Z_{2k/2^{n+1}}^{(n+1)} \\
\cdots, Z_{k/2^{n+1}}^{(n+1)}
\end{pmatrix}
+ \mathcal{N} \left( 0, \frac{1}{2}, Z_{(2k+2)/2^{n+1}}^{(n+1)} \right).
\]

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http://www.ntu.edu.sg/home/nprivault/index.html
On the other hand, the result of Question (c) shows that given that 
\( B_{2k/2^{n+1}} = x \) and \( B_{(2k+2)/2^{n+1}} = y \), the distribution of \( B_{(2k+1)/2^{n+1}} \) is

\[
\mathcal{N}\left( \frac{B_{2k/2^{n+1}} + B_{(2k+2)/2^{n+1}}}{2}, \frac{(2k+2)/2^{n+1} - (2k+2)/2^{n+1}}{2} \right) = \mathcal{N}\left( \frac{B_{2k/2^{n+1}} + B_{(2k+2)/2^{n+1}}}{2}, \frac{1}{2^{n+2}} \right). \tag{A.11}
\]

Given that \( Z^{(n)} \) and \( B^{(n)} \) have same distribution, we conclude by comparing (A.10) and (A.11) that \( Z^{(n+1)} \) and \( B^{(n+1)} \) also have same distribution.

f) We have

\[
P\left( \max_{t \in [0,1]} |Z_t^{(n+1)} - Z_t^{(n)}| \geq \varepsilon_n \right) = P\left( \sup_{t \in [0,1]} |Z_t^{(n+1)} - Z_t^{(n)}| \geq \varepsilon_n \right) \leq P\left( \bigcup_{k=0,1, \ldots, 2^n-1} \{|Z_t^{(n+1)} - Z_t^{(n)}| \geq \varepsilon_n \} \right) \leq \sum_{k=0}^{2^n-1} P\left( |Z_{1/2^{n+1}}^{(n+1)} - Z_{1/2^{n+1}}^{(n)}| \geq \varepsilon_n \right) = 2^n P\left( |Z_{1/2^{n+1}}^{(n+1)} - Z_{1/2^{n+1}}^{(n)}| \geq \varepsilon_n \right) = 2^n P\left( \left| \frac{Z_{0/2^n}^{(n)} + Z_{1/2^n}^{(n)}}{2} - Z_{1/2^{n+1}}^{(n+1)} \right| \geq \varepsilon_n \right).
\]

g) Since

\[ Z_{1/2^{n+1}}^{(n+1)} = \frac{Z_0^{(n)} + Z_{1/2^n}^{(n)}}{2} + \mathcal{N}(0, 1/2^{n+2}) = Z_{1/2^{n+1}}^{(n)} + \mathcal{N}(0, 1/2^{n+2}), \]

we have

\[
P\left( \sup_{t \in [0,1]} |Z_t^{(n+1)} - Z_t^{(n)}| \geq \varepsilon_n \right) \leq 2^n P\left( |Z_{1/2^{n+1}}^{(n+1)} - Z_{1/2^{n+1}}^{(n)}| \geq \varepsilon_n \right) = 2^n P\left( \left| Z_{1/2^{n+1}}^{(n+1)} - \frac{Z_0^{(n)} + Z_{1/2^n}^{(n)}}{2} \right| \geq \varepsilon_n \right).
\]
where we applied the bound of Question (a) to the Gaussian random variable
\[ Z_{\frac{n+1}{2^{n+1}}} - \frac{Z_0}{2^n} + \frac{Z_{n+1}}{2^n} \simeq \mathcal{N}(0, 1/2^n). \]

h) We have
\[
\sum_{n=0}^{\infty} \mathbb{P} \left( \| Z^{(n+1)} - Z^{(n)} \|_\infty \geq 2^{-n/4} \right) = \sum_{n=0}^{\infty} \mathbb{P} \left( \sup_{t \in [0,1]} | Z_t^{(n+1)} - Z_t^{(n)} | \geq \varepsilon_n \right)
\leq \sum_{n=0}^{\infty} \frac{2^{n/2}}{\varepsilon_n \sqrt{2\pi}} e^{-\varepsilon_n^2/2^n}
= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} 2^{3n/4} e^{-1/2^n + n/4} < \infty,
\]

since
\[
\lim_{n \to \infty} \frac{2^{3(n+1)/4} e^{-2^{1+(n+1)/2}}}{2^{3n/4} e^{-1/2^n + n/4}} = 2^{3/4} \lim_{n \to \infty} e^{-2^{1+n/2}(\sqrt{2}-1)} = 0.
\]
Hence the Borel-Cantelli lemma shows that
\[ \mathbb{P} \left( \| Z^{(n+1)} - Z^{(n)} \|_\infty \geq 2^{-n/4} \text{ for infinitely many } n \right) = 0, \]
therefore we have
\[ \mathbb{P} \left( \| Z^{(n+1)} - Z^{(n)} \|_\infty < 2^{-n/4} \text{ except for finitely many } n \right) = 1. \]

i) The result of Question (h) shows that with probability one we have
\[
\lim_{p,q \to \infty} \| Z^{(p)} - Z^{(q)} \|_\infty = \lim_{p,q \to \infty} \left\| \sum_{n=q}^{p-1} Z^{(n+1)} - Z^{(n)} \right\|_\infty
\leq \lim_{p,q \to \infty} \sum_{n=q}^{p-1} \| Z^{(n+1)} - Z^{(n)} \|_\infty
\leq \lim_{p \to \infty} \sum_{n=q}^{\infty} \| Z^{(n+1)} - Z^{(n)} \|_\infty
= 0,
\]
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hence the sequence \((Z^{(n)})_{n \geq 0}\) is Cauchy in \(C_0([0, 1])\) for the \(\| \cdot \|_\infty\) norm. Since \(C_0([0, 1])\) is a complete space for the \(\| \cdot \|_\infty\) norm, this implies that, with probability one, the sequence \((Z^{(n)})_{n \geq 0}\) admits a limit in \(C_0([0, 1])\).

j) 1. By construction we have \(Z^{(n)}_0 = 0\) for all \(n \in \mathbb{N}\), hence \(Z_0 = \lim_{n \to \infty} Z^{(n)}_0 = 0\), almost surely.

2. The sample trajectories \(t \mapsto Z_t\) are continuous, because the limit \(Z\) belongs to \(C_0([0, 1])\) with probability 1.

3. The result of Question (e) shows that for any fixed \(m \geq 1\), the sequences

\[ Z_{t_1} - Z_{t_0}, Z_{t_2} - Z_{t_1}, \ldots, Z_{t_m} - Z_{t_{m-1}} \]

and

\[ B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}} \]

have same distribution when the \(t_k\)s are dyadic rationals of the form \(t_k = i_n/2^n\), \(k = 0, 1, \ldots, n\). This property extends to any sequence \(t_0, t_1, \ldots, t_m\) of real numbers by approximation of each \(t_k > 0\) by a sequence \((i_n)_{n \in \mathbb{N}}\) such that \(t_k = \lim_{n \to \infty} i_n/2^n\) and taking the limit as \(n\) tends to infinity.

4. By a similar argument as in the above point 3, one can show that for any \(0 \leq s < t\), \(Z_t - Z_s\) has the Gaussian distribution \(\mathcal{N}(0, t - s)\).

Problem 4.22

a) We have

\[
\mathbb{E} \left[ Q_T^{(n)} \right] = \sum_{k=1}^{n} \mathbb{E} \left[ (B_{kT/n} - B_{(k-1)T/n})^2 \right] = \sum_{k=1}^{n} \left( \frac{T}{n} - \frac{(k-1)T}{n} \right) = T, \quad n \geq 1.
\]

b) We have

\[
\mathbb{E} \left[ (Q_T^{(n)})^2 \right] = \mathbb{E} \left[ \sum_{k=1}^{n} (B_{kT/n} - B_{(k-1)T/n})^2 \right] = \mathbb{E} \left[ \sum_{k,l=1}^{n} (B_{kT/n} - B_{(k-1)T/n})^2 (B_{lT/n} - B_{(l-1)T/n})^2 \right]
\]

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\[
\begin{align*}
= \sum_{k=1}^{n} \mathbb{E} \left[ (B_{kT/n} - B_{(k-1)T/n})^4 \right] \\
+ 2 \sum_{1 \leq k < l \leq n} \mathbb{E} \left[ (B_{kT/n} - B_{(k-1)T/n})^2 \right] \mathbb{E} \left[ (B_{lT/n} - B_{(l-1)T/n})^2 \right]
\end{align*}
\]

\[
= 3 \sum_{k=1}^{n} (kT/n - (k-1)T/n)^2 \\
+ 2 \sum_{1 \leq k < l \leq n} (kT/n - (k-1)T/n)(lT/n - (l-1)T/n)
\]

\[
= 3 \frac{T^2}{n} + \frac{n(n-1)T^2}{n^2}
\]

\[
= T^2 + \frac{2T^2}{n}, \quad n \geq 1,
\]

hence

\[
\text{Var}[Q^{(n)}_T] = \mathbb{E}\left[ (Q^{(n)}_T)^2 \right] - \left( \mathbb{E}\left[ Q^{(n)}_T \right] \right)^2 = \frac{2T^2}{n}, \quad n \geq 1.
\]

c) We have

\[
\left\| Q^{(n)}_T - T \right\|^2_{L^2(\Omega)} = \mathbb{E}\left[ (Q^{(n)}_T - \mathbb{E}[Q^{(n)}_T])^2 \right]
\]

\[
= \text{Var} \left[ Q^{(n)}_T \right]
\]

\[
= \frac{n(n+2)T^2}{n^2} - T^2
\]

\[
= \frac{2T^2}{n},
\]

hence

\[
\lim_{n \to \infty} \left\| Q^{(n)}_T - T \right\|^2_{L^2(\Omega)} = \lim_{n \to \infty} \frac{2T^2}{n} = 0,
\]

showing that

\[
\lim_{n \to \infty} Q^{(n)}_T = T
\]

in \( L^2(\Omega) \).

d) We have

\[
\sum_{k=1}^{n} (B_{kT/n} - B_{(k-1)T/n})B_{(k-1)T/n} = \frac{1}{2} \sum_{k=1}^{n} B_{kT/n}^2 - B_{(k-1)T/n}^2
\]

\[
- \frac{1}{2} \sum_{k=1}^{n} (B_{kT/n} - B_{(k-1)T/n})(B_{kT/n} - B_{(k-1)T/n})
\]
\[ \frac{1}{2}((B_T)^2 - (B_0)^2) \]
\[ - \frac{1}{2} \sum_{k=1}^{n} (B_{kT/n} - B_{(k-1)T/n})(B_{kT/n} - B_{(k-1)T/n}) \]
\[ = \frac{1}{2}((B_T)^2 - Q_T^{(n)}) , \]
which converges to \(((B_T)^2 - T)/2\) in \(L^2(\Omega)\) as \(n\) tends to infinity, hence
\[ \int_0^T B_t dB_t = \lim_{n \to \infty} \sum_{k=1}^{n} (B_{kT/n} - B_{(k-1)T/n})B_{(k-1)T/n} \]
\[ = \frac{(B_T)^2 - T}{2} . \]
e) We have
\[ \mathbb{E} \left[ Q_T^{(n)} \right] = \sum_{k=1}^{n} \mathbb{E} \left[ (B_{(k-1/2)T/n} - B_{(k-1)T/n})^2 \right] \]
\[ = \sum_{k=1}^{n} ((k - 1/2)T/n - (k - 1)T/n) \]
\[ = \frac{T}{2} , \quad n \geq 1. \]
Next, we have
\[ \mathbb{E} \left[ (Q_T^{(n)})^2 \right] = \mathbb{E} \left[ \left( \sum_{k=1}^{n} (B_{(k-1/2)T/n} - B_{(k-1)T/n})^2 \right)^2 \right] \]
\[ = \mathbb{E} \left[ \sum_{k,l=1}^{n} (B_{(k-1/2)T/n} - B_{(k-1)T/n})^2(B_{lT/n} - B_{(l-1)T/n})^2 \right] \]
\[ = \sum_{k=1}^{n} \mathbb{E} \left[ (B_{(k-1/2)T/n} - B_{(k-1)T/n})^4 \right] \]
\[ + 2 \sum_{1 \leq k < l \leq n} \mathbb{E} \left[ (B_{(k-1/2)T/n} - B_{(k-1)T/n})^2 \right] \mathbb{E} \left[ (B_{(l-1/2)T/n} - B_{(l-1)T/n})^2 \right] \]
\[ = 3 \sum_{k=1}^{n} ((k - 1/2)T/n - (k - 1)T/n)^2 \]
\[ + 2 \sum_{1 \leq k < l \leq n} ((k - 1/2)T/n - (k - 1)T/n)((l - 1/2)T/n - (l - 1)T/n) \]
\[
\begin{align*}
&= \frac{3 T^2}{4n} + \frac{n(n-1)T^2}{4n^2} \\
&= \frac{n(n+2)T^2}{4n^2}, \quad n \geq 1.
\end{align*}
\]

Finally we find
\[
\|\tilde{Q}^{(n)}_T - T/2\|^2_{L^2(\Omega)} = \mathbb{E} \left[ (\tilde{Q}^{(n)}_T - \mathbb{E}[\tilde{Q}^{(n)}_T])^2 \right] \\
= \text{Var} \left[ \tilde{Q}^{(n)}_T \right] \\
= \frac{n(n+2)T^2}{4n^2} - \frac{T^2}{4} \\
= \frac{T^2}{2n},
\]

hence
\[
\lim_{n \to \infty} \|\tilde{Q}^{(n)}_T - T/2\|^2_{L^2(\Omega)} = \lim_{n \to \infty} \frac{T^2}{2n} = 0,
\]

showing that
\[
\lim_{n \to \infty} \tilde{Q}^{(n)}_T = \frac{T}{2}
\]
in \(L^2(\Omega)\).

f) We have
\[
\sum_{k=1}^{n} (B_{kT}/n - B_{(k-1)T}/n)B_{(k-1/2)T}/n \\
= \sum_{k=1}^{n} (B_{kT}/n - B_{(k-1)T}/n)B_{(k-1/2)T}/n \\
+ \sum_{k=1}^{n} (B_{(k-1)T}/n - B_{(k-1)T}/n)B_{(k-1/2)T}/n \\
= \frac{1}{2} \sum_{k=1}^{n} B_{kT}^2/n - B_{(k-1/2)T}/n \\
- \frac{1}{2} \sum_{k=1}^{n} (B_{kT}/n - B_{(k-1/2)T}/n)(B_{kT}/n - B_{(k-1/2)T}/n) \\
+ \frac{1}{2} \sum_{k=1}^{n} B_{(k-1/2)T}/n - B_{(k-1)T}/n \\
+ \frac{1}{2} \sum_{k=1}^{n} (B_{(k-1)T}/n - B_{(k-1)T}/n)(B_{(k-1)T}/n - B_{(k-1)T}/n)
\]
$$g) \text{ We have}$$

$$= \frac{1}{2} (B_T)^2 - \frac{1}{2} \sum_{k=1}^{n} (B_{kT/n} - B_{(k-1/2)T/n})(B_{kT/n} - B_{(k-1/2)T/n})$$

$$+ \frac{1}{2} \sum_{k=1}^{n} (B_{(k-1/2)T/n} - B_{(k-1)T/n})(B_{(k-1/2)T/n} - B_{(k-1)T/n}),$$

which converges to \(((B_T)^2 - T + T)/2 = (B_T)^2/2\) in \(L^2(\Omega)\) as \(n\) tends to infinity, hence

$$\int_0^T B_t \circ dB_t = \lim_{n \to \infty} \sum_{k=1}^{n} (B_{kT/n} - B_{(k-1)T/n})B_{(k-1/2)T/n} = \frac{(B_T)^2}{2},$$

see Section 2.4 of [Mik98] for further details on the Stratonovich integral.

\(g)\) We have

$$\mathbb{E}\left[ \hat{Q}^{(n)}_T \right] = \sum_{k=1}^{n} \mathbb{E}[(B_{(k-\alpha)T/n} - B_{(k-1)T/n})^2]$$

$$= \sum_{k=1}^{n} ((k - \alpha)T/n - (k - 1)T/n)$$

$$= (1 - \alpha) \frac{T}{2}, \quad n \geq 1.$$

Next, we have

$$\mathbb{E}\left[ (\hat{Q}^{(n)}_T)^2 \right] = \mathbb{E}\left[ \left( \sum_{k=1}^{n} (B_{(k-\alpha)T/n} - B_{(k-1)T/n})^2 \right)^2 \right]$$

$$= \mathbb{E}\left[ \sum_{k,l=1}^{n} (B_{(k-\alpha)T/n} - B_{(k-1)T/n})^2(B_{lT/n} - B_{(l-1)T/n})^2 \right]$$

$$= \sum_{k=1}^{n} \mathbb{E}\left[ (B_{(k-\alpha)T/n} - B_{(k-1)T/n})^4 \right]$$

$$+ 2 \sum_{1 \leq k < l \leq n} \mathbb{E}\left[ (B_{(k-\alpha)T/n} - B_{(k-1)T/n})^2 \right] \mathbb{E}\left[ (B_{(l-\alpha)T/n} - B_{(l-1)T/n})^2 \right]$$

$$= 3 \sum_{k=1}^{n} ((k - \alpha)T/n - (k - 1)T/n)^2$$

$$+ 2 \sum_{1 \leq k < l \leq n} ((k - \alpha)T/n - (k - 1)T/n)((l - \alpha)T/n - (l - 1)T/n)$$

$$= 3(1 - \alpha)^2 \frac{T^2}{n} + (1 - \alpha)^2 \frac{n(n-1)T^2}{n^2}.$$
Finally we find
\[ \| Q_T^{(n)} - (1 - \alpha) T / 2 \|_{L^2(\Omega)}^2 = \mathbb{E} \left[ (Q_T^{(n)} - \mathbb{E}[Q_T^{(n)}])^2 \right] \]
\[ = \text{Var} \left[ Q_T^{(n)} \right] \]
\[ = (1 - \alpha) \frac{n(n + 2)T^2}{n^2} - (1 - \alpha)^2 T^2 \]
\[ = 2(1 - \alpha)^2 T^2 / n, \]
hence
\[ \lim_{n \to \infty} \| Q_T^{(n)} - (1 - \alpha) T \|_{L^2(\Omega)}^2 = (1 - \alpha)^2 \lim_{n \to \infty} \frac{T^2}{n} = 0. \]

Next we have
\[ \sum_{k=1}^{n} (B_{kT/n} - B_{(k-1)T/n})B_{(k-\alpha)T/n} \]
\[ = \sum_{k=1}^{n} (B_{kT/n} - B_{(k-\alpha)T/n})B_{(k-\alpha)T/n} + \sum_{k=1}^{n} (B_{(k-\alpha)T/n} - B_{(k-1)T/n})B_{(k-\alpha)T/n} \]
\[ = \frac{1}{2} \sum_{k=1}^{n} B_{kT/n}^2 - B_{(k-\alpha)T/n}^2 - \frac{1}{2} \sum_{k=1}^{n} (B_{kT/n} - B_{(k-\alpha)T/n})(B_{kT/n} - B_{(k-\alpha)T/n}) \]
\[ + \frac{1}{2} \sum_{k=1}^{n} B_{(k-\alpha)T/n}^2 - B_{(k-1)T/n}^2 \]
\[ + \frac{1}{2} \sum_{k=1}^{n} (B_{(k-\alpha)T/n} - B_{(k-1)T/n})(B_{(k-\alpha)T/n} - B_{(k-1)T/n}) \]
\[ = \frac{1}{2} (B_T)^2 - \frac{1}{2} \sum_{k=1}^{n} (B_{kT/n} - B_{(k-\alpha)T/n})(B_{kT/n} - B_{(k-\alpha)T/n}) \]
\[ + \frac{1}{2} \sum_{k=1}^{n} (B_{(k-\alpha)T/n} - B_{(k-1)T/n})(B_{(k-\alpha)T/n} - B_{(k-1)T/n}) \],
which converges to \(((B_T)^2 - \alpha T + (1 - \alpha) T) / 2 = ((B_T)^2 + (1 - 2\alpha) T) / 2 \)
in \( L^2(\Omega) \) as \( n \) tends to infinity, hence
\[ \int_0^T B_t \circ d^\alpha B_t = \lim_{n \to \infty} \sum_{k=1}^{n} (B_{kT/n} - B_{(k-1)T/n})B_{(k-\alpha)T/n} = \frac{(B_T)^2 + (1 - 2\alpha) T}{2}. \]
In particular we find

$$\int_0^T B_t \circ dB_t = \lim_{n \to \infty} \sum_{k=1}^n (B_{kT/n} - B_{(k-1)T/n}) B_{kT/n} = \frac{(B_T)^2 + T}{2},$$

and we note that

$$\int_0^T B_t \circ dB_t = \frac{1}{2} \left( \int_0^T B_t dB_t + \int_0^T B_t \circ dB_t \right).$$

h) We have

$$\lim_{n \to \infty} \sum_{k=1}^n (k - \alpha) \frac{T}{n} \left( \frac{k}{n} - (k - 1) \frac{T}{n} \right)$$

$$= \lim_{n \to \infty} \frac{T}{n} \sum_{k=1}^n (k - \alpha) \frac{T}{n}$$

$$= \lim_{n \to \infty} \frac{T}{n} \sum_{k=1}^n \frac{T}{n} - \alpha \lim_{n \to \infty} \frac{T}{n} \sum_{k=1}^n \frac{T}{n}$$

$$= T^2 \lim_{n \to \infty} \frac{n(n+1)}{2n^2} - \alpha \lim_{n \to \infty} \frac{T^2}{n}$$

$$= \frac{T^2}{2},$$

which does not depend on $\alpha \in [0, 1] < $ hence the stochastic phenomenon of the previous questions does not occur when approximating the deterministic integral $^T_0 tdt = T^2/2$ by Riemann sums.

In mathematical finance we choose to use the Itô integral (which corresponds to the choice $\alpha = 1$) because it is suitable for the modeling of market returns as

$$\frac{dS_t}{S_t} \sim S_{t+\Delta t} - S_t = \mu \Delta t + \sigma \Delta B_t = \mu \Delta t + \sigma (B_{t+\Delta t} - B_t)$$

or

$$dS_t \sim S_{t+\Delta t} - S_t = \mu S_t \Delta t + \sigma S_t \Delta B_t, = \mu S_t \Delta t + \sigma S_t (B_{t+\Delta t} - B_t),$$

based on the value $S_t$ at the the left endpoint of the discretized time interval $[t, t + \Delta t]$. 

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Chapter 5

Exercise 5.1 By the Itô formula we have
\[ dV_t = dg(t, S_t) \]
\[ = \frac{\partial g}{\partial t}(t, S_t)dt + \beta(\alpha - S_t)\frac{\partial g}{\partial x}(t, S_t)dt + \frac{1}{2} \sigma^2 S_t \frac{\partial^2 g}{\partial x^2}(t, S_t)dt + \sigma \sqrt{S_t} \frac{\partial g}{\partial x}(t, S_t)dB_t. \]  
(A.12)

By respective identification of the terms in \( dB_t \) and \( dt \) in (5.36) and (A.12) we get
\[
\begin{cases}
rg(t, S_t)dt + \beta(\alpha - S_t)\xi_t dt - r\xi_t S_t dt \\
= \frac{\partial g}{\partial t}(t, S_t)dt + \beta(\alpha - S_t)\frac{\partial g}{\partial x}(t, S_t)dt + \frac{1}{2} \sigma^2 S_t \frac{\partial^2 g}{\partial x^2}(t, S_t)dt,
\end{cases}
\]
\[
\sigma \xi_t \sqrt{S_t}dB_t = \sigma \sqrt{S_t} \frac{\partial g}{\partial x}(t, S_t)dB_t,
\]

hence
\[
\begin{cases}
rg(t, S_t) + \beta(\alpha - S_t)\xi_t - r\xi_t S_t = \frac{\partial g}{\partial t}(t, S_t) + \beta(\alpha - S_t)\frac{\partial g}{\partial x}(t, S_t) + \frac{1}{2} \sigma^2 S_t \frac{\partial^2 g}{\partial x^2}(t, S_t),
\end{cases}
\]
\[
\xi_t = \frac{\partial g}{\partial x}(t, S_t),
\]

and
\[
\begin{cases}
rg(t, S_t) + \beta(\alpha - S_t)\xi_t - r\xi_t S_t = \frac{\partial g}{\partial t}(t, S_t) + \frac{1}{2} \sigma^2 S_t \frac{\partial^2 g}{\partial x^2}(t, S_t),
\end{cases}
\]
\[
\xi_t = \frac{\partial g}{\partial x}(t, S_t),
\]

hence the function \( g(t, x) \) satisfies the PDE
\[
rg(t, x) = \frac{\partial g}{\partial t}(t, x) + rx \frac{\partial g}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x \frac{\partial^2 g}{\partial x^2}(t, x), \quad x > 0,
\]

and \( \xi_t \) is given by the partial derivative
\[
\xi_t = \frac{\partial g}{\partial x}(t, S_t), \quad t \in \mathbb{R}_+.
\]

Recall that if \( C \) is a contingent claim of the form \( C = \phi(S_T) \) such that \( (\xi_t, \eta_t)_{t \in [0, T]} \) hedges the claim \( C \), the arbitrage price of the claim \( C \) at time \( t \in [0, T] \) is given by
\[
\pi_t(X) = V_t = e^{-r(T-t)} E^*[\phi(S_T) \mid \mathcal{F}_t], \quad 0 \leq t \leq T,
\]
where \( E^* \) denotes expectation under the risk-neutral measure \( P^* \). Hence, from the noncentral Chi square probability density function

\[
f_{T-t}(x) = \frac{2\beta e^{-r(T-t)}}{\sigma^2(1 - e^{-\beta(T-t)})} \exp \left( - \frac{2\beta (x + r_t e^{-\beta(T-t)})}{\sigma^2(1 - e^{-\beta(T-t)})} \right) \left( \frac{x}{r_t e^{-\beta(T-t)}} \right)^{\alpha\beta/\sigma^2 - 1/2} \times I_{2\alpha\beta/\sigma^2 - 1} \left( \frac{4\beta \sqrt{r_t x e^{-\beta(T-t)}}}{\sigma^2(1 - e^{-\beta(T-t)})} \right),
\]

of \( S_T \) given \( S_t, x > 0 \), we find

\[
g(t, S_t) = e^{-r(T-t)} E^* [\phi(S_T) \mid \mathcal{F}_t] = \frac{2\beta e^{-r(T-t)}}{\sigma^2(1 - e^{-\beta(T-t)})} \int_0^\infty \phi(x) \exp \left( - \frac{2\beta (x + S_t e^{-\beta(T-t)})}{\sigma^2(1 - e^{-\beta(T-t)})} \right) \left( \frac{x}{S_t e^{-\beta(T-t)}} \right)^{\alpha\beta/\sigma^2 - 1/2} \times I_{2\alpha\beta/\sigma^2 - 1} \left( \frac{4\beta \sqrt{S_t x e^{-\beta(T-t)}}}{\sigma^2(1 - e^{-\beta(T-t)})} \right) dx
\]

\( 0 \leq t \leq T \), under the Feller condition \( 2\alpha\beta \geq \sigma^2 \).

Exercise 5.2

a) Let \( V_t := \xi_t S_t + \eta_t A_t \) denote the hedging portfolio value at time \( t \in [0, T] \). Since the dividend yield \( \delta S_t \) per share is continuously reinvested in the portfolio, the portfolio change \( dV_t \) decomposes as

\[
dV_t = \underbrace{\eta_t dA_t + \xi_t dS_t}_{\text{trading profit and loss}} + \underbrace{\delta \xi_t S_t dt}_{\text{dividend payout}}
\]

\[
= r\eta_t A_t dt + \xi_t ((\mu - \delta) S_t dt + \sigma S_t dB_t) + \delta \xi_t S_t dt
\]

\[
= r\eta_t A_t dt + \xi_t (\mu S_t dt + \sigma S_t dB_t)
\]

\[
= rV_t dt + (\mu - r)\xi_t S_t dt + \sigma \xi_t S_t dB_t, \quad t \in \mathbb{R}_+.
\]

b) By Itô’s formula we have

\[
dg(t, S_t) = \frac{\partial g}{\partial t}(t, S_t) dt + (\mu - \delta) S_t \frac{\partial g}{\partial x}(t, S_t) dt
\]

\[
+ \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 g}{\partial x^2}(t, S_t) dt + \sigma S_t \frac{\partial g}{\partial x}(t, S_t) dB_t,
\]

hence by identification of the terms in \( dB_t \) and \( dt \) in the expressions of \( dV_t \) and \( dg(t, S_t) \), we get
and we derive the Black-Scholes PDE with dividend
\[ rg(t, x) = \frac{\partial g}{\partial t}(t, x) + (r - \delta) x \frac{\partial g}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 g}{\partial x^2}(t, x). \]
\[ \text{(A.13)} \]

c) In order to solve (A.13) we note that letting \( g(t, x) := e^{-(T-t)\delta} f(t, x), \) the PDE (A.13) reads
\[ rf(t, x) = \delta f(t, x) + (r - \delta) x \frac{\partial f}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}(t, x), \]

hence \( f(t, x) := e^{(T-t)\delta} g(t, x), \) satisfies the standard Black-Scholes PDE with interest rate \( r - \delta, \) i.e. we have
\[ (r - \delta) f(t, x) = \frac{\partial f}{\partial t}(t, x) + (r - \delta) x \frac{\partial f}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}(t, x), \]

with the same terminal condition \( f(T, x) = g(T, x) = (x - K)^+, \) hence we have
\[ f(t, x) = \text{Bl}(K, x, \sigma, r - \delta, T - t) = x \Phi(d_+^\delta(T - t)) - K e^{-(r - \delta)(T - t)} \Phi(d_-^\delta(T - t)), \]

where
\[ d_\pm^\delta(T - t) := \log(x/K) + (r - \delta \pm \sigma^2/2)(T - t) \over \sigma \sqrt{T - t}. \]

Consequently, the pricing function of the European call option with dividend rate \( \delta \) is
\[ g(t, x) = e^{-(T-t)\delta} f(t, x) = e^{-(T-t)\delta} \text{Bl}(K, x, \sigma, r - \delta, T - t) = x e^{-(T-t)\delta} \Phi(d_+^\delta(T - t)) - K e^{-(T-t)r} \Phi(d_-^\delta(T - t)), \quad 0 \leq t \leq T. \]

Exercise 5.3

a) Substituting \( g(x, t) = x^2 f(t) \) in (5.37), we find \( f'(t) = -(r + \sigma^2) f(t), \) hence
\[ f(t) = f(0) e^{-(r + \sigma^2)t} = f(T) e^{(r + \sigma^2)(T - t)}, \]

hence \( g(x, t) = f(T) x^2 e^{(r + \sigma^2)(T - t)} = x^2 e^{(r + \sigma^2)(T - t)} \) due to the terminal condition \( g(x, T) = x^2. \)

b) We have \( \xi_t = \frac{\partial g}{\partial x} g(S_t, t) = 2 S_t e^{(r + \sigma^2)(T - t)}, \) and
\[ \eta_t = \frac{1}{A_t} (g(S_t, t) - \xi_t S_t) \]
\[ = \frac{1}{A_0 e^{rt}} \left( S_t^2 e^{(r+\sigma^2)(T-t)} - 2 S_t^2 e^{(r+\sigma^2)(T-t)} \right) \]
\[ = -\frac{S_t^2}{A_0} e^{(T-2t)r+\sigma^2(T-t)}, \quad t \in [0, T]. \]

Exercise 5.4

a) We have, counting approximately 46 days to maturity,
\[ d_-(T-t) = \frac{(r - \sigma^2/2)(T-t) + \log(S_t/K)}{\sigma \sqrt{T-t}} \]
\[ = \frac{(0.04377 - (0.9)^2/2)(46/365) + \log(17.2/36.08)}{0.9 \sqrt{46/365}} \]
\[ = -2.461179058, \]

and
\[ d_+(T-t) = d_-(T-t) + 0.9 \sqrt{46/365} = -2.14167602. \]

From the standard Gaussian cumulative distribution table we get
\[ \Phi(d_+(T-t)) = \Phi(-2.14) = 0.0161098 \]
and
\[ \Phi(d_-(T-t)) = \Phi(-2.46) = 0.00692406, \]

hence
\[ f(t, S_t) = S_t \Phi(d_+(T-t)) - K e^{-(T-t)r} \Phi(d_-(T-t)) \]
\[ = 17.2 \times 0.0161098 - 36.08 \times e^{-0.04377 \times 46/365} \times 0.00692406 \]
\[ = \text{HK} 0.028642744. \]

For comparison, running the corresponding Black-Scholes R script yields
\[ \text{BSCall}(17.2, 36.08, 0.04377, 46/365, 0.9) = 0.02864235. \]

b) We have
\[ \eta_t = \frac{\partial f}{\partial x}(t, S_t) = \Phi(d_+(T-t)) = \Phi(-2.14) = 0.0161098, \quad (A.14) \]

hence one should only hold a fractional quantity equal to 16.10 units in the risky asset in order to hedge 1000 such call options when \( \sigma = 0.90 \).

c) From the curve it turns out that when \( f(t, S_t) = 10 \times 0.023 = \text{HK} 0.23 \), the volatility \( \sigma \) is approximately equal to \( \sigma = 122\% \).
This approximate value of implied volatility can be found under the column “Implied Volatility (IV.)” on this set of market data from the Hong Kong Stock Exchange:

**Updated: 6 November 2008**

### Basic Data

<table>
<thead>
<tr>
<th>DW Code</th>
<th>Issuer</th>
<th>UL</th>
<th>Call /Put</th>
<th>Type</th>
<th>DW Listing</th>
<th>Maturity</th>
<th>Strike</th>
<th>Entitlement Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>▼ ▽ 01897 FB 00066</td>
<td>Call</td>
<td>Standard</td>
<td>18-12-2007</td>
<td>23-12-2008</td>
<td>36.08</td>
<td>10</td>
<td></td>
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### Market Data

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<th>Total Issue Size</th>
<th>O/S (%)</th>
<th>Delta (%)</th>
<th>IV (%)</th>
<th>Day High</th>
<th>Day Low</th>
<th>Closing Price #</th>
<th>T/O (%000)</th>
<th>UL Price ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>△ △ △ △ △ △ △</td>
<td>138,000,000</td>
<td>16.43</td>
<td>0.780</td>
<td>125.375</td>
<td>0.000</td>
<td>0.000</td>
<td>0.023</td>
<td>0</td>
</tr>
</tbody>
</table>

Fig. S.10: Market data for the warrant #01897 on the MTR Corporation.

**Remark:** a typical value for the volatility in standard market conditions would be around 20%. The observed volatility value $\sigma = 1.22$ per year is actually quite high.

**Exercise 5.5**

a) We find $h(x) = x - K$.

b) Letting $g(t, x)$, the PDE rewrites as

$$(x - \alpha(t))r = -\alpha'(t) + rx,$$

hence $\alpha(t) = \alpha(0) e^{rt}$ and $g(t, x) = x - \alpha(0) e^{rt}$. The final condition

$$g(T, x) = h(x) = x - K$$

yields $\alpha(0) = K e^{-rT}$ and $g(t, x) = x - K e^{-(T-t)r}$.

c) We have

$$\xi_t = \frac{\partial g}{\partial x}(t, S_t) = 1,$$

hence

$$\eta_t = \frac{V_t - \xi_t S_t}{A_t} = \frac{g(t, S_t) - S_t}{A_t} = \frac{S_t - K e^{-(T-t)r} - S_t}{A_t} = -K e^{-rT}.$$

Note that we could also have directly used the identification
\[ V_t = g(S_t, t) = S_t - K e^{-(T-t)r} = S_t - K e^{-rT}A_t = \xi_t S_t + \eta_t A_t, \]

which immediately yields \( \xi_t = 1 \) and \( \eta_t = -K e^{-rT} \).

d) It suffices to take \( K = 0 \), which shows that \( g(t, x) = x, \xi_t = 1 \) and \( \eta_t = 0 \).

Exercise 5.6

a) We develop two approaches.

(i) By financial intuition. We need to replicate a fixed amount of $1 at maturity \( T \), without risk. For this there is no need to invest in the stock. Simply invest \( g(t, S_t) := e^{-(T-t)r} \) at time \( t \in [0, T] \) and at maturity \( T \) you will have \( g(T, S_T) = e^{(T-t)r}g(t, S_t) = 1 \).

(ii) By analysis and the Black-Scholes PDE. Given the hint, we try plugging a solution of the form \( g(t, x) = f(t) \), not depending on the variable \( x \), into the Black-Scholes PDE (5.38). Given that here we have

\[
\frac{\partial g}{\partial x}(t, x) = 0, \quad \frac{\partial^2 g}{\partial x^2}(t, x) = 0, \quad \text{and} \quad \frac{\partial g}{\partial t}(t, x) = f'(t),
\]

we find that the Black-Scholes PDE reduces to \( rf(t) = f'(t) \) with the terminal condition \( f(T) = g(T, x) = 1 \). This equation has for solution \( f(t) = e^{-(T-t)r} \) and this is also the unique solution \( g(t, x) = f(t) = e^{-(T-t)r} \) of the Black-Scholes PDE (5.38) with terminal condition \( g(T, x) = 1 \).

b) We develop two approaches.

(i) By financial intuition. Since the terminal payoff $1 is risk-free we do not need to invest in the risky asset, hence we should keep \( \xi_t = 0 \). Our portfolio value at time \( t \) becomes

\[ V_t = g(t, S_t) = e^{-(T-t)r} = \xi_t S_t + \eta_t A_t = \eta_t A_t \]

with \( A_t = e^{rt} \), so that we find \( \eta_t = e^{-rT} \), \( t \in [0, T] \). This portfolio strategy remains constant over time, hence it is clearly self-financing.

(ii) By analysis. The Black-Scholes theory of Proposition 5.12 tells us that

\[ \xi_t = \frac{\partial g}{\partial x}(t, x) = 0, \]

and

\[ \eta_t = \frac{V_t - \xi_t S_t}{A_t} = \frac{V_t}{A_t} = \frac{e^{-(T-t)r}}{e^{rt}} = e^{-rT}. \]

Exercise 5.7 Log-contracts.

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a) Substituting the function \( g(x, t) := f(t) + \log x \) in the PDE (5.37) we have
\[
0 = f'(t) + r - \frac{\sigma^2}{2},
\]
hence
\[
f(t) = f(0) - \left( r - \frac{\sigma^2}{2} \right) t,
\]
with \( f(0) = \left( r - \frac{\sigma^2}{2} \right) T \) in order to match the terminal condition \( g(x, T) := \log x \), hence we have
\[
g(x, t) = \left( r - \frac{\sigma^2}{2} \right) (T - t) + \log x, \quad x > 0.
\]

b) Substituting the function
\[
h(x, t) := u(t) g(x, t) = u(t) \left( \left( r - \frac{\sigma^2}{2} \right) (T - t) + \log x \right)
\]
in the PDE (5.39), we find \( u'(t) = ru(t) \), hence \( u(t) = u(0) e^{rt} = e^{-(T-t)r} \), with \( u(T) = 1 \), and we conclude to
\[
h(x, t) = u(t) g(x, t) = e^{-(T-t)r} \left( \left( r - \frac{\sigma^2}{2} \right) (T - t) + \log x \right),
\]
\( x > 0, t \in [0, T] \).

c) We have
\[
\xi_t = \frac{\partial h(t, S_t)}{\partial x} = \frac{e^{-(T-t)r}}{S_t}, \quad 0 \leq t \leq T,
\]
and
\[
\eta_t = \frac{1}{A_t} (h(t, S_t) - \xi_t S_t)
\]
\[
= \frac{e^{-rT}}{A_0} \left( \left( r - \frac{\sigma^2}{2} \right) (T - t) + \log x - 1 \right),
\]
\( 0 \leq t \leq T \).

Exercise 5.8 Binary options.

a) From Proposition 5.12, the function \( C_d(t, x) \) solves the Black-Scholes PDE
\[
\begin{align*}
\frac{rC(t, x)}{\partial t} &= \frac{\partial C}{\partial t}(t, x) + r x \frac{\partial C}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 C}{\partial x^2}(t, x), \\
C(T, x) &= \mathbb{1}_{[K, \infty)}(x).
\end{align*}
\]
b) We check by direct differentiation that the Black-Scholes PDE is satisfied by the function \( C(t, x) \), together with the terminal condition \( C(T, x) = \mathbb{1}_{[K, \infty)}(x) \) as \( t \) tends to \( T \).

Exercise 5.9

a) By (4.34) we have

\[
S_t = S_0 e^{\alpha t} + \sigma \int_0^t e^{(t-s)\alpha} dB_s.
\]

b) By the self-financing condition (5.8) we have

\[
dV_t = \eta_t dA_t + \xi_t dB_t
\]

\[
= r \eta_t A_t dt + \alpha \xi_t S_t dt + \sigma \xi_t dB_t
\]

\[
= r V_t dt + (\alpha - r) \xi_t S_t dt + \sigma \xi_t dB_t, \tag{A.15}
\]

\( t \in \mathbb{R}_+ \). Rewriting (5.40) under the form of an Itô process

\[
S_t = S_0 + \int_0^t v_s ds + \int_0^t u_s dB_s, \quad t \in \mathbb{R}_+,
\]

with

\[
u_t = \sigma, \quad \text{and} \quad v_t = \alpha S_t, \quad t \in \mathbb{R}_+,
\]

the application of Itô’s formula Theorem 4.11 to \( V_t = C(t, S_t) \) shows that

\[
dC(t, S_t) = v_t \frac{\partial C}{\partial x}(t, S_t) dt + u_t \frac{\partial C}{\partial x}(t, S_t) dB_t
\]

\[
+ \frac{\partial C}{\partial t}(t, S_t) dt + \frac{1}{2} |u_t|^2 \frac{\partial^2 C}{\partial x^2}(t, S_t) dt
\]

\[
= \frac{\partial C}{\partial t}(t, S_t) dt + \alpha S_t \frac{\partial C}{\partial x}(t, S_t) dt + \frac{1}{2} \sigma^2 \frac{\partial^2 C}{\partial x^2}(t, S_t) dt + \sigma \frac{\partial C}{\partial x}(t, S_t) dB_t. \tag{A.16}
\]

Identifying the terms in \( dB_t \) and \( dt \) in (A.15) and (A.16) above, we get

\[
\left\{
\begin{array}{l}
   r C(t, S_t) = \frac{\partial C}{\partial t}(t, S_t) + r S_t \frac{\partial C}{\partial x}(t, S_t) + \frac{1}{2} \sigma^2 \frac{\partial^2 C}{\partial x^2}(t, S_t), \\
   \xi_t = \frac{\partial C}{\partial x}(t, S_t),
\end{array}
\right.
\]

hence the function \( C(t, x) \) satisfies the usual Black-Scholes PDE

\[
r C(t, x) = \frac{\partial C}{\partial t}(t, x) + r x \frac{\partial C}{\partial x}(t, x) + \frac{1}{2} \sigma^2 \frac{\partial^2 C}{\partial x^2}(t, x), \quad x > 0, \quad t \in [0, T], \tag{A.17}
\]

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http://www.ntu.edu.sg/home/nprivault/index.html
with the terminal condition $C(T, x) = e^x$, $x \in \mathbb{R}$.

c) By substituting (5.41) in the Black-Scholes PDE (A.17) we find the ordinary differential equation

$$xh'(t) + \frac{\sigma^2}{2r} h'(t)h(t) + rxh(t) + \frac{\sigma^2}{2} (h(t))^2 = 0, \quad x > 0, \quad t \in [0, T],$$

which is a Riccati equation with solution $h(t) = e^{(T-t)r}, t \in [0, T]$, which yields

$$C(t, x) = \exp \left( -(T - t)r + x e^{(T-t)r} + \frac{\sigma^2}{4r} (e^{2(T-t)r} - 1) \right).$$

d) We have

$$\xi_t = \frac{\partial C}{\partial x} (t, S_t) = \exp \left( S_t e^{(T-t)r} + \frac{\sigma^2}{4r} (e^{2(T-t)r} - 1) \right).$$

Exercise 5.10

a) Noting that $\varphi(x) = \Phi'(x) = (2\pi)^{-1/2} e^{-x^2/2}$, we have the

$$\frac{\partial h}{\partial d} (S, d) = S \varphi(d + \sigma \sqrt{T}) - K e^{-rt} \varphi(d)$$

$$= \frac{S}{\sqrt{2\pi}} e^{-\left(d + \sigma \sqrt{T}\right)^2/2} - \frac{K}{\sqrt{2\pi}} e^{-rT} e^{-d^2/2}$$

$$= \frac{S}{\sqrt{2\pi}} e^{-d^2/2 - \sigma \sqrt{T}d - \sigma^2 T/2} - \frac{K}{\sqrt{2\pi}} e^{-rT} e^{-d^2/2},$$

hence the vanishing of $\frac{\partial h}{\partial d} (S, d_*(S))$ at $d = d_*(S)$ yields

$$\frac{S}{\sqrt{2\pi}} e^{-d_*(S)^2/2 - \sigma \sqrt{T}d_*(S) - \sigma^2 T/2} - \frac{K}{\sqrt{2\pi}} e^{-rT} e^{-d_*(S)^2/2} = 0,$$

i.e. $d_*(S) = \frac{\log(S/K) + rT - \sigma^2 T/2}{\sigma \sqrt{T}}$. We can also check that

$$\frac{\partial^2 h}{\partial d^2} (S, d_*(S)) = \frac{\partial}{\partial d} \left( \frac{S}{\sqrt{2\pi}} e^{-\left(d_*(S) + \sigma \sqrt{T}\right)^2/2} - \frac{K}{\sqrt{2\pi}} e^{-rT} e^{-d_*(S)^2/2} \right)$$

$$= -(d_*(S) + \sigma \sqrt{T}) \frac{S}{\sqrt{2\pi}} e^{-\left(d_*(S) + \sigma \sqrt{T}\right)^2/2} + \frac{K}{\sqrt{2\pi}} d_*(S) e^{-rT} e^{-d_*(S)^2/2}$$

$$= -(d_*(S) + \sigma \sqrt{T}) \frac{K}{\sqrt{2\pi}} e^{-rT} e^{-d_*(S)^2/2} + \frac{K}{\sqrt{2\pi}} d_*(S) e^{-rT} e^{-d_*(S)^2/2}$$

$$= -\sigma \sqrt{T} \frac{K}{\sqrt{2\pi}} e^{-rT} e^{-d_*(S)^2/2} < 0,$$
hence the function \( d \mapsto h(S, d) := S \Phi(d + \sigma \sqrt{T}) - K e^{-rT} \Phi(d) \) admits a maximum at \( d = d_*(S) \), and

\[
h(S, d_*(S)) = S \Phi(d_*(S) + \sigma \sqrt{T}) - K e^{-rT} \Phi(d_*(S))
\]

\[
= S \Phi \left( \frac{\log(S/K) + (r + \sigma^2/2)T}{\sigma \sqrt{T}} \right) - K e^{-rT} \Phi \left( \frac{\log(S/K) + (r - \sigma^2/2)T}{\sigma \sqrt{T}} \right)
\]

is the Black-Scholes call option price.

b) Since \( \frac{\partial h}{\partial d}(S, d_*(S)) = 0 \), we find

\[
\Delta = \frac{d}{ds} h(S, d_*(S)) = \frac{\partial h}{\partial S}(S, d_*(S)) + d'_*(S) \frac{\partial h}{\partial d}(S, d_*(S))
\]

\[
= \Phi(d_*(S) + \sigma \sqrt{T}) = \Phi \left( \frac{\log(S/K) + rT + \sigma^2T/2}{\sigma \sqrt{T}} \right).
\]

Exercise 5.11

a) Given that

\[
p^* = \frac{b_N - a_N}{b_N - a_N} = \frac{1 - e^{-\sigma \sqrt{T/N}}}{e^{\sigma \sqrt{T/N}} - e^{-\sigma \sqrt{T/N}}}
\]

and

\[
q^* = \frac{b_N - r_N}{b_N - a_N} = \frac{e^{\sigma \sqrt{T/N}} - 1}{e^{\sigma \sqrt{T/N}} - e^{-\sigma \sqrt{T/N}}},
\]

Relation (3.13) reads

\[
\tilde{v}(t, x) = \frac{e^{\sigma \sqrt{T/N}} - 1}{e^{\sigma \sqrt{T/N}} - e^{-\sigma \sqrt{T/N}}} \tilde{v} \left( t + T/N, x(1 + rT/N) e^{-\sigma \sqrt{T/N}} \right)
\]

\[
+ \frac{1 - e^{-\sigma \sqrt{T/N}}}{e^{\sigma \sqrt{T/N}} - e^{-\sigma \sqrt{T/N}}} \tilde{v} \left( t + T/N, x(1 + rT/N) e^{\sigma \sqrt{T/N}} \right).
\]

From the equivalence

\[
\frac{e^{\sigma \sqrt{T/N}} - 1}{e^{\sigma \sqrt{T/N}} - e^{-\sigma \sqrt{T/N}}} \approx \frac{1 - e^{-\sigma \sqrt{T/N}}}{e^{\sigma \sqrt{T/N}} - e^{-\sigma \sqrt{T/N}}} \approx \frac{1}{2}
\]

as \( N \) tends to infinity this yields, after letting \( \Delta T := T/N \) and applying Taylor’s formula at the second order,

\[
0 = \frac{1}{2} \left( \tilde{v} \left( t + \Delta T, x(1 + r\Delta T - \sigma \sqrt{\Delta T}) \right) - \tilde{v}(t, x) \right)
\]

\[
+ \frac{1}{2} \left( \tilde{v} \left( t + \Delta T, x(1 + r\Delta T + \sigma \sqrt{\Delta T}) \right) - \tilde{v}(t, x) \right) + o(\Delta T)
\]

\[
= \frac{1}{2} \left( \Delta T \frac{\partial \tilde{v}}{\partial t}(t, x) + x(\Delta T - \sigma \sqrt{\Delta T}) \frac{\partial \tilde{v}}{\partial x}(t, x) \right)
\]

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b) Similarly, we have

\[ + \frac{x^2}{2} (r\Delta T - \sigma\sqrt{\Delta T})^2 \frac{\partial^2 \tilde{v}}{\partial x^2} (t, x) + o(\Delta T) \]

\[ + \frac{1}{2} \left( \Delta T \frac{\partial \tilde{v}}{\partial t} (t, x) + x(r\Delta T + \sigma\sqrt{\Delta T}) \frac{\partial \tilde{v}}{\partial x} (t, x) \right) + \frac{x^2}{2} (r\Delta T + \sigma\sqrt{\Delta T})^2 \frac{\partial^2 \tilde{v}}{\partial x^2} (t, x) + o(\Delta T) \]

\[ = \Delta T \frac{\partial \tilde{v}}{\partial t} (t, x) + rx\Delta T \frac{\partial \tilde{v}}{\partial x} (t, x) + \frac{x^2}{2} (\sigma\sqrt{\Delta T})^2 \frac{\partial^2 \tilde{v}}{\partial x^2} (t, x) + o(\Delta T), \]

which shows that

\[ \frac{\partial \tilde{v}}{\partial t} (t, x) + rx \frac{\partial \tilde{v}}{\partial x} (t, x) + x^2 \sigma^2 \frac{\partial^2 \tilde{v}}{\partial x^2} (t, x) = -\frac{o(\Delta T)}{\Delta T}, \]

hence as \( N \) tends to infinity (or as \( \Delta T \) tends to 0) we find*

\[ 0 = \frac{\partial \tilde{v}}{\partial t} (t, x) + rx \frac{\partial \tilde{v}}{\partial x} (t, x) + \frac{\sigma^2}{2} x^2 \frac{\partial^2 \tilde{v}}{\partial x^2} (t, x), \]

which shows that the function \( v(t, x) := e^{(T-t)r}\tilde{v}(t, x) \) solves the classical Black-Scholes PDE

\[ rv(t, x) = \frac{\partial v}{\partial t} (t, x) + rx \frac{\partial v}{\partial x} (t, x) + \frac{\sigma^2}{2} x^2 \frac{\partial^2 v}{\partial x^2} (t, x). \]

b) Similarly, we have

\[ \xi_t^{(1)} (x) = \frac{v(t, (1+bN)x) - v(t, (1+aN)x)}{x(bN-aN)} \]

\[ = \frac{v \left( t, (1 + r/N)e^{\sigma\sqrt{T/N}} x \right) - v \left( t, (1 + r/N)e^{-\sigma\sqrt{T/N}} x \right)}{x(1 + r/N)(e^{\sigma\sqrt{T/N}} - e^{-\sigma\sqrt{T/N}})} \]

\[ \rightarrow \frac{\partial v}{\partial x} (t, x), \]

as \( N \) tends to infinity.

Problem 5.12

a) When the risk-free rate is \( r = 0 \) the two possible returns are \( (5 - 4)/4 = 25\% \) and \( (2 - 4)/4 = -50\% \). Under the risk-neutral probability measure given by \( \mathbb{P}^*(S_1 = 5) = (4 - 2)/(5 - 2) = 2/3 \) and \( \mathbb{P}^*(S_1 = 2) = (5 - 4)/(5 - 2) = 1/3 \) the expected return is \( 2 \times 25\% / 3 - 50\% / 3 = 0\% \). In general the expected return can be shown to be equal to the risk-free rate \( r \).

* The notation \( o(\Delta T) \) denotes any function of \( \Delta T \) such that \( \lim_{\Delta T \to 0} o(\Delta T) / \Delta T = 0 \).
b) The two possible returns become \((3 \times 5 - 4 - 2 \times 4)/4 = 75\%\) and \((3 \times 2 - 4 - 2 \times 4)/4 = -150\%\). Under the risk-neutral probability measure given by \(\mathbb{P}^*(S_1 = 5) = (4 - 2)/(5 - 2) = 2/3\) and \(\mathbb{P}^*(S_1 = 2) = (5 - 4)/(5 - 2) = 1/3\) the expected return is \(2 \times 75\% / 3 - 150\% / 3 = 0\%\). Similarly to Question (a), the expected return can be shown to be equal to the risk-free rate \(r\) when \(r \neq 0\).

c) We decompose the amount \(F_t\) invested in one unit of the fund as

\[
F_t = \underbrace{\beta F_t}_{\text{purchased/sold}} - \underbrace{(\beta - 1)F_t}_{\text{borrowed/saved}},
\]

meaning that we invest the amount \(\beta F_t\) in the risky asset \(S_t\), and borrow/save the amount \(-(\beta - 1)F_t\) from/on the saving account.

d) We have

\[
F_t = \xi_t S_t + \eta_t A_t = \beta \frac{F_t}{S_t} S_t - (\beta - 1) \frac{F_t}{A_t} A_t, \quad t \in \mathbb{R}_+,
\]

with \(\xi_t = \beta F_t / S_t\) and \(\eta_t = -(\beta - 1) F_t / A_t\), \(t \in \mathbb{R}_+\).

e) We have

\[
dF_t = \xi_t dS_t + \eta_t dA_t
= \beta \frac{F_t}{S_t} dS_t - (\beta - 1) \frac{F_t}{A_t} dA_t
= \beta \frac{F_t}{S_t} dS_t - (\beta - 1)r F_t dt
= \beta F_t (rdt + \sigma dB_t) - (\beta - 1)r F_t dt
= r F_t dt + \beta \sigma F_t dB_t, \quad t \in \mathbb{R}_+.
\]

By (A.18), the return of the fund \(F_t\) is \(\beta\) times the return of the risky asset \(S_t\), up to the cost of borrowing \((\beta - 1)r\) per unit of time.

f) The discounted fund value \(e^{-rt}F_t\) is a martingale under the risk-neutral probability measure \(\mathbb{P}^*\) as we have

\[
d(e^{-rt}F_t) = \beta \sigma e^{-rt} F_t dB_t, \quad t \in \mathbb{R}_+.
\]

g) We have

\[
F_t = F_0 e^{\beta \sigma B_t + rt - \beta^2 \sigma^2 t/2} \quad \text{and} \quad S_t^\beta = \left(S_0 e^{\sigma B_t + rt - \sigma^2 t/2}\right)^\beta = F_0 e^{\beta \sigma B_t + \beta rt - \beta \sigma^2 t/2},
\]

hence

\[
F_t = S_t^\beta e^{-(\beta - 1)rt - \beta(\beta - 1)\sigma^2 t/2}, \quad t \in \mathbb{R}_+.
\]

Note that when \(\beta = 0\) we have \(F_t = e^{rt}\), i.e. in this case the fund \(F_t\) coincides with the money market account.
h) We have
\[ e^{-r(T-t)} \mathbb{E}^* \left[(F_T - K)^+ \mid \mathcal{F}_t \right] \]
\[ = F_t \varPhi \left( \frac{\log(F_t/K) + (r + \beta^2\sigma^2/2)(T-t)}{|\beta|\sigma\sqrt{T-t}} \right) \]
\[ - K e^{-r(T-t)} \varPhi \left( \frac{\log(F_t/K) + (r - \beta^2\sigma^2/2)(T-t)}{|\beta|\sigma\sqrt{T-t}} \right), \]
\[ t \in [0,T). \]

i) We have
\[ \varPhi \left( \frac{\log(F_t/K) + (r + \beta^2\sigma^2/2)(T-t)}{|\beta|\sigma\sqrt{T-t}} \right) \]
\[ = \varPhi \left( \frac{\log(S_t^\beta e^{-(\beta-1)rt-\beta(\beta-1)\sigma^2 t/2}/K) + (r + \beta^2\sigma^2/2)(T-t)}{|\beta|\sigma\sqrt{T-t}} \right) \]
\[ = \varPhi \left( \frac{\log(S_t^\beta/K) - (\beta-1)rt - \beta(\beta-1)\sigma^2 t/2 + (r + \beta^2\sigma^2/2)(T-t)}{|\beta|\sigma\sqrt{T-t}} \right) \]
\[ = \varPhi \left( \frac{\log(S_t^\beta/(K e^{(\beta-1)rt-\beta(\beta-1)\sigma^2 t/2-\beta^2\sigma^2 t/2}) + \beta r(T-t) + \beta \sigma^2(T-t)/2)}{|\beta|\sigma\sqrt{T-t}} \right) \]
\[ = \varPhi \left( \frac{\log(S_t/K_{\beta}(t)) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \right), \quad t \in [0,T), \]
if \( \beta > 0 \), with \( K_{\beta}(t) := K^{1/\beta} e^{(\beta-1)rT/\beta-\sigma^2(T/2-t)}. \)

j) When \( \beta < 0 \) we find that the Delta of the call option on \( F_T \) with strike price \( K \) is
\[ \varPhi \left( \frac{\log(F_t/K) + (r + \beta^2\sigma^2/2)(T-t)}{|\beta|\sigma\sqrt{T-t}} \right) \]
\[ = \varPhi \left( \frac{\log(S_t^\beta/K_{\beta}) + \beta r(T-t) + \beta \sigma^2(T-t)/2}{|\beta|\sigma\sqrt{T-t}} \right) \]
\[ = \varPhi \left( \frac{-\log(S_t/K_{\beta}(t)) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \right), \quad t \in [0,T), \]
which coincides, up to a negative sign, with the Delta of the put option on \( S_T \) with strike price \( K_{\beta}(t) := K^{1/\beta} e^{(\beta-1)rT/\beta-\sigma^2(T/2-t)}. \)

Problem 5.13

a) The option payoff is \((B_T - K)^+\) at maturity.
b) We can ignore what happens between two crossings as every crossing resets the portfolio to its state right before the previous crossing. Based on this,
It is clear that every of the four possible scenarios will lead to a portfolio value \((B_T - K)^+\) at maturity:

i) If \(B_0 < 1\) and \(B_T < 1\) we issue the option for free and finish with an empty portfolio and zero payoff.

ii) If \(B_0 < 1\) and \(B_T > 1\) we issue the option for free and finish with one AUD and one SGD to refund, which yields the payoff \(B_T - 1 = (B_T - 1)^+\).

iii) If \(B_0 > 1\) and \(B_T < 1\) we purchase one AUD and borrow one SGD at the start, however the AUD will be sold and the SGD refunded before maturity, resulting into an empty portfolio and zero payoff.

iv) If \(B_0 > 1\) and \(B_T > 1\) we purchase one AUD and borrow one SGD right before maturity, which yields the payoff \(B_T - 1 = (B_T - 1)^+\).

Therefore we are hedging the option in all cases. Note that \(P(B_T = K) = 0\) so the case \(B_T = 1\) can be ignored with probability one.

c) The portfolio strategy is given by

\[
\xi_t = 1_{[K,\infty)}(B_t) \quad \text{and} \quad \eta_t = -1_{[K,\infty)}(B_t), \quad t \in [0, T].
\]

It is called a stop-loss/start-gain strategy.

d) Noting that \(\int_0^t \eta_s dA_s = 0\) because \(A_t = A_0\) is constant, \(t \in [0, T]\), we find by (4.45) that

\[
\int_0^t \eta_s dA_s + \int_0^t \xi_s dB_s = \int_0^T 1_{[K,\infty)}(B_t)dB_t
\]

\[
= (B_T - K)^+ - (B_0 - K)^+ - \frac{1}{2} \mathcal{L}^K_{[0,T]}.
\]

e) Question (d) shows that

\[
(B_T - K)^+ = (B_0 - K)^+ + \int_0^t \eta_s dA_s + \int_0^t \xi_s dB_s + \frac{1}{2} \mathcal{L}^K_{[0,T]},
\]

i.e. the initial premium \((B_0 - K)^+\) plus the sum of portfolio profits and losses is not sufficient to cover the terminal payoff \((B_T - K)^+\), and that we fall short of this by the positive amount \(\frac{1}{2} \mathcal{L}^K_{[0,T]} > 0\). Therefore the portfolio allocation \((\xi_t, \eta_t)_{t \in [0,T]}\) is not self-financing.

Additional comments:
The stop-loss / start-gain strategy described here cannot implemented in practice because it would require infinitely many transactions when Brownian motion crosses the level \(K\), as illustrated in Figure S.11.
The arbitrage price of the option can in fact be computed as the expected discounted option payoff

\[
\pi_t = e^{-(T-t)r} \mathbb{E}^*[(B_T - K)^+ | \mathcal{F}_t] = e^{-(T-t)r} \mathbb{E}^*[(B_T - B_t + x - K)^+ | \mathcal{F}_t]_{x=B_t} = e^{-(T-t)r} \mathbb{E}^*[(B_T - B_t + x - K)^+]_{x=B_t} = e^{-(T-t)r} \int_{-\infty}^{\infty} (y + B_t - K)^+ e^{-y^2/(2(T-t))} \frac{dy}{\sqrt{2\pi(T-t)}} = e^{-(T-t)r} \int_{K-B_t}^{\infty} (y + B_t - K) e^{-y^2/(2(T-t))} \frac{dy}{\sqrt{2\pi(T-t)}}
\]

\[
= e^{-(T-t)r} \int_{(K-B_t)/\sqrt{T-t}}^{\infty} y e^{-y^2/2} \frac{dy}{\sqrt{2\pi}} + (B_t - K) e^{-(T-t)r} \Phi\left(\frac{B_t - K}{\sqrt{T-t}}\right)
\]

\[
= \frac{e^{-(T-t)r}}{\sqrt{2\pi}} \left[-e^{-y^2/2}\right]_{(K-B_t)/\sqrt{T-t}}^{\infty} + (B_t - K) e^{-(T-t)r} \Phi\left(\frac{B_t - K}{\sqrt{T-t}}\right)
\]

\[
= \frac{e^{-(T-t)r}}{\sqrt{2\pi}} e^{-(K-B_t)^2/(2(T-t))}
\]
\[ +(B_t - K) e^{-(T-t)r} \Phi \left( \frac{B_t - K}{\sqrt{T-t}} \right) \]
\[ =: g(t, B_t), \]

where the function
\[ g(t, x) := e^{-(T-t)r} \frac{1}{\sqrt{2\pi}} e^{-(K-x)^2/(2(T-t))} + (x - K) e^{-(T-t)r} \Phi \left( \frac{x - K}{\sqrt{T-t}} \right), \quad t \in [0, T), \]

solves the Black-Scholes heat equation
\[ \frac{\partial g}{\partial t} (t, x) + r \frac{\partial g}{\partial x} (t, x) + \frac{1}{2} \frac{\partial^2 g}{\partial^2 x} (t, x) = 0 \]

with terminal condition \( g(T, x) = (x - K)^+ \). The Delta gives the amount to be invested in AUD at time \( t \) and is given by
\[ \xi_t = \frac{\partial g}{\partial x} (t, B_t) \]
\[ = (K - B_t) \frac{e^{-(T-t)r}}{\sqrt{2\pi(T-t)}} e^{-(K-B_t)^2/(2(T-t))} \]
\[ + (B_t - K) \frac{e^{-(T-t)r}}{\sqrt{2\pi(T-t)}} e^{-(B_t-K)^2/(2(T-t))} + e^{-(T-t)r} \Phi \left( \frac{B_t - K}{\sqrt{T-t}} \right) \]
\[ = e^{-(T-t)r} \Phi \left( \frac{B_t - K}{\sqrt{T-t}} \right) \]
\[ =: h(t, B_t), \]

with
\[ h(t, x) := e^{-(T-t)r} \Phi \left( \frac{x - K}{\sqrt{T-t}} \right), \quad t \in [0, T), \]

and \( h(T, x) = \mathbb{1}_{[K, \infty)}(x) \).

\[ \text{Fig. S.12: Brownian path started at } B_0 > 1. \]
Fig. S.13: Risk-neutral pricing of the FX option by $\pi_t(B_t) = g(t, B_t)$ vs stop-loss / start-gain pricing.

Fig. S.14: Delta hedging of the FX option by $\xi_t = h(t, B_t)$ vs the stop-loss / start-gain strategy.

The “one or nothing” stop-loss / start-gain strategy is not self-financing because in practice there is an impossibility to buy/sell the AUD at exactly SGD1.00 to the existence of an order book that generates a gap between bid/ask prices as the sample of Figure S.15 with $383.16964 < 384.07141$.

Fig. S.15: Bitcoin XBT/USD order book.
The existence of the order book will force buying and selling within a certain range \([K - \varepsilon, K + \varepsilon]\), typically resulting into selling lower than \(K = 1.00\) and buying higher than \(K = 1.00\). This potentially results into a trading loss that can be proportional to the time

\[
\ell \left( \{ t \in [0, T] : K - \varepsilon < B_t < K + \varepsilon \} \right)
\]

spent by the exchange rate \((B_t)_{t \in [0,T]}\) within the range \([K - \varepsilon, K + \varepsilon]\).

The Itô-Tanaka formula (4.45)

\[
(B_T - K)^+ = (B_0 - K)^+ + \int_0^T 1_{[K, \infty)}(B_t) dB_t + \frac{1}{2} \mathcal{L}_{[0,T]}^K,
\]

precisely shows that the trading loss equals half the local time \(\mathcal{L}_{[0,T]}^K\) spent by \((B_t)_{t \in [0,T]}\) at the level \(K\). When \(\varepsilon\) is small we have

\[
\frac{1}{2} \mathcal{L}_{[0,T]}^K \approx \frac{1}{4\varepsilon} \ell(\{ t \in [0, T] : K - \varepsilon < B_t < K + \varepsilon \}),
\]

therefore the proportionality coefficient is \(1/(4\varepsilon)\).

Fig. S.16: Time spent by Brownian motion in the range \([K - \varepsilon, K + \varepsilon]\).

More generally, we could show that there is no self-financing (buy and hold) portfolio that can remain constant over time intervals, and that the self-financing portfolio has to be constantly re-adjusted in time as illustrated in Figure S.14. This invalidates the stop-loss / start-gain strategy as a self-financing portfolio strategy.

Chapter 6

Exercise 6.1 Since \(B_T \simeq \mathcal{N}(0, T)\), we have

\[
\mathbb{E}[\phi(S_T)] = \mathbb{E} \left[ \phi(S_0 e^{\sigma B_T + (r - \sigma^2/2)T} \right]
\]
Exercise 6.2 We have

\[
\begin{align*}
&= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} \phi(S_0 e^{\sigma y + (r - \sigma^2/2)T}) e^{-y^2/(2T)} dy \\
&= \frac{1}{\sqrt{2\pi\sigma^2 T}} \int_{-\infty}^{\infty} \phi(x) e^{-((\sigma^2/2 - r)T + \log x)^2/(2\sigma^2 T)} \frac{dx}{x} \\
&= \int_{-\infty}^{\infty} \phi(x) g(x) dx,
\end{align*}
\]

under the change of variable

\[
x = e^{\sigma y + (r - \sigma^2/2)T} \quad \text{and} \quad dx = \sigma e^{\sigma y + (r - \sigma^2/2)T} dy = \sigma x dy,
\]
i.e.

\[
y = \frac{(\sigma^2/2 - r)T + \log x}{\sigma} \quad \text{and} \quad dy = \frac{dx}{\sigma x},
\]

where

\[
g(x) := \frac{1}{x\sqrt{2\pi\sigma^2 T}} e^{-((\sigma^2/2 - r)T + \log x)^2/(2\sigma^2 T)}
\]
is the lognormal probability density function with location parameter \((r - \sigma^2/2)T\) and scale parameter \(\sigma \sqrt{T}\).

Remar: This type of technique is useful to get an upper price estimate from Black-Scholes when the actual option price is difficult to compute: here the closed-form computation would involve a double integration of the form

\[
\begin{align*}
\mathbb{E}^*[\phi(pS_{T_1} + qS_{T_2})] &= \mathbb{E}^*[\phi(pS_{T_1}) + q\phi(S_{T_2})] \\
&\leq p \mathbb{E}^*[\phi(S_{T_1})] + q \mathbb{E}^*[\phi(S_{T_2})] \\
&= p \mathbb{E}^*[\phi(S_{T_2})] + q \mathbb{E}^*[\phi(S_{T_2})] \\
&= \mathbb{E}^*[\phi(S_{T_2})],
\end{align*}
\]

since \(\phi\) is convex,

\[
\begin{align*}
\mathbb{E}^*[\phi(pS_{T_1} + qS_{T_2})] &= \mathbb{E}^*[\phi(pS_{T_1}) + q\phi(S_{T_2})] \\
&= p \mathbb{E}^*[\phi(S_{T_1})] + q \mathbb{E}^*[\phi(S_{T_2})] \\
&= p \mathbb{E}^*[\mathbb{E}^*[\phi(S_{T_2}) | \mathcal{F}_{T_1}]] + q \mathbb{E}^*[\phi(S_{T_2})] \\
&\leq p \mathbb{E}^*[\phi(S_{T_2})] + q \mathbb{E}^*[\phi(S_{T_2})] \\
&= \mathbb{E}^*[\phi(S_{T_2})],
\end{align*}
\]

because \((S_t)_{t \in \mathbb{R}_+}\) is a martingale, by Jensen’s inequality, and by the tower property, respectively.

Exercise 10.5 for an extension to arbitrary summations.
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( S_0 e^{\sigma x - \sigma^2 T_1/2} \left( p + q e^{\sigma y - \sigma^2 (T_2 - T_1)/2} \right) - K \right)^+ \\
\times e^{-x^2/(2T_1)-y^2(2(T_2-T_1))} \frac{dx\,dy}{\sqrt{T_1(T_2-T_1)}}
\]

\[
= \frac{1}{2\pi} \int_{(x,y) \in \mathbb{R}^2 : S_0 e^{\sigma x} (p + q e^{\sigma y - \sigma^2 (T_2 - T_1)/2}) \geq K e^{\sigma^2 T_1/2}} \left( S_0 e^{\sigma x - \sigma^2 T_1/2} \left( p + q e^{\sigma y - \sigma^2 (T_2 - T_1)/2} \right) - K \right) \\
\times e^{-x^2/(2T_1)-y^2(2(T_2-T_1))} \frac{dx\,dy}{\sqrt{T_1(T_2-T_1)}}
\]

Exercise 6.3

a) Using Jensen’s inequality and the martingale property of the discounted asset price process \((e^{-rt} S_t)_{t \in \mathbb{R}^+}\) under the risk-neutral probability measure \(\mathbb{P}^*\), we have

\[
e^{-(T-t)r} \mathbb{E}^*[(S_T - K)^+ | \mathcal{F}_t] \geq e^{-(T-t)r} \left( \mathbb{E}^*[S_T - K | \mathcal{F}_t] \right)^+
\]

\[
= e^{-(T-t)r} \left( e^{(T-t)r} S_t - K \right)^+
\]

\[
= (S_t - K e^{-(T-t)r})^+, \quad t \in [0, T].
\]

b) Similarly, by Jensen’s inequality and the martingale property we find

\[
e^{-(T-t)r} \mathbb{E}^*[(K - S_T)^+ | \mathcal{F}_t] \geq e^{-(T-t)r} \left( \mathbb{E}^*[K - S_T | \mathcal{F}_t] \right)^+
\]

\[
= e^{-(T-t)r} \left( K - e^{(T-t)r} S_t \right)^+
\]

\[
= (K e^{-(T-t)r} - S_t)^+, \quad t \in [0, T].
\]

Exercise 6.4

Fig. S.17: Lower bound vs Black-Scholes call price.
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Fig. S.18: Lower bound vs Black-Scholes put price.

a) (i) The bull spread option can be realized by purchasing one European call option with strike price $K_1$ and by short selling (or issuing) one European call option with strike price $K_2$, because the bull spread payoff function can be written as

$$x \mapsto (x - K_1)^+ - (x - K_2)^+.$$  

see http://optioncreator.com/st3cc7z.

(ii) The bear spread option can be realized by purchasing one European put option with strike price $K_2$ and by short selling (or issuing) one European put option with strike price $K_1$, because the bear spread payoff function can be written as

$$x \mapsto (K_2 - x)^+ - (K_1 - x)^+,$$

see http://optioncreator.com/stmomsb.

b) (i) The bull spread option can be priced at time $t \in [0, T)$ using the Black-Scholes formula as

$$\text{Bl}(K_1, S_t, \sigma, r, T - t) - \text{Bl}(K_2, S_t, \sigma, r, T - t).$$

(ii) The bear spread option can be priced at time $t \in [0, T)$ using the Black-Scholes formula as

$$\text{Bl}(K_2, S_t, \sigma, r, T - t) - \text{Bl}(K_1, S_t, \sigma, r, T - t).$$

Exercise 6.5

a) We have

$$C_t = e^{-(T-t)r} \mathbb{E}^*[S_T - K \mid \mathcal{F}_t]$$

$$= e^{-(T-t)r} \mathbb{E}^*[S_T \mid \mathcal{F}_t] - K e^{-(T-t)r}$$
We can check that the function \( g(x, t) = x - Ke^{-T}r \) satisfies the Black-Scholes PDE
\[
rg(x, t) = \frac{\partial g}{\partial t}(x, t) + r x \frac{\partial g}{\partial x}(x, t) + \frac{\sigma^2}{2} x^2 \frac{\partial^2 g}{\partial x^2}(x, t)
\]
with terminal condition \( g(x, T) = x - K \), since \( \frac{\partial g(x, t)}{\partial t} = -rKe^{-T}r \) and \( \frac{\partial g(x, t)}{\partial x} = 1 \).

b) We simply take \( \xi_t = 1 \) and \( \eta_t = -Ke^{-T}r \) in order to have

\[
C_t = \xi_t S_t + \eta_t e^{rt} = S_t - Ke^{-T}r, \quad t \in [0, T].
\]

Note again that this hedging strategy is constant over time, and the relation \( \xi_t = \frac{\partial g(S_t, t)}{\partial x} \) for the option Delta, cf. (A.14), is satisfied.

Exercise 6.6

a) Let \( \tilde{\mathbb{P}} \) denote the probability measure under which the process \( (\tilde{B}_t)_{t \in \mathbb{R}_+} \) defined by

\[
d\tilde{B}_t = \frac{\mu - (r - \delta)}{\sigma} dt + dB_t.
\]

The asset price process with added dividend yield is given by \( (e^{\delta t} S_t)_{t \in \mathbb{R}_+} \). Under absence of arbitrage the asset price process \( (S_t)_{t \in \mathbb{R}_+} \) has the dynamics

\[
dS_t = (r - \delta)S_t dt + \sigma S_t d\tilde{B}_t,
\]

and after discount the process

\[
(\tilde{S}_t)_{t \in \mathbb{R}_+} := (e^{-rt}e^{\delta t} S_t)_{t \in \mathbb{R}_+} = (e^{-(r-\delta)t} S_t)_{t \in \mathbb{R}_+}
\]
is a martingale under \( \tilde{\mathbb{P}} \). Since the dividend yield \( \delta S_t \) per share is continuously reinvested in the portfolio, we have

\[
dV_t = \underbrace{\eta_t dA_t + \xi_t dS_t}_\text{trading profit and loss} + \underbrace{\delta \xi_t S_t dt}_\text{dividend payout}
= r\eta_t A_t dt + \xi_t ((r - \delta)S_t dt + \sigma S_t d\tilde{B}_t) + \delta \xi_t S_t dt
\]

\[
= r\eta_t A_t dt + \xi_t (rS_t dt + \sigma S_t d\tilde{B}_t)
= rV_t dt + \sigma \xi_t S_t d\tilde{B}_t, \quad t \in \mathbb{R}_+,
\]
hence

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\[ d\tilde{V}_t = d( e^{-rt}V_t) \]
\[ = -re^{-rt}V_t dt + e^{-rt} dV_t \]
\[ = \sigma \xi_t e^{-rt}S_t d\tilde{B}_t \]
\[ = \sigma \xi_t \tilde{S}_t d\tilde{B}_t \]
\[ = \xi_t d\tilde{S}_t, \quad t \in \mathbb{R}_+, \]

which yields

\[ \tilde{V}_t - \tilde{V}_0 = \int_0^t d\tilde{V}_u = \int_0^t \xi_u d\tilde{S}_u, \quad t \in \mathbb{R}_+. \]

b) We have

\[ \tilde{V}_t = \tilde{V}_0 + \sigma \int_0^t \xi_u \tilde{S}_u d\tilde{B}_u = V_0 + \int_0^t \xi_u d\tilde{S}_u, \quad t \in \mathbb{R}_+, \]

which is a martingale under \( \hat{P} \) from Proposition 6.1, hence

\[ \tilde{V}_t = \hat{E}[V_T | \mathcal{F}_t] \]
\[ = e^{-rT} \hat{E}[V_T | \mathcal{F}_t] \]
\[ = e^{-rT} \hat{E}[C | \mathcal{F}_t], \]

which implies

\[ V_t = e^{rt} \tilde{V}_t = e^{-(T-t)r} \hat{E}[C | \mathcal{F}_t], \quad 0 \leq t \leq T. \]

c) After discounting the payoff \((S_T - K)^+\) at the rate \(r\) we conclude that

\[ V_t = e^{-(T-t)r} \hat{E}[(S_T - K)^+ | \mathcal{F}_t] \]
\[ = e^{-(T-t)r} \hat{E}[(S_0 e^{\sigma \tilde{B}_T + (r-\delta-\sigma^2/2)T} - K)^+ | \mathcal{F}_t] \]
\[ = e^{-(T-t)\delta} \left( e^{-(T-t)(r-\delta)} \hat{E}[(S_0 e^{\sigma \tilde{B}_T + (r-\delta-\sigma^2/2)T} - K)^+ | \mathcal{F}_t] \right) \]
\[ = e^{-(T-t)\delta} B(t, x, \sigma, r-\delta, T-t) \]
\[ = e^{-(T-t)\delta} S_t \Phi(d_+^\delta(T-t)) - K e^{-(T-t)(r-\delta)} \Phi(d_-^\delta(T-t)) \]
\[ = e^{-(T-t)\delta} S_t \Phi(d_+^\delta(T-t)) - K e^{-(T-t)r} \Phi(d_-^\delta(T-t)), \quad t \in [0, T), \]

where

\[ d_+^\delta(T-t) := \frac{\log(S_t/K) + (r - \delta + \sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}} \]

and

\[ d_-^\delta(T-t) := \frac{\log(S_t/K) + (r - \delta - \sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}. \]
Exercise 6.7 We start by pricing the “inner” at-the-money option with payoff \((S_{T_2} - S_{T_1})^+\) and strike price \(K = S_{T_1}\) at time \(T_1\) as

\[
e^{-(T_2 - T_1)r} \mathbb{E}^* [(S_{T_2} - S_{T_1})^+ | \mathcal{F}_{T_1}] = S_{T_1} \Phi \left( \frac{r + \sigma^2/2}{\sigma \sqrt{T_2 - T_1}} \right) - e^{-(T_2 - T_1)r} \mathbb{E}^* \left[ \frac{r + \sigma^2/2}{\sigma \sqrt{T_2 - T_1}} \right] \mathbb{E}^* [S_{T_1} | \mathcal{F}_{T_1}]
\]

where we applied (6.16) with \(T = T_2, t = T_1\), and \(K = S_{T_1}\). As a consequence, the forward start option can be priced as

\[
e^{-(T_1 - t)r} \mathbb{E}^* [(S_{T_2} - S_{T_1})^+ | \mathcal{F}_{T_1}] = e^{-(T_1 - t)r} \mathbb{E}^* \left[ \frac{r + \sigma^2/2}{\sigma \sqrt{T_2 - T_1}} \right] \Phi \left( \frac{r + \sigma^2/2}{\sigma \sqrt{T_2 - T_1}} \right) - e^{-(T_2 - T_1)r} \mathbb{E}^* \left[ \frac{r + \sigma^2/2}{\sigma \sqrt{T_2 - T_1}} \right] \mathbb{E}^* [S_{T_1} | \mathcal{F}_{T_1}]
\]

\(0 \leq t \leq T_1\).

Exercise 6.8 We have

\[
C(t, S_t) = e^{-(T-t)r} \mathbb{E}^* [\log S_T | \mathcal{F}_t] = e^{-(T-t)r} \mathbb{E}^* \left[ (\log S_t) + \sigma (\widehat{B}_T - \widehat{B}_t) + \left( r - \frac{\sigma^2}{2} \right) (T-t) | \mathcal{F}_t \right] = e^{-(T-t)r} \log S_t + e^{-(T-t)r} \left( r - \frac{\sigma^2}{2} \right) (T-t),
\]

\(t \in [0, T]\).

Exercise 6.9

a) For all \(t \in [0, T]\) we have
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\[ C(t, S_t) = e^{-(T-t)r} S_t^2 \mathbb{E} \left[ \frac{S_T^2}{S_t^2} \right] \]
\[ = e^{-(T-t)r} S_t^2 \mathbb{E} \left[ e^{2\sigma(B_T-B_t)-\sigma^2(T-t)+2(T-t)r} \right] \]
\[ = S_t^2 e^{(r+\sigma^2)(T-t)}. \]

b) For all \( t \in [0, T] \) we have

\[ \xi_t = \frac{\partial C}{\partial x}(t, x)|_{x=S_t} = 2S_t e^{(r+\sigma^2)(T-t)}, \]
i.e.

\[ \xi_t S_t = 2S_t^2 e^{(r+\sigma^2)(T-t)} = 2C(t, S_t), \]
and

\[ \eta_t = \frac{C(t, S_t) - \xi_t S_t}{A_t} = \frac{e^{-rt}}{A_0} (S_t^2 e^{(r+\sigma^2)(T-t)} - 2S_t^2 e^{(r+\sigma^2)(T-t)}) \]
\[ = -S_t^2 \frac{A_t}{A_0} e^{\sigma^2(T-t)+(T-2t)r}, \]
i.e.

\[ \eta_t A_t = -S_t^2 \frac{A_t}{A_0} e^{\sigma^2(T-t)+(T-2t)r} = -S_t^2 e^{\sigma^2(T-t)+(T-2t)r} = -C(t, S_t). \]

As for the self-financing condition, we have

\[ dC(t, S_t) = d(S_t^2 e^{(r+\sigma^2)(T-t)}) \]
\[ = -(r + \sigma^2) e^{(r+\sigma^2)(T-t)} S_t^2 dt + e^{(r+\sigma^2)(T-t)} d(S_t^2) \]
\[ = -(r + \sigma^2) e^{(r+\sigma^2)(T-t)} S_t^2 dt + e^{(r+\sigma^2)(T-t)} (2S_t dS_t + \sigma^2 S_t^2 dt) \]
\[ = -r e^{(r+\sigma^2)(T-t)} S_t^2 dt + 2S_t e^{(r+\sigma^2)(T-t)} dS_t, \]
and

\[ \xi_t dS_t + \eta_t dA_t = 2S_t e^{(r+\sigma^2)(T-t)} dS_t - r \frac{S_t^2}{A_0} e^{\sigma^2(T-t)+(T-2t)r} A_t dt \]
\[ = 2S_t e^{(r+\sigma^2)(T-t)} dS_t - r \frac{S_t^2}{A_0} e^{\sigma^2(T-t)+(T-2t)r} dt, \]
which recovers \( dC(t, S_t) = \xi_t dS_t + \eta_t dA_t, \) i.e. the portfolio strategy is self-financing.

Exercise 6.10

a) By (4.34) we have
b) The discounted process $X_t := e^{-rt} S_t$ satisfies
\[ dX_t = (\alpha - r) X_t dt + \sigma e^{-rs} dB_s, \]

hence
\[ X_t = S_0 + \sigma \int_0^t e^{-rs} dB_s, \]

which is a martingale when $\alpha = r$ by Proposition 6.1, as in this case it becomes a stochastic integral with respect to a standard Brownian motion. This fact can also be proved directly by computing the conditional expectation $\mathbb{E}[X_t \mid F_s]$ and showing it is equal to $X_s$, i.e.:
\[
\mathbb{E}[X_t \mid F_s] = \mathbb{E}
\left[
S_0 + \sigma \int_0^s e^{-ru} dB_u \mid F_s
\right]
\]
\[= \mathbb{E}[S_0] + \sigma \mathbb{E}\left[\int_0^t e^{-ru} dB_u \mid F_s\right]
\]
\[= S_0 + \sigma \mathbb{E}\left[\int_0^s e^{-ru} dB_u \mid F_s\right] + \sigma \mathbb{E}\left[\int_s^t e^{-ru} dB_u \mid F_s\right]
\]
\[= S_0 + \sigma \int_0^s e^{-ru} dB_u + \sigma \mathbb{E}\left[\int_s^t e^{-ru} dB_u \mid F_s\right]
\]
\[= S_0 + \sigma \int_0^s e^{-ru} dB_u
\]
\[= X_s, \quad 0 \leq s \leq t.
\]

c) We rewrite the stochastic differential equation satisfied by $(S_t)_{t \in \mathbb{R}^+}$ as
\[ dS_t = \alpha S_t dt + \sigma dB_t = rS_t dt + \sigma d\hat{B}_t, \]

where
\[ d\hat{B}_t := \frac{\alpha - r}{\sigma} S_t dt + dB_t, \]

which allows us to rewrite (4.34) with $\alpha = -r$ as
\[
S_t = e^{rt} \left(S_0 + \sigma \int_0^t e^{-rs} d\hat{B}_s\right) = S_0 e^{rt} + \sigma \int_0^t e^{(t-s)r} d\hat{B}_s. \quad (A.19)
\]

Taking
\[ \psi_t := \frac{\alpha - r}{\sigma} S_t, \quad 0 \leq t \leq T, \]

in the Girsanov Theorem 6.2, the process $(\hat{B}_t)_{t \in \mathbb{R}^+}$ is a standard Brownian motion under the probability measure $\mathbb{P}_\alpha$ defined by
\[
\frac{d\mathbb{P}_\alpha}{d\mathbb{P}} := \exp \left( - \int_0^T \psi_t dB_t - \frac{1}{2} \int_0^T \psi_t^2 dt \right) \\
= \exp \left( - \frac{\alpha - r}{\sigma} \int_0^T S_t dB_t - \frac{1}{2} \left( \frac{\alpha - r}{\sigma} \right)^2 \int_0^T S_t^2 dt \right),
\]

and \((X_t)_{t \in \mathbb{R}_+}\) is a martingale under \(\mathbb{P}_\alpha\).

d) Using \((A.19)\) under the risk-neutral probability measure \(\mathbb{P}^*\), we have

\[
C(t, S_t) = e^{-(T-t)r} \mathbb{E}_\alpha[\exp(S_T) \mid \mathcal{F}_t] \\
= e^{-(T-t)r} \mathbb{E}_\alpha \left[ \exp \left( e^{rt} S_0 + \sigma \int_0^t e^{(T-u)r} dB_u \right) \mid \mathcal{F}_t \right] \\
= e^{-(T-t)r} \mathbb{E}_\alpha \left[ \exp \left( e^{rt} S_0 + \sigma \int_0^t e^{(T-u)r} dB_u + \sigma \int_t^T e^{(T-u)r} dB_u \right) \mid \mathcal{F}_t \right] \\
= \exp \left( -(T-t)r + e^{(T-t)r} S_t \right) \mathbb{E}_\alpha \left[ \exp \left( \sigma \int_t^T e^{(T-u)r} dB_u \right) \mid \mathcal{F}_t \right] \\
= \exp \left( -(T-t)r + e^{(T-t)r} S_t \right) \mathbb{E}_\alpha \left[ \exp \left( \sigma \int_t^T e^{(T-u)r} dB_u \right) \right] \\
= \exp \left( -(T-t)r + e^{(T-t)r} S_t \right) \exp \left( \frac{\sigma^2}{2} \int_t^T (e^{(T-u)r})^2 du \right) \\
= \exp \left( -(T-t)r + e^{(T-t)r} S_t + \frac{\sigma^2}{4r} \left( e^{2(T-t)r} - 1 \right) \right).
\]

e) We have

\[
\xi_t = \frac{\partial C}{\partial x}(t, S_t) = \exp \left( S_t e^{(T-t)r} + \frac{\sigma^2}{4r} \left( e^{2(T-t)r} - 1 \right) \right)
\]

and

\[
\eta_t = \frac{C(t, S_t) - \xi_t S_t}{A_t} \\
= \frac{e^{-(T-t)r}}{A_t} \exp \left( S_t e^{(T-t)r} + \frac{\sigma^2}{4r} \left( e^{2(T-t)r} - 1 \right) \right) \\
- \frac{S_t}{A_t} \exp \left( S_t e^{(T-t)r} + \frac{\sigma^2}{4r} \left( e^{2(T-t)r} - 1 \right) \right).
\]

f) We have

\[
dC(t, S_t) = r e^{-(T-t)r} \exp \left( S_t e^{(T-t)r} + \frac{\sigma^2}{4r} \left( e^{2(T-t)r} - 1 \right) \right) dt \\
- r S_t \exp \left( S_t e^{(T-t)r} + \frac{\sigma^2}{4r} \left( e^{2(T-t)r} - 1 \right) \right) dt.
\]
\[-\frac{\sigma^2}{2} e^{(T-t)r} \exp \left( \frac{\sigma^2}{4r} e^{(T-t)r} + \frac{\sigma^2}{4r} (e^{2(T-t)r} - 1) \right) dt \]

+ \exp \left( \frac{\sigma^2}{4r} e^{(T-t)r} + \frac{\sigma^2}{4r} (e^{2(T-t)r} - 1) \right) dS_t

+ \frac{1}{2} e^{(T-t)r} \exp \left( \frac{\sigma^2}{4r} e^{(T-t)r} + \frac{\sigma^2}{4r} (e^{2(T-t)r} - 1) \right) \sigma^2 dt

= r e^{- (T-t)r} \exp \left( \frac{\sigma^2}{4r} e^{(T-t)r} + \frac{\sigma^2}{4r} (e^{2(T-t)r} - 1) \right) dt

- r S_t \exp \left( \frac{\sigma^2}{4r} e^{(T-t)r} + \frac{\sigma^2}{4r} (e^{2(T-t)r} - 1) \right) dt

+ \xi_t dS_t.

On the other hand we have

\[
\xi_t dS_t + \eta_t dA_t = \xi_t dS_t + r e^{- (T-t)r} \exp \left( \frac{\sigma^2}{4r} e^{(T-t)r} + \frac{\sigma^2}{4r} (e^{2(T-t)r} - 1) \right) \sigma^2 dt

- r S_t \exp \left( \frac{\sigma^2}{4r} e^{(T-t)r} + \frac{\sigma^2}{4r} (e^{2(T-t)r} - 1) \right) dt,
\]

showing that

\[dC(t, S_t) = \xi_t dS_t + \eta_t dA_t,\]

and confirming that the strategy \((\xi_t, \eta_t)_{t \in \mathbb{R}_+}\) is self-financing.

Exercise 6.11

a) We have

\[
\frac{\partial f}{\partial t}(t, x) = (r - \sigma^2 / 2) f(t, x), \quad \frac{\partial f}{\partial x}(t, x) = \sigma f(t, x),
\]

and

\[
\frac{\partial^2 f}{\partial x^2}(t, x) = \sigma^2 f(t, x),
\]

hence

\[dS_t = df(t, B_t) = \frac{\partial f}{\partial t}(t, B_t) dt + \frac{\partial f}{\partial x}(t, B_t) dB_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, B_t) dt \]

\[= \left( r - \frac{1}{2} \sigma^2 \right) f(t, B_t) dt + \sigma f(t, B_t) dB_t + \frac{1}{2} \sigma^2 f(t, B_t) dt \]

\[= r f(t, B_t) dt + \sigma f(t, B_t) dB_t\]
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\[ = rS_t dt + \sigma S_t dB_t. \]

b) We have

\[
\mathbb{E} \left[ e^{\sigma B_T} \mid \mathcal{F}_t \right] = \mathbb{E} \left[ e^{\sigma(B_T-B_t+B_t)} \mid \mathcal{F}_t \right]
\]

\[
= e^{\sigma B_t} \mathbb{E} \left[ e^{\sigma(B_T-B_t)} \mid \mathcal{F}_t \right]
\]

\[
= e^{\sigma B_t} \mathbb{E} \left[ e^{\sigma(B_T-B_t)} \right]
\]

\[
= e^{\sigma B_t + \sigma^2 (T-t)/2}.
\]

c) We have

\[
\mathbb{E} [S_T \mid \mathcal{F}_t] = \mathbb{E} \left[ e^{\sigma B_T + r T - \sigma^2 T/2} \mid \mathcal{F}_t \right]
\]

\[
= e^{r T - \sigma^2 T/2} \mathbb{E} \left[ e^{\sigma B_T} \mid \mathcal{F}_t \right]
\]

\[
= e^{r T - \sigma^2 T/2} e^{\sigma B_t + \sigma^2 (T-t)/2}
\]

\[
= e^{r T + \sigma B_t - \sigma^2 t/2}
\]

\[
= e^{(T-t) r + \sigma B_t + rt - \sigma^2 t/2}
\]

\[
= e^{(T-t) r} S_t, \quad t \in [0, T].
\]

d) We have

\[
V_t = e^{-(T-t) r} \mathbb{E} [C \mid \mathcal{F}_t]
\]

\[
= e^{-(T-t) r} \mathbb{E} [S_T - K \mid \mathcal{F}_t]
\]

\[
= e^{-(T-t) r} \mathbb{E} [S_T \mid \mathcal{F}_t] - e^{-(T-t) r} \mathbb{E} [K \mid \mathcal{F}_t]
\]

\[
= S_t - e^{-(T-t) r} K, \quad t \in [0, T].
\]

e) We take \( \xi_t = 1 \) and \( \eta_t = -K e^{-r T}/A_0 \), \( t \in [0, T] \).
f) We find

\[
V_T = \mathbb{E} [C \mid \mathcal{F}_T] = C.
\]

Exercise 6.12 Binary options. (Exercise 5.8 continued).

a) By definition of the indicator (or step) functions \( \mathbbm{1}_{[K, \infty)} \) and \( \mathbbm{1}_{[0, K]} \) we have

\[
\mathbbm{1}_{[K, \infty)} (x) = \begin{cases} 1 & \text{if } x \geq K, \\ 0 & \text{if } x < K, \end{cases}
\]

\[
\mathbbm{1}_{[0, K]} (x) = \begin{cases} 1 & \text{if } x \leq K, \\ 0 & \text{if } x > K, \end{cases}
\]

which shows the claimed result by the definition of \( C_b \) and \( P_b \).

b) We have
\[ \pi_t(C_b) = e^{-(T-t)r} \mathbb{E}[C_b \mid \mathcal{F}_t] = e^{-(T-t)r} \mathbb{E} \left[ \mathbb{1}_{[K,\infty)}(S_T) \mid S_t \right] = e^{-(T-t)r} \mathbb{P}(S_T \geq K \mid S_t) = C_b(t, S_t). \]

c) We have \( \pi_t(C_b) = C_b(t, S_t) \), where

\[ C_b(t, x) = e^{-(T-t)r} \mathbb{P}(S_T > K \mid S_t = x) = e^{-(T-t)r} \Phi \left( \frac{(r - \sigma^2/2)(T-t) + \log(S_t/K)}{\sigma\sqrt{T-t}} \right) = e^{-(T-t)r} \Phi \left( d_-(T-t) \right), \]

with \( d_-(T-t) = \frac{(r - \sigma^2/2)(T-t) + \log(S_t/K)}{\sigma\sqrt{T-t}} \).

d) The price of this modified contract with payoff

\[ C_\alpha = \mathbb{1}_{[K,\infty)}(S_T) + \alpha \mathbb{1}_{[0,K)}(S_T) \]

is given by

\[ \pi_t(C_\alpha) = e^{-(T-t)r} \mathbb{E} \left[ \mathbb{1}_{[K,\infty)}(S_T) + \alpha \mathbb{1}_{[0,K)}(S_T) \mid S_t \right] = e^{-(T-t)r} \mathbb{P}(S_T \geq K \mid S_t) + \alpha e^{-(T-t)r} \mathbb{P}(S_T \leq K \mid S_t) = e^{-(T-t)r} \mathbb{P}(S_T \geq K \mid S_t) + \alpha e^{-(T-t)r} (1 - \mathbb{P}(S_T \geq K \mid S_t)) = \alpha e^{-(T-t)r} e^{-(T-t)r} + (1 - \alpha) \mathbb{P}(S_T \geq K \mid S_t) = \alpha e^{-(T-t)r} + (1 - \alpha) e^{-(T-t)r} \Phi \left( \frac{(T-t) - \sigma^2(T-t)/2 + \log(S_t/K)}{\sigma\sqrt{T-t}} \right). \]

Fig. S.19: Price of a binary call option.
e) We note that
\[ 1 \mathbb{I}_{[K, \infty)}(S_T) + \mathbb{1}_{[0, K]}(S_T) = \mathbb{1}_{[0, \infty)}(S_T), \]
almost surely since \( P(S_T = K) = 0 \), hence
\[
\pi_t(C_b) + \pi_t(P_b) = e^{-(T-t)r} \mathbb{E}[C_b \mid \mathcal{F}_t] + e^{-(T-t)r} \mathbb{E}[P_b \mid \mathcal{F}_t]
= e^{-(T-t)r} \mathbb{E}[C_b + P_b \mid \mathcal{F}_t]
= e^{-(T-t)r} \mathbb{E}[\mathbb{1}_{[K, \infty)}(S_T) + \mathbb{1}_{[0, K]}(S_T) \mid \mathcal{F}_t]
= e^{-(T-t)r} \mathbb{E}[\mathbb{1}_{[0, \infty)}(S_T) \mid \mathcal{F}_t]
= e^{-(T-t)r} \mathbb{E} [1 \mid \mathcal{F}_t]
= e^{-(T-t)r}, \quad 0 \leq t \leq T.
\]

f) We have
\[
\pi_t(P_b) = e^{-(T-t)r} - \pi_t(C_b)
= e^{-(T-t)r} - e^{-(T-t)r} \Phi\left(\frac{(T-t)r - \sigma^2(T-t)/2 + \log(x/K)}{\sigma \sqrt{T-t}}\right)
= e^{-(T-t)r} \left(1 - \Phi\left(d_-(T-t)\right)\right)
= e^{-(T-t)r} \Phi\left(-d_-(T-t)\right).
\]

g) We have
\[
\xi_t = \frac{\partial C_b}{\partial x}(t, S_t)
= e^{-(T-t)r} \frac{\partial}{\partial x} \Phi\left(\frac{(T-t)r - \sigma^2(T-t)/2 + \log(x/K)}{\sigma \sqrt{T-t}}\right)_{x=S_t}
= e^{-(T-t)r} \frac{1}{\sigma S_t \sqrt{2\pi(T-t)}} e^{-d_-(T-t)^2/2}
> 0.
\]
The Black-Scholes hedging strategy of such a call option does not involve short selling because \( \xi_t > 0 \) for all \( t \), cf. Figure S.20 which represents the risky investment in the hedging portfolio of a binary call option.
h) Here we have

\[ \xi_t = \frac{\partial P_b}{\partial x}(t, S_t) \]

\[ = e^{-(T-t)r} \frac{\partial}{\partial x} \Phi \left( -\frac{(T-t)r - \sigma^2(T-t)/2 + \log(x/K)}{\sigma \sqrt{T-t}} \right)_{x=S_t} \]

\[ = -e^{-(T-t)r} \frac{1}{\sigma \sqrt{2\pi(T-t)}S_t} e^{-(d_1(T-t))^2/2} \]

\[ < 0. \]

The Black-Scholes hedging strategy of such a put option does involve short selling because \( \xi_t < 0 \) for all \( t \).

Exercise 6.13 Using Itô’s formula and the fact that the expectation of the stochastic integral with respect to \((W_t)_{t \in \mathbb{R}_+}\) is zero, cf. Relation (4.16), we

\[ \Rightarrow \]

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http://www.ntu.edu.sg/home/nprivault/index.html
have

\[ C(x, T) = e^{-rT} \mathbb{E} \left[ \phi(S_T) \mid S_0 = x \right] \]
\[ = \phi(x) - r \mathbb{E} \left[ \int_0^T e^{-rs} \phi(S_t) dt \mid S_0 = x \right] + \mathbb{E} \left[ S_t \int_0^T e^{-rt} \phi'(S_t) dt \mid S_0 = x \right] \]
\[ + \sigma \mathbb{E} \left[ S_t \int_0^T e^{-rt} \phi''(S_t) dB_t \mid S_0 = x \right] + \frac{1}{2} \mathbb{E} \left[ \int_0^T e^{-rt} \phi''(S_t) \sigma^2(S_t) dt \mid S_0 = x \right] \]
\[ = \phi(x) - r \mathbb{E} \left[ \int_0^T e^{-rs} \phi(S_t) dt \mid S_0 = x \right] + \mathbb{E} \left[ S_t \int_0^T e^{-rt} \phi'(S_t) dt \mid S_0 = x \right] \]
\[ + \frac{1}{2} \mathbb{E} \left[ \int_0^T e^{-rt} \phi''(S_t) \sigma^2(S_t) dt \mid S_0 = x \right] \]

hence by differentiation with respect to \( T \) we find

\[ \Theta_T = \frac{\partial}{\partial T} \left( e^{-rT} \mathbb{E} \left[ \phi(S_T) \mid S_0 = x \right] \right) \]
\[ = -re^{-rT} \mathbb{E} \left[ \phi(S_T) \mid S_0 = x \right] + re^{-rT} \mathbb{E} \left[ S_T \phi'(S_T) \mid S_0 = x \right] \]
\[ + \frac{1}{2} e^{-rT} \mathbb{E} \left[ \phi''(S_T) \sigma^2(S_T) \mid S_0 = x \right] . \]

Problem 6.14 Chooser options.

a) We take conditional expectations in the equality

\[ (S_T - K)^+ - (K - S_T)^+ = S_T - K \]

to find

\[ C(t, S_t, K, T) - P(t, S_t, K, T) \]
\[ = e^{-(T-t)r} \mathbb{E}^*[(S_T - K)^+ \mid \mathcal{F}_t] - e^{-(T-t)r} \mathbb{E}^*[(K - S_T)^+ \mid \mathcal{F}_t] \]
\[ = e^{-(T-t)r} \mathbb{E}^*[S_T - K \mid \mathcal{F}_t] \]
\[ = e^{-(T-t)r} \mathbb{E}^*[S_T \mid \mathcal{F}_t] - K e^{-(T-t)r} \]
\[ = S_t - K e^{-(T-t)r}, \quad t \in [0, T]. \]

b) The price this contract at time \( t \in [0, T] \) can be written as

\[ e^{-(T-t)r} \mathbb{E}^*[P(T, S_T, K, U) \mid \mathcal{F}_t] \]
\[ = e^{-(T-t)r} \mathbb{E}^*[e^{-(U-T)r} \mathbb{E}^*[(K - S_U)^+ \mid \mathcal{F}_T] \mid \mathcal{F}_t] \]
c) From the call-put parity (6.36) the payoff of this contract can be written as
\[
\max(P(T, S_t, K, U), C(T, S_t, K, U))
= \max(P(T, S_t, K, U), P(T, S_t, K, U) + S_t - K e^{-(U-T)r})
= P(T, S_t, K, U) + \max(S_t - K e^{-(U-T)r}, 0).
\]

d) The contract of Question (c) is priced at any time \( t \in [0, T] \) as
\[
e^{-(T-t)r} E^* \left[ \max(P(T, S_t, K, U), C(T, S_t, K, U)) \mid \mathcal{F}_t \right]
= e^{-(T-t)r} E^* [P(T, S_t, K, U) \mid \mathcal{F}_t]
+ e^{-(T-t)r} E^* \left[ \max(S_t - K e^{-(U-T)r}, 0) \mid \mathcal{F}_t \right]
= e^{-(T-t)r} E^* [e^{-(U-T)r} E^* [(K - S_u)^+ \mid \mathcal{T}_t] \mid \mathcal{F}_t]
+ e^{-(T-t)r} E^* \left[ \max(S_t - K e^{-(U-T)r}, 0) \mid \mathcal{F}_t \right]
= e^{-(U-t)r} E^* [(K - S_u)^+ \mid \mathcal{F}_t] + e^{-(T-t)r} E^* \left[ \max(S_t - K e^{-(U-T)r}, 0) \mid \mathcal{F}_t \right]
= P(t, S_t, K, U) + C(t, S_t, K e^{-(U-T)r}, T). \tag{A.20}
\]

Fig. S.22: Black-Scholes price of the maximum chooser option.

e) By (A.20) and Relation (5.14) in Proposition 5.12 we have
\[
\xi_t = \frac{\partial C}{\partial x}(t, S_t, K e^{-(U-T)r}, T) + \frac{\partial P}{\partial x}(t, S_t, K, U)
= \Phi \left( \frac{\log(e^{(U-T)r} S_t / K) + (r + \sigma^2 / 2)(T - t)}{\sigma \sqrt{T - t}} \right)
\]
f) From the call-put parity (6.36) the payoff of this contract can be written as

\[
\begin{align*}
\min(P(T, S_T, K, U), C(T, S_T, K, U)) \\
= \min(C(T, S_T, K, U) - S_T + K e^{-(U-T)r}, C(T, S_T, K, U)) \\
= C(T, S_T, K, U) + \min(-S_T + K e^{-(U-T)r}, 0) \\
= C(T, S_T, K, U) - \max(S_T - K e^{-(U-T)r}, 0).
\end{align*}
\]

\[\begin{align*}
\text{g) The contract of Question (f) is priced at any time } t \in [0, T] \text{ as }
\end{align*}\]

\[
e^{-r(T-t)} \mathbb{E}^* \left[ \min \left( P(T, S_T, K, U), C(T, S_T, K, U) \right) \mid \mathcal{F}_t \right]
= e^{-r(T-t)} \mathbb{E}^* [C(T, S_T, K, U) - S_T + K e^{-(U-T)r}] - e^{-r(T-t)} \mathbb{E}^* \left[ \max \left( S_T - K e^{-(U-T)r}, 0 \right) \mid \mathcal{F}_t \right]
= e^{-r(T-t)} \mathbb{E}^* [e^{-(U-T)r} \mathbb{E}^* [(S_U - K)^+ \mid \mathcal{F}_T] - S_T + K e^{-(U-T)r}] - e^{-r(T-t)} \mathbb{E}^* \left[ \max \left( S_T - K e^{-(U-T)r}, 0 \right) \mid \mathcal{F}_t \right]
= e^{-r(T-t)} \mathbb{E}^* [(S_U - K)^+ \mid \mathcal{F}_t] - e^{-r(T-t)} \mathbb{E}^* \left[ \max \left( S_T - K e^{-(U-T)r}, 0 \right) \mid \mathcal{F}_t \right] - S_T + K e^{-(U-T)r}
= C(t, S_t, K, U) - C(t, S_t, K e^{-(U-T)r} r, T). \\
\text{(A.21)}
\]
h) By (A.21) and Relation (5.14) in Proposition 5.12 we have

\[ \xi_t = \frac{\partial C}{\partial x}(t, S_t, K, U) - \frac{\partial C}{\partial x}(t, S_t, Ke^{-(U-T)r}, T) \]

\[ = \Phi \left( \frac{\log(S_t/K) + (r + \sigma^2/2)(U-t)}{\sigma \sqrt{U-t}} \right) \]

\[ - \Phi \left( \frac{\log(e^{(U-T)r}S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right) \]

\[ = \Phi \left( \frac{\log(S_t/K) + (r + \sigma^2/2)(U-t)}{\sigma \sqrt{U-t}} \right) \]

\[ - \Phi \left( \frac{\log(S_t/K) + (U-t)r + \sigma^2(T-t)/2}{\sigma \sqrt{T-t}} \right). \]

i) Such a contract is priced as the sum of a European call and a European put option with maturity \( U \), and is priced at time \( t \in [0, T] \) as \( P(t, S_t, K, U) + C(t, S_t, K, U) \). Its hedging strategy is the sum of the hedging strategies of Questions (e) and (h), i.e.

Fig. S.24: Black-Scholes price of the minimum chooser option.

Fig. S.25: Delta of the minimum chooser option.
\[ \xi_t = \Phi \left( \frac{\log(S_t/K) + (r + \sigma^2/2)(U - t)}{\sigma\sqrt{T-t}} \right) - \Phi \left( -\frac{\log(S_t/K) + (r + \sigma^2/2)(U - t)}{\sigma\sqrt{T-t}} \right) = 2\Phi \left( \frac{\log(S_t/K) + (r + \sigma^2/2)(U - t)}{\sigma\sqrt{T-t}} \right) - 1. \]

j) When \( U = T \), the contracts of Questions (c), (f) and (i) have respective payoffs
- \( \max((S_T - K)^+, (K - S_T)^+) = |S_T - K| \),
- \( \min((S_T - K)^+, (K - S_T)^+) = 0 \), and
- \( (S_T - K)^+ + (K - S_T)^+ = |S_T - K| \),
where \( |S_T - K| \) is known as the payoff of a straddle option.

Problem 6.15 [Pen10]

a) The self-financing condition reads
\[
dV_t = \eta_t dA_t + \xi_t dS_t = r\eta_t A_t dt + \mu \xi_t S_t dt + \sigma \xi_t S_t dB_t = rV_t dt + (\mu - r) \xi_t S_t dt + \sigma \xi_t S_t dB_t,
\]

hence
\[
V_T = V_0 + \int_0^T (rV_t + (\mu - r) \xi_t S_t) dt + \sigma \int_0^T \xi_t S_t dB_t
= V_t + \int_t^T (rV_s + (\mu - r) \xi_s S_s) ds + \sigma \int_t^T \xi_s S_s dB_s.
\]

b) The portfolio value \( V_t \) rewrites as
\[
V_t = V_T - \int_t^T \left( rV_s + \frac{\mu - r}{\sigma} \pi_s \right) ds - \int_t^T \pi_s dB_s
= V_T - r \int_t^T V_s ds - \int_t^T \pi_s dB_s.
\]

c) We have
\[
V_t = V_T - r \int_t^T V_s ds - \int_t^T \pi_s dB_s,
\]

hence
\[
dV_t = rV_t dt + \pi_t d\hat{B}_t,
\]
and after discounting we find

\[
d\tilde{V}_t = -r e^{-rt} V_t dt + e^{-rt} dV_t \\
= -r e^{-rt} V_t dt + e^{-rt} (r V_t dt + \pi_t d\tilde{B}_t) \\
= e^{-rt} \pi_t d\tilde{B}_t,
\]

which shows that

\[
\tilde{V}_T = V_0 + \int_0^T e^{-rt} \pi_t d\tilde{B}_t,
\]

after integration in \( t \in [0, T] \).

d) We have

\[
dV_t = du(t, S_t) = \frac{\partial u}{\partial t}(t, S_t) dt + \mu S_t \frac{\partial u}{\partial x}(t, S_t) dt + \sigma S_t \frac{\partial u}{\partial x}(t, S_t) d\tilde{B}_t + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 u}{\partial x^2}(t, S_t) dt.
\]

(A.22)
e) By matching the Itô formula (A.22) term by term to the BSDE (6.39) we find that \( V_t = u(t, S_t) \) satisfies the PDE

\[
\frac{\partial u}{\partial t}(t, x) + \mu \frac{\partial u}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 u}{\partial x^2}(t, x) + f \left( t, x, u(t, x), \sigma x \frac{\partial u}{\partial x}(t, x) \right) = 0.
\]
f) In this case we have

\[
\frac{\partial u}{\partial t}(t, x) + \mu x \frac{\partial u}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 u}{\partial x^2}(t, x) - ru(t, x) - (\mu - r) x \frac{\partial u}{\partial x}(t, x) = 0,
\]

which recovers the Black-Scholes PDE

\[
r u(t, x) = \frac{\partial u}{\partial t}(t, x) + \mu x \frac{\partial u}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 u}{\partial x^2}(t, x).
\]
g) In the Black-Scholes model the Delta of the European call option is given by

\[
\xi_t = \Phi \left( \frac{(r + \sigma^2/2)(T-t) + \log(S_t/K)}{\sigma \sqrt{T-t}} \right),
\]

hence

\[
\pi_t = \sigma \xi_t S_t = \sigma S_t \Phi \left( \frac{(r + \sigma^2/2)(T-t) + \log(S_t/K)}{\sigma \sqrt{T-t}} \right), \quad t \in [0, T].
\]
h) Replacing the self-financing condition with

\[
dV_t = \eta_t dA_t + \xi_t dS_t - \gamma S_t(\xi_t)^- dt \\
= r \eta_t A_t dt + \mu \xi_t S_t dt + \sigma \xi_t S_t d\tilde{B}_t - \gamma S_t(\xi_t)^- dt \\
= r V_t dt + (\mu - r) \xi_t S_t dt - \gamma S_t(\xi_t)^- dt + \sigma \xi_t S_t d\tilde{B}_t,
\]
we get the BSDE
\[ V_t = V_T - \int_t^T (rV_s + (\mu - r)\pi_s + \gamma(\pi_s)^-) \, ds - \int_t^T \pi_s dB_s. \]

i) In this case we have
\[ f(t, x, u, z) = -ru - \frac{\mu - r}{\sigma}z - \gamma z^- \]
and the BSDE reads
\[ dV_t = ru(t, S_t) dt + (\mu - r)\xi_t S_t dt - \gamma S_t (\xi_t)^- dt + \sigma \xi_t S_t dB_t. \]

j) We find the nonlinear PDE
\[ \frac{\partial u}{\partial t}(t, x) + \mu \frac{\partial u}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 u}{\partial x^2}(t, x) - \gamma \sigma x \left( \frac{\partial u}{\partial x}(t, x) \right)^- = ru(t, x), \]
with the terminal condition \( u(T, x) = g(x) \).

k) The self-financing condition reads
\[
\begin{align*}
  dV_t &= r\mathbb{1}_{\{\eta > 0\}} A_t \eta dt + R \mathbb{1}_{\{\eta < 0\}} A_t \eta dt + \xi_t dS_t \\
  &= rA_t \eta dt + (R - r) \mathbb{1}_{\{\eta < 0\}} A_t \eta dt + \xi_t dS_t \\
  &= rV_t dt - rS_t \xi_t dt - (R - r)(\eta_t A_t)^- dt + \xi_t dS_t \\
  &= rV_t dt + (\mu - r)S_t \xi_t dt - (R - r)(V_t - \xi_t S_t)^- dt + \sigma \xi_t S_t dB_t,
\end{align*}
\]
which yields the BSDE
\[ V_t = V_T - \int_t^T (rV_s + (\mu - r)\pi_s - (R - r)(V_s - \xi_s S_s)^-) \, ds - \int_t^T \pi_s dB_s, \]
hence we have
\[ f(t, x, u, z) = -ru - \frac{\mu - r}{\sigma}z + (R - r) \left( u - \frac{z}{\sigma} \right)^- \]
and the nonlinear PDE
\[
\begin{align*}
  \frac{\partial u}{\partial t}(t, x) + \mu \frac{\partial u}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 u}{\partial x^2}(t, x) + f(t, x, u(t, x), \sigma x \frac{\partial u}{\partial x}(t, x)) &= 0,
\end{align*}
\]
rewrites as
\[
\begin{align*}
  \frac{\partial u}{\partial t}(t, x) + r \frac{\partial u}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 u}{\partial x^2}(t, x) &= ru(t, x) + (r - R) \left( u(t, x) - x \frac{\partial u}{\partial x}(t, x) \right)^-.
\end{align*}
\]

l) The sum of profits and losses of the portfolio \((\xi_t, \eta_t)_{t \in \mathbb{R}_+}\) is
\[ V_0 + \int_0^T \eta_t dA_t + \int_0^T \xi_t dS_t = V_0 + \int_0^T dV_t + \int_0^T dU_t = V_T + U_T - U_0 > V_T = C, \]

hence the corresponding portfolio strategy superhedging the claim \( V_T = C \).

Exercise 6.16 Girsanov theorem. For all \( n \geq 1 \), let

\[ \psi_t^{(n)} := \mathbb{1}_{\{ \psi_t \in [-n,n] \}} \psi_t, \quad 0 \leq t \leq T. \]

Since \( (\psi_t^{(n)})_{t \in [0,T]} \) is a bounded process it satisfies the Novikov integrability condition (6.8), hence for all \( n \geq 1 \) and random variable \( F \in L^1(\Omega) \) we have

\[ \mathbb{E}[F] = \mathbb{E} \left[ F \left( B + \int_0^T \psi_s^{(n)} ds \right) \exp \left( - \int_0^T \psi_s^{(n)} dB_s - \frac{1}{2} \int_0^T (\psi_s^{(n)})^2 ds \right) \right], \]

which yields

\[ \mathbb{E}[F] = \lim_{n \to \infty} \mathbb{E} \left[ F \left( B + \int_0^T \psi_s^{(n)} ds \right) \exp \left( - \int_0^T \psi_s^{(n)} dB_s - \frac{1}{2} \int_0^T (\psi_s^{(n)})^2 ds \right) \right] \]

\[ \geq \mathbb{E} \left[ \liminf_{n \to \infty} F \left( B + \int_0^T \psi_s^{(n)} ds \right) \exp \left( - \int_0^T \psi_s^{(n)} dB_s - \frac{1}{2} \int_0^T (\psi_s^{(n)})^2 ds \right) \right] \]

\[ = \mathbb{E} \left[ F \left( B + \int_0^T \psi_s ds \right) \exp \left( - \int_0^T \psi_s dB_s - \frac{1}{2} \int_0^T (\psi_s)^2 ds \right) \right], \]

where we applied Fatou’s Lemma 18.1.

Chapter 7

Exercise 7.1 We need to compute the average

\[ \frac{1}{T} \mathbb{E} \left[ \int_0^T v_t dt \right] = \frac{1}{T} \int_0^T \mathbb{E}[v_t] dt = \frac{1}{T} \int_0^T u(t) dt, \]

where \( u(t) := \mathbb{E}[v_t] \). Taking expectation on both sides of the equation

\[ v_t = v_0 - \lambda \int_0^t (v_s - m) ds + \eta \int_0^t \sqrt{v_s} dB_s, \]

we find

\[ u(t) = \mathbb{E}[v_t] \]

\[ = \mathbb{E} \left[ v_0 - \lambda \int_0^t (v_s - m) ds + \eta \int_0^t \sqrt{v_s} dB_s \right] \]

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\[ \begin{align*}
&= v_0 - \lambda \mathbb{E} \left[ \int_0^t (v_s - m) \, ds \right] \\
&= v_0 - \lambda \int_0^t (\mathbb{E}[v_s] - m) \, ds \\
&= v_0 - \lambda \int_0^t (u(s) - m) \, ds, \quad t \geq 0,
\end{align*} \]

hence by differentiation with respect to \( t \in \mathbb{R} \) we find the ordinary differential equation
\[
 u'(t) = \lambda m - \lambda u(t),
\]
cf. e.g. Exercise 4.18-(b). This equation can be rewritten as
\[
(\lambda t u(t))' = \lambda e^{\lambda t} u(t) + e^{\lambda t} u'(t) = \lambda m e^{\lambda t},
\]
which can be integrated as
\[
e^{\lambda t} u(t) = \left( u(0) + \lambda \int_0^t e^{\lambda s} ds \right) \\
= \mathbb{E}[v_0] + m (e^{\lambda t} - 1) \\
= m e^{\lambda t} + \mathbb{E}[v_0] - m \quad t \in \mathbb{R}_+,
\]
from which we conclude that
\[
u(t) = m + (\mathbb{E}[v_0] - m) e^{-\lambda t}, \quad t \in \mathbb{R}_+,
\]
and
\[
\frac{1}{T} \mathbb{E} \left[ \int_0^T v_t \, dt \right] = \frac{1}{T} \int_0^T u(t) \, dt \\
= \frac{1}{T} \int_0^T (m + (\mathbb{E}[v_0] - m) e^{-\lambda t}) \, dt \\
= m + \frac{\mathbb{E}[v_0] - m}{T} \int_0^T e^{-\lambda t} \, dt \\
= m + (\mathbb{E}[v_0] - m) \frac{1 - e^{-\lambda T}}{\lambda T}.
\]

Exercise 7.2
a) We have
\[
\text{VS}_T = \frac{1}{T} \mathbb{E} \left[ \int_0^T \frac{1}{S_t^2} (dS_t)^2 \right] \\
= \frac{1}{T} \mathbb{E} \left[ \int_0^T \frac{1}{S_t^2} \left( (r - \alpha v_t)S_t \, dt + S_t \sqrt{\beta + v_t d_{t(1)}} \right)^2 \right]
\]
\[
= \frac{1}{T} \mathbb{E} \left[ \int_0^T (\beta + v_t) dt \right]
= \beta + \frac{1}{T} \int_0^T \mathbb{E}[v_t] dt,
\]

with
\[
\mathbb{E}[v_t] = \mathbb{E}[v_0] e^{-\lambda t} + m(1 - e^{-\lambda t}), \quad t \in \mathbb{R}_+,
\]
cf. \textit{e.g.} Exercise 4.18-(b), hence
\[
VST = \beta + \frac{1}{T} \int_0^T (\mathbb{E}[v_0] e^{-\lambda t} + m(1 - e^{-\lambda t})) dt
= \beta + \frac{1}{T} \int_0^T \mathbb{E}[v_0] e^{-\lambda t} + m(1 - e^{-\lambda t}) dt
= \beta + m + \frac{1}{T} (\mathbb{E}[v_0] - m) \int_0^T e^{-\lambda t} dt
= \beta + m + (\mathbb{E}[v_0] - m) \frac{e^{\lambda T} - 1}{\lambda T}.
\]

Note that if the process \((v_t)_{t \in \mathbb{R}_+}\) is started in the gamma stationary distribution then we have \(\mathbb{E}[v_0] = \mathbb{E}[v_t] = m, \ t \in \mathbb{R}_+,\) and the variance swap rate \(VST = \beta + m\) becomes independent of the time \(T.\)

b) We have
\[
VST = \frac{1}{T} \mathbb{E} \left[ \int_0^T \frac{1}{S_t^2} (dS_t)^2 \right]
= \frac{1}{T} \mathbb{E} \left[ \int_0^T \frac{1}{S_t^2} \left( \sigma_t S_t dB_t^{(1)} \right)^2 \right]
= \frac{1}{T} \mathbb{E} \left[ \int_0^T \sigma_t^2 dt \right]
= \frac{\sigma_0^2}{T} \int_0^T \mathbb{E} \left[ e^{2\alpha B_t^{(2)} - \alpha^2 t} \right] dt
= \frac{\sigma_0^2}{T} \int_0^T e^{-\alpha^2 t} \mathbb{E} \left[ e^{2\alpha B_t^{(2)}} \right] dt
= \frac{\sigma_0^2}{T} \int_0^T e^{-\alpha^2 t + 2\alpha^2 t} dt
= \frac{\sigma_0^2}{T} \int_0^T e^{\alpha^2 t} dt
= \frac{\sigma_0^2}{\alpha^2 T} (e^{\alpha^2 T} - 1).
\]

Exercise 7.3

a) We have

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\[ \frac{\partial M}{\partial K}(K, S, r, \tau) = \frac{\partial C}{\partial K}(K, S, \sigma_{\text{imp}}(K), r, \tau) + \sigma'_{\text{imp}}(K) \frac{\partial C}{\partial \sigma}(K, S, \sigma_{\text{imp}}(K), r, \tau). \]

b) We have

\[ \frac{\partial C}{\partial K}(K, S, \sigma_{\text{imp}}(K), r, \tau) + \sigma'_{\text{imp}}(K) \frac{\partial C}{\partial \sigma}(K, S, \sigma_{\text{imp}}(K), r, \tau) \leq 0, \]

which shows that

\[ \sigma'_{\text{imp}}(K) \leq -\frac{\frac{\partial C}{\partial K}(K, S, \sigma_{\text{imp}}(K), r, \tau)}{\frac{\partial C}{\partial \sigma}(K, S, \sigma_{\text{imp}}(K), r, \tau)}. \]

c) We have

\[ \frac{\partial P}{\partial K}(K, S, \sigma_{\text{imp}}(K), r, \tau) + \sigma'_{\text{imp}}(K) \frac{\partial P}{\partial \sigma}(K, S, \sigma_{\text{imp}}(K), r, \tau) \geq 0, \]

which shows that

\[ \sigma'_{\text{imp}}(K) \geq -\frac{\frac{\partial P}{\partial K}(K, S, \sigma_{\text{imp}}(K), r, \tau)}{\frac{\partial P}{\partial \sigma}(K, S, \sigma_{\text{imp}}(K), r, \tau)}. \]

Exercise 7.4

a) We have

\[ \sigma_{\text{imp}}(K, S) \simeq \sigma_{\text{loc}}((K + S)/2) \]
\[ = \sigma_0 + \beta((K + S)/2 - S_0)^2 \]
\[ = \sigma_0 + \frac{\beta}{4}(K - (2S_0 - S))^2. \]

(b) Out of the money \( K > S_0. \)

Fig. S.26: Implied vs local volatility.

b) We find

\[ \sigma'_{\text{imp}}(K) \leq -\frac{\frac{\partial C}{\partial K}(K, S, \sigma_{\text{imp}}(K), r, \tau)}{\frac{\partial C}{\partial \sigma}(K, S, \sigma_{\text{imp}}(K), r, \tau)}. \]
\[
\frac{\partial}{\partial S} \left( (S, K, T, \sigma_{\text{imp}}(K, S), r) \right) = \frac{\partial \mathcal{B}}{\partial x} (x, K, T, \sigma_{\text{imp}}(K, S), r)_{x=S} \]
\[
+ \frac{\partial \sigma_{\text{imp}}}{\partial S} \frac{\partial \mathcal{B}}{\partial \sigma} (x, K, T, \sigma, r)_{\sigma=\sigma_{\text{imp}}(K, S)}
\]
\[
= \Delta + \nu \frac{\beta}{2} (K - (2S_0 - S)),
\]
where
\[
\Delta = \frac{\partial \mathcal{B}}{\partial x} (x, K, T, \sigma_{\text{imp}}(K, S), r)_{x=S}
\]
is the Black-Scholes Delta and
\[
\nu = \frac{\partial \mathcal{B}}{\partial \sigma} (S, K, T, \sigma, r)_{\sigma=\sigma_{\text{imp}}(K, S)}
\]
is the Black-Scholes Vega, cf. §2.2 of [HKLW02].

Exercise 7.5
a) By the Itô formula we have
\[
\log \frac{S_T}{S_0} = \log S_T - \log S_0 = \int_0^T \frac{dS_t}{S_t} - \frac{1}{2} \int_0^T \frac{\sigma_t^2}{S_t^2} dt.
\]

b) By (7.48) we have
\[
\mathbb{E}^* \left[ \int_0^T \sigma_t^2 dt \mid \mathcal{F}_t \right] = 2 \mathbb{E}^* \left[ \int_0^T \frac{dS_t}{S_t} \mid \mathcal{F}_t \right] - 2 \mathbb{E}^* \left[ \log \frac{S_T}{S_0} \mid \mathcal{F}_t \right]
\]
\[
= 2 \int_0^T \frac{dS_u}{S_u} + 2r(T-t) - 2 \mathbb{E}^* \left[ \log \frac{S_T}{S_0} \mid \mathcal{F}_t \right].
\]

c) At time \( t \in [0, T] \) we check that
\[
L_t + e^{-(T-t)r} \frac{2}{S_t} S_t + 2 e^{-rT} \left( \int_0^t \frac{dS_u}{S_u} + (T-t)r - 1 \right) A_t
\]
\[
= L_t + 2r(T-t) e^{-(T-t)r} + 2 e^{-(T-t)r} \int_0^t \frac{dS_u}{S_u}
\]
\[
= V_t.
\]
d) By (7.49) we have
\[
dV_t = d \left( L_t + 2r(T-t) e^{-(T-t)r} + 2 e^{-(T-t)r} \int_0^t \frac{dS_u}{S_u} \right)
\]
\[
= dL_t - 2r e^{-(T-t)r} dt + 2r^2 (T-t) e^{-(T-t)r} dt
\]

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\[
+2r e^{-(T-t)r} \int_0^t \frac{dS_u}{S_u} dt + 2 e^{-(T-t)r} \frac{dS_t}{S_t} = dL_t + e^{-(T-t)r} \frac{2}{S_t} dS_t + 2e^{-rT} \left( \int_0^t \frac{dS_u}{S_u} + (T-t) - 1 \right) dA_t,
\]

with \( dA_t = r e^{rt} dt \), hence the portfolio is self-financing.

**Chapter 8**

**Exercise 8.1**

a) We have \( S_t = S_0 e^{\sigma B_t} \), \( t \in \mathbb{R}_+ \).

b) We have
\[
\mathbb{E}[S_T] = S_0 \mathbb{E}[e^{\sigma B_T}] = S_0 e^{\sigma^2 T/2}.
\]

c) We have
\[
\mathbb{P} \left( \sup_{t \in [0,T]} B_t \geq a \right) = 2 \int_a^\infty e^{-x^2/(2T)} \frac{dx}{\sqrt{2 \pi T}}, \quad a > 0,
\]
i.e. the probability density function \( \varphi \) of \( \sup_{t \in [0,T]} B_t \) is given by
\[
\varphi(a) = \sqrt{\frac{2}{\pi T}} e^{-a^2/(2T)} 1_{[0,\infty)}(a), \quad a \in \mathbb{R}.
\]

d) We have
\[
\mathbb{E}\left[ \hat{S}_T \right] = S_0 \mathbb{E}\left[ \exp \left( \sigma \sup_{t \in [0,T]} B_t \right) \right] = S_0 \int_0^\infty e^{\sigma x} \varphi(x) dx
\]
\[
= \frac{2S_0}{\sqrt{2\pi T}} \int_0^\infty e^{\sigma x - x^2/(2T)} dx = \frac{2S_0}{\sqrt{2\pi \sigma^2 T}} \int_0^\infty e^{-(x-\sigma T)^2/(2T) + \sigma^2 T/2} dx
\]
\[
= \frac{2S_0}{\sqrt{2\pi T}} e^{\sigma^2 T/2} \int_{\sigma T}^\infty e^{-x^2/(2T)} dx = \frac{2S_0}{\sqrt{2\pi}} e^{\sigma^2 T/2} \int_{\sigma T}^\infty e^{x^2/2} dx
\]
\[
= \frac{2S_0}{\sqrt{2\pi T}} e^{\sigma^2 T/2} \int_{-\infty}^{\sigma \sqrt{T}} e^{-x^2/2} dx
\]
\[
= 2S_0 e^{\sigma^2 T/2} \Phi(\sigma \sqrt{T})
\]
\[
= 2 \mathbb{E}[S_T] \Phi(\sigma \sqrt{T}).
\]

**Remarks:**

(i) From the inequality
\[
0 \leq \mathbb{E}[(B_T - \sigma T)^+] \]
\[
\begin{align*}
&= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} (x - \sigma T)^+ e^{-x^2/(2T)} dx \\
&= - \frac{1}{\sqrt{2\pi T}} \int_{\sigma T}^{\infty} (x - \sigma T)^+ e^{-x^2/(2T)} dx \\
&= \frac{1}{\sqrt{2\pi T}} \int_{\sigma T}^{\infty} xe^{-x^2/(2T)} dx - \frac{\sigma T}{\sqrt{2\pi T}} \int_{\sigma T}^{\infty} e^{-x^2/(2T)} dx \\
&= \sqrt{\frac{T}{2\pi}} \int_{\sigma \sqrt{T}}^{\infty} xe^{-x^2/2} dx - \frac{\sigma T}{\sqrt{2\pi}} \int_{\sigma \sqrt{T}}^{\infty} e^{-x^2/2} dx \\
&= \sqrt{\frac{T}{2\pi}} \left[ e^{-x^2/2} \right]_{\sigma \sqrt{T}}^{\infty} - \sigma T \Phi(-\sigma \sqrt{T}) \\
&= \sqrt{\frac{T}{2\pi}} e^{-\sigma^2 T/2} - \sigma T \left( 1 - \Phi(\sigma \sqrt{T}) \right),
\end{align*}
\]

we get
\[
\Phi(\sigma \sqrt{T}) \geq 1 - \frac{e^{-\sigma^2 T/2}}{\sigma \sqrt{2\pi T}},
\]

hence
\[
\mathbb{E} \left[ \hat{S}_T \right] = 2S_0 e^{\sigma^2 T/2} \Phi(\sigma \sqrt{T}) \\
\geq 2S_0 e^{\sigma^2 T/2} \left( 1 - \frac{e^{-\sigma^2 T/2}}{\sigma \sqrt{2\pi T}} \right) \\
= 2 \mathbb{E}[S_T] \left( 1 - \frac{e^{-\sigma^2 T/2}}{\sigma \sqrt{2\pi T}} \right) \\
= 2S_0 \left( e^{\sigma^2 T/2} - \frac{1}{\sigma \sqrt{2\pi T}} \right).
\]

(ii) We observe that the ratio between the expected gains by selling at the maximum and selling at time \( T \) is given by \( 2\Phi(\sigma \sqrt{T}) \), which cannot be greater than 2.
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Fig. S.27: Average return by selling at the maximum vs selling at maturity.

e) By a symmetry argument we have

$$
\begin{align*}
\mathbb{P} \left( \inf_{t \in [0,T]} B_t \leq a \right) &= \mathbb{P} \left( - \sup_{t \in [0,T]} (-B_t) \leq a \right) \\
&= \mathbb{P} \left( - \sup_{t \in [0,T]} B_t \leq a \right) \\
&= \mathbb{P} \left( \sup_{t \in [0,T]} B_t \geq -a \right) \\
&= 2 \int_{-a}^{\infty} e^{-x^2/(2T)} \frac{dx}{\sqrt{2\pi T}}, \quad a < 0,
\end{align*}
$$

i.e. the probability density function $\varphi$ of $\inf_{t \in [0,T]} B_t$ is given by

$$
\varphi(a) = \sqrt{\frac{2}{\pi T}} e^{-a^2/(2T)} \mathbb{I}_{(-\infty,0]}(a), \quad a \in \mathbb{R}.
$$

f) We have

$$
\begin{align*}
\mathbb{E} \left[ \tilde{S}_T \right] &= S_0 \mathbb{E} \left[ \exp \left( \sigma \inf_{t \in [0,T]} B_t \right) \right] \\
&= S_0 \int_{-\infty}^{0} e^{\sigma x} \varphi(x) dx \\
&= \frac{2S_0}{\sqrt{2\pi T}} \int_{-\infty}^{0} e^{\sigma x - x^2/(2T)} dx \\
&= \frac{2S_0}{\sqrt{2\pi T}} \int_{-\infty}^{0} e^{-(x-\sigma T)^2/(2T) + \sigma^2 T/2} dx \\
&= \frac{2S_0}{\sqrt{2\pi T}} e^{\sigma^2 T/2} \int_{-\infty}^{-\sigma T} e^{-x^2/(2T)} dx
\end{align*}
$$
\[ N. \text{Privault} \]

\[
\begin{align*}
&= \frac{2S_0}{\sqrt{2\pi}} e^{\sigma^2 T/2} \int_{-\infty}^{-\sigma\sqrt{T}} e^{-x^2/2} dx \\
&= 2S_0 e^{\sigma^2 T/2} \Phi(-\sigma\sqrt{T}) \\
&= 2 \mathbb{E}[S_T] \Phi(-\sigma\sqrt{T}).
\end{align*}
\]

Remarks:

(i) From the inequality

\[
0 \leq \mathbb{E}[(\sigma T - B_T)^+] \\
= \frac{1}{\sqrt{2 \pi T}} \int_{-\infty}^{\infty} (\sigma T - x)^+ e^{-x^2/(2T)} dx \\
= - \frac{1}{\sqrt{2 \pi T}} \int_{-\infty}^{-\sigma T} (\sigma T + x) e^{-x^2/(2T)} dx \\
= - \frac{\sigma T}{\sqrt{2 \pi T}} \int_{-\infty}^{-\sigma T} e^{-x^2/(2T)} dx - \frac{1}{\sqrt{2 \pi T}} \int_{-\infty}^{-\sigma T} x e^{-x^2/(2T)} dx \\
= -\sigma T \Phi(-\sigma\sqrt{T}) + \sqrt{\frac{T}{2 \pi}} \left[ e^{-x^2/2} \right]_{-\infty}^{-\sigma\sqrt{T}} \\
= -\sigma T \Phi(-\sigma\sqrt{T}) + \sqrt{\frac{T}{2 \pi}} e^{-\sigma^2 T/2},
\]

we get

\[ e^{\sigma^2 T/2} \Phi(-\sigma\sqrt{T}) \leq \frac{1}{\sigma \sqrt{2 \pi T}}, \quad \text{hence} \quad \mathbb{E}[\tilde{S}_T] \leq \frac{2S_0}{\sigma \sqrt{2 \pi T}}. \]

(ii) The ratio between the expected gains by maturity \( T \) vs selling at the minimum is given by \( 2 \Phi(-\sigma\sqrt{T}) \), which is at most 1 and tends to 0 as \( \sigma \) and \( T \) tend to infinity.

Fig. S.28: Average returns by selling at the minimum vs selling at maturity.

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(iii) Given that \( \mathbb{E} [\hat{S}_T] = 2 \mathbb{E}[S_T] \Phi(\sigma \sqrt{T}) \), we find the bound

\[
2 \mathbb{E}[S_T] \Phi(-\sigma \sqrt{T}) \leq \mathbb{E}[S_T] \leq 2 \mathbb{E}[S_T] \Phi(\sigma \sqrt{T}),
\]

with equality if \( \sigma = 0 \) or \( T = 0 \). We also have

\[
2 \mathbb{E}[S_T] - \mathbb{E} [\hat{S}_T] = 2 e^{\sigma^2 T/2} (1 - \Phi(\sigma \sqrt{T})) = 2 e^{\sigma^2 T/2} \Phi(-\sigma \sqrt{T}) = \mathbb{E} [\hat{S}_T],
\]

hence

\[
\mathbb{E} [\hat{S}_T] + \mathbb{E} [\hat{S}_T] = 2 \mathbb{E}[S_T], \quad \text{or} \quad \mathbb{E}[S_T] - \mathbb{E} [\hat{S}_T] = \mathbb{E} [\hat{S}_T] - \mathbb{E}[S_T],
\]

and

\[
2 \mathbb{E}[S_T] - \frac{2S_0}{\sigma \sqrt{2\pi T}} \leq \mathbb{E} [\hat{S}_T] \leq 2 \mathbb{E}[S_T].
\]

g) Regarding call option prices we have, assuming \( K \geq S_0 \),

\[
\mathbb{E} [((\hat{S}_T - K)^+) = S_0 \mathbb{E} \left[ \left( \exp \left( \sigma \max_{t \in [0,T]} B_t \right) - K \right)^+ \right]
\]

\[
= \int_0^\infty (S_0 e^{\sigma x} - K)^+ \varphi(x) dx
\]

\[
= \frac{2}{\sqrt{2\pi T}} \int_0^\infty (S_0 e^{\sigma x} - K)^+ e^{-x^2/(2T)} dx
\]

\[
= \frac{2}{\sqrt{2\pi T}} \int_{\sigma^{-1} \log(K/S_0)}^\infty (S_0 e^{\sigma x} - K) e^{-x^2/(2T)} dx
\]

\[
= \frac{2S_0}{\sqrt{2\pi T}} \int_{\sigma^{-1} \log(K/S_0)}^\infty e^{\sigma x - x^2/(2T)} dx - \frac{2K}{\sqrt{2\pi T}} \int_{\sigma^{-1} \log(K/S_0)}^\infty e^{-x^2/(2T)} dx
\]

\[
= \frac{2S_0}{\sqrt{2\pi T}} \int_{\sigma^{-1} \log(K/S_0)}^\infty e^{-(x-\sigma T)^2/(2T)+\sigma^2 T/2} dx - \frac{2K}{\sqrt{2\pi T}} \int_{\sigma^{-1} \log(K/S_0)}^\infty e^{-x^2/(2T)} dx
\]

\[
= \frac{2S_0}{\sqrt{2\pi T}} e^{\sigma^2 T/2} \int_{-\sigma T+\sigma^{-1} \log(K/S_0)}^\infty e^{-x^2/(2T)} dx - \frac{2K}{\sqrt{2\pi T}} \int_{\sigma^{-1} \log(K/S_0)}^\infty e^{-x^2/(2T)} dx
\]

\[
= 2S_0 e^{\sigma^2 T/2} \Phi((\sigma T + \sigma^{-1} \log(S_0/K))/\sqrt{T}) - 2K \Phi(\sigma^{-1} \log(S_0/K)/\sqrt{T}),
\]

with

\[
e^{-\sigma^2 T/2} \mathbb{E} [((\hat{S}_T - K)^+] = 2S_0 \Phi(\sigma \sqrt{T}) - K e^{-\sigma^2 T/2}
\]

if \( K \leq S_0 \). Recall that when \( r = \sigma^2/2 \) the price of the finite expiration American call option price is the Black-Scholes price with maturity \( T \), with

\[
\text{BlCall}(S_0, K, \sigma, r, T)
\]

\[
= S_0 \Phi((\sigma T + \sigma^{-1} \log(S_0/K))/\sqrt{T}) - K e^{-\sigma^2 T/2} \Phi(\sigma^{-1} \log(S_0/K)/\sqrt{T})
\]
\[
\begin{cases}
2S_0 \Phi\left((\sigma T + \sigma^{-1} \log(S_0/K))/\sqrt{T}\right) - 2K e^{-\sigma^2 T/2} \Phi\left(\sigma^{-1} \log(S_0/K)/\sqrt{T}\right) \\
2S_0 \Phi(\sigma \sqrt{T}) - Ke^{-\sigma^2 T/2} 
\end{cases}
\]

if \(K \geq S_0\),

\[
\begin{cases}
2 \times \text{BlCall}(S_0, K, \sigma, r, T) \\
2S_0 \Phi(\sigma \sqrt{T}) - Ke^{-\sigma^2 T/2}
\end{cases}
\]

if \(K \leq S_0\).

\[
= \max\left(2 \times \text{BlCall}(S_0, K, \sigma, r, T), 2S_0 \Phi(\sigma \sqrt{T}) - Ke^{-\sigma^2 T/2}\right).
\]

---

h) Regarding put option prices we have, assuming \(S_0 \geq K\),

\[
\mathbb{E}\left[(K - \hat{S}_T)^+\right] = S_0 \mathbb{E}\left[(K - \exp\left(\sigma \min_{t \in [0, T]} B_t^\prime\right))^+\right]
\]

\[
= \int_0^{\infty} (K - S_0 e^{\sigma x})^+ \varphi(x)dx
\]

\[
= \frac{2}{\sqrt{2\pi T}} \int_{-\infty}^{0} (K - S_0 e^{\sigma x})^+ e^{-x^2/(2T)}dx
\]

\[
= \frac{2}{\sqrt{2\pi T}} \int_{-\infty}^{\sigma^{-1} \log(K/S_0)} (K - S_0 e^{\sigma x}) e^{-x^2/(2T)}dx
\]

\[
= \frac{2K}{\sqrt{2\pi T}} \int_{-\infty}^{\sigma^{-1} \log(K/S_0)} e^{-x^2/(2T)}dx - \frac{2S_0}{\sqrt{2\pi T}} \int_{-\infty}^{\sigma^{-1} \log(K/S_0)} e^{\sigma x - x^2/(2T)}dx
\]

\[
= \frac{2K}{\sqrt{2\pi T}} \int_{-\infty}^{\sigma^{-1} \log(K/S_0)} e^{-x^2/(2T)}dx - \frac{2S_0}{\sqrt{2\pi T}} \int_{-\infty}^{\sigma^{-1} \log(K/S_0)} e^{-(x-\sigma T)^2/(2T)+\sigma^2 T/2}dx
\]

\[
= \frac{2K}{\sqrt{2\pi T}} \int_{-\infty}^{\sigma^{-1} \log(K/S_0)} e^{-x^2/(2T)}dx.
\]

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\[ = 2K \Phi(-\sigma^{-1} \log(S_0/K)/\sqrt{T}) - 2S_0 e^{\sigma^2 T/2} \Phi(-(\sigma T + \sigma^{-1} \log(S_0/K))/\sqrt{T}), \]

with

\[ e^{-\sigma^2 T/2} \mathbb{E} [(K - \tilde{S}_T)^+] = K e^{-\sigma^2 T/2} - 2S_0 \Phi(-\sigma \sqrt{T}) \]

if \( S_0 \leq K \). Therefore we deduce the bounds

\[ \text{Bl}_{\text{Put}}(S_0, K, \sigma, r, T) = K e^{-\sigma^2 T/2} \Phi(-\sigma^{-1} \log(S_0/K)/\sqrt{T}) - S_0 \Phi(-(\sigma T + \sigma^{-1} \log(S_0/K))/\sqrt{T}) \]

\[ \leq \text{American put option price} \]

\[ \leq \left\{ \begin{array}{ll}
2K e^{-\sigma^2 T/2} \Phi(-\sigma^{-1} \log(S_0/K)/\sqrt{T}) - 2S_0 \Phi(-(\sigma T + \sigma^{-1} \log(S_0/K))/\sqrt{T}) & \text{if } S_0 \geq K, \\
K e^{-\sigma^2 T/2} - 2S_0 \Phi(-\sigma \sqrt{T}) & \text{if } S_0 \leq K,
\end{array} \right. \]

\[ = \left\{ \begin{array}{ll}
2 \times \text{Bl}_{\text{Put}}(S_0, K, \sigma, r, T) & \text{if } S_0 \geq K, \\
K e^{-\sigma^2 T/2} - 2S_0 \Phi(-\sigma \sqrt{T}) & \text{if } S_0 \leq K,
\end{array} \right. \]

\[ = \max \left( 2 \times \text{Bl}_{\text{Put}}(S_0, K, \sigma, r, T), K e^{-\sigma^2 T/2} - 2S_0 \Phi(-\sigma \sqrt{T}) \right) \]

for the finite expiration American put option price when \( r = \sigma^2/2 \).

Fig. S.30: Black-Scholes put upper bound with \( S_0 = 1 \).

Exercise 8.2

a) We have
\[ P(\tau_a \geq t) = P(X_t > a) = \int_a^\infty \varphi_{X_t}(x) \, dx = \sqrt{\frac{2}{\pi t}} \int_y^\infty e^{-x^2/(2t)} \, dx, \]

\( y > 0. \)

b) We have

\[ \varphi_{\tau_a}(t) = \frac{d}{dt} P(\tau_a \leq t) = \frac{d}{dt} \int_a^\infty \varphi_{X_t}(x) \, dx \]

\[ = -\frac{1}{2} \sqrt{\frac{2}{\pi}} t^{-3/2} \int_a^\infty e^{-x^2/(2t)} \, dx + \frac{1}{2} \sqrt{\frac{2}{\pi}} t^{-3/2} \int_a^\infty \frac{x^2}{t} e^{-x^2/(2t)} \, dx \]

\[ = \frac{1}{2} \sqrt{\frac{2}{\pi}} t^{-3/2} \left( -\int_a^\infty e^{-x^2/(2t)} \, dx + a e^{-a^2/(2t)} + \int_a^\infty e^{-x^2/(2t)} \, dx \right) \]

\[ = \frac{a}{\sqrt{2\pi t^3}} e^{-a^2/(2t)}, \quad t > 0. \]

c) We have

\[ \mathbb{E} \left[ (\tau_a)^{-2} \right] = \frac{a}{\sqrt{2\pi}} \int_0^\infty t^{-5/2} e^{-a^2/(2t)} \, dt \]

\[ = \frac{2a}{\sqrt{2\pi}} \int_0^\infty x^2 e^{-a^2x^2/2} \, dx \]

\[ = \frac{1}{a^2}, \]

by the change of variable \( x = t^{-1/2}, \, x^2 = 1/t, \, t = x^{-2}, \, dt = -2x^{-3} \, dx. \)

Remark: We have

\[ \mathbb{E}[\tau_a] = \frac{a}{\sqrt{2\pi}} \int_0^\infty t^{-1/2} e^{-a^2/(2t)} \, dt = +\infty. \]

Exercise 8.3 Barrier options.

a) By (9.17) and (8.34) we find

\[ \xi_t = \frac{\partial g}{\partial y}(t, S_t) = \Phi \left( \delta_{+}^{T-t} \left( \frac{S_t}{K} \right) \right) - \Phi \left( \delta_{+}^{T-t} \left( \frac{S_t}{B} \right) \right) \]

\[ + \frac{K}{B} e^{-(T-t)r} \left( 1 - \frac{2r}{\sigma^2} \right) \left( \frac{S_t}{B} \right)^{-2r/\sigma^2} \left( \Phi \left( \delta_{+}^{T-t} \left( \frac{B^2}{KS_t} \right) \right) - \Phi \left( \delta_{+}^{T-t} \left( \frac{B}{S_t} \right) \right) \right) \]

\[ + \frac{2r}{\sigma^2} \left( \frac{S_t}{B} \right)^{-1-2r/\sigma^2} \left( \Phi \left( \delta_{+}^{T-t} \left( \frac{B^2}{KS_t} \right) \right) - \Phi \left( \delta_{+}^{T-t} \left( \frac{B}{S_t} \right) \right) \right) \]
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\[- \frac{2}{\sigma \sqrt{2\pi (T - t)}} \left( 1 - \frac{K}{B} \right) \exp \left( - \frac{1}{2} \left( \frac{\delta_{T-t} \left( \frac{S_t}{B} \right)}{\sigma} \right)^2 \right), \]

\(0 < S_t \leq B, \ 0 \leq t \leq T,\) cf. also Exercise 7.1-(ix) of [Shr04] and Figure 8.19 above.

b) We find

\[\mathbb{P}(Y_T \leq a \ & \ B_T \geq b) = \mathbb{P}(B_T \leq 2a - b), \quad a < b < 0,\]

hence

\[f_{Y_T, B_T}(a, b) = \frac{d\mathbb{P}(Y_T \leq a \ & \ B_T \leq b)}{dadb} = - \frac{d\mathbb{P}(Y_T \leq a \ & \ B_T \geq b)}{dadb},\]

\(a, b \in \mathbb{R},\) satisfies

\[f_{Y_T, B_T}(a, b) = \sqrt{\frac{2}{\pi T}} 1_{(-\infty, \min(0,b)]}(a) \left( \frac{b - 2a}{T} \right) e^{-\frac{(2a-b)^2}{2T}}, \]

\[= \begin{cases} \sqrt{\frac{2}{\pi T}} \left( \frac{b - 2a}{T} \right) e^{-\frac{(2a-b)^2}{2T}}, & a < \min(0,b), \\ 0, & a > \min(0,b). \end{cases}\]

c) We find

\[f_{\tilde{Y}_T, \tilde{B}_T}(a, b) = 1_{(-\infty, \min(0,b)]}(a) \frac{1}{T} \sqrt{\frac{2}{\pi T}} (b - 2a) e^{-\mu^2 T/2 + \mu b - (2a-b)^2/(2T)}\]

\[= \begin{cases} \frac{1}{T} \sqrt{\frac{2}{\pi T}} (2a - b) e^{-\mu^2 T/2 + \mu b - (2a-b)^2/(2T)}, & a < \min(0,b), \\ 0, & a > \min(0,b). \end{cases}\]

d) The function \(g(t, x)\) is given by the Relations (8.24) and (8.25) above.

Exercise 8.4 Barrier forward contracts.

a) Up-and-in barrier long forward contract. We have

\[e^{-(T-t)r} \mathbb{E}[C \mid \mathcal{F}_t] = e^{-(T-t)r} \mathbb{E} \left[ (S_T - K) 1_{\max_{0 \leq u \leq T} S_u > B} \right] \mathcal{F}_t \]
\[
= \mathbb{1}\left\{ \max_{0 \leq u \leq t} S_u > B \right\} (S_t - K e^{-(T-t)r}) + \mathbb{1}\left\{ \max_{0 \leq u \leq t} S_u \leq B \right\} \phi(t, S_t),
\]

(A.24)

where the function
\[
\phi(t, x) := x \Phi\left( \delta^T_t(x/B) \right) - K e^{-(T-t)r} \Phi\left( \delta^T_t(x/B) \right)
+ B (B/x)^{2r/\sigma^2} \Phi\left( -\delta^T_t(B/x) \right)
- K e^{-(T-t)r} (B/x)^{-1+2r/\sigma^2} \Phi\left( -\delta^T_t(B/x) \right)
\]
solves the Black-Scholes PDE with the terminal condition
\[
\phi(T, x) = \left( x - K + \left( B \frac{2r}{x} \phi - \frac{K B}{B} \right) \right) \mathbb{1}_{[B, \infty)}(x),
\]
as in the proof of Proposition 8.4. Note that only the values of \( \phi(t, x) \) with \( x \in [0, B] \) are used for pricing.

Fig. S.31: Price of the up-and-in long forward contract with \( K = 60 < B = 80 \).

As for the hedging strategy, we find
\[
\xi_t = \frac{\partial \phi}{\partial x}(t, S_t) = \Phi\left( \delta^T_t(x/B) \right) + \frac{1}{\sqrt{2\pi}} e^{-\left( \delta^T_t(x/B) \right)^2/2}
- \frac{1}{x \sqrt{2\pi}} K e^{-(T-t)r - \left( \delta^T_t(x/B) \right)^2/2} - \frac{2r}{\sigma^2} (B/x)^{1+2r/\sigma^2} \Phi\left( -\delta^T_t(B/x) \right)
+ \frac{1}{\sqrt{2\pi}} (B/x)^{1+2r/\sigma^2} e^{-\left( \delta^T_t(B/x) \right)^2/2}
- \frac{K (1 - 2r/\sigma^2)}{B} e^{-(T-t)r (B/x)^{2r/\sigma^2} \Phi\left( -\delta^T_t(B/x) \right)}
- \frac{K}{B \sqrt{2\pi}} (B/x)^{2r/\sigma^2} e^{-\left( \delta^T_t(B/x) \right)^2/2}
\]
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\[
= \Phi \left( \delta^{T-t}(x/B) \right) - \frac{2r}{\sigma^2} \left( B/x \right)^{1+2r/\sigma^2} \Phi \left( -\delta^{T-t}(B/x) \right) \\
+ \frac{1}{\sqrt{2\pi}} \left( 1 - K/B \right) \left( e^{-\left( \delta^{T-t}(x/B) \right)^2/2} + \frac{B}{x} e^{-(T-t)r-\left( \delta^{T-t}(x/B) \right)^2/2} \right) \\
- \frac{K}{B} \left( 1 - 2r/\sigma^2 \right) e^{-(T-t)r} \left( B/x \right)^{2r/\sigma^2} \Phi \left( -\delta^{T-t}(B/x) \right),
\]

since by (9.22) we have

\[
e^{-(\delta^-_{T-t}(B/x))^2/2} = e^{r(T-t)}(x/B)^{2r/\sigma^2} e^{-(\delta_-^{T-t}(x/B))^2/2}
\]

and

\[
e^{-(\delta^+_{T-t}(B/x))^2/2} = e^{r(T-t)}(B/x)^{2r/\sigma^2} e^{-(\delta_+^{T-t}(B/x))^2/2}.
\]

Fig. S.32: Delta of the down-and-in long forward contract with \( K = 60 < B = 80 \).

b) Up-and-out barrier long forward contract. We have

\[
e^{-(T-t)r} \mathbb{E}[C | F_t] = e^{-(T-t)r} \mathbb{E} \left[ (S_T - K) 1_{\left\{ \max_{0\leq u\leq T} S_u < B \right\}} \bigg| F_t \right] \\
= 1_{\left\{ \max_{0\leq u\leq T} S_u \leq B \right\}} \phi(t, S_t),
\]

where the function

\[
\phi(t, x) := x \Phi \left( -\delta^+_{T-t}(x/B) \right) - K e^{-(T-t)r} \Phi \left( -\delta^-_{T-t}(x/B) \right) \\
- B(B/x)^{2r/\sigma^2} \Phi \left( -\delta^+_{T-t}(B/x) \right) \\
+ K e^{-(T-t)r} \left( B/x \right)^{-1+2r/\sigma^2} \Phi \left( -\delta^-_{T-t}(B/x) \right)
\]

solves the Black-Scholes PDE with the terminal condition.
\[ \phi(T, x) = (x - K) \mathbb{1}_{[0, B]}(x) - \left( \frac{B}{x} \right)^{2r/\sigma^2} \left( B - x \frac{K}{B} \right) \mathbb{1}_{[B, \infty)}(x). \]

Note that only the values of \( \phi(t, x) \) with \( x \in [B, \infty) \) are used for pricing.

As for the hedging strategy, we find

\[
\xi_t = \frac{\partial \phi}{\partial x}(t, S_t) = \Phi \left( -\delta^T_{+t}(x/B) \right) - \frac{1}{\sqrt{2\pi}} e^{-(\delta^T_{+t}(x/B))^2/2}
+ \frac{1}{x \sqrt{2\pi}} Ke^{-(T-t)r-(\delta^T_{+t}(x/B))^2/2} + \frac{2r}{\sigma^2} (B/x) 1+2r/\sigma^2 \Phi \left( -\delta^T_{+t}(B/x) \right)
- \frac{1}{\sqrt{2\pi}} (B/x)^{1+2r/\sigma^2} e^{-(\delta^T_{+t}(B/x))^2/2}
+ \frac{K}{B(2\pi)} e^{(T-t)r(B/x)2r/\sigma^2} \Phi \left( -\delta^T_{+t}(B/x) \right)
+ \frac{K}{B(2\pi)} e^{-(T-t)r-(\delta^T_{+t}(B/x))^2/2}
= \Phi \left( -\delta^T_{+t}(x/B) \right) + \frac{2r}{\sigma^2} (B/x) 1+2r/\sigma^2 \Phi \left( -\delta^T_{+t}(B/x) \right)
- \frac{1}{\sqrt{2\pi}} e^{-(\delta^T_{+t}(x/B))^2/2} - \frac{1}{x \sqrt{2\pi}} B e^{-(T-t)r-(\delta^T_{+t}(x/B))^2/2}
+ \frac{K}{B(2\pi)} e^{-(\delta^T_{+t}(x/B))^2/2} + \frac{1}{\sqrt{2\pi}} B e^{-(T-t)r-(\delta^T_{+t}(x/B))^2/2}
+ \frac{K}{B(2\pi)} (1-2r/\sigma^2) e^{(T-t)r(B/x)2r/\sigma^2} \Phi \left( -\delta^T_{+t}(B/x) \right)
= \Phi \left( -\delta^T_{+t}(x/B) \right) + \frac{2r}{\sigma^2} (B/x) 1+2r/\sigma^2 \Phi \left( -\delta^T_{+t}(B/x) \right)
- \frac{1}{\sqrt{2\pi}} \left( 1-K/B \right) \left( e^{-(\delta^T_{+t}(x/B))^2/2} + \frac{B}{x} e^{-(T-t)r-(\delta^T_{+t}(x/B))^2/2} \right)
+ \frac{K}{B(2\pi)} \left( 1-2r/\sigma^2 \right) e^{-(T-t)r(B/x)2r/\sigma^2} \Phi \left( -\delta^T_{+t}(B/x) \right),
\]

Fig. S.33: Price of the up-and-out long forward contract with \( K = 60 < B = 80 \).
by (9.22).

Fig. S.34: Delta of the up-and-out long forward contract price with $K = 60 < B = 80$.

c) Down-and-in barrier long forward contract. We have

\[
e^{-r(T-t)} \mathbb{E}[C \mid F_t] = e^{-r(T-t)} \mathbb{E}
\left[
(S_T - K) \mathbb{1}\left\{\min_{0 \leq u \leq T} S_u < B\right\} \bigg| F_t
\right]
\]

\[
= \mathbb{1}\left\{\min_{0 \leq u \leq T} S_u < B\right\} (S_t - K e^{-r(T-t)}) + \mathbb{1}\left\{\min_{0 \leq u \leq T} S_u \geq B\right\} \phi(t, S_t)
\]

(A.26)

where the function

\[
\phi(t, x) := x \Phi\left(-\delta^+T-t(x/B)\right) - K e^{-r(T-t)} \Phi\left(-\delta^-T-t(x/B)\right) + B(B/x)^{2r/\sigma^2} \Phi\left(\delta^+T-t(B/x)\right)
\]

\[
- K e^{-r(T-t)} (B/x)^{-1+2r/\sigma^2} \Phi\left(\delta^-T-t(B/x)\right)
\]

solves the Black-Scholes PDE with the terminal condition

\[
\phi(T, x) = \left(x - K + \left(B/x\right)^{2r/\sigma^2} \left(B - x \frac{K}{B}\right)\right) \mathbb{1}_{[0,B]}(x).
\]
As for the hedging strategy, we find
\[ \xi_t = \frac{\partial \phi}{\partial x}(t, S_t) \]
\[ = \Phi \left( -\delta^T_t(x/B) \right) + \frac{2r}{\sigma^2}(B/x)^{1+2r/\sigma^2} \Phi \left( \delta^T_t(B/x) \right) \]
\[ - \frac{1}{\sqrt{2\pi}} \left( 1 - K/B \right) \left( e^{-(\delta^T_t(x/B))^2/2} + \frac{B}{x} e^{-(T-t)r-(\delta^T_t(x/B))^2/2} \right) \]
\[ + \frac{K}{B} (1 - 2r/\sigma^2) e^{-(T-t)r}(B/x)^{2r/\sigma^2} \Phi \left( \delta^T_t(B/x) \right). \]

**Fig. S.35:** Price of the down-and-in long forward contract with \( K = 60 < B = 80 \).

**d) Down-and-out barrier long forward contract.** We have
\[ e^{-(T-t)r} \mathbb{E}[C \mid \mathcal{F}_t] = e^{-(T-t)r} \mathbb{E} \left[ \left( S_T - K \right) 1 \left\{ \min_{0\leq u\leq T} S_u > B \right\} \mid \mathcal{F}_t \right] \]
\[ = 1 \left\{ \min_{0\leq u\leq t} S_u \geq B \right\} \phi(t, S_t) \]  
(A.27)

where the function

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\[ \phi(t, x) := x \Phi \left( \frac{\delta^T - t}{\delta^T - t} \left( \frac{x}{B} \right) \right) - K e^{-(T-t) \left( \frac{x}{B} \right)} \]

\[-B \left( \frac{B}{x} \right)^{2r/\sigma^2} \Phi \left( \frac{\delta^T - t}{\delta^T - t} \left( \frac{B}{x} \right) \right) \]

\[+K e^{-(T-t) \left( \frac{B}{x} \right)} \left( \frac{B}{x} \right)^{2r/\sigma^2} \Phi \left( \frac{\delta^T - t}{\delta^T - t} \left( \frac{B}{x} \right) \right) \]

solves the Black-Scholes PDE with the terminal condition

\[ \phi(T, x) = (x - K) \mathbb{1}_{[B, \infty)}(x) - \left( B - x \frac{K}{B} \right) \left( \frac{B}{x} \right)^{2r/\sigma^2} \mathbb{1}_{[0, B]}(x). \]

Note that \( \phi(t, x) \) above coincides with the price of (8.25) of the standard down-and-out barrier call option in the case \( K < B \), cf. Exercise 8.3-(d).

Fig. S.37: Price of the down-and-out long forward contract with \( K = 60 < B = 80 \).

As for the hedging strategy, we find

\[ \xi_t = \frac{\partial \phi}{\partial x} (t, S_t) \]

\[ = \Phi \left( \frac{\delta^T - t}{\delta^T - t} \left( \frac{x}{B} \right) \right) - \frac{2r}{\sigma^2} \left( B/x \right)^{1+2r/\sigma^2} \Phi \left( \frac{\delta^T - t}{\delta^T - t} (B/x) \right) \]

\[+ \frac{1}{\sqrt{2\pi}} \left( 1 - \frac{K}{B} \right) \left( e^{-\left( \frac{\delta^T - t (x/B)}{2} \right)^2/2} + \frac{B}{x} e^{-(T-t)(\delta^T - t (x/B))^2/2} \right) \]

\[-\frac{K}{B} \left( 1 - \frac{2r}{\sigma^2} \right) e^{-(T-t)r \left( \frac{B}{x} \right)^{2r/\sigma^2}} \Phi \left( \frac{\delta^T - t}{\delta^T - t} \left( \frac{B}{x} \right) \right). \]
Fig. S.38: Delta of the down-and-out long forward contract with $K = 60 < B = 80$.

e) Up-and-in barrier short forward contract. The price of the up-and-in barrier short forward contract is identical to (A.24) with a negative sign.

f) Up-and-out barrier short forward contract. The price of the up-and-out barrier short forward contract is identical to (A.25) with a negative sign. Note that $\phi(t, x)$ coincides with the price of (8.23) of the standard up-and-out barrier put option in the case $B < K$.

g) Down-and-in barrier short forward contract. The price of the down-and-in barrier short forward contract is identical to (A.26) with a negative sign.

h) Down-and-out barrier short forward contract. The price of the down-and-out barrier short forward contract is identical to (A.27) with a negative sign.

Exercise 8.5 When $B < K$ we find

$$\text{Vega}_{\text{down-and-out-call}} = S_t \sqrt{\frac{T-t}{2\pi}} e^{-\left(\delta_+^{-t}(S_t/K)\right)^2/2} - \frac{4r}{\sigma^3} \left(\frac{S_t}{B}\right)^{1-2r/\sigma^2} \left(\frac{B^2}{S_t} \Phi \left(\frac{\delta_+^{-t}}{\left(\frac{B^2}{KS_t}\right)}\right) - Ke^{-(T-t)r} \Phi \left(\frac{\delta_+^{-t}}{\left(\frac{B^2}{KS_t}\right)}\right)\right) \log \frac{S_t}{B} - \sqrt{\frac{T-t}{2\pi}} \frac{B^2}{S_t} \left(\frac{S_t}{B}\right)^{1-2r/\sigma^2} e^{-\left(\delta_+^{-t}(B^2/K/S_t)\right)^2/2}.$$ 

When $B > K$ we find

$$\text{Vega}_{\text{down-and-out-call}} = \frac{S_t}{\sqrt{2\pi}} e^{-\left(\delta_+^{-t}(S_t/K)\right)^2/2} \left(\frac{K}{B} - 1\right) \left(\frac{\delta_+^{-t}(S_t/B)}{2} + \sqrt{T-t} \right) + \sqrt{T-t}.$$
hence, letting 

\[ \Phi \left( \frac{\delta_{t}^{T-t} (t/2) \sigma}{\delta_{t}^{T-t} (B/S_t)} \right) \log \frac{S_t}{B} \]

\[ - \frac{1}{\sqrt{2\pi} S_t} B^2 e^{-\left(\delta_{t}^{T-t} (S_t/B)\right)^2/2} \left( \frac{K}{B} - 1 \right) \left( \frac{\delta_{t}^{T-t} (B/S_t)}{\sigma} + \sqrt{T-t} \right) + \sqrt{T-t} \).

The corresponding formulas for the down-and-in call can be obtained from

the parity relation (8.16) and the value \( S_t \sqrt{\frac{T-t}{2\pi}} e^{-\left(\delta_{t}^{T-t} (S_t/K)\right)^2/2} \) of the Black-Scholes Vega, see Table 5.1.

Exercise 8.6 We have

\[ \mathbb{E}^*[C] = \mathbb{E}^* \left[ \mathbb{I}_{\{S_T \geq K\}} \mathbb{I}_{\{M^*_0 \leq B\}} \right] \]

\[ = \mathbb{E}^* \left[ \mathbb{I}_{\{S_0 e^{\sigma B_T \geq K}\}} \mathbb{I}_{\{S_0 e^{\sigma X_T \leq B}\}} \right] \]

\[ = \int_{-\infty}^{\infty} \int_{y \geq 0} \mathbb{I}_{\{\hat{S}_0 \geq K\}} \mathbb{I}_{\{S_0 e^{\sigma x \leq B}\}} d\mathbb{P}(\hat{S}_0 \leq x, B_T \leq y) \]

\[ = \int_{-\infty}^{\infty} \int_{y \geq 0} \mathbb{I}_{\{\hat{S}_0 \geq K\}} \mathbb{I}_{\{S_0 e^{\sigma x \leq B}\}} f_{\hat{S}_0, B_T}(x, y) dx dy \]

\[ = \frac{1}{T} \sqrt{\frac{2}{\pi T}} \int_{-\infty}^{\sigma^{-1} \log(B/S_0)} \int_{y \geq 0} \mathbb{I}_{\{\hat{S}_0 \geq K\}} \mathbb{I}_{\{S_0 e^{\sigma x \leq B}\}} (2x - y) e^{-\mu^2 T/2 + \mu y - (2x - y)^2 / (2T)} dx dy \]

\[ = \frac{1}{T} \sqrt{\frac{2}{\pi T}} \int_{\sigma^{-1} \log(K/S_0)}^{\sigma^{-1} \log(B/S_0)} \int_{y \geq 0} \mathbb{I}_{\{\hat{S}_0 \geq K\}} (2x - y) e^{-\mu^2 T/2 + \mu y - (2x - y)^2 / (2T)} dx dy, \]

if \( B \geq S_0 \) (otherwise the option price is 0), with \( \mu = r / \sigma - \sigma / 2 \) and \( y \geq 0 = \max(y, 0) \). Next, letting \( a = y \geq 0 \) and \( b = \sigma^{-1} \log(B/S_0) \), we have

\[ \int_a^b (2x - y) e^{2x(y-x)/T} dx = \frac{T}{2} (1 - e^{2b(y-b)/T}), \]

hence, letting \( c = \sigma^{-1} \log(K/S_0) \), we have

\[ \mathbb{E}^*[C] = e^{-\mu^2 T/2} \frac{1}{\sqrt{2\pi T}} \int_c^b e^{\mu y - y^2 / (2T)} \left( 1 - e^{2b(y-b)/T} \right) dy \]

\[ = e^{-\mu^2 T/2} \frac{1}{\sqrt{2\pi T}} \int_c^b e^{\mu y - y^2 / (2T)} dy \]

\[ - e^{-\mu^2 T/2 - 2b^2 / T} \frac{1}{\sqrt{2\pi T}} \int_c^b e^{y(\mu + 2b/T) - y^2 / (2T)} dy. \]

Using the relation

\[ \frac{1}{\sqrt{2\pi T}} \int_c^b e^{\gamma y - y^2 / (2T)} dy = e^{\gamma^2 T / 2} \left( \Phi \left( \frac{-c + \gamma T}{\sqrt{T}} \right) - \Phi \left( \frac{-b + \gamma T}{\sqrt{T}} \right) \right), \]

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we find
\[ \mathbb{E}^*[C] = \mathbb{E}^* \left[(S_T - K)^+ 1_{\{M_0^T \leq B\}} \right] \]
\[ = \Phi \left( \frac{-c + \mu T}{\sqrt{T}} \right) - \Phi \left( \frac{-b + \mu T}{\sqrt{T}} \right) \]
\[ - e^{-\mu^2 T/2 - 2b^2 T + (\mu + 2b/T)^2 T/2} \left( \Phi \left( \frac{-c + (\mu + 2b/T) T}{\sqrt{T}} \right) - \Phi \left( \frac{-b + (\mu + 2b/T) T}{\sqrt{T}} \right) \right) \]
\[ = \Phi \left( \frac{\delta T}{K} \right) - \Phi \left( \frac{\delta T}{B} \right) \]
\[ - e^{-\mu^2 T/2 - 2b^2 T + (\mu + 2b/T)^2 T/2} \left( \Phi \left( \frac{B^2}{KS_0} \right) \right) - \Phi \left( \frac{B}{S_0} \right) \),
\]
\[ 0 \leq x \leq B. \text{ Given the relation} \]
\[ - \mu^2 T/2 - 2b^2 T + \left( \mu + 2b/T \right)^2 = -1 + \frac{2r}{\sigma^2} \log \frac{B}{S_0}, \]
\]
we get
\[ e^{-rT} \mathbb{E}^*[C] = e^{-rT} \mathbb{E}^* \left[1_{\{S_T \geq K\}} 1_{\{M_0^T \leq B\}} \right] \]
\[ = e^{-rT} \left( \Phi \left( \frac{\delta T}{K} \right) - \Phi \left( \frac{\delta T}{B} \right) \right) \]
\[ - \left( \frac{S_0}{B} \right)^{1 - 2r/\sigma^2} \left( \Phi \left( \frac{B^2}{KS_0} \right) \right) - \Phi \left( \frac{B}{S_0} \right) \bigg). \]

\[ \text{Chapter 9} \]

Exercise 9.1

a) This probability density function is given by
\[ x \mapsto \sqrt{\frac{2}{\pi T}} e^{-(x-\sigma T/2)^2/(2T)} - \sigma e^{\sigma x} \Phi \left( \frac{-x - \sigma T/2}{\sqrt{T}} \right), \quad x \in \mathbb{R}_+. \]

b) We have
\[ E \left[ \min_{t \in [0,T]} S_t \right] = S_0 E \left[ \min_{t \in [0,T]} e^{\sigma B_t - \sigma^2 t/2} \right] \]
\[ = S_0 E \left[ e^{-\sigma \max_{t \in [0,T]}(B_t + \sigma t/2)} \right] \]

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\begin{align*}
S_0 & \int_0^\infty e^{-x} \left( \sqrt{\frac{2}{\pi T}} e^{-(x-\sigma T/2)^2/(2T)} - \sigma e^{\sigma x} \Phi \left( \frac{-x-\sigma T/2}{\sqrt{T}} \right) \right) dx \\
& = \frac{2S_0}{\sqrt{2\pi T}} \int_0^\infty e^{-(x+\sigma T/2)^2/(2T)} dx - S_0 \sigma \int_0^\infty \Phi \left( \frac{-x-\sigma T/2}{\sqrt{T}} \right) dx \\
& = \frac{2S_0}{\sqrt{2\pi T}} \int_0^\frac{T}{\sqrt{2T}} e^{-x^2/(2T)} dx - \frac{\sigma S_0}{\sqrt{2\pi T}} \int_0^\frac{T}{\sqrt{2T}} xe^{-(x+\sigma T/2)^2/(2T)} dx \\
& = \frac{2S_0}{\sqrt{2\pi T}} \int_0^\frac{T}{\sqrt{2T}} e^{-x^2/(2T)} dx - \frac{\sigma S_0}{\sqrt{2\pi T}} \int_0^\frac{T}{\sqrt{2T}} \left( x - \sigma T/2 \right) e^{-x^2/(2T)} dx \\
& = 2S_0 \left( 1 + \sigma^2 T/4 \right) \Phi \left( -\sigma \sqrt{T}/2 \right) - S_0 \sigma \sqrt{\frac{T}{2\pi}} e^{-\sigma^2 T/8}. \tag{A.28}
\end{align*}

c) We have
\begin{align*}
E \left[ \left( K - \min_{t \in [0,T]} S_t \right)^+ \right] &= E \left[ K - \min_{t \in [0,T]} S_t \right] \\
& = K - S_0 \left( 2(1 + \sigma^2 T/4) \Phi \left( -\sigma \sqrt{T}/2 \right) - \sigma \sqrt{\frac{T}{2\pi}} e^{-\sigma^2 T/8} \right).
\end{align*}

---

![Fig. S.39: Expected minimum of geometric Brownian motion over [0, T].](image)

The derivative with respect to time is given by
\begin{align*}
\frac{\partial}{\partial T} E \left[ \min_{t \in [0,T]} S_t \right] &= S_0 (\sigma^2/2) \Phi \left( -\sigma \sqrt{T}/2 \right) - 2S_0 \left( 1 + \frac{\sigma^2 T}{4} \right) \frac{\sigma}{4\sqrt{2\pi T}} e^{-\sigma^2 T/8} \\
& \quad - \frac{\sigma S_0}{\sqrt{8\pi T}} e^{-\sigma^2 T/8} + \frac{S_0 \sigma^3}{8} \sqrt{\frac{T}{2\pi}} e^{-\sigma^2 T/8} \\
& = \frac{S_0 \sigma^2}{2} \Phi \left( -\sigma \sqrt{T}/2 \right) - \frac{S_0 \sigma}{\sqrt{2\pi T}} e^{-\sigma^2 T/8} \left( 1 + \frac{3\sigma^2 T}{4} \right).
\end{align*}
Fig. S.40: Time derivative of the expected minimum of geometric Brownian motion.

On the other hand, when $r > 0$ we have

$$
\mathbb{E}^* \left[ m^T_0 \mid \mathcal{F}_t \right] = m^t_0 \Phi \left( \delta^T-t \left( \frac{S^t_t}{m^t_0} \right) \right) - S^t_t \frac{\sigma^2}{2r} \left( \frac{m^t_0}{S^t_t} \right)^{2r/\sigma^2} \Phi \left( \delta^T_t \left( \frac{m^t_0}{S^t_t} \right) \right) + S^t_t e^{(T-t)r} \left( 1 + \frac{\sigma^2}{2r} \right) \Phi \left( -\delta^T_t \left( \frac{S^t_t}{m^t_0} \right) \right).
$$

When $r$ tends to 0, this minimum tends to

$$
m^t_0 \Phi \left( \frac{\log(S^t_t/m^t_0) - \sigma^2 T/2}{\sigma \sqrt{T}} \right) + S^t_t \Phi \left( -\frac{\log(S^t_t/m^t_0) + \sigma^2 T/2}{\sigma \sqrt{T}} \right) + \sigma^2 S^t_t \lim_{r \to 0} \frac{1}{2r} \left( e^{(T-t)r} \Phi \left( -\delta^T_t \left( \frac{S^t_t}{m^t_0} \right) \right) - \left( \frac{m^t_0}{S^t_t} \right)^{2r/\sigma^2} \Phi \left( \delta^T_t \left( \frac{m^t_0}{S^t_t} \right) \right) \right),
$$

where

$$
\lim_{r \to 0} \frac{1}{2r} \left( e^{(T-t)r} \Phi \left( -\delta^T_t \left( \frac{S^t_t}{m^t_0} \right) \right) - \left( \frac{m^t_0}{S^t_t} \right)^{2r/\sigma^2} \Phi \left( \delta^T_t \left( \frac{m^t_0}{S^t_t} \right) \right) \right) = \lim_{r \to 0} \frac{1}{2r} \left( (1 + (T-t)r) \Phi \left( -\frac{\log(S^t_t/m^t_0) + \sigma^2 T/2 + rT}{\sigma \sqrt{T}} \right) - \left( 1 + \frac{2r}{\sigma^2} \log \frac{m^t_0}{S^t_t} \right) \Phi \left( \frac{\log(m^t_0/S^t_t) - \sigma^2 T/2 + rT}{\sigma \sqrt{T}} \right) \right) = \frac{1}{2} \left( T - t + \frac{2}{\sigma^2} \log \frac{m^t_0}{S^t_t} \right) \Phi \left( -\frac{\log(S^t_t/m^t_0) + \sigma^2 T/2}{\sigma \sqrt{T}} \right) + \lim_{r \to 0} \frac{1}{r \sqrt{8\pi}} \left( \int_{-\infty}^{-\left(\log(S^t_t/m^t_0) + \sigma^2 T/2 + rT\right)/(\sigma \sqrt{T})} e^{-y^2/2} \, dy - \int_{-\infty}^{-\left(\log(S^t_t/m^t_0) - \sigma^2 T/2 + rT\right)/(\sigma \sqrt{T})} e^{-y^2/2} \, dy \right).
$$
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\[
= \frac{1}{2} \left( T - t + \frac{2}{\sigma^2} \log \frac{m_0^t}{S_t} \right) \Phi \left( - \frac{\log(S_t/m_0^t) + \sigma^2 T/2}{\sigma \sqrt{T}} \right) \\
- \lim_{r \to 0} \frac{1}{r \sqrt{2\pi}} \int_{(-\log(S_t/m_0^t) - \sigma^2 T/2 + rT)/\sigma \sqrt{T}} e^{-y^2/2} \, dy \\
= \frac{1}{2} \left( T - t + \frac{2}{\sigma^2} \log \frac{m_0^t}{S_t} \right) \Phi \left( - \frac{\log(S_t/m_0^t) + \sigma^2 T/2}{\sigma \sqrt{T}} \right) \\
- \frac{\sqrt{T}}{\sigma \sqrt{2\pi}} e^{-((\log(S_t/m_0^t) + \sigma^2 T/2)/(\sigma \sqrt{T}))^2/2},
\]

hence

\[
\mathbb{E}^* [m_0^T | \mathcal{F}_t] = m_0^t \Phi \left( \frac{\log(S_t/m_0^t) - \sigma^2 T/2}{\sigma \sqrt{T}} \right) + S_t \Phi \left( - \frac{\log(S_t/m_0^t) + \sigma^2 T/2}{\sigma \sqrt{T}} \right) \\
+ \frac{S_t}{2} \left( \sigma^2 (T - t) + 2 \log \frac{m_0^t}{S_t} \right) \Phi \left( - \frac{\log(S_t/m_0^t) + \sigma^2 T/2}{\sigma \sqrt{T}} \right) \\
- \sigma S_t \sqrt{\frac{T}{2\pi}} e^{-((\log(S_t/m_0^t) + \sigma^2 T/2)/(\sigma \sqrt{T}))^2/2}.
\]

In particular, when \( T \) tends to infinity we find that

\[
\lim_{T \to \infty} \frac{\mathbb{E}^* [m_0^T | \mathcal{F}_t]}{\mathbb{E}^* [S_T | \mathcal{F}_t]} = 0, \quad r \geq 0.
\]

When \( t = 0 \) we have \( S_0 = m_0^0 \), and we recover

\[
\mathbb{E}^* [m_0^T] = 2 \left( S_0 + \frac{\sigma^2 T}{4} \right) \Phi \left( -\sigma \sqrt{T}/2 \right) - \sigma S_0 \sqrt{\frac{T}{2\pi}} e^{-\sigma^2 T/8}.
\]

Exercise 9.2

a) By (A.28), we have

\[
E \left[ \max_{t \in [0,1]} S_t \right] = E \left[ e^{\sigma \max_{t \in [0,1]} (B_t - \sigma t/2)} \right] \\
= S_0 E \left[ e^{(-\sigma) \max_{t \in [0,T]} (B_t - (-\sigma) t/2)} \right] \\
= 2S_0 (1 + \frac{\sigma^2 T}{4}) \Phi (\sigma \sqrt{T}/2) + S_0 \sigma \sqrt{\frac{T}{2\pi}} e^{-\sigma^2 T/8}.
\]

b) We have

\[
E \left[ \left( S_0 \max_{t \in [0,1]} e^{\sigma B_t - \sigma^2 t/2 - K} \right)^+ \right] = E \left[ S_0 \max_{t \in [0,1]} e^{\sigma B_t - \sigma^2 t/2} \right] - K
\]
\[ \frac{\partial}{\partial T} E \left[ \max_{t \in [0,T]} S_t \right] = \frac{S_0 \sigma^2}{2} \Phi \left( \frac{\sigma \sqrt{T}}{2} \right) + \frac{S_0 \sigma}{\sqrt{2 \pi T}} e^{-\sigma^2 T/8} \left( 1 + \frac{3 \sigma^2 T}{4} \right). \]

Fig. S.41: Expected maximum of geometric Brownian motion over \([0,T]\).

The derivative with respect to time is given by

\[ \frac{\partial}{\partial T} E \left[ \max_{t \in [0,T]} S_t \right] = \frac{S_0 \sigma^2}{2} \Phi \left( \frac{\sigma \sqrt{T}}{2} \right) + \frac{S_0 \sigma}{\sqrt{2 \pi T}} e^{-\sigma^2 T/8} \left( 1 + \frac{3 \sigma^2 T}{4} \right). \]

Fig. S.42: Time derivative of the expected maximum of geometric Brownian motion.

Note that when \(r > 0\) we have

\[ E^* \left[ M_0^T \mid F_t \right] = M_0^t \Phi \left( -\delta_-^{T-t} \left( \frac{S_t}{M_0^t} \right) \right) + S_t e^{(T-t)r} \left( 1 + \frac{\sigma^2}{2r} \right) \Phi \left( \delta_+^{T-t} \left( \frac{S_t}{M_0^t} \right) \right) \]

\[ -S_t \frac{\sigma^2}{2r} \left( \frac{M_0^t}{S_t} \right)^{2r/\sigma^2} \Phi \left( -\delta_-^{T-t} \left( \frac{M_0^t}{S_t} \right) \right). \]

When \(r\) tends to 0, this maximum tends to
In particular, when $T$ tends to infinity we find that

$$
\mathbb{E}^* \left[ M_0^T \mid \mathcal{F}_t \right] = M_0^t \Phi \left( -\frac{\log(S_t/M_0^t) - \sigma^2 T/2}{\sigma \sqrt{T}} \right) + S_t \Phi \left( \frac{\log(S_t/M_0^t) + \sigma^2 T/2}{\sigma \sqrt{T}} \right) + \frac{S_t}{2} \left( \sigma^2 (T - t) + 2 \log \frac{M_0^t}{S_t} \right) \Phi \left( \frac{\log(S_t/M_0^t) + \sigma^2 T/2}{\sigma \sqrt{T}} \right) + \sigma S_t \sqrt{\frac{T}{2\pi}} e^{-\left( (\log(S_t/M_0^t) + \sigma^2 T/2) / (\sigma \sqrt{T}) \right)^2 / 2}.
$$

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\[
\lim_{T \to \infty} \frac{\mathbb{E}^*[M_0^T | \mathcal{F}_t]}{\mathbb{E}^*[S_T | \mathcal{F}_t]} = \begin{cases} 
1 + \frac{\sigma^2}{2r} & \text{if } r > 0, \\
+\infty & \text{if } r = 0.
\end{cases}
\]

When \( t = 0 \) we have \( S_0 = M_0^0 \), and we recover
\[
\mathbb{E}^*[M_0^T] = 2 \left( S_0 + \frac{\sigma^2 T}{4} \right) \Phi \left( \frac{\sigma \sqrt{T}}{2} \right) + \sigma S_0 \sqrt{\frac{T}{2\pi}} e^{-\sigma^2 T/8}.
\]

Exercise 9.3

a) We have
\[
P \left( \min_{t \in [0,T]} B_t \leq a \right) = 2 \int_{-\infty}^{a} e^{-x^2/(2T)} \frac{dx}{\sqrt{2\pi T}}, \quad a < 0,
\]
\text{i.e. the probability density function } \varphi \text{ of } \sup_{t \in [0,T]} B_t \text{ is given by }
\[
\varphi(a) = \sqrt{\frac{2}{\pi T}} e^{-a^2/(2T)} \mathbb{1}_{(-\infty,0)}(a), \quad a \in \mathbb{R}.
\]

b) We have
\[
\mathbb{E} \left[ \min_{t \in [0,T]} S_t \right] = S_0 \mathbb{E} \left[ \exp \left( \sigma \min_{t \in [0,T]} B_t \right) \right]
= \frac{2S_0}{\sqrt{2\pi T}} \int_{-\infty}^{0} e^{\sigma x - x^2/(2T)} dx = \frac{2S_0}{\sqrt{2\pi \sigma^2 T}} \int_{-\infty}^{0} e^{-(x-\sigma T)^2/(2T)+\sigma^2 T/2} dx
= \frac{2S_0}{\sqrt{2\pi T}} e^{\sigma^2 T/2} \int_{-\infty}^{-\sigma \sqrt{T}} e^{-x^2/(2T)} dx = \frac{2S_0}{\sqrt{2\pi}} e^{\sigma^2 T/2} \int_{-\infty}^{-\sigma \sqrt{T}} e^{-x^2/2} dx
= 2S_0 e^{\sigma^2 T/2} \Phi \left( -\sigma \sqrt{T} \right) = 2 \mathbb{E}[S_T] \left( 1 - \Phi \left( \sigma \sqrt{T} \right) \right),
\]
hence
\[
\mathbb{E} \left[ S_T - \min_{t \in [0,T]} S_t \right] = \mathbb{E}[S_T] - \mathbb{E} \left[ \min_{t \in [0,T]} S_t \right] = \mathbb{E}[S_T] - 2 \mathbb{E}[S_T] \left( 1 - \Phi \left( \sigma \sqrt{T} \right) \right)
= \mathbb{E}[S_T] \left( 2 \Phi \left( \sigma \sqrt{T} \right) - 1 \right) = 2S_0 e^{\sigma^2 T/2} \left( \Phi \left( \sigma \sqrt{T} \right) - \frac{1}{2} \right),
\]
and
\[
e^{-\sigma^2 T/2} \mathbb{E} \left[ S_T - \min_{t \in [0,T]} S_t \right] = S_0 \left( 2 \Phi \left( \sigma \sqrt{T} \right) - 1 \right) = S_0 \left( 1 - 2 \Phi \left( -\sigma \sqrt{T} \right) \right).
\]
Remark: We note that as $T$ goes to infinity, the price of the lookback option converges to $S_0$.

Fig. S.43: Lookback call option price as a function of $T$ with $S_0 = 1$.

Exercise 9.4 Lookback options. By (9.6) we find

$$
\xi_t = \frac{\partial f}{\partial x}(t, S_t, M_t^0)
= -1 + \left(1 + \frac{2r}{\sigma^2}\right) \Phi\left(\delta_{+}^{T-t} \left(\frac{S_t}{M_t^0}\right)\right)
+ e^{-(T-t)r} \left(\frac{M_t^0}{S_t}\right)^{2r/\sigma^2} \left(1 - \frac{\sigma^2}{2r}\right) \Phi\left(-\delta_{+}^{T-t} \left(\frac{M_t^0}{S_t}\right)\right), \quad t \in [0, T],
$$

and

$$
\eta_t A_t = f(t, S_t, M_t^0) - \xi_t S_t
= M_t^0 e^{-(T-t)r} \Phi\left(-\delta_{+}^{T-t} \left(\frac{S_t}{M_t^0}\right)\right) - e^{-(T-t)r} \left(\frac{M_t^0}{S_t}\right)^{-1+2r/\sigma^2} \Phi\left(-\delta_{+}^{T-t} \left(\frac{M_t^0}{S_t}\right)\right),
$$

$t \in [0, T]$.

Exercise 9.5 We have

$$
\mathbb{E}^*\left[ e^{-r\tau} \mathbbm{1}_{\{\tau \leq T\}} \mathbbm{1}_{\{M_0^\tau - S_\tau \geq K\}} \right]
= \int_1^T \int_0^\infty \int_{K+x}^\infty e^{-rt} f_{(\tau, S_\tau, M_\tau)}(t, x, y) dy dx dt
= \int_1^T \int_0^\infty \int_K^{y-K} e^{-rt} f_{(\tau, S_\tau, M_\tau)}(t, x, y) dx dy dt
$$

for $T \geq 1$, and $\mathbb{E}^*\left[ e^{-r\tau} \mathbbm{1}_{\{\tau \leq T\}} \mathbbm{1}_{\{M_0^\tau - S_\tau \geq K\}} \right] = 0$ if $T \in [0, 1]$. 

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Chapter 10

Exercise 10.1

a) The integral \( \int_{0}^{T} r_s ds \) is centered Gaussian with variance

\[
\mathbb{E} \left[ \left( \int_{0}^{T} r_s ds \right)^2 \right] = \sigma^2 \mathbb{E} \left[ \int_{0}^{T} \int_{0}^{T} B_s B_t ds dt \right]
\]

\[
= \sigma^2 \int_{0}^{T} \int_{0}^{T} \mathbb{E}[B_s B_t] ds dt
\]

\[
= \sigma^2 \int_{0}^{T} \int_{0}^{T} \min(s, t) ds dt
\]

\[
= 2\sigma^2 \int_{0}^{T} \int_{0}^{t} s ds dt
\]

\[
= \sigma^2 \int_{0}^{T} t^2 dt
\]

\[
= T^3 \sigma^2/3.
\]

b) Since the integral \( \int_{0}^{T} r_s ds \) is a random variable with probability density

\[
\varphi(x) = \frac{1}{\sqrt{2\pi T^3/3}} e^{-3x^2/(2\pi T^3)},
\]

we have

\[
e^{-rT} \mathbb{E} \left[ \left( \int_{0}^{T} r_s du - \kappa \right)^+ \right] = e^{-rT} \int_{-\infty}^{\infty} (x - \kappa)^+ \varphi(x) \, dx
\]

\[
= \frac{e^{-rT}}{\sqrt{2\pi \sigma^2 T^3/3}} \int_{-\infty}^{\infty} (x - \kappa) e^{-3x^2/(2\sigma^2 T^3)} \, dx
\]

\[
= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{\kappa/\sqrt{\sigma^2 T^3/3}}^{\infty} (x \sqrt{\sigma^2 T^3/3} - \kappa) e^{-x^2/2} \, dx
\]

\[
= \frac{e^{-rT} \sqrt{\sigma^2 T^3/3}}{\sqrt{2\pi}} \int_{\kappa/\sqrt{\sigma^2 T^3/3}}^{\infty} x e^{-x^2/2} \, dx - \kappa \frac{e^{-rT}}{\sqrt{2\pi}} \int_{\kappa/\sqrt{\sigma^2 T^3/3}}^{\infty} e^{-x^2/2} \, dx
\]

\[
= -\frac{e^{-rT} \sqrt{\sigma^2 T^3/3}}{\sqrt{2\pi}} \left[ e^{-x^2/2} \right]_{\kappa/\sqrt{\sigma^2 T^3/3}}^{\infty} - \kappa \frac{e^{-rT}}{\sqrt{2\pi}} \left( 1 - \Phi(\kappa/\sqrt{\sigma^2 T^3/3}) \right)
\]

\[
= \frac{e^{-rT} \sqrt{\sigma^2 T^3/3}}{\sqrt{2\pi}} e^{-3\kappa^2/(2\sigma^2 T^3)} - \kappa \frac{e^{-rT}}{\sqrt{2\pi}} \left( 1 - \Phi(\kappa/\sqrt{\sigma^2 T^3/3}) \right)
\]

\[
= e^{-rT} \sqrt{\sigma^2 T^3/6\pi} e^{-3\kappa^2/(2\sigma^2 T^3)} - \kappa \frac{e^{-rT}}{\sqrt{2\pi}} \Phi\left( -\kappa \sqrt{\frac{3}{\sigma^2 T^3}} \right).
\]
Exercise 10.3 The geometric mean price $G$ satisfies

$$G = \exp \left( \frac{1}{T} \int_0^T \log S_u du \right) = \exp \left( \frac{1}{T} \int_0^T \log S_u du + \frac{1}{T} \int_t^T \log S_u du \right)$$

$$= \exp \left( \frac{1}{T} \int_0^T \log S_u du + \frac{T-t}{T} \log S_t + \frac{1}{T} \int_t^T \log S_u du \right)$$

$$= \exp \left( \frac{1}{T} \int_0^T \log S_u du + \frac{T-t}{T} \log S_t \right) + \frac{1}{T} \int_t^T \left( r(u-t) + \sigma (B_u - B_t) - \sigma^2 (u-t)/2 \right) du$$

$t \in [0, T]$, cf. [GY93] page 361. We check that the function $f(t,x,y) = e^{-(T-t) r}(y/T - \kappa) + x(1 - e^{-(T-t) r})/rT$ satisfies the PDE

$$rf(t,x,y) = \frac{\partial f}{\partial t} (t,x,y) + x \frac{\partial f}{\partial x} (t,x,y) + r x \frac{\partial f}{\partial x} (t,x,y) + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial x^2} (t,x,y),$$

t, x > 0$, and the boundary conditions $f(t,0,y) = e^{-(T-t) r}(y/T - \kappa)$, $0 \leq t \leq T$, $y \in \mathbb{R}_+$, and $f(T,x,y) = y/T - \kappa$, $x, y \in \mathbb{R}_+$. However, the condition $\lim_{y \to -\infty} f(t,x,y) = 0$ is not satisfied because we need to take $y > 0$ in the above calculation.
\[
\begin{align*}
&= \exp \left( \frac{1}{T} \int_0^t \log S_u \, du + \frac{T - t}{T} \log S_t \right) \\
&\quad + \frac{1}{T} \int_0^{T-t} (ru - \sigma^2 u/2) \, du + \frac{\sigma}{T} \int_t^T (B_u - B_t) \, du \\
&= (S_t)^{(T-t)/T} \exp \left( \frac{1}{T} \int_0^t \log S_u \, du + \frac{(T-t)^2}{2T} (r - \sigma^2/2) + \frac{\sigma}{T} \int_t^T (B_u - B_t) \, du \right)
\end{align*}
\]

where \( \int_t^T B_u \, du \) is centered Gaussian with conditional variance

\[
\mathbb{E} \left[ \left( \int_t^T B_u \, du \right)^2 \mid \mathcal{F}_t \right] = \mathbb{E} \left[ \left( \int_t^T (B_u - B_t) \, du \right)^2 \mid \mathcal{F}_t \right]
\]

\[
= \mathbb{E} \left[ \left( \int_t^T (B_u - B_t) \, du \right)^2 \right]
\]

\[
= \mathbb{E} \left[ \left( \int_0^{T-t} (B_u - B_t) \, du \right)^2 \right] = \int_0^{T-t} \int_0^{T-t} \mathbb{E} [B_s B_u] \, ds \, du
\]

\[
= 2 \int_0^{T-t} \int_0^u s \, ds \, du = \int_0^{T-t} u^2 \, du = \frac{(T-t)^3}{3}.
\]

Hence, letting

\[
m := \frac{1}{T} \int_0^t \log S_u \, du + \frac{T - t}{T} \log S_t + \frac{(T-t)^2}{2T} (r - \sigma^2/2), \quad X := \frac{\sigma}{T} \int_t^T B_u \, du,
\]

and \( v^2 = \sigma^2(T-t)/3 \), we find

\[
e^{-(T-t)r} \mathbb{E}^* \left[ \left( \exp \left( \frac{1}{T} \int_0^T \log S_u \, du \right) - K \right)^+ \mid \mathcal{F}_t \right]
\]

\[
= (S_t)^{(T-t)/T} e^{-(T-t)r} \exp \left( \frac{1}{T} \int_0^t \log S_u \, du + \frac{(T-t)^2}{4T} (2r - \sigma^2) + \frac{\sigma^2}{6} (T-t) \right)
\]

\[
\times \Phi \left( \frac{\sigma^2(T-t)/3 + \frac{1}{T} \int_0^t \log S_u \, du + \log \frac{S_t(T-t)/T}{K} + \frac{(T-t)^2}{2T} (r - \sigma^2/2)}{\sigma \sqrt{(T-t)/3}} \right)
\]

\[
- K e^{-(T-t)r} \Phi \left( \frac{\frac{1}{T} \int_0^t \log S_u \, du + \log \frac{S_t(T-t)/T}{K} + \frac{(T-t)^2}{2T} (r - \sigma^2/2)}{\sigma \sqrt{(T-t)/3}} \right),
\]

\( 0 \leq t \leq T \). In case \( t = 0 \), we get

\[
e^{-rT} \mathbb{E}^* \left[ \left( \exp \left( \frac{1}{T} \int_0^T \log S_u \, du \right) - K \right)^+ \right]
\]

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Exercise 10.4 Under the above condition we have, taking \( t \in [\tau, T] \),
\[
e^{-T-t}r \mathbb{E}^{*} \left[ \left( \frac{1}{T-\tau} \int_{\tau}^{T} r_s ds - K \right)^+ \mid \mathcal{F}_t \right] = e^{-(T-t)r} \mathbb{E}^{*} \left[ \left( \Lambda_t + \frac{1}{T-\tau} \int_{t}^{T} r_s ds - K \right)^+ \mid \mathcal{F}_t \right]
\]
\[
= e^{-(T-t)r} \mathbb{E}^{*} \left[ \Lambda_t + \frac{1}{T-\tau} \int_{t}^{T} r_s ds - K \mid \mathcal{F}_t \right]
\]
\[
= e^{-(T-t)r} \mathbb{E}^{*} (\Lambda_t - K) + \frac{e^{-(T-t)r}}{T-\tau} \mathbb{E}^{*} \left[ \int_{t}^{T} r_s ds \mid \mathcal{F}_t \right]
\]
\[
= e^{-(T-t)r} (\Lambda_t - K) + \frac{e^{-(T-t)r}}{T-\tau} \int_{t}^{T} \mathbb{E}^{*}[r_s \mid \mathcal{F}_t] ds, \quad t \in [\tau, T],
\]
where
\[
\mathbb{E}^{*}[r_s \mid \mathcal{F}_t] = v_t e^{-\lambda(s-t)} + m(1 - e^{-\lambda(s-t)}), \quad 0 \leq s \leq t,
\]
hence
\[
\mathbb{E}^{*} \left[ \left( \frac{1}{T-\tau} \int_{\tau}^{T} r_s ds - K \right)^+ \mid \mathcal{F}_t \right] = \mathbb{E}^{*} \left[ \left( \Lambda_t + \frac{1}{T-\tau} \int_{t}^{T} r_s ds - K \right)^+ \mid \mathcal{F}_t \right]
\]
\[
= \Lambda_t - K + \frac{1}{T-\tau} \int_{t}^{T} \mathbb{E}^{*}[r_s \mid \mathcal{F}_t] ds
\]
\[
= \Lambda_t - K + \frac{1}{T-\tau} \int_{t}^{T} \left( r_t e^{-\lambda(s-t)} + m(1 - e^{-\lambda(s-t)}) \right) ds
\]
\[
= \Lambda_t - K + \frac{1}{T-\tau} (r_t - m) \int_{0}^{T-t} e^{-\lambda s} ds + m(T-t) \frac{e^{-(T-t)r}}{T-\tau}
\]
\[
= \Lambda_t - K + (r_t - m) \frac{1}{T-\tau} \int_{0}^{T-t} e^{-\lambda s} ds + m \frac{T-t}{T-\tau}
\]
\[
= \Lambda_t - K + \frac{1 - e^{-\lambda(T-t)}}{\lambda(T-\tau)} (r_t - m) + m \frac{T-t}{T-\tau}.
\]

Exercise 10.5 This question extends Exercise 6.2 to \( n \geq 3 \). If \((S_t)_{t \in \mathbb{R}_+}\) is a martingale then for any convex payoff function \( \phi \) we can write
\[
\mathbb{E}^{*} \left[ \phi \left( \frac{S_{T_1} + \cdots + S_{T_n}}{n} \right) \right] \leq \mathbb{E}^{*} \left[ \phi \left( \frac{S_{T_1} + \cdots + S_{T_n}}{n} \right) \right] \quad \text{since } \phi \text{ is convex},
\]
Exercise 10.7 The Asian option price can be written as

\[
\mathbb{E}^*[\phi(S_{T_n})] + \cdots + \mathbb{E}^*[\phi(S_{T_1})]
\]

\[
= \mathbb{E}^*[\phi(\mathbb{E}^*[S_{T_n} \mid F_{T_1}])] + \cdots + \mathbb{E}^*[\phi(\mathbb{E}^*[S_{T_1} \mid F_{T_1}])]
\]

because \((S_t)_{t \in \mathbb{R}_+}\) is a martingale,

\[
\leq \mathbb{E}^*[\mathbb{E}^*[\phi(S_{T_n}) \mid F_{T_1}]] + \cdots + \mathbb{E}^*[\mathbb{E}^*[\phi(S_{T_1}) \mid F_{T_1}]]
\]

by Jensen's inequality,

\[
= \mathbb{E}^*[\phi(S_{T_n})] + \cdots + \mathbb{E}^*[\phi(S_{T_1})]
\]

by the tower property,

\[
= \mathbb{E}^*[\phi(S_{T_n})].
\]

On the other hand, if \((S_t)_{t \in \mathbb{R}_+}\) is only a submartingale then the above argument still applies to a convex nondecreasing payoff function \(\phi\) such as \(\phi(x) = (x - K)^+\).

Exercise 10.6 Taking \(t \in [\tau, T]\), under the condition

\[
A_t := \frac{1}{T - \tau} \int_{\tau}^{t} S_s ds \geq K,
\]

we have

\[
e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T - \tau} \int_{\tau}^{T} S_s ds - K \right)^+ \mid F_t \right]
\]

\[
= e^{-(T-t)r} \mathbb{E}^* \left[ \left( A_t + \frac{1}{T - \tau} \int_{t}^{T} S_s ds - K \right)^+ \mid F_t \right]
\]

\[
= e^{-(T-t)r} \mathbb{E}^* \left[ A_t + \frac{1}{T - \tau} \int_{t}^{T} S_s ds - K \mid F_t \right]
\]

\[
= e^{-(T-t)r} (A_t - K) + \frac{e^{-(T-t)r}}{T - \tau} \mathbb{E}^* \left[ \int_{t}^{T} S_s ds \mid F_t \right]
\]

\[
= e^{-(T-t)r} (A_t - K) + \frac{e^{-(T-t)r}}{T - \tau} \int_{t}^{T} \mathbb{E}^*[S_s \mid F_t] ds
\]

\[
= e^{-(T-t)r} (A_t - K) + S_t \frac{e^{-(T-t)r}}{T - \tau} \int_{t}^{T} e^{(s-t)r} ds
\]

\[
= e^{-(T-t)r} (A_t - K) + S_t \frac{e^{-(T-t)r}}{T - \tau} \int_{0}^{T-t} e^{s} ds
\]

\[
= e^{-(T-t)r} (A_t - K) + S_t \frac{e^{-(T-t)r}}{(T - \tau)r} (e^{(T-t)r} - 1)
\]

\[
= e^{-(T-t)r} (A_t - K) + S_t \frac{1 - e^{-(T-t)r}}{(T - \tau)r}, \quad t \in [\tau, T].
\]
\[
\begin{align*}
\frac{e^{-r(T-t)}}{T} \mathbb{E}^* \left[ \frac{1}{T} \int_0^T S_u du - K \right]^+ \bigg|_{F_t} &= S_t \mathbb{E}^* \left[ (U_T)^+ \big| U_t \right] \\
&= S_t h(t, U_t) = S_t g(t, Z_t),
\end{align*}
\]

which shows that
\[
g(t, Z_t) = h(t, U_t),
\]
and it remains to use the relation
\[
U_t = \frac{1 - e^{-r(T-t)}}{rT} + e^{-r(T-t)} Z_t, \quad t \in [0, T].
\]

Exercise 10.8

a) When \( \Lambda_t / T \geq K \) we have
\[
f(t, S_t, \Lambda_t) = e^{-(T-t)r} \left( \frac{\Lambda_t}{T} - K \right) + S_t \frac{1 - e^{-r(T-t)}}{rT},
\]

see Exercise 10.6.

b) When \( \Lambda_t / T \geq K \) we have
\[
\xi_t = \frac{1 - e^{-r(T-t)}}{rT} \quad \text{and} \quad \eta_t = e^{(T-t)r} \left( \frac{\Lambda_t}{T} - K \right), \quad t \in [0, T].
\]

c) At maturity we have \( f(T, S_T, \Lambda_T) = (\Lambda_T / T - K)^+ \), hence \( \xi_T = 0 \) and
\[
\eta_T \Lambda_T = A_T \frac{e^{-rT}}{A_0} \left( \frac{\Lambda_T}{T} - K \right) 1_{(\Lambda_T > KT)} = \left( \frac{\Lambda_T}{T} - K \right)^+.
\]

d) By Proposition 10.9 we have
\[
\xi_t = \frac{1}{S_t} \left( f(t, S_t, \Lambda_t) - \left( \frac{\Lambda_t}{T} - K \right) \frac{\partial g}{\partial z} \left( t, \frac{1}{S_t} \left( \frac{\Lambda_t}{T} - K \right) \right) \right)
\]
where the function \( g(t, z) \) satisfies \( f(t, x, y) = x g(t, (y/T - K)/x) \) and
\[
g(t, z) = z e^{-(T-t)r} + \frac{1 - e^{-r(T-t)}}{rT}, \quad z > 0,
\]

and solves the PDE
\[
\frac{\partial g}{\partial t}(t, z) + \left( \frac{1}{T} - rz \right) \frac{\partial g}{\partial z}(t, z) + \frac{1}{2} \sigma^2 z^2 \frac{\partial^2 g}{\partial z^2}(t, z) = 0,
\]

\( \diamond \)

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under the terminal condition \( g(T, z) = z^+ \), hence letting \( h(t, z) := e^{(T-t)r} \frac{\partial g}{\partial z}(t, z) \), we have

\[
e^{(T-t)r} \frac{\partial g}{\partial t}(t, z) + e^{(T-t)r} \left( \frac{1}{T} - rz \right) \frac{\partial g}{\partial z}(t, z) + \frac{1}{2} \sigma^2 z^2 \frac{\partial^2 g}{\partial z^2}(t, z) = 0,
\]

with \( h(t, z) = 1, z > 0 \), hence

\[
e^{(T-t)r} \frac{\partial^2 g}{\partial t \partial z}(t, z) - r e^{(T-t)r} \frac{\partial g}{\partial z}(t, z) + e^{(T-t)r} \left( \frac{1}{T} - rz \right) \frac{\partial^2 g}{\partial z^2}(t, z) + \sigma^2 z e^{(T-t)r} \frac{\partial^2 g}{\partial z^2}(t, z) + \frac{1}{2} e^{(T-t)r} \sigma^2 z^2 \frac{\partial^3 g}{\partial z^3}(t, z) = 0,
\]

or

\[
\frac{\partial h}{\partial t}(t, z) + \left( \frac{1}{T} + (\sigma^2 - r)z \right) \frac{\partial h}{\partial z}(t, z) + \frac{1}{2} \sigma^2 z^2 \frac{\partial^2 h}{\partial z^2}(t, z) = 0,
\]

with the terminal condition \( h(T, z) = \mathbb{1}_{\{z > 0\}} \). On the other hand, we have

\[
\eta_t = \frac{1}{A_t} \left( f(t, S_t, \Lambda_t) - \xi_t S_t \right) = \frac{1}{A_t} \left( \frac{\Lambda_t}{T} - K \right) \frac{\partial g}{\partial z} \left( t, \frac{1}{S_t} \left( \frac{\Lambda_t}{T} - K \right) \right) = e^{-(T-t)r} \frac{\Lambda_t}{T} - K \right) h \left( t, \frac{1}{S_t} \left( \frac{\Lambda_t}{T} - K \right) \right).
\]

Exercise 10.9 We have

\[
\mathbb{E} \left[ \int_T^\tau S_t \, dt \right] = S_0 \frac{e^{rT} - e^{r\tau}}{r},
\]

and

\[
\mathbb{E} \left[ \left( \int_T^\tau S_t \, dt \right)^2 \right] = 2S_0^2 r \frac{e^{(\sigma^2+2r)T} - (\sigma^2 + 2r) e^{rT + (\sigma^2 + r)\tau} + (\sigma^2 + r) e^{(\sigma^2 + 2r)\tau}}{(\sigma^2 + r)(\sigma^2 + 2r) r}.
\]

Chapter 11

Exercise 11.1

a) This process is a convex function \( x \mapsto (2 - x)^+ \) of the Brownian martingale, hence it is a submartingale.
b) This process can be written as $e^{\sigma B_t - \sigma^2 t/2 + \mu t}$ with $\sigma = 1$ and $\mu = \sigma^2/2 > 0$, hence it is a submartingale.

c) When $t > 0$, the question “is $\nu > t$?” cannot be answered at time $t$ without waiting to know the value of $B_{2t}$ at time $2t > t$. Therefore $\nu$ is not a stopping time.

d) For any $t \in \mathbb{R}_+$, the question “is $\tau > t$?” can be answered based on the observation of the paths of $(B_s)_{0 \leq s \leq t}$ and of the (deterministic) curve $(e^{s/2} + \alpha e^{s/2})_{0 \leq s \leq t}$ up to the time $t$. Therefore $\tau$ is a stopping time.

e) Since $\tau$ is a stopping time and $(e^{B_{t-t/2}})_{t \in \mathbb{R}_+}$ is a martingale, the stopping time theorem shows that $(e^{B_{t \wedge \tau} - (t \wedge \tau)/2})_{t \in \mathbb{R}_+}$ is also a martingale and, in particular, its expectation

$$
\mathbb{E}[e^{B_{t \wedge \tau} - (t \wedge \tau)/2}] = \mathbb{E}[e^{B_{0 \wedge \tau} - (0 \wedge \tau)/2}] = \mathbb{E}[e^{B_0 - 0/2}] = 1
$$

is constantly equal to 1 for all $t \geq 0$. This shows that

$$
\mathbb{E}[e^{B_{t - \tau} - t/2}] = \mathbb{E}\left[\lim_{t \to \infty} e^{B_{t \wedge \tau} - (t \wedge \tau)/2}\right] = \lim_{t \to \infty} \mathbb{E}\left[(B_{t \wedge \tau}^2 - (t \wedge \tau))\right] = 1.
$$

Next, we note that we have $e^{B_{t - \tau} - t/2} = 1 + \alpha \tau$, hence

$$
1 + \alpha \mathbb{E}[\tau] = \mathbb{E}[1 + \alpha \tau] = \mathbb{E}[e^{B_{\tau} - t/2}] = 1,
$$

i.e. $\mathbb{E}[\tau] = (1 - \alpha)/\beta$.

Remark: This argument also recovers $\mathbb{E}[\tau] = 0$ when $\alpha = 1$, however it fails when $(\alpha > 1$ and $\beta > 0)$ or $(\alpha < 1$ and $\beta < 0)$ because $\tau$ is not a.s. finite in those cases.

Exercise 11.2 Stopping times.

a) When $0 \leq t < 1$ the question “is $\nu > 1$?” cannot be answered at time $t$ without waiting to know the value of $B_1$ at time 1. Therefore $\nu$ is not a stopping time.

b) For any $t \in \mathbb{R}_+$, the question “is $\tau > t$?” can be answered based on the observation of the paths of $(B_s)_{0 \leq s \leq t}$ and of the (deterministic) curve $(\alpha e^{-s/2})_{0 \leq s \leq t}$ up to the time $t$. Therefore $\tau$ is a stopping time.

Since $\tau$ is a stopping time and $(B_t)_{t \in \mathbb{R}_+}$ is a martingale, the stopping time theorem shows that $(e^{B_{t \wedge \tau} - (t \wedge \tau)/2})_{t \in \mathbb{R}_+}$ is also a martingale and in particular its expectation

$$
\mathbb{E}\left[e^{B_{t \wedge \tau} - (t \wedge \tau)/2}\right] = \mathbb{E}\left[e^{B_{0 \wedge \tau} - (0 \wedge \tau)/2}\right] = \mathbb{E}\left[e^{B_0 - 0/2}\right] = 1
$$

is constantly equal to 1 for all $t$. This shows that

* We let $t \wedge \tau = \min(t, \tau)$.
Exercise 11.3  

a) By the stopping time theorem, for all $t \to \infty$  

$$E \left[ e^{B_\tau - \tau^2/2} \right] = E \left[ \lim_{t \to \infty} e^{B_{t \wedge \tau} - (t \wedge \tau)/2} \right] = \lim_{t \to \infty} E \left[ e^{B_{t \wedge \tau} - (t \wedge \tau)/2} \right] = 1.$$  

Next, we note that we have $e^{B_\tau} = \alpha e^{-\tau/2}$, hence  

$$\alpha E[e^{-\tau}] = E[e^{B_\tau - \tau^2/2}] = 1, \quad \text{i.e.} \quad E[e^{-\tau}] = 1/\alpha \leq 1.$$  

Remark: note that this argument fails when $\alpha < 1$ because in that case $\tau$ is not a.s. finite.  

c) For any $t \in \mathbb{R}_+$, the question "is $\tau > t$?" can be answered based on the observation of the paths of $(B_s)_{0 \leq s \leq t}$ and of the (deterministic) curve $(1 + \alpha s)_{0 \leq s \leq t}$ up to the time $t$. Therefore $\tau$ is a stopping time.  

Since $\tau$ is a stopping time and $(B_t)_{t \in \mathbb{R}_+}$ is a martingale, the stopping time theorem shows that $(B^2_{t \wedge \tau} - (t \wedge \tau))_{t \in \mathbb{R}_+}$ is also a martingale and in particular its expectation  

$$E[B^2_{t \wedge \tau} - (t \wedge \tau)] = E[B^2_{0 \wedge \tau} - (0 \wedge \tau)] = E[B^2_0 - 0] = 0$$  

is constantly equal to 0 for all $t$. This shows that  

$$E[B^2_{\tau} - \tau] = E \left[ \lim_{t \to \infty} (B^2_{t \wedge \tau} - (t \wedge \tau)) \right] = \lim_{t \to \infty} E[(B^2_{t \wedge \tau} - (t \wedge \tau))] = 0.$$  

Next, we note that we have $B^2_{\tau} = 1 + \alpha \tau$, hence  

$$1 + \alpha E[\tau] = E[1 + \alpha \tau] = E[B^2_{\tau}] - E[\tau] = 0,$$  

i.e.  

$$E[\tau] = 1/(1 - \alpha).$$  

Remark: Note that this argument is valid whenever $\alpha \leq 1$ and yields $E[\tau] = +\infty$ when $\alpha = 1$, however it fails when $\alpha > 1$ because in that case $\tau$ is not a.s. finite.

Exercise 11.3  

a) By the stopping time theorem, for all $n \geq 0$ we have  

$$1 = E \left[ e^{\sqrt{2r} B_{\tau_L} - n \tau_L \wedge n} \right]$$  

$$= E \left[ e^{\sqrt{2r} B_{\tau_L} - n \tau_L \wedge n} \mathbb{1}_{\{\tau_L < n\}} \right] + E \left[ e^{\sqrt{2r} B_{\tau_L} - n \tau_L \wedge n} \mathbb{1}_{\{\tau_L \geq n\}} \right]$$  

$$= E \left[ e^{\sqrt{2r} B_{\tau_L} - n \tau_L \wedge n} \right] + E \left[ e^{\sqrt{2r} B_{\tau_L} - n \tau_L \wedge n} \mathbb{1}_{\{\tau_L \geq n\}} \right]$$  

$$= e^{L \sqrt{2r}} E \left[ e^{-r \tau_L \wedge n} \mathbb{1}_{\{\tau_L < n\}} \right] + E \left[ e^{\sqrt{2r} B_{\tau_L} - n \tau_L \wedge n} \right].$$  

The first term above converges to  

$$e^{L \sqrt{2r}} E \left[ e^{-r \tau_L \wedge n} \mathbb{1}_{\{\tau_L \leq n\}} \right] = e^{L \sqrt{2r}} E \left[ e^{-r \tau_L \wedge n} \mathbb{1}_{\{\tau_L \leq n\}} \right]$$  

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as $n$ tends to infinity, by dominated or monotone convergence and the fact that $r > 0$. The second term can be bounded as

$$0 \leq \mathbb{E} \left[ e^{\sqrt{2r}B_n - rn} \cdot 1_{\{\tau_L \leq n\}} \right] \leq e^{-rn} \mathbb{E} \left[ e^{L\sqrt{2r}} \cdot 1_{\{\tau_L \geq n\}} \right] \leq e^{-rn} e^{L\sqrt{2r}},$$

which tends to 0 as $n$ tends to infinity because $r > 0$. Therefore we have

$$1 = \lim_{n \to \infty} \mathbb{E} \left[ e^{\sqrt{2r}B_{\tau_L \wedge n} - r(\tau_L \wedge n)} \right] = e^{L\sqrt{2r}} \mathbb{E} [e^{-r\tau_L}],$$

which yields $\mathbb{E} [e^{-r\tau_L}] = e^{-L\sqrt{2r}}$ for any $r \geq 0$. When $r < 0$ we could in fact show that $\mathbb{E} [e^{-r\tau_L}] = +\infty$.

b) In order to maximize

$$\mathbb{E} \left[ e^{-r\tau_L B_{\tau_L}} \right] = L \mathbb{E} \left[ e^{-r\tau_L} \right] = L e^{-L\sqrt{2r}},$$

we differentiate

$$\frac{\partial}{\partial L} (L e^{-L\sqrt{2r}}) = e^{-L\sqrt{2r}} - L\sqrt{2r} e^{-L\sqrt{2r}} = 0,$$

which yields the optimal level $L^* = 1/\sqrt{2r}$.

This shows that when the value of $r$ is “large” the better strategy is to opt for a “small gain” at the level $L^* = 1/\sqrt{2r}$ rather than to wait for a longer time.

Exercise 11.4

a) Letting $A_0 := 0$,

$$A_{n+1} := A_n + \mathbb{E} [M_{n+1} - M_n \mid \mathcal{F}_n], \quad n \geq 0,$$

and

$$N_n := M_n - A_n, \quad n \in \mathbb{N}, \quad \text{(A.29)}$$

we have

(i) for all $n \in \mathbb{N}$,

$$\mathbb{E} [N_{n+1} \mid \mathcal{F}_n] = \mathbb{E} [M_{n+1} - A_{n+1} \mid \mathcal{F}_n]$$

$$= \mathbb{E} [M_{n+1} - A_n - \mathbb{E} [M_{n+1} - M_n \mid \mathcal{F}_n] \mid \mathcal{F}_n]$$

$$= \mathbb{E} [M_{n+1} - A_n \mid \mathcal{F}_n] - \mathbb{E} [\mathbb{E} [M_{n+1} - M_n \mid \mathcal{F}_n] \mid \mathcal{F}_n]$$

$$= \mathbb{E} [M_{n+1} - A_n \mid \mathcal{F}_n] - \mathbb{E} [M_{n+1} - M_n \mid \mathcal{F}_n]$$

$$= - \mathbb{E} [A_n \mid \mathcal{F}_n] + \mathbb{E} [M_n \mid \mathcal{F}_n]$$

$$= M_n - A_n$$

$$= N_n.$$
hence $(N_n)_{n \in \mathbb{N}}$ is a martingale with respect to $(\mathcal{F}_n)_{n \in \mathbb{N}}$.

(ii) We have

\[
A_{n+1} - A_n = \mathbb{E}[M_{n+1} - M_n \mid \mathcal{F}_n] \\
= \mathbb{E}[M_{n+1} \mid \mathcal{F}_n] - \mathbb{E}[M_n \mid \mathcal{F}_n] \\
= \mathbb{E}[M_{n+1} \mid \mathcal{F}_n] - M_n \geq 0, \quad n \in \mathbb{N},
\]

since $(M_n)_{n \in \mathbb{N}}$ is a submartingale.

(iii) By induction we have

\[
A_n = A_{n-1} + \mathbb{E}[M_n - M_{n-1} \mid \mathcal{F}_{n-1}], \quad n \geq 1,
\]

which is $\mathcal{F}_{n-1}$-measurable provided that $A_n$ is $\mathcal{F}_{n-1}$-measurable, $n \geq 1$.

(iv) This property is obtained by construction in (A.29).

b) For all bounded stopping times $\sigma$ and $\tau$ such that $\sigma \leq \tau$ a.s., we have

\[
\mathbb{E}[M_\sigma] = \mathbb{E}[N_\sigma] + \mathbb{E}[A_\sigma] \\
\leq \mathbb{E}[N_\sigma] + \mathbb{E}[A_\tau] \\
= \mathbb{E}[N_\tau] + \mathbb{E}[A_\tau] \\
= \mathbb{E}[M_\tau],
\]

by (11.11), since $(M_n)_{n \in \mathbb{N}}$ is a martingale and $(A_n)_{n \in \mathbb{N}}$ is nondecreasing.

Exercise 11.5 The option payoffs at immediate exercise are given as follows:

\[
p^* = \frac{2}{3} \quad (K - S_2)^+ = 0
\]

\[
(K - S_1)^+ = 0.05
\]

\[
(K - S_0)^+ = 0.3
\]

\[
g^* = \frac{1}{3} \quad (K - S_2)^+ = 0.17
\]

\[
(K - S_1)^+ = 0.35
\]

\[
g^* = \frac{1}{3} \quad (K - S_2)^+ = 0.44
\]

The expected payoffs are given by
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\[ (K - S_2)^+ = 0 \]

\[ \mathbb{E}^*[(K - S_2)^+ | S_1 = 1.2] = \frac{0.17}{3} \]

\[ (K - S_2)^+ = 0.17 \]

\[ \mathbb{E}^*[(K - S_2)^+ | S_1 = 0.9] = 0.26 \]

\[ (K - S_2)^+ = 0.44 \]

Consequently, at time \( t = 1 \) we would exercise immediately if \( S_1 = 0.9 \), and wait if \( S_1 = 1.2 \). At time \( t = 0 \) with \( S_0 = 1 \) the initial value of the option is \((0.34/3 + 0.35)/3 = 1.39/9 \approx 0.154 < 0.25\) so we would exercise immediately as well.

Exercise 11.6

a) Taking \( f(x) := Cx^{-2r/\sigma^2} \), we have

\[
rx f'(x) + \frac{1}{2} \sigma^2 x^2 f''(x) = -C \frac{2r^2}{\sigma^2} x^{-2r/\sigma^2} + Cr \left( 1 + \frac{2r}{\sigma^2} \right) x^{-2r/\sigma^2}
\]

\[
= C r x^{-2r/\sigma^2}
\]

\[
= rf(x),
\]

and the condition \( \lim_{x \to \infty} f(x) = 0 \) is satisfied since \( r > 0 \).

b) The conditions \( f(L^*) = K - L^* \) and \( f'(L^*) = -1 \) read

\[
\begin{align*}
C(L^*)^{-2r/\sigma^2} &= K - L^*, \\
-\frac{2r}{\sigma^2} C(L^*)^{-1-2r/\sigma^2} &= -1,
\end{align*}
\]

i.e.

\[
\begin{align*}
C(L^*)^{-2r/\sigma^2} &= K - L^* \\
\frac{2r}{\sigma^2} (K - L^*) &= L^*,
\end{align*}
\]

hence

\[
\begin{align*}
L^* &= \frac{2rK}{2r + \sigma^2} \\
C &= \frac{K\sigma^2}{2r + \sigma^2} \left( \frac{2rK}{2r + \sigma^2} \right)^{2r/\sigma^2} = \frac{\sigma^2}{2r} \left( \frac{2rK}{2r + \sigma^2} \right)^{1+2r/\sigma^2}.
\end{align*}
\]
Exercise 11.7

a) Given the value $-\Phi(-d_+(S^*, T))$ of the Black-Scholes put Delta, the smooth fit condition states that at $x = S^*$, the left derivative of (11.43) should match the right derivative of (11.44), i.e.

$$-1 = -\Phi(-d_+(S^*, T)) - \frac{2r \alpha}{\sigma^2} (S^*)^{-1},$$

which yields

$$\alpha(S^*) = \frac{\sigma^2 S^*}{2r} (1 - \Phi(-d_+(S^*, T))) = \frac{\sigma^2 S^*}{2r} \Phi(d_+(S^*, T)),$$

and

$$f(x, T) \approx \begin{cases} 
    \text{BS}_p(x, T) + \frac{\sigma^2(S^*)^{1+2r/\sigma^2}}{2rx^{2r/\sigma^2}} \Phi(d_+(S^*, T)), & x > S^*, \\
    K - x, & x \leq S^*.
\end{cases}$$

Note that at maturity ($T = 0$ here) we have $d_+(S^*, 0) = -\infty$ since $S^* < K$, hence $\Phi(d_+(S^*, 0)) = 0$ and $f(x, 0) = K - x$ as expected.

b) Equating (11.43) to (11.44) at $x = S^*$ yields the equation

$$K - S^* = \text{BS}_p(x, T) + \alpha(S^*),$$

i.e.

$$1 = e^{-rT} \Phi(-d_-(S^*, T)) + \frac{S^*}{K} \left(1 + \frac{\sigma^2}{2r}\right) \Phi(-d_+(S^*, T)),$$

which can be used to determine the value of $S^*$, and then the corresponding value of $\alpha$. The proposed strategy is to exercise the put option as soon as the underlying price reaches the critical level $S^*$.

Fig. S.44: Perpetual vs finite expiration American put option price.
The plot in Figure S.44 yields a finite expiration critical price $S^* = 87.3$ which is expectedly higher than the perpetual critical price $L^* = 85.71$, with $K = 100$, $\sigma = 10\%$, and $r = 3\%$. The perpetual price, however, appears higher than the finite expiration price.

Exercise 11.8

a) We have

$$\tau_\epsilon = \begin{cases} 
\epsilon & \text{if } Z = 1, \\
+\infty & \text{if } Z = 0.
\end{cases}$$

b) First, we note that

$$\mathcal{F}_t = \begin{cases} 
\{\emptyset, \Omega\} & \text{if } t = 0, \\
\{\emptyset, \Omega, \{Z = 0\}, \{Z = 1\}\} & \text{if } t > 0.
\end{cases}$$

Next, we have

$$\{\tau_\epsilon > 0\} = \{Z = 0\},$$

hence

$$\{\tau_\epsilon > 0\} \notin \mathcal{F}_0 = \{\emptyset, \Omega\},$$

and therefore $\tau_0$ is not an $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$-stopping time.

c) i) For $t = 0$ we have $\{\tau_\epsilon > 0\} = \{Z = 0\} \cup \{Z = 1\} = \Omega$, hence

$$\{\tau_\epsilon > 0\} \in \mathcal{F}_0 = \{\emptyset, \Omega\}.$$

ii) For $0 < t < \epsilon$ we have $\{\tau_\epsilon > t\} = \Omega$, hence

$$\{\tau_\epsilon > t\} \in \mathcal{F}_t = \{\emptyset, \Omega, \{Z = 0\}, \{Z = 1\}\}.$$

iii) For $t > \epsilon$ we have $\{\tau_\epsilon > t\} = \{Z = 0\}$, hence

$$\{\tau_\epsilon > t\} \in \mathcal{F}_t = \{\emptyset, \Omega, \{Z = 0\}, \{Z = 1\}\}.$$

Therefore $\tau_\epsilon$ is an $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$-stopping time when $\epsilon > 0$.

Note that here the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$ is not right-continuous, as

$$\{\emptyset, \Omega\} = \mathcal{F}_0 \neq \mathcal{F}_{0+} := \bigcap_{t > 0} \mathcal{F}_t = \{\emptyset, \Omega, \{Z = 0\}, \{Z = 1\}\}.$$

Exercise 11.9

a) This intrinsic payoff is $\kappa - S_0$. 

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b) We note that

\[ Z_t = \left( \frac{S_t}{S_0} \right)^\lambda e^{-(r-\delta)t+\lambda \sigma^2 t/2 - \lambda^2 \sigma^2 t/2} = e^{\lambda \sigma B_t - \lambda^2 \sigma^2 t/2}, \quad t \in \mathbb{R}_+, \]

is a geometric Brownian motion without drift, hence a martingale, under the risk-neutral probability measure \( \mathbb{P}^* \).

c) The parameter \( \lambda \) should satisfy the equation

\[ r = (r-\delta)\lambda - \frac{\sigma^2}{2}\lambda(1-\lambda), \]

i.e.

\[ \lambda^2 \sigma^2 / 2 + \lambda(r-\delta - \sigma^2 / 2) - r = 0. \]

This equation admits two solutions

\[ \lambda_{\pm} = -\frac{(r-\delta - \sigma^2 / 2) \pm \sqrt{(r-\delta - \sigma^2 / 2)^2 + 4r\sigma^2 / 2}}{\sigma^2}, \]

d) Relation (11.23) follows from the fact that \( S_t < L \) and \( \lambda_- < 0 \).

e) By the stopping time theorem we have

\[ \mathbb{E}^*[Z_{\tau_L}] = \mathbb{E}^*[Z_0] = 1, \]

which rewrites as

\[ \mathbb{E}^* \left[ \left( \frac{S_{\tau_L}}{S_0} \right)^\lambda e^{-(r-\delta)\lambda - \lambda \sigma^2 / 2 + \lambda^2 \sigma^2 / 2 \tau_L} \right] = 1, \]

or, given the relation \( S_{\tau_L} = L \),

\[ \left( \frac{L}{S_0} \right)^\lambda \mathbb{E}^* \left[ e^{-(r-\delta)\lambda - \lambda \sigma^2 / 2 + \lambda^2 \sigma^2 / 2 \tau_L} \right] = 1, \]

i.e.

\[ \mathbb{E}^* \left[ e^{-r\tau_L} \right] = \left( \frac{S_0}{L} \right)^\lambda, \]

provided that we choose \( \lambda \) such that

\[ -((r-\delta)\lambda - \lambda \sigma^2 / 2 + \lambda^2 \sigma^2 / 2) = -r, \quad (A.30) \]

i.e.

\[ \lambda = \frac{-(r-\delta - \sigma^2 / 2) \pm \sqrt{(r-\delta - \sigma^2 / 2)^2 + 4r\sigma^2 / 2}}{\sigma^2}, \]

and we choose the negative solution.
\[ \lambda := \frac{-(r - \delta - \sigma^2/2) - \sqrt{(r - \delta - \sigma^2/2)^2 + 4r\sigma^2/2}}{\sigma^2} \]

since \( S_0/L = x/L > 1 \) and the expectation \( \mathbb{E}^*[e^{-r\tau L}] \) is lower than 1 as \( r \geq 0 \).

f) This follows from (11.23) and the fact that \( r > 0 \). Using the fact that

\[ S_{\tau L} = L < K \quad \text{when} \quad \tau L < \infty, \]

we find

\[ \mathbb{E}\left[ e^{-r\tau L} \mid S_0 = x \right] = \mathbb{E}\left[ e^{-r\tau L} (K - S_{\tau L})^+ \mathbb{I}_{\{\tau L < x\}} \mid S_0 = x \right] \]

\[ = (K - L) \mathbb{E}\left[ e^{-r\tau L} \mathbb{I}_{\{\tau L < x\}} \mid S_0 = x \right] \]

Next, noting that \( \tau L = 0 \) if \( S_0 \leq L \), for all \( L \in (0, K) \) we have

\[ \mathbb{E}\left[ e^{-r\tau L} (K - S_{\tau L})^+ \mid S_0 = x \right] \]

\[ = \begin{cases} 
K - x, & 0 < x \leq L, \\
E\left[ e^{-r\tau L} (K - L)^+ \mid S_0 = x \right], & x > L.
\end{cases} \]

\[ = \begin{cases} 
K - x, & 0 < x \leq L, \\
(K - L) E\left[ e^{-r\tau L} \mid S_0 = x \right], & x > L.
\end{cases} \]

\[ = \begin{cases} 
K - x, & 0 < x \leq L, \\
(K - L) \left( \frac{x}{L} \right)^{-\frac{-(r-a-a^2/2)-\sqrt{(r-a-a^2/2)^2+4ra^2/2}}{\sigma^2}}, & x \geq L.
\end{cases} \]

g) In order to compute \( L^* \) we observe that, geometrically, the slope of \( x \mapsto f_L(x) = (K - L)(x/L)^{\lambda^-} \) at \( x = L^* \) is equal to \(-1\), \textit{i.e.}

\[ f'_{L^*}(L^*) = \lambda^- (K - L^*) \left( \frac{L^*}{\lambda^-} \right)^{\lambda^- - 1} = -1, \]

hence

\[ \lambda^- (K - L^*) = L^*, \quad \text{or} \quad L^* = \frac{\lambda^-}{\lambda^- - 1} K < K. \quad (A.31) \]

Equivalently we may recover the value of \( L^* \) from the optimality condition

\[ \frac{\partial f_L(x)}{\partial L} = -(x/L)^{\lambda^-} - \lambda^- x (K - L)(x/L)^{\lambda^- + 1} = 0, \]

at \( L = L^* \), hence

\[ \bigcirc \]
\[- \left( \frac{x}{L} \right)^{\lambda_-} - \lambda_- (K - L) x^{\lambda_-} L^{-\lambda_- - 1} = 0,\]

hence

\[L^* = \frac{\lambda_-}{1 - \lambda_-} K = \frac{1}{1 - 1/\lambda_-} K,\]

and

\[\sup_{L \in (0, K)} \mathbb{E}^* \left[ e^{-r t_L} (K - S_{t_L})^+ \mid S_0 = x \right] = -\frac{1}{\lambda_-} \left( \frac{K}{1 - 1/\lambda_-} \right)^{1 - \lambda_-} x^{\lambda_-}.\]

h) For \(x \geq L\) we have

\[f_{L^*}(x) = (K - L^*) \left( \frac{x}{L^*} \right)^{\lambda_-}\]

\[= \left( K - \frac{\lambda_-}{\lambda_- - 1} K \right) \left( \frac{x}{\lambda_- - 1} K \right)^{\lambda_-}\]

\[= \left( - \frac{K}{\lambda_- - 1} \right) \left( x (\lambda_- - 1)^{\lambda_-} \left( \frac{x}{\lambda_-} \right)^{\lambda_-} \left( \frac{1}{\lambda_-} \right) \right)^{\lambda_- - 1}\]

\[= \left( \frac{x}{-\lambda_-} \right)^{\lambda_-} \left( \frac{\lambda_- - 1}{-K} \right)^{\lambda_- - 1} \frac{K}{1 - \lambda_-}. \quad (A.32)\]

i) Let us check that the relation

\[f_{L^*}(x) \geq (K - x)^+ \quad (A.33)\]

holds. For all \(x \leq K\) we have

\[f_{L^*}(x) - (K - x) = \left( \frac{x}{K} \right)^{\lambda_-} \left( \frac{\lambda_- - 1}{\lambda_-} \right)^{\lambda_-} \left( K \frac{1}{1 - \lambda_-} + x \right) - K\]

\[= K \left( \left( \frac{x}{K} \right)^{\lambda_-} \left( \frac{\lambda_- - 1}{\lambda_-} \right)^{\lambda_-} \left( \frac{1}{1 - \lambda_-} + \frac{x}{K} - 1 \right). \right.\]

Hence it suffices to take \(K = 1\) and to show that for all

\[L^* = \frac{\lambda_-}{\lambda_- - 1} \leq x \leq 1\]

we have

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\[ f_{L^*}(x) - (1 - x) = \frac{x^\lambda}{1 - \lambda} \left( \frac{\lambda - 1}{\lambda} \right)^\lambda + x - 1 \geq 0. \]

Equality to 0 holds for \( x = \lambda_-(\lambda_- - 1) \). By differentiation of this relation we get

\[ f'_{L^*}(x) - (1 - x)' = \lambda_- x^{\lambda_- - 1} \left( \frac{\lambda_- - 1}{\lambda_-} \right)^\lambda_- \frac{1}{1 - \lambda_-} + 1 \]

\[ = x^{\lambda_- - 1} \left( \frac{\lambda_- - 1}{\lambda_-} \right)^\lambda_- + 1 \]

\[ \geq 0, \]

hence the function \( f_{L^*}(x) - (1 - x) \) is nondecreasing and the inequality holds throughout the interval \( [\lambda_-/(\lambda_- - 1), K] \).

On the other hand, using (A.30) it can be checked by hand that \( f_{L^*} \) given by (A.32) satisfies the equality

\[ (r - \delta)xf'_{L^*}(x) + \frac{1}{2} \sigma^2 x^2 f''_{L^*}(x) = rf_{L^*}(x) \quad (A.34) \]

for \( x \geq L^* = \frac{\lambda_-}{\lambda_- - 1} K \). In case

\[ 0 \leq x \leq L^* = \frac{\lambda_-}{\lambda_- - 1} K \]

we have

\[ f_{L^*}(x) = K - x = (K - x)^+, \]

hence the relation

\[ \left( rf_{L^*}(x) - (r - \delta)xf_{L^*}'(x) - \frac{1}{2} \sigma^2 x^2 f_{L^*}''(x) \right) (f_{L^*}(x) - (K - x)^+) = 0 \]

always holds. On the other hand, in that case we also have

\[ (r - \delta)xf'_{L^*}(x) + \frac{1}{2} \sigma^2 x^2 f''_{L^*}(x) = -(r - \delta)x, \]

and to conclude we need to show that

\[ (r - \delta)xf'_{L^*}(x) + \frac{1}{2} \sigma^2 x^2 f''_{L^*}(x) \leq rf_{L^*}(x) = r(K - x), \quad (A.35) \]

which is true if

\[ \delta x \leq rK. \]

Indeed by (A.30) we have
\[(r - \delta)\lambda_\nu = r + \lambda_\nu (\lambda_\nu - 1)\sigma^2 / 2 \geq r,\]

hence

\[\delta \frac{\lambda_\nu}{\lambda_\nu - 1} \leq r,\]

since \(\lambda_\nu < 0\), which yields

\[\delta x \leq \delta L_* \leq \delta \frac{\lambda_\nu}{\lambda_\nu - 1} K \leq rK.\]

j) By Itô’s formula and the relation

\[dS_t = (r - \delta)S_t dt + \sigma S_t d\tilde{B}_t\]

we have

\[d(f_{L*}(S_t)) = -re^{-rt}f_{L*}(S_t) dt + e^{-rt}dL_* (S_t)\]

\[= -re^{-rt}f_{L*}(S_t) dt + e^{-rt}f'_{L*}(S_t) dS_t + \frac{1}{2} e^{-rt} \sigma^2 S_t^2 f''_{L*}(S_t)\]

\[= e^{-rt} \left(-rf_{L*}(S_t) + (r - \delta)S_t f'_{L*}(S_t) + \frac{1}{2} \sigma^2 S_t^2 f''_{L*}(S_t)\right) dt\]

\[+ e^{-rt} \sigma S_t f'_{L*}(S_t) d\tilde{B}_t,\]

and from Equations (A.34) and (A.35) we have

\[(r - \delta)xf'_{L*}(x) + \frac{1}{2} \sigma^2 x^2 f''_{L*}(x) \leq rf_{L*}(x),\]

hence

\[t \mapsto e^{-rt}f_{L*}(S_t)\]

is a supermartingale.

k) By the supermartingale property of

\[t \mapsto e^{-rt}f_{L*}(S_t),\]

for all stopping times \(\tau\) we have

\[f_{L*}(S_0) \geq \mathbb{E}^* \left[ e^{-r\tau} f_{L*}(S_\tau) \mid S_0 \right] \geq \mathbb{E}^* \left[ e^{-r\tau} (K - S_\tau)^+ \mid S_0 \right],\]

by (A.33), hence

\[f_{L*}(S_0) \geq \sup_{\tau \text{ stopping time}} \mathbb{E}^* \left[ e^{-r\tau} (K - S_\tau)^+ \mid S_0 \right]. \quad (A.36)\]

l) The stopped process

\[\hat{S}_t = \mathbb{E}^* \left[ e^{-r\tau} (K - S_\tau)^+ \mid S_0 \right].\]
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\[ t \mapsto e^{-rt \wedge \tau L^*} f_{L^*}(S_{t \wedge \tau L^*}) \]

is a martingale since it has vanishing drift up to time \( \tau L^* \) by (A.34), and it is constant after time \( \tau L^* \), hence by the martingale stopping time Theorem (11.7) we find

\[
\begin{align*}
    f_{L^*}(S_0) &= \mathbb{E}^* \left[ e^{-r \tau} f_{L^*}(S_{\tau L^*}) \mid S_0 \right] \\
    &= \mathbb{E}^* \left[ e^{-r \tau} f_{L^*} \left( L^* \right) \mid S_0 \right] \\
    &= \mathbb{E}^* \left[ e^{-r \tau} (K - S_{\tau L^*})^+ \mid S_0 \right] \\
    &\leq \sup_{\tau \text{ stopping time}} \mathbb{E}^* \left[ e^{-r \tau} (K - S_{\tau})^+ \mid S_0 \right].
\end{align*}
\]

m) By combining the above results and conditioning at time \( t \) instead of time \( 0 \) we deduce that

\[
\begin{align*}
f_{L^*}(S_t) &= \mathbb{E}^* \left[ e^{-r(\tau L^* - t)}(K - S_{\tau L^*})^+ \mid S_t \right] \\
    &= \begin{cases} 
        K - S_t, & 0 < S_t \leq \frac{\lambda_-}{\lambda_- - 1} K, \\
        \left( \frac{\lambda_- - 1}{-K} \right)^{\lambda_- - 1} \left( \frac{-S_t}{\lambda_-} \right)^{\lambda_-}, & S_t \geq \frac{\lambda_-}{\lambda_- - 1} K,
    \end{cases}
\end{align*}
\]

for all \( t \in \mathbb{R}_+ \), where

\[
\tau_{L^*} = \inf \{ u \geq t : S_u \leq L \}.
\]

We note that the perpetual put option price does not depend on the value of \( t \geq 0 \).

Exercise 11.10

a) We have

\[
Z^{(\lambda)}_t = (S_t)^{\lambda} e^{-t((r-\delta)\lambda-\lambda(1-\lambda)\sigma^2/2)} = (S_0)^{\lambda} e^{\lambda \sigma B_t - \lambda^2 \sigma^2 t / 2},
\]

which is a driftless geometric Brownian motion, and therefore a martingale under \( \mathbb{P}^* \).

b) The condition is \( r = (r-\delta)\lambda - \lambda(1-\lambda)\sigma^2/2 \), with solutions

\[
\lambda_- = \frac{\delta - r + \sigma^2/2 - \sqrt{(\delta - r + \sigma^2/2)^2 + 2r \sigma^2}}{\sigma^2} \leq 0,
\]

\[
\lambda_+ = \frac{\delta - r + \sigma^2/2 + \sqrt{(\delta - r + \sigma^2/2)^2 + 2r \sigma^2}}{\sigma^2} \geq 1.
\]
c) The inequality
\[ 0 \leq Z_t^{(\lambda_+)} = (S_t)^{\lambda_+} e^{-rt} \leq L^{\lambda_+} \]
holds because \( \lambda_+ > 0 \) and \( S_t \leq L, 0 \leq t < \tau_L \). Hence, since \( \lim_{t \to \infty} Z_t^{(\lambda_+)} = 0 \), we have
\[
L^{\lambda_+} \mathbb{E}^* \left[ e^{-r\tau_L} \right] = \mathbb{E}^* \left[ (S_{\tau_L})^{\lambda_+} e^{-r\tau_L} \mathbb{1}_{\{\tau_L < \infty\}} \right] = \mathbb{E}^* \left[ Z_{\tau_L}^{(\lambda_+)} \mathbb{1}_{\{\tau_L < \infty\}} \right] = \mathbb{E}^* \left[ Z_{\tau_L}^{(\lambda_+)} \right] = (S_0)^{\lambda_+},
\]
hence
\[
\mathbb{E}^* \left[ e^{-r\tau_L} (S_{\tau_L} - K)^+ \mid S_0 = x \right] = \mathbb{E}^* \left[ (S_{\tau_L} - K) e^{-r\tau_L} \mathbb{1}_{\{\tau_L < \infty\}} \mid S_0 = x \right]
= (L - K) \mathbb{E}^* \left[ e^{-r\tau_L} \mathbb{1}_{\{\tau_L < \infty\}} \mid S_0 = x \right]
= (L - K) \mathbb{E}^* \left[ e^{-r\tau_L} \mid S_0 = x \right]
= (L - K) \left( \frac{x}{L} \right)^{\lambda_+},
\]
when \( S_0 = x > L \). In order to maximize
\[
\sup_{L \in (0,K)} \mathbb{E}^* \left[ e^{-r\tau_L} (K - S_{\tau_L})^+ \mid S_0 = x \right],
\]
we differentiate \( L \mapsto (L - K) (x/L)^{\lambda_+} \) with respect to \( L \), to find
\[
\left( \frac{x}{L} \right)^{\lambda_+} - \lambda_+ (L - K) x^{\lambda_+} L^{-\lambda_+ - 1} = 0,
\]
hence
\[
L_\delta^* = \frac{\lambda_+}{\lambda_+ - 1} K = \frac{K}{1 - 1/\lambda_+},
\]
and
\[
\sup_{L \in (0,K)} \mathbb{E}^* \left[ e^{-r\tau_L} (K - S_{\tau_L})^+ \mid S_0 = x \right] = \frac{1}{\lambda_+} \left( \frac{K}{1 - 1/\lambda_+} \right)^{1-\lambda_+} x^{\lambda_+}.
\]
We note that as \( \delta \searrow 0 \) we have \( \lambda_+ \searrow 1 \) and \( L_\delta^* \nearrow \infty \), and since
\[
\left( \frac{K}{1 - 1/\lambda_+} \right)^{1-\lambda_+} = \exp \left( (\lambda_+ - 1) \log \frac{\lambda_+ - 1}{\lambda_+ K} \right) \to 1,
\]
we find that the perpetual American call price without dividend (\( \delta = 0 \)) is \( S_0 = x \).
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Exercise 11.11

a) By the definition (11.48) of $S_1(t)$ and $S_2(t)$ we have

$$Z_t = e^{-rt} S_2(t) \left( \frac{S_1(t)}{S_2(t)} \right) \alpha$$

$$= e^{-rt} S_1(t)^\alpha S_2(t)^{1-\alpha}$$

$$= S_1(0)^\alpha S_2(0)^{1-\alpha} e^{(\alpha \sigma_1 + (1-\alpha) \sigma_2) W_t - \sigma_2^2 t/2},$$

which is a martingale when

$$\sigma_2^2 = (\alpha \sigma_1 + (1-\alpha) \sigma_2)^2,$$

i.e.

$$\alpha \sigma_1 + (1-\alpha) \sigma_2 = \pm \sigma_2,$$

which yields either $\alpha = 0$ or

$$\alpha = \frac{2 \sigma_2}{\sigma_2 - \sigma_1} > 1,$$

since $0 \leq \sigma_1 < \sigma_2$.

b) We have

$$\mathbb{E}[e^{-r\tau L} (S_1(\tau_L) - S_2(\tau_L))^+] = \mathbb{E}[e^{-r\tau L} (LS_2(\tau_L) - S_2(\tau_L))^+]$$

$$= (L - 1)^+ \mathbb{E}[e^{-r\tau L} S_2(\tau_L)].$$  \( \text{(A.37)} \)

c) Since $\tau_L \wedge t$ is a bounded stopping time we can write

$$S_2(0) \left( \frac{S_1(0)}{S_2(0)} \right)^\alpha = \mathbb{E} \left[ e^{-r(\tau_L \wedge t)} S_2(\tau_L \wedge t) \left( \frac{S_1(\tau_L \wedge t)}{S_2(\tau_L \wedge t)} \right)^\alpha \right]$$

$$= \mathbb{E} \left[ e^{-r\tau L} S_2(\tau_L) \left( \frac{S_1(\tau_L)}{S_2(\tau_L)} \right)^\alpha \mathbb{1}_{\{\tau_L \leq t\}} \right] + \mathbb{E} \left[ e^{-rt} S_2(t) \left( \frac{S_1(t)}{S_2(t)} \right)^\alpha \mathbb{1}_{\{\tau_L > t\}} \right]$$

(A.38)

We have

$$e^{-rt} S_2(t) \left( \frac{S_1(t)}{S_2(t)} \right)^\alpha \mathbb{1}_{\{\tau_L > t\}} \leq e^{-rt} S_2(t) L^\alpha \mathbb{1}_{\{\tau_L > t\}} \leq e^{-rt} S_2(t) L^\alpha,$$

hence by a uniform integrability argument,

$$\lim_{t \to \infty} \mathbb{E} \left[ e^{-rt} S_2(t) \left( \frac{S_1(t)}{S_2(t)} \right)^\alpha \mathbb{1}_{\{\tau_L > t\}} \right] = 0,$$

and letting $t$ go to infinity in (A.38) shows that

$$S_2(0) \left( \frac{S_1(0)}{S_2(0)} \right)^\alpha = \mathbb{E} \left[ e^{-r\tau L} S_2(\tau_L) \left( \frac{S_1(\tau_L)}{S_2(\tau_L)} \right)^\alpha \right] = L^\alpha \mathbb{E} \left[ e^{-r\tau L} S_2(\tau_L) \right],$$

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since $S_1(\tau_L)/S_2(\tau_L) = L/L = 1$. The conclusion
\[\mathbb{E}[e^{-rL}(S_1(\tau_L) - S_2(\tau_L))^+] = (L - 1)^+L^{-\alpha}S_2(0)\left(\frac{S_1(0)}{S_2(0)}\right)\alpha\] (A.39)
then follows by an application of (A.37).
d) In order to maximize (A.39) as a function of $L$ we consider the derivative
\[\frac{\partial}{\partial L} \frac{L - 1}{L^\alpha} = \frac{1}{L^\alpha} - \alpha(L - 1)L^{-\alpha - 1} = 0,
\]
which vanishes for
\[L^* = \frac{\alpha}{\alpha - 1},\]
and we substitute $L$ in (A.39) with the value of $L^*$.
e) In addition to $r = \sigma^2/2$ it is sufficient to let $S_1(0) = \kappa$ and $\sigma_1 = 0$ which yields $\alpha = 2$, $L^* = 2$, and we find
\[\sup_{\tau \text{ stopping time}} \mathbb{E}[e^{-r\tau}(\kappa - S_2(\tau))^+] = \frac{1}{S_2(0)}\left(\frac{\kappa}{2}\right)^2,
\]
which coincides with the result of Proposition 11.9.

Exercise 11.12
a) It suffices to check the sign of the quantity
\[(\lambda - 1)(\lambda + 2r/\sigma^2),\] (A.40)
in (11.50), which is positive when $\lambda \in (-\infty, -2r/\sigma^2] \cup [1, \infty)$, and negative when $-2r/\sigma^2 \leq \lambda \leq 1$.
b) The sign of (A.40) is positive when $\lambda \in (-\infty, 1] \cup [-2r/\sigma^2, \infty)$, and negative when $1 \leq \lambda \leq -2r/\sigma^2$.
c) By the stopping time theorem, for any $n \geq 0$ we have
\[x^\lambda = \mathbb{E}^*\left[e^{-r(\tau_L \wedge n)}Z_{\tau_L \wedge n}^{(\lambda)} | S_0 = x\right]
= \mathbb{E}^*\left[Z_{\tau_L}^{(\lambda)} \mathbb{1}_{\{\tau_L < n\}} | S_0 = x\right] + e^{-rn} \mathbb{E}^*\left[Z_n^{(\lambda)} \mathbb{1}_{\{\tau_L > n\}} | S_0 = x\right]
\geq \mathbb{E}^*\left[e^{-r\tau_L}(S_{\tau_L})^{\lambda} \mathbb{1}_{\{\tau_L < n\}} | S_0 = x\right]
= L^\lambda \mathbb{E}^*\left[e^{-r\tau_L} \mathbb{1}_{\{\tau_L < n\}} | S_0 = x\right],
\]
and by letting $n$ to infinity and applying monotone convergence this yields
\[\mathbb{E}^*\left[e^{-r\tau_L} \mathbb{1}_{\{\tau_L < \infty\}} | S_0 = x\right] \leq \left(\frac{x}{L}\right)^\lambda \leq \left\{\begin{array}{ll}
(x/L)^{\min(1, -2r/\sigma^2)}, & x \geq L, \\
(x/L)^{\max(1, -2r/\sigma^2)}, & 0 < x \leq L.
\end{array}\right\}
\]
d) We note that \( P^*(\tau_L < \infty) = 1 \) by (11.14), hence if \(-\sigma^2/2 \leq r < 0\) we have
\[
\mathbb{E}^* \left[ e^{-r\tau_L} (K - S_{\tau_L})^+ \mathbb{1}_{\{\tau_L < \infty\}} \mid S_0 = x \right] = (K - L) \mathbb{E}^* \left[ e^{-r\tau_L} \mid S_0 = x \right] \leq \begin{cases} (K - L)(x/L)^{-2r/\sigma^2}, & x \geq L, \\ (K - L)x/L, & 0 < x \leq L. \end{cases}
\]
Similarly, if \( r \leq -\sigma^2/2 \) we have
\[
\mathbb{E}^* \left[ e^{-r\tau_L} (S_{\tau_L} - K)^+ \mathbb{1}_{\{\tau_L < \infty\}} \mid S_0 = x \right] = (L - K) \mathbb{E}^* \left[ e^{-r\tau_L} \mathbb{1}_{\{\tau_L < \infty\}} \mid S_0 = x \right] \leq \begin{cases} (L - K)(x/L)^{-2r/\sigma^2}, & x \geq L, \\ (L - K)x/L, & 0 < x \leq L. \end{cases}
\]

e) This follows by noting that \((K - L)(x/L) = (K/L - 1)x\) increases to \(\infty\) when \(L\) tends to zero.

f) If \(-\sigma^2/2 \leq r < 0\) we have
\[
\mathbb{E}^* \left[ e^{-r\tau_L} (S_{\tau_L} - K)^+ \mathbb{1}_{\{\tau_L < \infty\}} \mid S_0 = x \right] = (L - K) \mathbb{E}^* \left[ e^{-r\tau_L} \mathbb{1}_{\{\tau_L < \infty\}} \mid S_0 = x \right] \leq \begin{cases} (L - K)(x/L)^{-2r/\sigma^2}, & x \geq L, \\ (L - K)x/L, & 0 < x \leq L. \end{cases}
\]
If \( r \leq -\sigma^2/2 \) we have
\[
\mathbb{E}^* \left[ e^{-r\tau_L} (S_{\tau_L} - K)^+ \mathbb{1}_{\{\tau_L < \infty\}} \mid S_0 = x \right] = (L - K) \mathbb{E}^* \left[ e^{-r\tau_L} \mathbb{1}_{\{\tau_L < \infty\}} \mid S_0 = x \right] \leq \begin{cases} (L - K)(x/L)^{-2r/\sigma^2}, & x \geq L, \\ (L - K)(x/L)^{-2r/\sigma^2}, & 0 < x \leq L. \end{cases}
\]

g) This follows by noting that for fixed \( x > 0 \), \((L - K)x/L = (1 - K/L)x\) increases to \(x\) when \(L\) tends to infinity.

Exercise 11.13 American binary options.

a) The optimal strategy is as follows:
   (i) if \( S_t \geq K \), then exercise immediately.
   (ii) if \( S_t < K \), then wait.

b) The optimal strategy is as follows:
(i) if \( S_t > K \), then wait.
(ii) if \( S_t \leq K \), exercise immediately.

c) Based on the answers to Question (a) we set

\[
C_{d Am}^\Lambda(t, K) = 1, \quad 0 \leq t < T,
\]

and

\[
C_{d Am}^\Lambda(T, x) = 0, \quad 0 \leq x < K.
\]

d) Based on the answers to Question (b), we set

\[
P_{d Am}^\Lambda(t, K) = 1, \quad 0 \leq t < T,
\]

and

\[
P_{d Am}^\Lambda(T, x) = 0, \quad x > K.
\]

e) Starting from \( S_t \leq K \), the maximum possible payoff is clearly reached as soon as \( S_t \) hits the level \( K \) before the expiration date \( T \), hence the discounted optimal payoff of the option is \( e^{-r(\tau_K - t)}\mathbb{1}_{\{\tau_K < T\}} \).

f) From Relation (8.8) we find

\[
\mathbb{P}(\tau_a \leq u) = \Phi \left( \frac{a - \mu u}{\sqrt{u}} \right) - e^{2\mu a} \Phi \left( \frac{-a - \mu u}{\sqrt{u}} \right), \quad u > 0,
\]

and by differentiation with respect to \( u \) this yields the probability density function

\[
f_{\tau_a}(u) = \frac{\partial}{\partial u} \mathbb{P}(\tau_a \leq u) = \frac{a}{\sqrt{2\pi u^3}} e^{-(a-\mu u)^2/(2u)} \mathbb{1}_{[0, \infty)}(u)
\]

of the first hitting time of level \( a \) by Brownian motion with drift \( \mu \). Given the relation

\[
S_u = S_t e^{\sigma(B_u - B_t) - \sigma^2(u-t)/2 + \mu(u-t)}, \quad u \geq t,
\]

we find that the probability density function of the first hitting time of level \( K \) after time \( t \) by \( (S_u)_{u \in [t, \infty)} \) is given by

\[
s \mapsto \frac{a}{\sqrt{2\pi(s-t)^3}} e^{-(a-\mu(s-t))^2/(2(s-t))}, \quad s > t,
\]

with

\[
\mu := \sigma^{-1} \left( r - \frac{\sigma^2}{2} \right) \quad \text{and} \quad a := \frac{1}{\sigma} \log \frac{K}{x},
\]

given that \( S_t = x \). Hence for \( x \in (0, K) \) we have

\[
C_{d Am}^\Lambda(t, x) = \mathbb{E}\left[ e^{-r(\tau_K - t)} \mathbb{1}_{\{\tau_K < T\}} \mid S_t = x \right]
\]
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\[ \begin{align*}
&= \int_t^T e^{-r(s-t)} \frac{a}{\sqrt{2\pi(s-t)^3}} e^{-(a-\mu(s-t))^2/(2(s-t))} ds \\
&= \int_0^{T-t} e^{-rs} \frac{a}{\sqrt{2\pi s^3}} e^{-(a-\mu s)^2/(2s)} ds \\
&= \int_0^{T-t} \frac{\log(K/x)}{\sigma \sqrt{2\pi s^3}} \exp\left(-rs - \frac{1}{2\sigma^2 s} \left(-\left(r - \frac{\sigma^2}{2}\right) s + \log \frac{K}{x}\right)^2\right) ds \\
&= \left(\frac{K}{x}\right)^{(r/\sigma^2-1/2)\pm(r/\sigma^2+1/2)} \\
&\times \int_0^{T-t} \frac{\log(K/x)}{\sigma \sqrt{2\pi s^3}} \exp\left(-\frac{1}{2\sigma^2 s} \left(\pm\left(r + \frac{\sigma^2}{2}\right) s + \log \frac{K}{x}\right)^2\right) ds \\
&= \frac{1}{\sqrt{2\pi}} \frac{x}{\sigma \sqrt{T-t}} \int_{y-}^{\infty} e^{-y^2/2} dy + \frac{1}{\sqrt{2\pi}} \left(\frac{K}{x}\right)^{2r/\sigma^2} \int_{y+}^{\infty} e^{-y^2/2} dy \\
&= \frac{x}{K} \Phi\left(\frac{(r + \sigma^2/2)(T-t) + \log(x/K)}{\sigma \sqrt{T-t}}\right) \\
&\quad + \left(\frac{x}{K}\right)^{-2r/\sigma^2} \Phi\left(\frac{-(r + \sigma^2/2)(T-t) + \log(x/K)}{\sigma \sqrt{T-t}}\right), \quad 0 < x < K,
\end{align*} \]

where
\[ y\pm = \frac{1}{\sigma \sqrt{T-t}} \left(\pm \left(r + \frac{\sigma^2}{2}\right) (T-t) + \log \frac{K}{x}\right), \]

and we used the decomposition

\[ \log \frac{K}{x} = \frac{1}{2} \left(\left(r + \frac{\sigma^2}{2}\right) s + \log \frac{K}{x}\right) + \frac{1}{2} \left(-\left(r + \frac{\sigma^2}{2}\right) s + \log \frac{K}{x}\right). \]

We check that
\[ C_d^{Am}(t, K) = \Phi(\infty) + \Phi(-\infty) = 1, \]

and
\[ C_d^{Am}(T, x) = \frac{x}{K} \Phi(-\infty) + \left(\frac{x}{K}\right)^{-2r/\sigma^2} \Phi(-\infty) = 0, \quad x < K, \]

since \( t = T \), which is consistent with the answers to Question (c).

\( g) \) Starting from \( S_t \geq K \), the maximum possible payoff is clearly reached as soon as \( S_t \) hits the level \( K \) before the expiration date \( T \), hence the discounted optimal payoff of the option is \( e^{-r(t_K-t)}\mathbb{1}_{\{t_K<T\}} \).

\( h) \) Using the notation and answer to Question (f), for \( x > K \) we find

\[ P_d^{Am}(t, x) = \mathbb{E}[e^{-r(t_K-t)}\mathbb{1}_{\{t_K<T\}} | S_t = x] \]

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\[
\begin{align*}
&\int_0^{T-t} e^{-rs} \frac{a}{\sqrt{2\pi s^3}} e^{-\frac{(a-\mu s)^2}{2s}} \, ds \\
&= \int_0^{T-t} \frac{\log(x/K)}{\sigma\sqrt{2\pi s^3}} \exp \left( -rs - \frac{1}{2\sigma^2} \left( \left( r - \frac{\sigma^2}{2} \right) s + \log \frac{x}{K} \right)^2 \right) \, ds \\
&= \left( \frac{K}{x} \right) \left( \frac{\sigma^2}{s} \right) \left( \frac{1}{\sigma^2} \right) \exp \left( -\frac{1}{2\sigma^2} \left( \left( r + \frac{\sigma^2}{2} \right) s + \log \frac{x}{K} \right)^2 \right) \, ds \\
&= \frac{1}{\sqrt{2\pi}} \int_{y_-}^{y_+} e^{-y^2/2} \, dy \\
&= \frac{x}{K} \Phi \left( \frac{-(r + \sigma^2/2)(T-t) - \log(x/K)}{s\sqrt{T-t}} \right) \\
&+ \left( \frac{x}{K} \right)^{-2r/\sigma^2} \Phi \left( \frac{(r + \sigma^2/2)(T-t) - \log(x/K)}{s\sqrt{T-t}} \right), \quad x > K,
\end{align*}
\]

with
\[
y_{\pm} = \frac{1}{\sigma\sqrt{T-t}} \left( \mp \left( r + \frac{\sigma^2}{2} \right) (T-t) + \log \frac{x}{K} \right),
\]

We check that
\[
P_d^{Am}(t, K) = \Phi(-\infty) + \Phi(\infty) = 1,
\]

and
\[
P_d^{Am}(T, x) = \frac{x}{K} \Phi(-\infty) + \left( \frac{x}{K} \right)^{-2r/\sigma^2} \Phi(-\infty) = 0, \quad 0 < x < K,
\]

since \( t = T \), which is consistent with the answers to Question (c).

i) The call-put parity does not hold for American binary options since for \( x \in (0, K) \) we have
\[
C_d^{Am}(t, x) + P_d^{Am}(t, x) = 1 + \frac{x}{K} \Phi \left( \frac{(r + \sigma^2/2)(T-t) + \log(x/K)}{s\sqrt{T-t}} \right) \\
+ \left( \frac{x}{K} \right)^{-2r/\sigma^2} \Phi \left( \frac{-(r + \sigma^2/2)(T-t) + \log(x/K)}{s\sqrt{T-t}} \right),
\]

while for \( x > K \) we find
\[
C_d^{Am}(t, x) + P_d^{Am}(t, x) = 1 + \frac{x}{K} \Phi \left( \frac{-(r + \sigma^2/2)(T-t) - \log(x/K)}{s\sqrt{T-t}} \right) \\
+ \left( \frac{x}{K} \right)^{-2r/\sigma^2} \Phi \left( \frac{(r + \sigma^2/2)(T-t) - \log(x/K)}{s\sqrt{T-t}} \right).
\]
Exercise 11.14 American forward Contracts.

a) For all stopping times $\tau$ such that $t \leq \tau \leq T$ we have

$$\mathbb{E}^* \left[ e^{-r(\tau-t)} (K - S_\tau) \mid S_t \right] = K \mathbb{E}^* \left[ e^{-r(\tau-t)} \mid S_t \right] - \mathbb{E}^* \left[ e^{-r(\tau-t)} S_\tau \mid S_t \right]
$$

$$= e^{-r(\tau-t)} K - S_t,$$

since $\tau \in [t, T]$ is bounded and $(e^{-rt} S_t)_{t \in \mathbb{R}^+}$ is a martingale, and the above quantity is clearly maximized by taking $\tau = t$. Hence we have

$$f(t, S_t) = \sup_{t \leq \tau \leq T} \mathbb{E}^* \left[ e^{-r(\tau-t)} (K - S_\tau) \mid S_t \right] = K - S_t,$$

and the optimal strategy is to exercise immediately (or avoiding to buy the option) at time $t$.

b) Similarly we have

$$\mathbb{E}^* \left[ e^{-r(\tau-t)} (S_\tau - K) \mid S_t \right] = \mathbb{E}^* \left[ e^{-r(\tau-t)} S_\tau \mid S_t \right] - K \mathbb{E}^* \left[ e^{-r(\tau-t)} \mid S_t \right]
$$

$$= S_t - K \mathbb{E}^* \left[ e^{-r(\tau-t)} \mid S_t \right],$$

since $\tau \in [t, T]$ is bounded and $(e^{-rt} S_t)_{t \in \mathbb{R}^+}$ is a martingale, and the above quantity is clearly maximized by taking $\tau = T$. Hence we have

$$f(t, S_t) = \sup_{t \leq \tau \leq T} \mathbb{E}^* \left[ e^{-r(\tau-t)} (S_\tau - K) \mid S_t \right] = S_t - e^{-r(T-t)} K,$$

and the optimal strategy is to wait until the maturity time $T$ in order to exercise.

c) Concerning the perpetual American long forward contract, since $u \mapsto e^{-r(u-t)} S_u$ is a martingale, for all stopping times $\tau$ we have

$$\mathbb{E}^* \left[ e^{-r(\tau-t)} (S_\tau - K) \mid S_t \right] = \mathbb{E}^* \left[ e^{-r(\tau-t)} S_\tau \mid S_t \right] - K \mathbb{E}^* \left[ e^{-r(\tau-t)} \mid S_t \right]
$$

$$= S_t - K \mathbb{E}^* \left[ e^{-r(\tau-t)} \mid S_t \right]
$$

$$\leq S_t, \quad t \geq 0.$$

On the other hand, for all fixed $T > 0$ we have

$$\mathbb{E}^* \left[ e^{-r(T-t)} (S_T - K) \mid S_t \right] = e^{-r(T-t)} \mathbb{E}^* \left[ S_T \mid S_t \right] - e^{-r(T-t)} \mathbb{E}^* \left[ K \mid S_t \right]
$$

$$= S_t - e^{-r(T-t)} K, \quad t \in [0, T],$$

hence

* by Fatou’s Lemma 18.1.
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\[
\sup_{\tau \leq \tau \leq T} \mathbb{E}^* \left[ e^{-r(\tau-t)} (S_\tau - K) \mid S_t \right] \geq (S_t - e^{-r(T-t)} K), \quad T \in [t, \infty),
\]

and letting \( T \to \infty \) we get

\[
\sup_{\tau \geq t} \mathbb{E}^* \left[ e^{-r(\tau-t)} (S_\tau - K) \mid S_t \right] \geq \lim_{T \to \infty} (S_t - e^{-r(T-t)} K) = S_t,
\]
hence we have

\[
f(t, S_t) = \sup_{\tau \leq \tau \leq T} \mathbb{E}^* \left[ e^{-r(\tau-t)} (S_\tau - K) \mid S_t \right] = S_t,
\]
and the optimal strategy \( \tau^* = +\infty \) is to wait indefinitely.

Concerning the perpetual American short forward contract we have

\[
f(t, S_t) = \sup_{\tau \leq \tau \leq T} \mathbb{E}^* \left[ e^{-r(\tau-t)} (K - S_\tau) \mid S_t \right] 
\leq \sup_{\tau \leq \tau \leq T} \mathbb{E}^* \left[ e^{-r(\tau-t)} (K - S_\tau)^+ \mid S_t \right] = f_{L^*}(S_t),
\]
On the other hand, for \( \tau = \tau_{L^*} \) we have

\[
(K - S_{\tau_{L^*}}) = (K - L^*) = (K - L^*)^*
\]
since \( 0 < L^* = 2Kr/(2r + \sigma^2) < K \), hence

\[
f_{L^*}(S_t) = \mathbb{E}^* \left[ e^{-r(\tau_{L^*}-t)} (K - S_{\tau_{L^*}})^+ \mid S_t \right] 
= \mathbb{E}^* \left[ e^{-r(\tau_{L^*}-t)} (K - S_{\tau_{L^*}}) \mid S_t \right] 
\leq \sup_{\tau \leq \tau \leq T} \mathbb{E}^* \left[ e^{-r(\tau-t)} (K - S_\tau) \mid S_t \right] = f(t, S_t),
\]
which shows that

\[
f(t, S_t) = f_{L^*}(S_t),
\]
\textit{i.e.} the perpetual American short forward contract has same price and exercise strategy as the perpetual American put option.

Exercise 11.15
a) We have

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\[ Y_t = e^{-rt} \left( S_0 e^{rt + \sigma \tilde{B}_t - \sigma^2 t/2} \right)^{-2r/\sigma^2} \]
\[ = S_0^{-2r/\sigma^2} e^{-rt - 2r^2 t/\sigma^2 + 2r \tilde{B}_t} \]
\[ = S_0^{-2r/\sigma^2} e^{2r \tilde{B}_t / \sigma - (2r/\sigma)^2 t/2} \]

and

\[ Z_t = e^{-rt} S_t = S_0 e^{\sigma \tilde{B}_t - \sigma^2 t/2}, \]

which are both martingales under \( P^* \) because they are standard geometric Brownian motions with respective volatilities \( \sigma \) and \( 2r/\sigma \).

b) Since \( Y_t \) and \( Z_t \) are both martingales and \( \tau_L \) is a stopping time we have

\[ S_0^{-2r/\sigma^2} = \mathbb{E}^*[Y_0] \]
\[ = \mathbb{E}^*[Y_{\tau_L}] \]
\[ = \mathbb{E}^*[e^{-r\tau_L} S_{\tau_L}^{-2r/\sigma^2}] \]
\[ = \mathbb{E}^*[e^{-r\tau_L} L^{-2r/\sigma^2}] \]
\[ = L^{-2r/\sigma^2} \mathbb{E}^*[e^{-r\tau_L}], \]

hence

\[ \mathbb{E}^*[e^{-r\tau_L}] = \left( \frac{x}{L} \right)^{-2r/\sigma^2} \]

if \( S_0 = x \geq L \) (note that in this case \( Y_{\tau_L \wedge t} \) remains bounded by \( L^{-2r/\sigma^2} \)),

and

\[ S_0 = \mathbb{E}^*[Z_0] = \mathbb{E}^*[Z_{\tau_L}] = \mathbb{E}^*[e^{-r\tau_L} S_{\tau_L}] = \mathbb{E}^*[e^{-r\tau_L} L] = L \mathbb{E}^*[e^{-r\tau_L}], \]

hence

\[ \mathbb{E}^*[e^{-r\tau_L}] = \frac{x}{L} \]

if \( S_0 = x \leq L \). Note that in this case \( Z_{\tau_L \wedge t} \) remains bounded by \( L \).

c) We find

\[ \mathbb{E} \left[ e^{-r\tau_L} (K - S_{\tau_L}) \mid S_0 = x \right] = (K - L) \mathbb{E}^* \left[ e^{-r\tau_L} \mid S_0 = x \right] \]
\[ = \begin{cases} \frac{x}{L} \left( \frac{K - L}{L} \right), & 0 < x \leq L, \\ (K - L) \left( \frac{x}{L} \right)^{-2r/\sigma^2}, & x \geq L. \end{cases} \]

(A.41)

d) By differentiating

\[ \frac{\partial}{\partial L} \mathbb{E} \left[ e^{-r\tau_L} (K - S_{\tau_L}) \mid S_0 = x \right] \]
\[
= \begin{cases} 
(x/L)^{-2r/\sigma^2} \left( \frac{2r}{\sigma^2} \left( \frac{K}{L} - 1 \right) - 1 \right), & 0 < L < x, \\
-\frac{Kx}{L^2}, & L > x,
\end{cases}
\]

and check that the minimum occurs for \( L^* = x \).

e) The value \( L^* = x \) shows that the optimal strategy for the American finite expiration short forward contract is to exercise immediately starting from \( S_0 = x \), which is consistent with the result of Exercise 11.14-(a), since given any stopping time \( \tau \) upper bounded by \( T \) we have

\[
\mathbb{E}[e^{-r\tau}(K - S_\tau)] = K \mathbb{E}[e^{-r\tau}] - \mathbb{E}[e^{-r\tau}S_\tau] = K \mathbb{E}[e^{-r\tau}] - S_0 \leq K - S_0.
\]

Exercise 11.16

a) The option payoff equals \((\kappa - S_t)^p\) if \( S_t \leq L \).
b) We have

\[
f_L(S_t) = \mathbb{E}^* \left[ e^{-r(\tau_L - t)}((\kappa - S_{\tau_L})^+)^p \mid S_t \right] = \mathbb{E}^* \left[ e^{-r(\tau_L - t)}((\kappa - L)^+)^p \mid S_t \right] = (\kappa - L)^p \mathbb{E}^* \left[ e^{-r(\tau_L - t)} \mid S_t \right].
\]

c) We have

\[
f_L(x) = \mathbb{E}^* \left[ e^{-r(\tau_L - t)}(\kappa - S_{\tau_L})^+ \mid S_t = x \right] = \begin{cases} 
(\kappa - x)^p, & 0 < x \leq L, \\
(\kappa - L)^p \left( \frac{L}{x} \right)^{2r/\sigma^2}, & x \geq L.
\end{cases}
\]

(A.42)
d) By the differentiation \( \frac{d}{dx}(\kappa - x)^p = -p(\kappa - x)^{p-1} \) we find

\[
\frac{\partial f_L(x)}{\partial L} = \frac{2r}{\sigma^2 L} (\kappa - L)^p \left( \frac{L}{x} \right)^{2r/\sigma^2} - p(\kappa - L)^{p-1} \left( \frac{L}{x} \right)^{2r/\sigma^2},
\]

hence the condition \( \frac{\partial f_{L^*}(x)}{\partial L} \bigg|_{x=L^*} = 0 \) reads

\[
\frac{2r}{\sigma^2 L^*} (\kappa - L^*) - p = 0, \quad \text{or} \quad L^* = \frac{2r}{2r + p\sigma^2} \kappa < \kappa.
\]

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e) By (A.42) the price can be computed as

\[ f(t, S_t) = f_{L^*}(S_t) = \begin{cases} 
(\kappa - S_t)^p, & 0 < S_t \leq L^*, \\
\left( \frac{p^2 \kappa}{2r + p^2} \right)^p \left( \frac{2r + p^2}{2r} \frac{S_t}{\kappa} \right)^{-2r/\sigma^2}, & S_t > L^*, 
\end{cases} \]

using (11.12) as in the proof of Proposition 11.9, since the process

\[ u \mapsto e^{-ru} f_{L^*}(S_u), \quad u \geq t, \]

is a nonnegative supermartingale.

**Exercise 11.17**

a) The option payoff is \( \kappa - (S_t)^p \).

b) We have

\[ f_{L}(S_t) = \mathbb{E}^* \left[ e^{-r(\tau_L - t)(\kappa - (S_{\tau_L})^p)} \mid S_t \right] \\
= \mathbb{E}^* \left[ e^{-r(\tau_L - t)(\kappa - L^p)} \mid S_t \right] \\
= (\kappa - L^p) \mathbb{E}^* \left[ e^{-r(\tau_L - t)} \mid S_t \right]. \]

c) We have

\[ f_{L}(x) = \mathbb{E}^* \left[ e^{-r(\tau_L - t)(\kappa - (S_{\tau_L})^p)} \mid S_t = x \right] \\
= \begin{cases} 
\kappa - x^p, & 0 < x \leq L, \\
(\kappa - L^p) \left( \frac{x}{L} \right)^{-2r/\sigma^2}, & x > L.
\end{cases} \]

d) We have

\[ f'_{L^*}(L^*) = -\frac{2r}{\sigma^2} (\kappa - (L^*)^p) \left( \frac{(L^*)^{-2r/\sigma^2 - 1}}{(L^*)^{-2r/\sigma^2}} \right) = -p(L^*)^{p-1}, \]
i.e.

\[ \frac{2r}{\sigma^2} (\kappa - (L^*)^p) = p(L^*)^p, \]
or

\[ L^* = \left( \frac{2r\kappa}{2r + p^2} \right)^{1/p} < (\kappa)^{1/p}. \quad (A.43) \]
Remark: We may also compute $L^*$ by maximizing $L \mapsto f_L(x)$ for all fixed $x$. The derivative $\partial f_L(x)/\partial L$ can be computed as

$$\frac{\partial f_L(x)}{\partial L} = \frac{\partial}{\partial L} \left( (\kappa - L^p) \left( \frac{L^2}{x} \right)^{2r/\sigma^2} \right)$$

$$= -pL^{p-1} \left( \frac{L^2}{x} \right)^{2r/\sigma^2} + 2r \sigma^2 L^{-1}(\kappa - L^p) \left( \frac{L^2}{x} \right)^{2r/\sigma^2},$$

and equating $\partial f_L(x)/\partial L$ to 0 at $L = L^*$ yields

$$-p(L^*)^{p-1} + \frac{2r}{\sigma^2}(L^*)^{-1}(\kappa - (L^*)^p) = 0,$$

which recovers (A.43).

e) We have

$$f_{L^*}(S_t) = \begin{cases} \kappa - (S_t)^p, & 0 < S_t \leq L^*, \\ (\kappa - (L^*)^p)^{(S_t)^{-2r/\sigma^2}}/(L^*)^{-2r/\sigma^2}, & S_t > L^* \end{cases}$$

$$= \begin{cases} \kappa - (S_t)^p, & 0 < S_t \leq L^*, \\ \frac{\kappa^2}{2r(p)^{2r/\sigma^2}}(S_t)^{p+2r/\sigma^2}, & S_t > L^* \end{cases}$$

$$= \begin{cases} \frac{p\sigma^2\kappa}{2r + p\sigma^2} \left( \frac{2r + p\sigma^2}{2r} \frac{S_t^p}{\kappa} \right)^{-2r/(p\sigma^2)} \kappa, & S_t > L^*, \end{cases}$$

however we cannot conclude as in Exercise 11.16-(e) since the process

$$u \mapsto e^{-ru}f_{L^*}(S_u), \quad u \geq t,$$

does not remain nonnegative when $p > 1$, so that (11.12) cannot be applied as in the proof of Proposition 11.9.

Chapter 12

Exercise 12.1

a) We have
By change of numéraire we have

\[ d\hat{X}_t = d\left(\frac{X_t}{N_t}\right) = \frac{X_0}{N_0} d\left( e^{(\sigma - \eta)B_t - (\sigma^2 - \eta^2)t/2} \right) \]

\[ = \frac{X_0}{N_0} (\sigma - \eta) e^{(\sigma - \eta)B_t - (\sigma^2 - \eta^2)t/2} dB_t + \frac{X_0}{2N_0} (\sigma - \eta)^2 e^{(\sigma - \eta)B_t - (\sigma^2 - \eta^2)t/2} dt \]

\[ - \frac{X_0}{2N_0} (\sigma^2 - \eta^2) e^{(\sigma - \eta)B_t - (\sigma^2 - \eta^2)t/2} dt \]

\[ = -\frac{X_t}{2N_t} (\sigma^2 - \eta^2) dt + \frac{X_t}{N_t} (\sigma - \eta) dB_t + \frac{X_t}{2N_t} (\sigma - \eta)^2 dt \]

\[ = -\frac{X_t}{N_t} (\sigma - \eta)(dB_t - \eta dt) \]

\[ = (\sigma - \eta) \frac{X_t}{N_t} dB_t = (\sigma - \eta) \hat{X}_t dB_t, \]

where \( dB_t = dB_t - \eta dt \) is a standard Brownian motion under \( \hat{P} \).

b) By change of numéraire we have

\[ e^{-rT} \mathbb{E}[(X_T - \lambda N_T)^+] = \mathbb{E} \left[ \frac{N_0}{N_T} (X_T - \lambda N_T)^+ \right] = N_0 \mathbb{E}[(\hat{X}_T - \lambda)^+]. \]

Next, by the result of Question (a), \( \hat{X}_t \) is a driftless geometric Brownian motion with volatility \( \sigma - \eta \) under \( \hat{P} \), hence we have

\[ \mathbb{E}[(\hat{X}_T - \lambda)^+] = \hat{X}_0 \Phi \left( \frac{\log(\hat{X}_0/\lambda)}{\hat{\sigma} \sqrt{T}} + \frac{\hat{\sigma} \sqrt{T}}{2} \right) - \lambda \Phi \left( \frac{\log(\hat{X}_0/\lambda)}{\hat{\sigma} \sqrt{T}} - \frac{\hat{\sigma} \sqrt{T}}{2} \right), \]

by the Black-Scholes formula with zero interest rate and volatility parameter \( \hat{\sigma} = \sigma - \eta \). By multiplication by \( N_0 \) and the relation \( X_0 = N_0 \hat{X}_0 \) we conclude to (12.35), i.e.

\[ e^{-rT} \mathbb{E}[(X_T - \lambda N_T)^+] = N_0 \mathbb{E}[(\hat{X}_T - \lambda)^+] \]

\[ = N_0 \hat{X}_0 \Phi(d_+) - \lambda N_0 \Phi(d^-) \]

\[ = X_0 \Phi(d_+) - \lambda N_0 \Phi(d^-). \]

c) We have \( \hat{\sigma} = \sigma - \eta \).

Exercise 12.2 We have \( N_t = P(t, T) \) and from (13.18) and the relations \( P(t, T) = F(t, r_t) \) and \( P(t, S) = G(t, r_t) \) we find
Exercise 12.3 Forward contract. Taking \( N_t := P(t, T), \ t \in [0, T], \) we have

\[
\mathbb{E}^* \left[ \exp \left( - \int_t^T r_s ds \right) \left( P(T, S) - K \right) \bigg| \mathcal{F}_t \right] = N_t \mathbb{E} \left[ \frac{(P(T, S) - K)}{P(T, T)} \bigg| \mathcal{F}_t \right]
\]

hence

\[
\frac{dP(t, S)}{P(t, S)} = r_t dt + \sigma(t, r_t) \frac{\partial \log F}{\partial x}(t, r_t) dW_t,
\]

\[
\frac{dN_t}{N_t} = \frac{dP(t, T)}{P(t, T)} = r_t dt + \sigma(t, r_t) \frac{\partial \log F}{\partial x}(t, r_t) dW_t.
\]

By the Girsanov theorem (12.10) we also have

\[
dW_t = dW_t - \frac{dN_t}{N_t} \cdot dW_t = dW_t - \sigma(t, r_t) \frac{\partial \log F}{\partial x}(t, r_t) dt,
\]

hence

\[
\frac{dP(t, S)}{P(t, S)} = r_t dt + \sigma^2(t, r_t) \frac{\partial \log F}{\partial x}(t, r_t)^2 \frac{\partial \log G}{\partial x}(t, r_t) dt + \sigma(t, r_t) \frac{\partial \log G}{\partial x}(t, r_t) d\hat{W}_t.
\]

Using the relation \( P(t, S) = G(t, r_t) \) we can also write

\[
dP(t, S) = r_t P(t, S) dt + \sigma^2(t, r_t) \frac{\partial \log F}{\partial x}(t, r_t)^2 \frac{\partial G}{\partial x}(t, r_t) dt + \sigma(t, r_t) \frac{\partial G}{\partial x}(t, r_t) d\hat{W}_t.
\]

Remark: The above result can also be obtained by a direct argument using the tower property:

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b) From (A.44) we have

\[ \mathbb{E}^* \left[ \exp \left( - \int_t^T r_s ds \right) (P(T, S) - K) \bigg| \mathcal{F}_t \right] \]

\[ = \mathbb{E}^* \left[ \exp \left( - \int_t^T r_s ds \right) \left( \mathbb{E}^* \left[ \exp \left( - \int_T^S r_s ds \right) \bigg| \mathcal{F}_T \right) - K \right) \bigg| \mathcal{F}_t \right] \]

\[ = \mathbb{E}^* \left[ \exp \left( - \int_t^T r_s ds \right) \mathbb{E}^* \left[ \exp \left( - \int_T^S r_s ds \right) - K \bigg| \mathcal{F}_T \right) \bigg| \mathcal{F}_t \right] \]

\[ = \mathbb{E}^* \left[ \mathbb{E}^* \left[ \exp \left( - \int_t^T r_s ds \right) \bigg| \mathcal{F}_T \right) \bigg| \mathcal{F}_t \right] \]

\[ = \mathbb{E}^* \left[ \exp \left( - \int_t^T r_s ds \right) - K \exp \left( - \int_t^T r_s ds \right) \bigg| \mathcal{F}_T \right) \bigg| \mathcal{F}_t \right] \]

\[ = P(t, S) - KP(t, T), \quad t \in [0, T]. \]

Exercise 12.4 Bond options.

a) Itô’s formula yields

\[ d \left( \frac{P(t, S)}{P(t, T)} \right) = \frac{P(t, S)}{P(t, T)} (\zeta^S(t) - \zeta^T(t)) (dW_t - \zeta^T(t) dt) \]

\[ = \frac{P(t, S)}{P(t, T)} (\zeta^S(t) - \zeta^T(t)) d\hat{W}_t, \quad (A.44) \]

where \((\hat{W}_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion under \(\hat{P}\) by the Girsanov theorem.

b) From (A.44) we have

\[ \frac{P(t, S)}{P(t, T)} = \frac{P(0, S)}{P(0, T)} \exp \left( \int_0^t (\zeta^S(s) - \zeta^T(s)) d\hat{W}_s - \frac{1}{2} \int_0^t (\zeta^S(s) - \zeta^T(s))^2 ds \right), \]

hence

\[ \frac{P(u, S)}{P(u, T)} = \frac{P(t, S)}{P(t, T)} \exp \left( \int_t^u (\zeta^S(s) - \zeta^T(s)) d\hat{W}_s - \frac{1}{2} \int_t^u (\zeta^S(s) - \zeta^T(s))^2 ds \right), \]

\[ t \in [0, u], \text{ and for } u = T \text{ this yields} \]

\[ P(T, S) = \frac{P(t, S)}{P(t, T)} \exp \left( \int_t^T (\zeta^S(s) - \zeta^T(s)) d\hat{W}_s - \frac{1}{2} \int_t^T (\zeta^S(s) - \zeta^T(s))^2 ds \right), \]

since \(P(T, T) = 1\). Let \(\hat{P}\) denote the forward measure associated to the numéraire

\[ N_t := P(t, T), \quad 0 \leq t \leq T. \]

c) For all \(S \geq T > 0\) we have

\[ \mathbb{E} \left[ e^{-\int_t^T r_s ds} (P(T, S) - K)^+ \bigg| \mathcal{F}_t \right] \]
\[
\mathbb{E}\left[ \left( \frac{P(t, S)}{P(t, T)} \exp \left( X - \frac{1}{2} \int_t^T |\zeta^S(s) - \zeta^T(s)|^2 ds \right) - K \right)^+ \mid \mathcal{F}_t \right] = P(t, T) \mathbb{E}\left[ \left( e^{X + m(t, T, S)} - K \right)^+ \mid \mathcal{F}_t \right],
\]

where \( X \) is a centered Gaussian random variable with variance
\[
v^2(t, T, S) = \int_t^T |\zeta^S(s) - \zeta^T(s)|^2 ds
\]
given \( \mathcal{F}_t \), and
\[
m(t, T, S) = -\frac{1}{2} v^2(t, T, S) + \log \frac{P(t, S)}{P(t, T)}.
\]

Recall that when \( X \) is a centered Gaussian random variable with variance \( v^2 \), the expectation of \((e^{m+X} - K)^+\) is given, as in the standard Black-Scholes formula, by
\[
\mathbb{E}[(e^{m+X} - K)^+] = e^{m + \frac{v^2}{2}} \Phi(v + (m - \log K) / v) - K \Phi((m - \log K) / v),
\]
where
\[
\Phi(z) = \int_{-\infty}^z e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}, \quad z \in \mathbb{R},
\]
denotes the Gaussian cumulative distribution function and for simplicity of notation we dropped the indices \( t, T, S \) in \( m(t, T, S) \) and \( v^2(t, T, S) \).

Consequently we have
\[
\mathbb{E} \left[ e^{-\int_t^T r_s ds} (P(T, S) - K)^+ \mid \mathcal{F}_t \right] = P(t, S) \Phi \left( \frac{v}{2} + \frac{1}{v} \log \frac{P(t, S)}{KP(t, T)} \right) - KP(t, T) \Phi \left( -\frac{v}{2} + \frac{1}{v} \log \frac{P(t, S)}{KP(t, T)} \right).
\]

d) The self-financing hedging strategy that hedges the bond option is obtained by holding a (possibly fractional) quantity
\[
\Phi \left( \frac{v}{2} + \frac{1}{v} \log \frac{P(t, S)}{KP(t, T)} \right)
\]
of the bond with maturity \( S \), and by shorting a quantity
\[
K \Phi \left( -\frac{v}{2} + \frac{1}{v} \log \frac{P(t, S)}{KP(t, T)} \right)
\]
of the bond with maturity \( T \).
Exercise 12.5

a) The process

\[ e^{-rt} S_2(t) = S_2(0) e^{\sigma_2 W_t + (\mu - r)t} \]

is a martingale if

\[ r - \mu = \frac{1}{2} \sigma_2^2. \]

b) We note that

\[ e^{-rt} X_t = e^{-rt} e^{(r - \mu) t - \sigma_1^2 t/2} S_1(t) \]

\[ = e^{-rt} e^{(\sigma_2^2 - \sigma_1^2) t/2} S_1(t) \]

\[ = e^{-\mu t - \sigma_1^2 t/2} S_1(t) \]

\[ = S_1(0) e^{\mu t - \sigma_1^2 t/2} e^{\sigma_1 W_t + \mu t} \]

\[ = S_1(0) e^{\sigma_1 W_t - \sigma_1^2 t/2} \]

is a martingale, where

\[ X_t = e^{(r - \mu) t - \sigma_1^2 t/2} S_1(t) = e^{(\sigma_2^2 - \sigma_1^2) t/2} S_1(t). \]

c) By (12.37) we have

\[ \hat{X}(t) = \frac{X_t}{N_t} \]

\[ = e^{(\sigma_2^2 - \sigma_1^2) t/2} \frac{S_1(t)}{S_2(t)} \]

\[ = \frac{S_1(0)}{S_2(0)} e^{(\sigma_2^2 - \sigma_1^2) t/2 + (\sigma_1 - \sigma_2) W_t} \]

\[ = \frac{S_1(0)}{S_2(0)} e^{(\sigma_2^2 - \sigma_1^2) t/2 + (\sigma_1 - \sigma_2) \hat{W}_t + \sigma_2 (\sigma_1 - \sigma_2) t} \]

\[ = \frac{S_1(0)}{S_2(0)} e^{(\sigma_1 - \sigma_2) W_t + \sigma_2 \sigma_1 t - (\sigma_2^2 + \sigma_1^2) t/2} \]

where

\[ \hat{W}_t := W_t - \sigma_2 t \]

is a standard Brownian motion under the forward measure \( \hat{P} \) defined by

\[ \frac{d\hat{P}}{dP} = e^{-\int_0^T r_s ds} \frac{N_T}{N_0} \]

\[ = e^{-rT} \frac{S_2(T)}{S_2(0)} \]
d) Given that $X_t = e^{(\sigma^2_2 - \sigma^2_1)t/2}S_1(t)$ and $\hat{X}(t) = X_t / N_t = X_t / S_2(t)$, we have

$$e^{-rT} \mathbb{E}[\{S_1(T) - \kappa S_2(T)\}^+] = e^{-rT} \mathbb{E}[\{e^{-(\sigma^2_2 - \sigma^2_1)t/2}X_T - \kappa S_2(T)\}^+]$$

$$= e^{-rT} e^{-(\sigma^2_2 - \sigma^2_1)t/2} \mathbb{E}[\{X_T - \kappa e^{(\sigma^2_2 - \sigma^2_1)t/2}S_2(T)\}^+]$$

$$= S_2(0) e^{-(\sigma^2_2 - \sigma^2_1)t/2} \mathbb{E}[\{\hat{X}_0 e^{(\sigma_1 - \sigma_2)W_T - (\sigma_1 - \sigma_2)^2T/2 - \kappa e^{(\sigma^2_2 - \sigma^2_1)t/2}}\}^+]$$

$$= S_2(0) e^{-(\sigma^2_2 - \sigma^2_1)t/2} \left( \hat{X}_0 \Phi^0_+(T, \hat{X}_0) - \kappa e^{(\sigma^2_2 - \sigma^2_1)t/2} \Phi^0_-(T, \hat{X}_0) \right)$$

$$= S_2(0) e^{-(\sigma^2_2 - \sigma^2_1)t/2} \hat{X}_0 \Phi^0_+(T, \hat{X}_0) - \kappa S_2(0) \Phi^0_-(T, \hat{X}_0)$$

where

$$\Phi^0_+(T, x) = \Phi \left( \frac{\log(x/\kappa)}{\sigma_1 - \sigma_2} + \frac{(\sigma_1 - \sigma_2)^2}{2|\sigma_1 - \sigma_2|} \sqrt{T} \right)$$

$$= \left\{ \begin{array}{ll}
\Phi \left( \frac{\log(x/\kappa)}{\sigma_1 - \sigma_2} + \sigma_1 \sqrt{T} \right), & \sigma_1 > \sigma_2, \\
\Phi \left( \frac{\log(x/\kappa)}{\sigma_1 - \sigma_2} - \sigma_1 \sqrt{T} \right), & \sigma_1 < \sigma_2,
\end{array} \right.$$

and

$$\Phi^0_-(T, x) = \Phi \left( \frac{\log(x/\kappa)}{\sigma_1 - \sigma_2} - \frac{(\sigma_1 - \sigma_2)^2}{2|\sigma_1 - \sigma_2|} \sqrt{T} \right)$$

$$= \left\{ \begin{array}{ll}
\Phi \left( \frac{\log(x/\kappa)}{\sigma_1 - \sigma_2} + \sigma_2 \sqrt{T} \right), & \sigma_1 > \sigma_2, \\
\Phi \left( \frac{\log(x/\kappa)}{\sigma_1 - \sigma_2} - \sigma_2 \sqrt{T} \right), & \sigma_1 < \sigma_2,
\end{array} \right.$$

if $\sigma_1 \neq \sigma_2$. In case $\sigma_1 = \sigma_2$ we find

$$e^{-rT} \mathbb{E}[\{S_1(T) - \kappa S_2(T)\}^+] = e^{-rT} \mathbb{E}[S_1(T)(1 - \kappa S_2(0)/S_1(0))^+]$$
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\[ \left( 1 - \kappa S_2(0)/S_1(0) \right)^+ e^{-rT} \mathbb{E}[S_1(T)] = (S_1(0) - \kappa S_2(0)) \mathbb{1}_{\{S_1(0) > \kappa S_2(0)\}}. \]

Exercise 12.6 We have

\[
e^{-(T-t)r} \mathbb{E}^* \left[ \mathbb{1}_{\{R_T \geq \kappa\}} \big| R_t \right] = e^{-(T-t)r} \mathbb{P}^* \left( \frac{R_T}{R_t} \geq \kappa \right) \big| R_t \big)
\]

\[
= e^{-(T-t)r} \mathbb{P}^* \left( R_t e^{\sigma(W_T - W_t) + (r - r^f)(T-t) - \sigma^2(T-t)/2} \geq \kappa \right) \big| R_t \big)
\]

\[
= e^{-(T-t)r} \mathbb{P}^* \left( x e^{\sigma(W_T - W_t) + (r - r^f)(T-t) - \sigma^2(T-t)/2} \geq \kappa \right) \big| R_t \big)
\]

\[
= e^{-(T-t)r} \Phi \left( \frac{(r - r^f)(T-t) - \sigma^2(T-t)/2 - \log(\kappa/R_t)}{\sigma \sqrt{T-t}} \right),
\]

after applying the hint provided, with

\[
\eta^2 := \sigma^2(T-t) \quad \text{and} \quad \mu := (r - r^f)(T-t) - \sigma^2(T-t)/2.
\]

Remark: IQOption™ proposes at the money binary options with a very short time to maturity. In this case we have \( \kappa = R_t \) and

\[ T - t \simeq 30 \text{ seconds} = 0.000000951 = 9.51 \times 10^{-7} \text{ year}^{-1} \]

is small, hence

\[
e^{-(T-t)r} \mathbb{E}^* \left[ \mathbb{1}_{\{R_T \geq \kappa\}} \big| R_t \right] = e^{-(T-t)r} \Phi \left( \frac{(r - r^f) - \sigma}{\sigma \sqrt{T-t}} \right) \approx \Phi(0) = \frac{1}{2}.
\]

Taking for example \( r - r^f = 0.02 = 2\% \) and \( \sigma = 0.3 = 30\% \), we have

\[
\left( \frac{r - r^f}{\sigma} - \frac{\sigma}{2} \right) \sqrt{T-t} = \left( \frac{0.02}{0.3} - \frac{0.3}{2} \right) \sqrt{9.51 \times 10^{-7}} = -0.000081279
\]

and

\[
e^{-(T-t)r} \mathbb{E}^* \left[ \mathbb{1}_{\{R_T \geq \kappa\}} \big| R_t \right] = e^{-(T-t)r} \Phi(-0.000081279)
\]

\[
= e^{-r \times 0.000000951 \times 0.499968}
\]

\[
= 0.49996801
\]

\[
\approx \frac{1}{2},
\]

with \( r = 0.02 = 2\% \).

Exercise 12.7

a) It suffices to check that the definition of \((W_t^N)_{t \in \mathbb{R}_+}\) implies the correlation identity \(dW_t^S \cdot dW_t^N = \rho dt\) by Itô’s calculus.

\[ \textcircled{Q} \]

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http://www.ntu.edu.sg/home/nprivault/index.html
b) We let
\[ \hat{\sigma}_t = \sqrt{(\sigma_S^t)^2 - 2\rho\sigma_R^t\sigma_S^t + (\sigma_R^t)^2} \]
and
\[ dW_t^X = \frac{\sigma_S^t - \rho\sigma_R^t}{\hat{\sigma}_t} dW_t^S - \sqrt{1 - \rho^2}\frac{\sigma_R^t}{\hat{\sigma}_t} dW_t, \quad t \in \mathbb{R}_+, \]
which defines a standard Brownian motion under \( P^* \) due to the definition of \( \hat{\sigma}_t \).

Exercise 12.8

a) We have
\[ \hat{\sigma} = \sqrt{(\sigma_S^t)^2 - 2\rho\sigma_R^t\sigma_S^t + (\sigma_R^t)^2}. \]

b) Letting \( \tilde{X}_t = e^{-rt}X_t = e^{(a-r)t}S_t/R_t, t \in \mathbb{R}_+ \), we have
\[
\mathbb{E}^*\left[ \left( \frac{S_T}{R_T} - \kappa \right)^+ | \mathcal{F}_t \right] = e^{-aT} \mathbb{E}^*\left[ \left( X_T - e^{aT}\kappa \right)^+ | \mathcal{F}_t \right]
\]
\[
= e^{-(a-r)T} \left( \tilde{X}_T \Phi \left( \frac{(r-a+\hat{\sigma}^2/2)(T-t)}{\hat{\sigma}\sqrt{T-t}} \right) + \frac{1}{\hat{\sigma}\sqrt{T-t}} \log \frac{S_t}{\kappa R_t} \right) - \kappa e^{(a-r)T} \Phi \left( \frac{(r-a-\hat{\sigma}^2/2)(T-t)}{\hat{\sigma}\sqrt{T-t}} \right) + \frac{1}{\hat{\sigma}\sqrt{T-t}} \log \frac{S_t}{\kappa R_t} \right)
\]
\[
= \frac{S_t}{R_t} e^{(r-a)(T-t)} \Phi \left( \frac{(r-a+\hat{\sigma}^2/2)(T-t)}{\hat{\sigma}\sqrt{T-t}} \right) + \frac{1}{\hat{\sigma}\sqrt{T-t}} \log \frac{S_t}{\kappa R_t} \right),
\]
hence the price of the quanto option is
\[
e^{-r(T-t)} \mathbb{E}^*\left[ \left( \frac{S_T}{R_T} - \kappa \right)^+ | \mathcal{F}_t \right]
\]
\[
= \frac{S_t}{R_t} e^{-a(T-t)} \Phi \left( \frac{(r-a+\hat{\sigma}^2/2)(T-t)}{\hat{\sigma}\sqrt{T-t}} \right) + \frac{1}{\hat{\sigma}\sqrt{T-t}} \log \frac{S_t}{\kappa R_t} \right) - \kappa e^{-r(T-t)} \Phi \left( \frac{(r-a-\hat{\sigma}^2/2)(T-t)}{\hat{\sigma}\sqrt{T-t}} \right) + \frac{1}{\hat{\sigma}\sqrt{T-t}} \log \frac{S_t}{\kappa R_t} \right).
\]

Chapter 13

Exercise 13.1 We have
\[ dr_t = r_0 e^{-bt} + \frac{a}{b} d(1 - e^{-bt}) + \sigma d \left( e^{-bt} \int_0^t e^{bs} dB_s \right) \]

\[ = -br_0 e^{-bt} dt + a e^{-bt} dt + \sigma e^{-bt} \int_0^t e^{bs} dB_s + \sigma \int_0^t e^{bs} dB_s d e^{-bt} \]

\[ = -br_0 e^{-bt} dt + a e^{-bt} dt + \sigma e^{-bt} e^{bs} dB_t - \sigma b \int_0^t e^{bs} dB_s e^{-bt} dt \]

\[ = -br_0 e^{-bt} dt + a e^{-bt} dt + \sigma dB_t - \sigma b \int_0^t e^{bs} dB_s e^{-bt} dt \]

\[ = -br_0 e^{-bt} dt + a e^{-bt} dt + \sigma dB_t - b \left( r_t - r_0 e^{-bt} - \frac{a}{b} (1 - e^{-bt}) \right) dt \]

\[ = (a - br_t) dt + \sigma dB_t, \]

which shows that \( r_t \) solves (13.72).

**Exercise 13.2** We have

\[ P(0, T_2) = \exp \left( - \int_0^{T_2} f(t, s) ds \right) = e^{-r_1 T_1 - r_2 (T_2 - T_1)}, \quad t \in [0, T_2], \]

and

\[ P(T_1, T_2) = \exp \left( - \int_{T_1}^{T_2} f(t, s) ds \right) = e^{-r_2 (T_2 - T_1)}, \quad t \in [0, T_2], \]

from which we deduce

\[ r_2 = -\frac{1}{T_2 - T_1} \log P(T_1, T_2), \]

and

\[ r_1 = -r_2 \frac{T_2 - T_1}{T_1} - \frac{1}{T_1} \log P(0, T_2) \]

\[ = \frac{1}{T_1} \log P(T_1, T_2) - \frac{1}{T_1} \log P(0, T_2) \]

\[ = -\frac{1}{T_1} \log \frac{P(0, T_2)}{P(T_1, T_2)}. \]

**Exercise 13.3**

a) We have \( r_t = r_0 + at + \sigma B_t \), and

\[ F(t, r_t) = F(t, r_0 + at + \sigma B_t), \]

hence by Proposition 13.2 the PDE satisfied by \( F(t, x) \) is

\[ -xF(t, x) + \frac{\partial F}{\partial t}(t, x) + a \frac{\partial F}{\partial x}(t, x) + \frac{1}{2} \sigma^2 \frac{\partial^2 F}{\partial x^2}(t, x) = 0, \quad (A.45) \]

\[ \square \]
with terminal condition \(F(T, x) = 1\).

b) We have \(r_t = r_0 + at + \sigma B_t\) and

\[
F(t, r_t) = \mathbb{E}^{*} \left[ \exp \left( - \int_t^T r_s ds \right) \bigg| \mathcal{F}_t \right]
\]

\[
= \mathbb{E}^{*} \left[ \exp \left( -r_0 (T - t) - a \int_t^T sds - \sigma \int_t^T B_s ds \right) \bigg| \mathcal{F}_t \right]
\]

\[
= \mathbb{E}^{*} \left[ e^{-r_0 (T - t) - a (T^2 - t^2) / 2} \exp \left( - (T - t) B_t - \sigma \int_t^T (T - s) dB_s \right) \bigg| \mathcal{F}_t \right]
\]

\[
= e^{-r_0 (T - t) - a (T^2 - t^2) / 2 - \sigma (T - t) B_t} \mathbb{E}^{*} \left[ \exp \left( - \sigma \int_t^T (T - s) dB_s \right) \bigg| \mathcal{F}_t \right]
\]

\[
= \exp \left( -(T - t) r_t - a (T - t)^2 / 2 + \frac{\sigma^2}{2} \int_t^T (T - s)^2 ds \right)
\]

\[
= \exp \left( -(T - t) r_t - a (T - t)^2 / 2 + \sigma^2 (T - t)^3 / 6 \right),
\]

hence \(F(t, x) = \exp \left( -(T - t) x - a (T - t)^2 / 2 + \sigma^2 (T - t)^3 / 6 \right)\).

Note that the PDE (A.45) can also be solved by looking for a solution of the form \(F(t, x) = e^{A(T - t) + xC(T - t)}\), in which case one would find \(A(s) = -as^2 / 2 + \sigma^2 s^3 / 6\) and \(C(s) = -s\).

c) We check that the function \(F(t, x)\) of Question (b) satisfies the PDE (A.45) of Question (a), since \(F(T, x) = 1\) and

\[
-xF(t, x) + \left( x + a(T - t) - \frac{\sigma^2}{2} (T - t)^2 \right) F(t, x) - a(T - t) F(t, x)
\]

\[
+ \frac{1}{2} \sigma^2 (T - t)^2 F(t, x) = 0.
\]

d) We have

\[
f(t, T, S) = \frac{1}{S - T} \left( \log P(t, T) - \log P(t, S) \right)
\]

\[
= \frac{1}{S - T} \left( \left( -(T - t) r_t + \frac{\sigma^2}{6} (T - t)^3 \right) - \left( -(S - t) r_t + \frac{\sigma^2}{6} (S - t)^3 \right) \right)
\]

\[
= r_t + \frac{1}{S - T} \frac{\sigma^2}{6} ((T - t)^3 - (S - t)^3).
\]

e) We have

\[
f(t, T) = - \frac{\partial}{\partial T} \log P(t, T) = r_t - \frac{\sigma^2}{2} (T - t)^2.
\]

f) We have

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\[ d_t f(t, T) = \sigma^2 (T - t) dt + adt + \sigma dB_t. \]

g) The HJM condition (13.58) is satisfied since the drift of \( d_t f(t, T) \) equals \( \sigma \int_t^T \sigma ds \).

Exercise 13.4 We have

\[
d_r = \alpha \beta d \left( S_t \int_0^t \frac{1}{S_u} du \right) + r_0 dt S_t \\
= \alpha \beta S_t d \left( \int_0^t \frac{1}{S_u} du \right) + \alpha \beta \int_0^t \frac{1}{S_u} dS_u S_t + r_0 dt S_t \\
= \alpha \beta S_t dt + \alpha \beta \int_0^t \frac{S_t}{S_u} du S_t + r_0 dt S_t \\
= \alpha \beta dt + (r_t - r_0 S_t) \frac{dS_t}{S_t} + r_0 dt S_t \\
= \alpha \beta dt + r_t \frac{dS_t}{S_t}, \quad t \in \mathbb{R}_+. \\
\]

Exercise 13.5 By Itô’s formula we have

\[
d \left( e^{-\int_0^t r_s ds} P(t, T) \right) = -r_t e^{-\int_0^t r_s ds} P(t, T) dt + e^{-\int_0^t r_s ds} dP(t, T) \\
= -r_t e^{-\int_0^t r_s ds} F(t, r_t) dt + e^{-\int_0^t r_s ds} dF(t, r_t) \\
= -r_t e^{-\int_0^t r_s ds} F(t, r_t) dt + e^{-\int_0^t r_s ds} \frac{\partial F}{\partial t}(t, r_t) ((\beta r_t^{-1-\gamma}) + \alpha r_t) dt + \sigma r_t^{\gamma/2} dB_t \\
+ e^{-\int_0^t r_s ds} \left( \frac{1}{2} \sigma^2 r_t^\gamma \frac{\partial^2 F}{\partial x^2}(t, r_t) + \frac{\partial F}{\partial t}(t, r_t) \right) dt \\
= e^{-\int_0^t r_s ds} \sigma r_t^{\gamma/2} \frac{\partial F}{\partial x}(t, r_t) dB_t \\
+ e^{-\int_0^t r_s ds} \left( -r_t F(t, r_t) + (\beta r_t^{-1-\gamma}) + \alpha r_t \frac{\partial F}{\partial x}(t, r_t) + \frac{1}{2} \sigma^2 r_t^\gamma \frac{\partial^2 F}{\partial x^2}(t, r_t) + \frac{\partial F}{\partial t}(t, r_t) \right) dt. \tag{A.46} \]

Given that \( t \mapsto e^{-\int_0^t r_s ds} P(t, T) \) is a martingale, the above expression (A.46) should only contain terms in \( dB_t \) and all terms in \( dt \) should vanish inside (A.46). This leads to the identities
\begin{align*}
\begin{cases}
    r_t F(t, r_t) = (\beta r_t^{-(1-\gamma)} + \alpha r_t) \frac{\partial F}{\partial x}(t, r_t) + \frac{1}{2} \sigma^2 r_t^\gamma \frac{\partial^2 F}{\partial x^2}(t, r_t) + \frac{\partial F}{\partial t}(t, r_t) \\
    d\left(e^{-\int_0^t rs\,ds} P(t, T)\right) = \sigma e^{-\int_0^t rs\,ds} \gamma/2 \frac{\partial F}{\partial x}(t, r_t) dB_t,
\end{cases}
\end{align*}

and to the PDE

\[ x F(t, x) = \frac{\partial F}{\partial t}(t, x) + (\beta x^{-(1-\gamma)} + \alpha x) \frac{\partial F}{\partial x}(t, x) + \frac{\sigma^2}{2} x^\gamma \frac{\partial^2 F}{\partial x^2}(t, x). \]

**Exercise 13.6**

a) The process \( e^{-\int_0^t rs\,ds} F(t, r_t) \) is a martingale and we have

\[
d\left(e^{-\int_0^t rs\,ds} F(t, r_t)\right) \\
= e^{-\int_0^t rs\,ds} \left[ -r_t F(t, r_t) dt + \frac{\partial F}{\partial t}(t, r_t) dt + \frac{\partial F}{\partial x}(t, r_t) dr_t + \frac{\sigma^2}{2} \frac{\partial^2 F}{\partial x^2}(t, r_t)(dr_t)^2 \right] \\
= e^{-\int_0^t rs\,ds} \left[ -r_t F(t, r_t) dt + \frac{\partial F}{\partial t}(t, r_t) dt + \frac{\partial F}{\partial x}(t, r_t)(-ar_t dt + \sigma \sqrt{r_t} dB_t) \right] \\
+ r_t e^{-\int_0^t rs\,ds} \frac{\sigma^2}{2} \frac{\partial^2 F}{\partial x^2}(t, r_t) dt,
\]

hence

\[-xF(t, x) + \frac{\partial F}{\partial t}(t, x) - ax \frac{\partial F}{\partial x}(t, x) + x \frac{\sigma^2}{2} \frac{\partial^2 F}{\partial x^2}(t, x) = 0. \quad (A.48)\]

b) Plugging \( F(t, x) = e^{A(T-t)+xC(T-t)} \) into the PDE (A.48) shows that

\[
e^{A(T-t)+xC(T-t)} \left[ -x - A'(T-t) - xC'(T-t) - axC(T-t) + \frac{\sigma^2}{2} x^2 C^2(T-t) \right] \\
= 0,
\]

hence

\[
\begin{cases}
    -1 - C'(T-t) - aC(T-t) + \frac{\sigma^2}{2} C^2(T-t) = 0, \\
    A'(T-t) = 0.
\end{cases}
\]

Remark: The initial condition \( A(0) = 0 \) shows that \( A(s) = 1 \), and it can be shown from the condition \( C(0) = 0 \) that

\[
C(T-t) = \frac{2(1 - e^{\gamma(T-t)})}{2\gamma + (a + \gamma)(e^{\gamma(T-t)} - 1)}, \quad t \in [0, T],
\]

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http://www.ntu.edu.sg/home/nprivault/index.html
with \( \gamma = \sqrt{a^2 + 2\sigma^2} \), see e.g. Eq. (3.25) page 66 of [BM06].

Exercise 13.7

a) The payoff of the convertible bond is given by \( \max(\alpha S_T, P(\tau, T)) \).

b) We have

\[
\max(\alpha S_T, P(\tau, T)) = P(\tau, T) \mathbb{1}_{\{\alpha S_T \leq P(\tau, T)\}} + \alpha S_T \mathbb{1}_{\{\alpha S_T > P(\tau, T)\}}
\]

\[
= P(\tau, T) + (\alpha S_T - P(\tau, T)) \mathbb{1}_{\{\alpha S_T > P(\tau, T)\}}
\]

\[
= P(\tau, T) + (\alpha S_T - P(\tau, T))^+
\]

where the latter European call payoff has the strike price \( K := P(\tau, T)/\alpha \).

c) From the Markov property applied at time \( t \in [0, \tau] \), we will write the corporate bond price as a function \( C(t, S_t, r_t) \) of the underlying asset price and interest rate, hence we have

\[
C(t, S_t, r_t) = \mathbb{E} \left[ e^{-\int_t^\tau r_s \, ds} \max(\alpha S_T, P(\tau, T)) \bigg| F_t \right].
\]

The martingale property follows from the equalities

\[
e^{-\int_0^t r_s \, ds} C(t, S_t, r_t) = e^{-\int_0^t r_s \, ds} \mathbb{E} \left[ e^{-\int_t^\tau r_s \, ds} \max(\alpha S_T, P(\tau, T)) \bigg| F_t \right]
\]

\[
= \mathbb{E} \left[ e^{-\int_0^t r_s \, ds} \max(\alpha S_T, P(\tau, T)) \bigg| F_t \right].
\]

d) We have

\[
d \left( e^{-\int_0^t r_s \, ds} C(t, S_t, r_t) \right)
\]

\[
= -r_t e^{-\int_0^t r_s \, ds} C(t, S_t, r_t) dt + e^{-\int_0^t r_s \, ds} \frac{\partial C}{\partial x}(t, S_t, r_t) (r S_t dt + \sigma S_t dB_t^{(1)})
\]

\[
+ e^{-\int_0^t r_s \, ds} \frac{\partial C}{\partial y}(t, S_t, r_t) (\gamma(t, r_t) dt + \eta(t, S_t) dB_t^{(2)})
\]

\[
+ e^{-\int_0^t r_s \, ds} \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 C}{\partial x^2}(t, S_t, r_t) dt + e^{-\int_0^t r_s \, ds} \eta^2(t, r_t) \frac{1}{2} \frac{\partial^2 C}{\partial y^2}(t, S_t, r_t) dt
\]

\[
+ \rho \sigma S_t \eta(t, r_t) e^{-\int_0^t r_s \, ds} \frac{\partial C}{\partial x \partial y}(t, S_t, r_t) dt,
\]

hence by the martingale property of \( \left( e^{-\int_0^t r_s \, ds} C(t, S_t, r_t) \right)_{t \in \mathbb{R}^+} \), the associated PDE reads

\[
0 = -y C(t, x, y) + ry \frac{\partial C}{\partial x}(t, x, y) + \gamma(t, y) \frac{\partial C}{\partial y}(t, x, y)
\]
\[
\frac{\sigma^2}{2} x^2 \frac{\partial^2 C}{\partial x^2}(t, x, y) + \eta^2(t, y) \frac{1}{2} \frac{\partial^2 C}{\partial y^2}(t, x, y) + \rho \sigma x \eta(t, y) \frac{\partial^2 C}{\partial x \partial y}(t, x, y),
\]

with the terminal condition
\[
C(\tau, x, y) = \max(\alpha x, F(\tau, y)), \quad \text{where} \quad F(\tau, r_\tau) = P(\tau, T)
\]
is the bond pricing function.

e) We have
\[
\mathbb{E}^* \left[ e^{-\int_t^\tau \rho s ds} \max(\alpha S_\tau, P(\tau, T)) \right] 
= \mathbb{E}^* \left[ e^{-\int_t^\tau \rho s ds} P(\tau, T) \right] + \alpha \mathbb{E}^* \left[ e^{-\int_t^\tau \rho s ds} (S_\tau - P(\tau, T)/\alpha)^+ \right] 
= P(t, T) + \alpha P(t, T) \mathbb{E}_T \left[ (S_\tau/P(\tau, T) - 1/\alpha)^+ \right].
\]
f) We find
\[
dZ_t = (\sigma - \sigma B(t)) Z_t d\hat{W}_t,
\]
where \((\hat{W}_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion under the forward measure \(\hat{P}_T\).
g) By modeling \((Z_t)_{t \in \mathbb{R}_+}\) as
\[
dZ_t = \sigma(t) Z_t d\hat{W}_t,
\]
we find the price
\[
P(t, T) + S_t \Phi(d_+) - P(t, T) \Phi(d_-),
\]
where
\[
d_+ = \frac{1}{v(t, T)} \left( \log \frac{S_t}{P(t, T)} + \frac{v^2(t, \tau)}{2} \right), \quad d_- = \frac{1}{v(t, T)} \left( \log \frac{S_t}{P(t, T)} - \frac{v^2(t, \tau)}{2} \right),
\]
and
\[
v^2(t, T) = \int_t^T \sigma^2(s, T) ds, \quad 0 < t < T.
\]

Exercise 13.8
a) We have \(\sigma(t, s) = \sigma s\) and we check that
\[
\alpha(t, T) = \sigma^2 T(T^2 - t^2)/2 = \sigma T \int_t^T \sigma s ds = \sigma(t, T) \int_t^T \sigma(s, T) ds.
\]
b) We have
\[
f(t, T) = f(0, T) + \int_0^t ds f(s, T)
\]
\[\begin{align*}
&= f(0, T) + \frac{\sigma^2}{2} T \int_0^t (T^2 - s^2) ds + \sigma T \int_0^t dB_s \\
&= f(0, T) + \frac{\sigma^2}{2} T^3 \int_0^t ds - \frac{\sigma^2}{2} T \int_0^t s^2 ds + \sigma T \int_0^t dB_s \\
&= f(0, T) + \sigma^2 T^3 t/2 - \sigma^2 T t^3/6 + \sigma TB_t \\
&= f(0, T) + \sigma^2 T t(T^2/2 - t^2/6) + \sigma TB_t.
\end{align*}\]

c) We have
\[r_t = f(t, t) = f(0, t) + \sigma^2 t^2(t^2/2 - t^2/6) + \sigma t B_t = f(0, t) + \sigma^2 t^4/3 + \sigma t B_t.\]
d) We have
\[dr_t = \frac{4}{3}\sigma^2 t^3 dt + \sigma B_t dt + \sigma t dB_t\]
\[= 4\sigma^2 t^3/3 dt + \frac{1}{t}(r_t - f(0, t) - \sigma^2 t^4/3) dt + \sigma t dB_t\]
\[= \frac{1}{t}(r_t - f(0, t) + \sigma^2 t^4) dt + \sigma t dB_t\]
\[= \sigma^2 t^3 dt + \frac{1}{t}(r_t - f(0, t)) dt + \sigma t dB_t,\]

which is short term interest rate model of the Hull-White type with the time-dependent deterministic coefficients \(\eta(t) = \sigma^2 t^3\), \(\psi(t) = 1/t\) and \(\xi(t) = \sigma t\). Note that \(t \mapsto f(0, t)\) is the initial rate curve data.

Exercise 13.9

a) We have
\[P(t, T) = P(s, T) \exp \left( \int_s^t r_u du + \int_s^t \sigma_u^T dB_u - \frac{1}{2} \int_s^t \sigma_u^2 du \right),\]
\[0 \leq s \leq t \leq T.\]
b) We have
\[d \left( e^{-\int_0^t r_s ds} P(t, T) \right) = e^{-\int_0^t r_s ds} \sigma_t^T P(t, T) dB_t,\]

which gives a martingale after integration, from the properties of the Itô integral.
c) By the martingale property of the previous question we have
\[\mathbb{E} \left[ e^{-\int_0^T r_s ds} \big| \mathcal{F}_t \right] = \mathbb{E} \left[ P(T, T) e^{-\int_0^T r_s ds} \big| \mathcal{F}_t \right] = P(t, T) e^{-\int_0^t r_s ds}, \quad 0 \leq t \leq T.\]
d) By the previous question we have
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\[ P(t, T) = e^{\int_0^t r_s ds} \mathbb{E} \left[ e^{-\int_0^T r_s ds} \bigg| \mathcal{F}_t \right] \]

\[ = \mathbb{E} \left[ e^{\int_0^t r_s ds} e^{-\int_0^T r_s ds} \bigg| \mathcal{F}_t \right] \]

\[ = \mathbb{E} \left[ e^{-\int_0^t r_s ds} \bigg| \mathcal{F}_t \right], \quad 0 \leq t \leq T. \]

since \( e^{-\int_0^t r_s ds} \) is an \( \mathcal{F}_t \)-measurable random variable.

e) We have

\[
\frac{P(t, S)}{P(t, T)} = \frac{P(s, S)}{P(s, T)} \exp \left( \int_s^t (\sigma_u^S - \sigma_u^T) dB_u - \frac{1}{2} \int_s^t (|\sigma_u^S|^2 - |\sigma_u^T|^2) du \right)
\]

\[ = \frac{P(s, S)}{P(s, T)} \exp \left( \int_s^t (\sigma_u^S - \sigma_u^T) dB_u^T - \frac{1}{2} \int_s^t (\sigma_u^S - \sigma_u^T)^2 du \right), \quad 0 \leq t \leq T, \]

hence letting \( s = t \) and \( t = T \) in the above expression we have

\[
P(T, S) = \frac{P(t, S)}{P(t, T)} \exp \left( \int_t^T (\sigma_s^S - \sigma_s^T) dB_s^T - \frac{1}{2} \int_t^T (\sigma_s^S - \sigma_s^T)^2 ds \right).
\]

f) We have

\[
P(t, T) \mathbb{E}_T \left[ (P(T, S) - \kappa)^+ \right] = P(t, T) \mathbb{E}_T \left[ \left( \frac{P(t, S)}{P(t, T)} e^{\int_t^T (\sigma_s^S - \sigma_s^T) dB_s^T - \int_t^T (\sigma_s^S - \sigma_s^T)^2 ds / 2 - \kappa} \right)^+ \right]
\]

\[ = P(t, T) \mathbb{E}[(e^{X} - \kappa)^+ | \mathcal{F}_t] \]

\[ = P(t, T) e^{m_t + v_t^2 / 2} \Phi \left( \frac{v_t}{2} + \frac{1}{v_t} \left( m_t + v_t^2 / 2 - \log \kappa \right) \right) - \kappa P(t, T) \Phi \left( -\frac{v_t}{2} + \frac{1}{v_t} \left( m_t + v_t^2 / 2 - \log \kappa \right) \right), \]

with

\[ m_t = \log(P(t, S) / P(t, T)) - \frac{1}{2} \int_t^T (\sigma_s^S - \sigma_s^T)^2 ds \]

and

\[ v_t^2 = \int_t^T (\sigma_s^S - \sigma_s^T)^2 ds, \]

i.e.

\[
P(t, T) \mathbb{E}_T \left[ (P(T, S) - \kappa)^+ \right] = P(t, S) \Phi \left( \frac{v_t}{2} + \frac{1}{v_t} \log \frac{P(t, S)}{\kappa P(t, T)} \right) - \kappa P(t, T) \Phi \left( -\frac{v_t}{2} + \frac{1}{v_t} \log \frac{P(t, S)}{\kappa P(t, T)} \right).
\]
Exercise 13.10

a) We check that $P(T, T) = e^{X_T^T} = 1$.

b) We have

$$
f(t, T, S) = -\frac{1}{S - T} \left( X_t^S - X_t^T - \mu(S - T) \right)
= \mu - \sigma \frac{1}{S - T} \left( (S - t) \int_0^t \frac{1}{S - s} dB_s - (T - t) \int_0^T \frac{1}{T - s} dB_s \right)
= \mu - \sigma \frac{1}{S - T} \int_0^t \left( \frac{S - t}{S - s} - \frac{T - t}{T - s} \right) dB_s
= \mu + \frac{\sigma}{S - T} \int_0^t \left( \frac{s - t}{S - s} - \frac{T - t}{T - s} \right) \frac{1}{(S - s)(T - s)} dB_s.
$$

c) We have

$$
f(t, T) = \mu - \sigma \int_0^t \frac{t - s}{(T - s)^2} dB_s.
$$

d) We note that

$$
\lim_{T \searrow t} f(t, T) = \mu - \sigma \int_0^t \frac{1}{t - s} dB_s
$$

does not exist in $L^2(\Omega)$.

e) By Itô’s calculus we have

$$
\frac{dP(t, T)}{P(t, T)} = \sigma dB_t + \frac{1}{2} \sigma^2 dt + \mu dt - \frac{X_t^T}{T - t} dt
= \sigma dB_t + \frac{1}{2} \sigma^2 dt - \log P(t, T) \frac{1}{T - t} dt,
$$

$t \in [0, T]$.

f) Let

$$
r_t^S = \mu + \frac{1}{2} \sigma^2 - \frac{X_t^S}{S - t}
= \mu + \frac{1}{2} \sigma^2 - \sigma \int_0^t \frac{1}{S - s} dB_s,
$$

and apply the result of Exercise 13.9-(c).

g) We have

$$
\mathbb{E} \left[ \frac{dP_T}{dP} \mid \mathcal{F}_t \right] = e^{\sigma B_t - \sigma^2 t/2}.
$$
h) By the Girsanov theorem, the process \( \tilde{B}_t := B_t - \sigma t \) is a standard Brownian motion under \( \mathbb{P}_T \).

i) We have

\[
\log P(T, S) = -\mu(S - T) + \sigma(S - T) \int_0^T \frac{1}{S - s} dB_s \\
= -\mu(S - T) + \sigma(S - T) \int_0^T \frac{1}{S - s} dB_s + \sigma(S - T) \int_T^T \frac{1}{S - s} dB_s \\
= \frac{S - T}{S - t} \log p(t, S) + \sigma(S - T) \int_t^T \frac{1}{S - s} d\tilde{B}_s \\
+ \sigma^2(S - T) \int_t^T \frac{1}{S - s} ds \\
= \frac{S - T}{S - t} \log p(t, S) + \sigma(S - T) \int_t^T \frac{1}{S - s} d\tilde{B}_s \\
+ \sigma^2(S - T) \log \frac{S - t}{S - T},
\]

for \( 0 < T < S \).

j) We have

\[
P(t, T) \mathbb{E}_T \left[ (P(T, S) - K)^+ \mid \mathcal{F}_t \right] \\
= P(t, T) \mathbb{E}[(e^X - \kappa)^+ \mid \mathcal{F}_t] \\
= P(t, T) e^{m_t + v_t^2/2} \Phi \left( \frac{v_t}{2} + \frac{1}{v_t} (m_t + v_t^2/2 - \log \kappa) \right) \\
- \kappa P(t, T) \Phi \left( -\frac{v_t}{2} + \frac{1}{v_t} (m_t + v_t^2/2 - \log \kappa) \right) \\
= P(t, T) e^{m_t + v_t^2/2} \Phi \left( v_t + \frac{1}{v_t} (m_t - \log \kappa) \right) - \kappa P(t, T) \Phi \left( \frac{1}{v_t} (m_t - \log \kappa) \right),
\]

with

\[
m_t = \frac{S - T}{S - t} \log p(t, S) + \sigma^2(S - T) \log \frac{S - t}{S - T}
\]

and

\[
v_t^2 = \sigma^2(S - T)^2 \int_t^T \frac{1}{(S - s)^2} ds \\
= \sigma^2(S - T)^2 \left( \frac{1}{S - T} - \frac{1}{S - t} \right) \\
= \sigma^2(S - T) \frac{(T - t)}{(S - t)},
\]

hence

\[
P(t, T) \mathbb{E}_T \left[ (P(T, S) - K)^+ \mid \mathcal{F}_t \right]
\]

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\[ P(t, T) = P(t, S)^{(S-T)(S-t)} \left( \frac{S-t}{S-T} \right)^{\sigma^2(S-T)} e^{\nu t^2/2} \times \Phi \left( \nu t + \frac{1}{\nu t} \log \left( \frac{(P(t, S))^{(S-T)(S-t)}}{\kappa} \left( \frac{S-t}{S-T} \right)^{\sigma^2(S-T)} \right) \right) \]

Exercise 13.11 From Proposition 13.2 the bond pricing PDE is

\[
\begin{cases}
\frac{\partial F}{\partial t}(t, x) = xF(t, x) - (\alpha - \beta x) \frac{\partial F}{\partial x}(t, x) - \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 F}{\partial x^2}(t, x) \\
F(T, x) = 1.
\end{cases}
\]

We search for a solution of the form

\[ F(t, x) = e^{A(T-t) - xB(T-t)}, \]

with \( A(0) = B(0) = 0 \), which implies

\[
\begin{cases}
A'(s) = 0 \\
B'(s) + \beta B(s) + \frac{1}{2} \sigma^2 B^2(s) = 1,
\end{cases}
\]

hence in particular \( A(s) = 0, \ s \in \mathbb{R} \), and \( B(s) \) solves a Riccati equation, whose solution is easily checked to be

\[ B(s) = \frac{2(e^{\gamma s} - 1)}{2\gamma + (\beta + \gamma)(e^{\gamma s} - 1)}, \]

with \( \gamma = \sqrt{\beta^2 + 2\sigma^2} \).

Exercise 13.12

a) We have

\[
P(t, T) = \mathbb{E} \left[ \exp \left( - \int_t^T r_s ds \right) \right] = \mathbb{E} \left[ \exp \left( - \int_t^T h(s) ds - \int_t^T X_s ds \right) \right] = \exp \left( - \int_t^T h(s) ds \right) \mathbb{E} \left[ \exp \left( - \int_t^T X_s ds \right) \right]
\]
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\[ P(t, T) = \exp \left( - \int_t^T h(s)ds + A(T - t) + X_t C(T - t) \right), \]

hence, since \( X_0 = 0 \) we find \( P(0, T) = \exp \left( - \int_0^T h(s)ds + A(T) \right) \).

b) By the identification

\[ P(t, T) = \exp \left( - \int_t^T h(s)ds + A(T - t) + X_t C(T - t) \right) = \exp \left( - \int_t^T f(t, s)ds \right) \]

we find

\[ \int_t^T h(s)ds = \int_t^T f(t, s)ds + A(T - t) + X_t C(T - t), \]

and by differentiation with respect ot \( T \) this yields

\[ h(T) = f(t, T) + A'(T - t) + X_t C'(T - t), \quad t \in [t, T], \]

where

\[ A(T - t) = \frac{4ab - 3\sigma^2}{4b^3} + \frac{\sigma^2 - 2ab}{2b^2} (T - t) + \frac{\sigma^2 - ab}{b^3} e^{-b(T-t)} - \frac{\sigma^2}{4b^3} e^{-2b(T-t)}. \]

Given an initial market data curve \( f^M(0, T) \), the matching \( f^M(0, T) = f(0, T) \) can be achieved at time \( t = 0 \) by letting

\[ h(T) := f^M(0, T) + A'(T) = f^M(0, T) + \frac{\sigma^2 - 2ab}{2b^2} - \frac{\sigma^2 - ab}{b^2} e^{-bT} + \frac{\sigma^2}{2b^2} e^{-2bT}, \]

\( T > 0 \). Note however that in general, at time \( t \in (0, T] \) we will have

\[ h(T) = f(t, T) + A'(T - t) + X_t C'(T - t) = f^M(0, T) + A'(T), \]

and the relation

\[ f(t, T) = f^M(0, T) + A'(T) - A'(T - t) - X_t C'(T - t), \quad t \in [0, T], \]

will allow us to match market data at time \( t = 0 \) only, i.e. for the initial curve. In any case, model calibration is to be done at time \( t = 0 \).

Exercise 13.13 From the definition

\[ L(t, t, T) = \frac{1}{T - t} \left( \frac{1}{P(t, T)} - 1 \right), \]

we have

\[ P(t, T) = \frac{1}{1 + (T - t)L(t, t, T)}, \]

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and similarly
\[ P(t, S) = \frac{1}{1 + (S-t)L(t,t,S)}. \]

Hence we get
\[
L(t, T, S) = \frac{1}{S-T} \left( \frac{P(t,T)}{P(t,S)} - 1 \right)
= \frac{1}{S-T} \left( \frac{1 + (S-t)L(t,t,S)}{1 + (T-t)L(t,t,T)} - 1 \right)
= \frac{1}{S-T} \left( \frac{(S-t)L(t,t,S) - (T-t)L(t,t,T)}{1 + (T-t)L(t,t,T)} \right).
\]

When \( T = \) one year and \( L(0,0,T) = 2\% \), \( L(0,0,2T) = 2.5\% \) we find
\[
L(t, T, S) = \frac{1}{T} \left( \frac{2TL(0,0,2T) - TL(0,0,T)}{1 + TL(0,0,T)} \right) = \frac{2 \times 0.025 - 0.02}{1 + 0.02} = 2.94\%,
\]
so we would not prefer a spot rate at \( L(T, T, 2T) = 2\% \) over a forward contract with rate \( L(0, T, 2T) = 2.94\% \).

Exercise 13.14
a) We have
\[
\begin{align*}
y_{0,1} &= -\frac{1}{T_1} \log P(0, T_1) = 9.53\%, \\
y_{0,2} &= -\frac{1}{T_2} \log P(0, T_2) = 9.1\%, \\
y_{1,2} &= -\frac{1}{T_2 - T_1} \log \frac{P(0, T_2)}{P(T_1, T_2)} = 8.6%,
\end{align*}
\]
with \( T_1 = 1 \) and \( T_2 = 2 \).

b) We have
\[
P_c(1, 2) = (1 + 0.1) \times P_0(1, 2) = (1 + 0.1) \times e^{-(T_2-T_2)y_{1,2}} = 1.00914,
\]
and
\[
P_c(0, 2) = (1 + 0.1) \times P_0(0, 2) + 0.1 \times P_0(0, 1)
= (1 + 0.1) \times e^{-(T_2-T_2)y_{0,2}} + 0.1 \times e^{-(T_2-T_2)y_{0,1}}
= 1.00831.
\]

Exercise 13.15 We write
\[
\mathbb{E}[\Delta r_t] = (a - br_t) \Delta t
= \sigma p(r_t) \sqrt{\Delta t} - \sigma q(r_t) \sqrt{\Delta t}
\]
\[ \sigma p(r_t) \sqrt{\Delta t} - \sigma (1 - p(r_t)) \sqrt{\Delta t}, \]

which yields
\[ p(r_t) = \frac{1}{2} + \frac{a - br_t}{2\sigma} \sqrt{\Delta t} \quad \text{and} \quad q(r_t) = \frac{1}{2} - \frac{a - br_t}{2\sigma} \sqrt{\Delta t}. \]

Similarly, we have
\[ \mathbb{E}[\Delta r_t] = (a - br_t) \Delta t \]
\[ = \sigma p(r_t) \sqrt{\Delta t} - \sigma q(r_t) \sqrt{\Delta t} \]
\[ = \sigma p(r_t) \sqrt{\Delta t} - \sigma (1 - p(r_t)) \sqrt{\Delta t}, \]

which yields
\[ p(r_t) = \frac{1}{2} + \frac{a - br_t}{2\sigma} \sqrt{\Delta t} \quad \text{and} \quad q(r_t) = \frac{1}{2} - \frac{a - br_t}{2\sigma} \sqrt{\Delta t}. \]

Remarks:

a) The use of binomial (or recombining) trees can make the implementation of the Monte Carlo method easier as their size grows linearly instead of exponentially.

b) One could take \( p(r_t) = p(r_t) = 1/2 \) and use the discretization \( r_{tk+1} := r_t + (a - br_t) \Delta t \pm \sigma \sqrt{\Delta t} \), however this would not lead to a binomial tree as \( r_{t2} \) could be obtained in four different ways as
\[
\begin{align*}
    r_{t2} &= r_{t1} (1 - b\Delta t) + a\Delta t \pm \sigma \sqrt{\Delta t} \\
    &= \begin{cases}
    (r_t0 (1 - b\Delta t) + a\Delta t + \sigma \sqrt{\Delta t}) (1 - b\Delta t) + a\Delta t + \sigma \sqrt{\Delta t} \\
    (r_t0 (1 - b\Delta t) + a\Delta t + \sigma \sqrt{\Delta t}) (1 - b\Delta t) + a\Delta t - \sigma \sqrt{\Delta t} \\
    (r_t0 (1 - b\Delta t) + a\Delta t - \sigma \sqrt{\Delta t}) (1 - b\Delta t) + a\Delta t + \sigma \sqrt{\Delta t} \\
    (r_t0 (1 - b\Delta t) + a\Delta t - \sigma \sqrt{\Delta t}) (1 - b\Delta t) + a\Delta t - \sigma \sqrt{\Delta t).
    \end{cases}
\end{align*}
\]

c) Note that by the Girsanov theorem, the process \((r_t/\sigma)_{t \in [0,T]}\) with
\[
\frac{dr_t}{\sigma} = \frac{a - br_t}{\sigma} dt + dB_t
\]
is a standard Brownian motion under the probability measure \( Q \) with density
\[
\frac{dQ}{dP} = \exp \left( -\frac{1}{\sigma} \int_0^T (a - br_t) dB_t - \frac{1}{2\sigma^2} \int_0^T (a - br_t)^2 dt \right)
\]
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\[ \simeq \exp \left( -\frac{1}{\sigma^2} \int_0^T (a - br_t)(dr_t - (a - br_t)dt) - \frac{1}{2\sigma^2} \int_0^T (a - br_t)^2 dt \right) \]

\[ = \exp \left( -\frac{1}{\sigma^2} \int_0^T (a - br_t)dr_t + \frac{1}{2\sigma^2} \int_0^T (a - br_t)^2 dt \right). \]

In other words, if we generate \((r_t/\sigma)_{t\in[0,T]}\) and the increments \(\sigma^{-1}dr_t \simeq \pm \sqrt{\Delta t}\) as a standard Brownian motion under \(Q\), then, under the probability measure \(P\) such that

\[
\frac{dP}{dQ} = \exp \left( \frac{1}{\sigma^2} \int_0^T (a - br_t)dr_t - \frac{1}{2\sigma^2} \int_0^T (a - br_t)^2 dt \right)
\]

\[ \simeq 2^{T/\Delta T} \prod_{0 < t < T} \left( \frac{1}{2} + \frac{a - br_t}{2\sigma} \sqrt{\Delta t} \right), \]

the process

\[ dB_t = \frac{dr_t}{\sigma} - \frac{a - br_t}{\sigma} dt \]

will be a standard Brownian motion under \(P\), and the samples

\[ dr_t = (a - br_t)dt + \sigma dB_t \]

of \((r_t)_{t\in[0,T]}\) will be distributed as a Vasicek process under \(P\).

Exercise 13.16

a) We have

\[ P(1, 2) = \mathbb{E}^* \left[ \frac{100}{1 + r_1} \right] = \frac{100}{2(1 + r_1^u)} + \frac{100}{2(1 + r_1^d)}. \]

b) We have

\[ P(0, 2) = \frac{100}{2(1 + r_0)(1 + r_1^u)} + \frac{100}{2(1 + r_0)(1 + r_1^d)}. \]

c) We have \(P(0, 1) = 91.74 = 100/(1 + r_0)\), hence

\[ r_0 = \frac{100 - P(0, 1)}{P(0, 1)} = 100/91.74 - 1 = 0.090037061 \simeq 9\%. \]

d) We have

\[ 83.40 = P(0, 2) = \frac{P(0, 1)}{2(1 + r_1^u)} + \frac{P(0, 1)}{2(1 + r_1^d)}. \]

and \(r_1^u/r_1^d = e^{2\sigma\sqrt{\Delta t}}\), hence
or
\[
e^{2\sigma\sqrt{\Delta t}} (r_1^d)^2 + 2r_1^d e^{\sigma\sqrt{\Delta t}} \cosh (\sigma\sqrt{\Delta t}) \left(1 - \frac{P(0,1)}{2P(0,2)} \right) + 1 - \frac{P(0,1)}{P(0,2)} = 0,
\]

and
\[
\begin{align*}
\frac{r_1^d}{r_0} &= e^{-\sigma\sqrt{\Delta t}} \left( \cosh (\sigma\sqrt{\Delta t}) \left( \frac{P(0,1)}{2P(0,2)} - 1 \right) \right) \\
&\pm \sqrt{\left( \frac{P(0,1)}{2P(0,2)} - 1 \right)^2 \cosh^2 (\sigma\sqrt{\Delta t}) + \frac{P(0,1)}{P(0,2)} - 1} \\
&= 0.078684844 \approx 7.87%.
\end{align*}
\]

and
\[
\begin{align*}
\frac{r_1^u}{r_0} &= r_1^d e^{2\sigma\sqrt{\Delta t}} \\
&= e^{\sigma\sqrt{\Delta t}} \left( \cosh (\sigma\sqrt{\Delta t}) \left( \frac{P(0,1)}{2P(0,2)} - 1 \right) \right) \\
&\pm \sqrt{\left( \frac{P(0,1)}{2P(0,2)} - 1 \right)^2 \cosh^2 (\sigma\sqrt{\Delta t}) + \frac{P(0,1)}{P(0,2)} - 1} \\
&= 0.122174525 \approx 12.22%.
\end{align*}
\]

We also find
\[
\mu = \frac{1}{\Delta t} \left( \sigma\sqrt{\Delta t} + \log \frac{r_1^d}{r_0} \right) = \frac{1}{\Delta t} \left( -\sigma\sqrt{\Delta t} + \log \frac{r_1^u}{r_0} \right) = 0.08522918 \approx 8.52%.
\]

Exercise 13.17

a) When \( n = 1 \) the relation (13.78) shows that \( \tilde{f}(t,t,T_1) = f(t,t,T_1) \) with \( F(t,x) = c_1 e^{-(T_1-1)x} \) and \( P(t,T_1) = c_1 e^{f(t,t,T_1)} \), hence
\[
D(t,T_1) := \frac{1}{P(t,T_1)} \frac{\partial F}{\partial x}(t,f(t,t,T_1)) = T_1 - t, \quad 0 \leq t \leq T_1.
\]

b) In general, we have
\[
D(t,T_n) = \frac{1}{P(t,T_n)} \frac{\partial F}{\partial x}(t,\tilde{f}(t,t,T_n))
\]
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\[
P(t, T_n) \sum_{k=1}^{n} (T_k - t) c_k e^{-(T_k - t)f(t, t, T_n)} = \sum_{k=1}^{n} w_k(T_k - t),
\]

where

\[
w_k := \frac{c_k}{P(t, T_n)} e^{-(T_k - t)f(t, t, T_n)} = \frac{c_k e^{-(T_k - t)f(t, t, T_n)}}{\sum_{l=1}^{n} c_l e^{-(T_l - t)f(t, t, T_l)}} = \frac{c_k e^{-(T_k - t)f(t, t, T_n)}}{\sum_{l=1}^{n} c_l e^{-(T_l - t)f(t, t, T_n)}}, \quad k = 1, 2, \ldots, n,
\]

and the weights \(w_1, w_2, \ldots, w_n\) satisfy

\[
\sum_{k=1}^{n} w_k = 1.
\]

c) We have

\[
C(t, T_n) = \frac{1}{P(t, T_n)} \frac{\partial^2 F}{\partial x^2}(t, \tilde{f}(t, t, T_n))
\]

\[
= \sum_{k=1}^{n} w_k(T_k - t)^2
\]

\[
= \sum_{k=1}^{n} w_k(T_k - t - D(t, T_n))^2 + 2D(t, T_n) \sum_{k=1}^{n} w_k(T_k - t) - (D(t, T_n))^2
\]

\[
= \sum_{k=1}^{n} w_k(T_k - t - D(t, T_n))^2 + (D(t, T_n))^2
\]

\[
= (S(t, T_n))^2 + (D(t, T_n))^2,
\]

with

\[
(S(t, T_n))^2 := \sum_{k=1}^{n} w_k(T_k - t - D(t, T_n))^2.
\]

d) We have
\[ D(t, T_n) = \frac{1}{P(t, T_n)} \sum_{k=1}^{n} B(T_k - t)c_k e^{A(T_k - t) + B(T_k - t)\tilde{f}_\alpha(t,t,T_n)} \]
\[ = \frac{1}{e^{A(T_k - t) + B(T_k - t)\tilde{f}_\alpha(t,t,T_n)}} \sum_{k=1}^{n} B(T_k - t)c_k e^{A(T_k - t) + B(T_k - t)\tilde{f}_\alpha(t,t,T_n)} \]
\[ = \sum_{k=1}^{n} B(T_k - t)c_k e^{(B(T_k - t) - B(T_n - t))\tilde{f}_\alpha(t,t,T_n)}. \]

e) We have
\[ D(t, T_n) = \frac{1}{b} \sum_{k=1}^{n} (1 - e^{-b(T_k - t)})c_k e^{(e^{-b(T_n - t)} - e^{-b(T_k - t)})\tilde{f}_\alpha(t,t,T_n)/b} \]
\[ = \frac{1}{b} \sum_{k=1}^{n} (1 - e^{-b(T_k - t)})c_k (P(t, t + \alpha(T_n - t)) \frac{e^{-b(T_n - t)} - e^{-b(T_k - t)}}{\alpha b(T_n - t)}). \]

Exercise 13.18

a) We have
\[ f(t, x) = f(0, x) + \alpha \int_0^t x^2 ds + \sigma \int_0^t ds B(s, x) = r + \alpha tx^2 + \sigma B(t, x). \]

b) We have
\[ r_t = f(t, 0) = r + B(t, 0) = r. \]

c) We have
\[ P(t, T) = \exp \left( -\int_t^T f(t, s) ds \right) \]
\[ = \exp \left( -r(T - t) - \alpha t \int_0^{T-t} s^2 ds - \sigma \int_0^{T-t} B(t, x) dx \right) \]
\[ = \exp \left( -r(T - t) - \frac{\alpha}{3} t(T - t)^3 - \sigma \int_0^{T-t} B(t, x) dx \right), \quad t \in [0, T]. \]

d) We have
\[ \mathbb{E} \left[ \left( \int_0^{T-t} B(t, x) dx \right)^2 \right] = \mathbb{E} \left[ \int_0^{T-t} B(t, x) dx \int_0^{T-t} B(t, y) dy \right] \]
\[ = \mathbb{E} \left[ \int_0^{T-t} \int_0^{T-t} B(t, y) B(t, x) dx dy \right] \]
\[ = \int_0^{T-t} \int_0^{T-t} \mathbb{E} [B(t, y) B(t, x)] dx dy \]
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\[
= \int_{0}^{T-t} \int_{0}^{T-t} \mathbb{E}[B(t,x)B(t,y)]dxdy
= t \int_{0}^{T-t} \int_{0}^{T-t} \min(x,y)dxdy
= 2t \int_{0}^{T-t} \int_{0}^{y} x dxdy = \frac{1}{3} t(T-t)^3.
\]

e) We have

\[
\mathbb{E}[P(t,T)] = \mathbb{E} \left[ \exp \left( -r(T-t) - \frac{\alpha}{3} t(T-t)^3 - \sigma \int_{0}^{T-t} B(t,x) dx \right) \right]
= \exp \left( -r(T-t) - \frac{\alpha}{3} t(T-t)^3 + \frac{\sigma^2}{6} t(T-t)^3 \right), \quad t \in [0,T].
\]

f) We need to take \( \alpha = \frac{\sigma^2}{2} \).

Remark: In order to derive an analog of the HJM absence of arbitrage condition in this stochastic string model, one would have to check whether the discounted bond price \( e^{-rt}P(t,T) \) can be a martingale by doing stochastic calculus with respect to the Brownian sheet \( B(t,x) \).

g) We have

\[
\mathbb{E} \left[ \exp \left( -\int_{0}^{T} r_s ds \right) (P(T,S) - K)^+ \right]
= e^{-rT} \mathbb{E} \left[ \left( \exp \left( -r(S-T) - \frac{\alpha}{3} T(S-T)^3 + \sigma \int_{0}^{S-T} B(T,x) dx \right) - K \right)^+ \right]
= e^{-rT} \mathbb{E} \left[ (e^m + X - K)^+ \right],
\]

where \( x = e^{-r(S-T)} \), \( m = -\alpha T(S-T)^3/3 \), and

\[
X = \sigma \int_{0}^{S-T} B(T,x) dx \simeq \mathcal{N}(0, \sigma^2 t(T-t)^3/3).
\]

Given the relation \( \alpha = \frac{\sigma^2}{2} \) this yields

\[
\mathbb{E} \left[ \exp \left( -\int_{0}^{T} r_s ds \right) (P(T,S) - K)^+ \right]
= e^{-rS} \Phi \left( \frac{\sigma \sqrt{T(S-T)^3/12} + \log \left( \frac{e^{-r(S-T)}/K}{\sigma \sqrt{T(S-T)^3/3}} \right)}{\sigma \sqrt{T(S-T)^3/3}} \right)
-K e^{-rT} \Phi \left( -\frac{\sigma \sqrt{T(S-T)^3/12} + \log \left( \frac{e^{-r(S-T)}/K}{\sigma \sqrt{T(S-T)^3/3}} \right)}{\sigma \sqrt{T(S-T)^3/3}} \right)
\]
\[
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\]

\[
= P(0, S) \Phi \left( \frac{\sigma \sqrt{T(S-T)^3/12} + \frac{\log(e^{-r(S-T)/K})}{\sigma \sqrt{T(S-T)^3/3}}}{\sigma \sqrt{T(S-T)^3/3}} \right) \\
\]

\[-KP(0, T) \Phi \left( -\sigma \sqrt{T(S-T)^3/12} + \frac{\log(e^{-r(S-T)/K})}{\sigma \sqrt{T(S-T)^3/3}} \right) .
\]

Chapter 14

Exercise 14.1

a) We take \( t = 0 \), \( T_1 = 9 \) months, \( T_2 = 1 \) year, \( \kappa = 4.5\% \), and the LIBOR rate \( (L(t, T_1, T_2))_{t \in [0, T_1]} \) is modeled as a driftless geometric Brownian motion with volatility coefficient \( \hat{\sigma} = \sigma_{1,2}(t) = 0.1 \) under the forward measure \( P_2 \). The discount factors are given by \( P(0, T_1) = (1 + r)^{-9/12} \) and \( P(0, T_2) = 1/(1 + r) \), with \( r = 4.95\% \).

b) By (14.16) the price of the floorlet is

\[
\mathbb{E}^* \left[ e^{-\int_0^{T_2} r_s ds} (\kappa - L(T_1, T_1, T_2))^+ \right] = P(0, T_2) (\kappa \Phi(-d_-(T_1)) - L(0, T_1, T_2) \Phi(-d_+(T_1))), \quad (A.49)
\]

where \( d_+(T_1) \) and \( d_-(T_1) \) are given in Proposition 14.4, and the LIBOR rate \( L(0, T_1, T_2) \) is given by

\[
L(0, T_1, T_2) = \frac{P(0, T_1) - P(0, T_2)}{(T_2 - T_1) P(0, T_2)} = \frac{(1 + r)^{-3/4} - (1 + r)^{-1}}{0.3(1 + r)^{-1}} = \frac{(1 + r)^{1/4} - 1}{0.3}.
\]

Finally, we need to multiply (A.49) by the notional principal amount of $1 million, i.e. to multiply by 10,000 when (A.49) is quoted in percentage points.

Exercise 14.2

a) We take \( t = 0 \), \( T_1 = 4 \) years, \( T_2 = 5 \) years, \( T_3 = 6 \) years, \( T_4 = 7 \) years, \( \kappa = 5\% \), and the swap rate \( (S(t, T_1, T_4))_{t \in [0, T_1]} \) is modeled as a driftless geometric Brownian motion with volatility coefficient \( \hat{\sigma} = \sigma_{1,4}(t) = 0.2 \) under the forward swap measure \( P_{1,4} \). The discount factors are given by \( P(0, T_1) = (1 + r)^{-4} \), \( P(0, T_2) = (1 + r)^{-5} \), \( P(0, T_3) = (1 + r)^{-6} \), \( P(0, T_4) = (1 + r)^{-7} \), where \( r = 5\% \).
b) By Proposition 14.10 the price of the swaption is
\[
P(0, T_1) - P(0, T_4)\Phi_+(dT_1 - t)) - \kappa \Phi_-(d(T_1)) (P(0, T_2) + P(0, T_3) + P(0, T_4)),
\]
(A.50)
where \(d_+(T_1)\) and \(d_-(T_1)\) are given in Proposition 14.10, and the LIBOR swap rate \(S(0, T_1, T_4)\) is given by
\[
S(0, T_1, T_4) = \frac{P(0, T_1) - P(0, T_4)}{P(0, T_2) + P(0, T_3) + P(0, T_4)}
\]
\[
= \frac{P(0, T_1) - P(0, T_4)}{(1 + r)^{-4} - (1 + r)^{-7}}
\]
\[
= \frac{(1 + r)^{-5} + (1 + r)^{-6} + (1 + r)^{-7}}{(1 + r)^2 + (1 + r) + 1}.
\]
Finally, we need to multiply (A.50) by the notional principal amount of $10 million, i.e. to multiply by 100,000 when (A.50) is quoted in percentage points.

Exercise 14.3
a) We have
\[
d \left( \frac{P(t, T_2)}{P(t, T_1)} \right) = \frac{dP(t, T_2)}{P(t, T_1)} - P(t, T_2) \frac{dP(t, T_1)}{(P(t, T_1))^2}
\]
\[
+ \frac{2}{2} P(t, T_2) \frac{(dP(t, T_1))^2}{(P(t, T_1))^3} - \frac{dP(t, T_1) \cdot dP(t, T_2)}{(P(t, T_1))^2}
\]
\[
= r_t P(t, T_2) dt + \zeta_2(t) P(t, T_2) dW_t
\]
\[
- P(t, T_2) r_t P(t, T_1) dt + \zeta_1(t) P(t, T_1) dW_t
\]
\[
+ P(t, T_2) \frac{(r_t P(t, T_1) dt + \zeta_1(t) P(t, T_1) dW_t)^2}{(P(t, T_1))^2}
\]
\[
- \frac{(r_t P(t, T_1) dt + \zeta_1(t) P(t, T_1) dW_t) \cdot (r_t P(t, T_2) dt + \zeta_2(t) P(t, T_2) dW_t)}{(P(t, T_1))^2}
\]
\[
= \zeta_2(t) \frac{P(t, T_2)}{P(t, T_1)} dW_t - \zeta_1(t) \frac{P(t, T_2)}{P(t, T_1)} dt
\]
\[
+ (\zeta_1(t))^2 \frac{P(t, T_2)}{P(t, T_1)} dt - \zeta_1(t) \zeta_2(t) \frac{P(t, T_2)}{P(t, T_1)} dt
\]
\[ P(T_1, T_2) = \frac{P(T_1, T_2)}{P(T_1, T_1)} \]
\[ = \frac{P(t, T_2)}{P(t, T_1)} \exp \left( \int_t^{T_1} (\zeta_2(s) - \zeta_1(s)) d\hat{W}_s - \frac{1}{2} \int_t^{T_1} (\zeta_2(s) - \zeta_1(s))^2 ds \right) \]
\[ = \frac{P(t, T_2)}{P(t, T_1)} \exp \left( X - \frac{v^2}{2} \right), \]
where \( X \) is a centered Gaussian random variable with variance \( v^2 = \int_t^{T_1} (\zeta_2(s) - \zeta_1(s))^2 ds \), independent of \( \mathcal{F}_t \) under \( \hat{P} \). Hence by the hint below we find
\[
\mathbb{E}^* \left[ e^{-\int_0^{T_1} r_s ds} (K - P(T_1, T_2))^+ | \mathcal{F}_t \right] = P(t, T_1) \mathbb{E} \left[ (K - P(T_1, T_2))^+ | \mathcal{F}_t \right]
\]
\[ = P(t, T_1) \left( \kappa \Phi \left( \frac{v}{2} + (\log(\kappa/x))/v \right) - \frac{P(t, T_2)}{P(t, T_1)} \Phi \left( -\frac{v}{2} + (\log(\kappa/x))/v \right) \right) \]
\[ = P(t, T_1) \kappa \Phi \left( \frac{v}{2} + (\log(\kappa/x))/v \right) - P(t, T_2) \Phi \left( -\frac{v}{2} + (\log(\kappa/x))/v \right), \]
with \( x = P(t, T_2)/P(t, T_1) \).

Exercise 14.4

a) The forward measure \( \hat{P}_S \) is defined from the numéraire \( N_t := P(t, S) \) and this gives
\[ F_t = P(t, S) \mathbb{E}[(\kappa - L(T, T, S))^+ | \mathcal{F}_t]. \]

b) The LIBOR rate \( L(t, T, S) \) is a driftless geometric Brownian motion with volatility \( \sigma \) under the forward measure \( \hat{P}_S \). Indeed, the LIBOR rate \( L(t, T, S) \) can be written as the forward price \( L(t, T, S) = \hat{X}_t = X_t/N_t \) where \( X_t = (P(t, T) - P(t, S))/(S - T) \) and \( N_t = P(t, S) \). Since both discounted bond prices \( e^{-\int_0^t r_s ds} P(t, T) \) and \( e^{-\int_0^t r_s ds} P(t, S) \) are martingales under \( \mathbb{P}^* \), the same is true of \( X_t \). Hence \( L(t, T, S) = X_t/N_t \) becomes a martingale under the forward measure \( \hat{P}_S \) by Proposition 2.1, and computing its dynamics under \( \hat{P}_S \) amounts to removing any “\( dt \)” term in

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http://www.ntu.edu.sg/home/nprivault/index.html
Exercise 14.5 The swaption can be priced as

\[
dL(t, T, S) = \sigma L(t, T, S) d\hat{W}_t, \quad 0 \leq t \leq T,
\]

hence \( L(t, T, S) = L(0, T, S) e^{\sigma \hat{W}_t - \sigma^2 t/2} \), where \((\hat{W}_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion under \(\hat{P}_S\).

c) We find

\[
F_t = P(t, S) \hat{E}[(\kappa - L(T, T, S))^+ | \mathcal{F}_t]
\]

\[
= P(t, S) \hat{E}[(\kappa - L(T, T, S)) e^{-\sigma^2(T-t)/2 + \sigma(\hat{W}_T - \hat{W}_t)} + | \mathcal{F}_t]
\]

\[
= P(t, S)(\kappa \Phi(-d_-(T-t)) - \hat{X}_t \Phi(-d_+(T-t)))
\]

\[
= \kappa P(t, S) \Phi(-d_-(T-t)) - P(t, S)L(t, T, S)\Phi(-d_+(T-t))
\]

\[
= \kappa P(t, S) \Phi(-d_-(T-t)) - (P(t, T) - P(t, S))\Phi(-d_+(T-t))/(S-T),
\]

where \( e^n = L(t, T, S) e^{-\sigma^2(T-t)/2} \), \( v^2 = (T-t)\sigma^2 \), and

\[
d_+(T-t) = \frac{\log(L(t, T, S)/\kappa)}{\sigma \sqrt{T-t}} + \frac{\sigma \sqrt{T-t}}{2},
\]

and

\[
d_-(T-t) = \frac{\log(L(t, T, S)/\kappa)}{\sigma \sqrt{T-t}} - \frac{\sigma \sqrt{T-t}}{2},
\]

because \( L(t, T, S) \) is a driftless geometric Brownian motion with volatility \( \sigma \) under the forward measure \(\hat{P}_S\).

Exercise 14.5 The swaption can be priced as

\[
\mathbb{E}^* \left[ e^{-\int_t^{T_i} r_s ds} \left( P(T_i, T_i) - P(T_i, T_j) - \kappa P(T_i, T_i, T_j) \right)^+ | \mathcal{F}_t \right]
\]

\[
= \mathbb{E}^* \left[ e^{-\int_t^{T_i} r_s ds} \left( 1 - \kappa \sum_{k=i}^{j-1} c_{k+1} P(T_i, T_{k+1}) \right)^+ | \mathcal{F}_t \right]
\]

\[
= \kappa \mathbb{E}^* \left[ e^{-\int_t^{T_i} r_s ds} \left( \sum_{k=i}^{j-1} c_{k+1} F_{k+1}(T_i, \gamma_k) - \sum_{k=i}^{j-1} c_{k+1} F_{k+1}(T_i, r_{T_i}) \right)^+ | \mathcal{F}_t \right]
\]

\[
= \kappa \mathbb{E}^* \left[ e^{-\int_t^{T_i} r_s ds} \left( \sum_{k=i}^{j-1} c_{k+1} (F_{k+1}(T_i, \gamma_k) - F_{k+1}(T_i, r_{T_i})) \right)^+ | \mathcal{F}_t \right]
\]

\[
= \kappa \mathbb{E}^* \left[ e^{-\int_t^{T_i} r_s ds} \sum_{k=i}^{j-1} c_{k+1} 1_{\{r_{T_i} \leq \gamma_k\}} (F_{k+1}(T_i, \gamma_k) - F_{k+1}(T_i, r_{T_i})) | \mathcal{F}_t \right]
\]
b) Approximating the random process

Exercise 14.6

a) We have

\[\frac{dX_t}{X_t} = \sum_{i=2}^{n} c_i d\hat{P}(t, T_i) = \sum_{i=2}^{n} c_i \sigma_i(t) \hat{P}(t, T_i) \, d\hat{W}_t = \sigma_t X_t d\hat{W}_t,\]

from which we obtain

\[\sigma_t = \frac{1}{X_t} \sum_{i=2}^{n} c_i \sigma_i(t) \hat{P}(t, T_i) = \frac{\sum_{i=2}^{n} c_i \sigma_i(t) \hat{P}(t, T_i)}{\sum_{i=2}^{n} c_i \hat{P}(t, T_i)} = \frac{\sum_{i=2}^{n} c_i \sigma_i(t) P(t, T_i)}{\sum_{i=2}^{n} c_i P(t, T_i)}.\]

b) Approximating the random process \(\sigma_t\) by the deterministic function

\[\hat{\sigma}(t) := \frac{1}{X_0} \sum_{i=2}^{n} c_i \sigma_i(t) \hat{P}(0, T_i) = \frac{\sum_{i=2}^{n} c_i \sigma_i(t) \hat{P}(0, T_i)}{\sum_{i=2}^{n} c_i \hat{P}(0, T_i)} = \frac{\sum_{i=2}^{n} c_i \sigma_i(t) P(0, T_i)}{\sum_{i=2}^{n} c_i P(0, T_i)},\]

we find

\[P(0, T_1) \hat{\mathbb{E}} \left[ (X_{T_0} - \kappa)^+ \right] \]

\[\simeq P(0, T_1) \hat{\mathbb{E}} \left[ \left( X_0 e^{\int_0^{T_0} |\hat{\sigma}(t)|^2 \, dW_t^{(1)} - \frac{1}{2} \int_0^{T_0} |\hat{\sigma}(t)|^2 \, dt - \kappa \right)^+ \right] \]

\[= P(0, T_1) \text{Bl}_{\text{put}} \left( \kappa, X_0, \sqrt{\frac{1}{T_0} \int_0^{T_0} |\hat{\sigma}(t)|^2 \, dt}, 0, T \right) \]

\[= P(0, T_1) \left( X_0 \Phi \left( v/2 + (\log(\kappa/X_0))/v \right) - \kappa \Phi \left( -v/2 + (\log(\kappa/X_0))/v \right) \right)\]
\[
\begin{align*}
\Phi(v) &= -\frac{\kappa}{\rho} + \frac{\kappa}{\rho^2} \left(1 - \frac{1}{\rho^2} \log \left( \frac{\kappa}{\rho} \right) \right) \\
&= -\kappa P(0,T) \Phi \left( -\frac{\kappa}{\rho} + \frac{\kappa}{\rho^2} \left(1 - \frac{1}{\rho^2} \log \left( \frac{\kappa}{\rho} \right) \right) \right),
\end{align*}
\]

with \( v := \sqrt{\int_0^{T_0} |\hat{\sigma}(t)|^2 dt} \).

**Exercise 14.7**

a) We have
\[
\frac{dP(t,T_i)}{P(t,T_i)} = r_t dt + \zeta_t^i dB_t, \quad i = 1, 2,
\]
and
\[
P(T,T_i) = P(t,T_i) \exp \left( \int_t^T r_s ds + \int_t^T \zeta_s^i dB_s - \frac{1}{2} \int_t^T |\zeta_s^i|^2 ds \right),
\]
for \( 0 \leq t \leq T \leq T_i, \ i = 1, 2 \), hence
\[
\log P(T,T_i) = \log P(t,T_i) + \int_t^T r_s ds + \int_t^T \zeta_s^i dB_s - \frac{1}{2} \int_t^T |\zeta_s^i|^2 ds,
\]
for \( 0 \leq t \leq T \leq T_i, \ i = 1, 2 \), and
\[
d\log P(t,T_i) = r_t dt + \zeta_t^i dB_t - \frac{1}{2} |\zeta_t^i|^2 dt, \quad i = 1, 2.
\]

In the present model
\[
dr_t = \sigma dB_t,
\]
where \((B_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion under \( P \), by the the solution of Exercise 13.3 and (13.18) we have
\[
\zeta_t^i = -\sigma (T_i - t), \quad 0 \leq t \leq T_i, \ i = 1, 2.
\]

Letting
\[
 dB_t^i = dB_t - \zeta_t^i dt,
\]
defines a standard Brownian motion under \( P_i, \ i = 1, 2 \), and we have
\[
\frac{P(T,T_1)}{P(T,T_2)} = \frac{P(t,T_1)}{P(t,T_2)} \exp \left( \int_t^T (\zeta_s^1 - \zeta_s^2) dB_s - \frac{1}{2} \int_t^T (|\zeta_s^1|^2 - |\zeta_s^2|^2) ds \right)
\]
\[
= \frac{P(t,T_1)}{P(t,T_2)} \exp \left( \int_t^T (\zeta_s^1 - \zeta_s^2)^2 dB_s^2 - \frac{1}{2} \int_t^T (\zeta_s^1 - \zeta_s^2)^2 ds \right),
\]
which is an \( \mathcal{F}_t \)-martingale under \( P_2 \) and under \( P_{1,2} \), and

\[ \diamond \]
\[
P(T, T_2) = \frac{P(T, T_2)}{P(t, T_1)} \exp \left( - \int_t^T (\zeta^s_s - \zeta^s_t) dB^s_s - \frac{1}{2} \int_t^T (\zeta^s_s - \zeta^s_t)^2 ds \right)
\]

which is an \( \mathcal{F}_t \)-martingale under \( \mathbb{P}_1 \).

b) We have

\[
f(t, T_1, T_2) = -\frac{1}{T_2 - T_1} \left( \log P(t, T_2) - \log P(t, T_1) \right)
= r_t + \frac{1}{T_2 - T_1} \sigma^2 \left( (T_1 - t)^3 - (T_2 - t)^3 \right).
\]

c) We have

\[
df(t, T_1, T_2) = -\frac{1}{T_2 - T_1} d \log \left( \frac{P(t, T_2)}{P(t, T_1)} \right)
= -\frac{1}{T_2 - T_1} \left( (\zeta^2_t - \zeta^1_t) dB^2_t - \frac{1}{2} (|\zeta^2_t|^2 - |\zeta^1_t|^2) dt \right)
= -\frac{1}{T_2 - T_1} \left( (\zeta^2_t - \zeta^1_t) dB^2_t + \zeta^2_t dt - \frac{1}{2} (|\zeta^2_t|^2 - |\zeta^1_t|^2) dt \right)
= -\frac{1}{T_2 - T_1} \left( (\zeta^2_t - \zeta^1_t) dB^2_t - \frac{1}{2} (\zeta^2_t - \zeta^1_t)^2 dt \right).
\]

d) We have

\[
f(T, T_1, T_2) = -\frac{1}{T_2 - T_1} \log \left( \frac{P(T, T_2)}{P(T, T_1)} \right)
= f(t, T_1, T_2) - \frac{1}{T_2 - T_1} \left( \int_t^T (\zeta^2_s - \zeta^1_s) dB^1_s - \frac{1}{2} (|\zeta^2_s|^2 - |\zeta^1_s|^2) ds \right)
= f(t, T_1, T_2) - \frac{1}{T_2 - T_1} \left( \int_t^T (\zeta^2_s - \zeta^1_s) dB^2_s + \frac{1}{2} \int_t^T (\zeta^2_s - \zeta^1_s)^2 ds \right)
= f(t, T_1, T_2) - \frac{1}{T_2 - T_1} \left( \int_t^T (\zeta^2_s - \zeta^1_s) dB^1_s + \frac{1}{2} \int_t^T (\zeta^2_s - \zeta^1_s)^2 ds \right).
\]

Hence \( f(T, T_1, T_2) \) has a Gaussian distribution given \( \mathcal{F}_t \) with conditional mean

\[
m = f(t, T_1, T_2) + \frac{1}{2} \int_t^T (\zeta^2_s - \zeta^1_s)^2 ds
\]

under \( \mathbb{P}_2 \), resp.

\[
m = f(t, T_1, T_2) - \frac{1}{2} \int_t^T (\zeta^2_s - \zeta^1_s)^2 ds
\]

under \( \mathbb{P}_1 \), and variance

\[
v^2 = \frac{1}{(T_2 - T_1)^2} \int_t^T (\zeta^2_s - \zeta^1_s)^2 ds.
\]
Hence

\[(T_2 - T_1) \mathbb{E}^* \left[ e^{-\int_t^{T_2} r_s ds} (f(T_1, T_1, T_2) - \kappa) \bigg| \mathcal{F}_t \right] \]

\[= (T_2 - T_1) P(t, T_2) \mathbb{E}_2 \left[ (f(T_1, T_1, T_2) - \kappa)^+ \bigg| \mathcal{F}_t \right] \]

\[= (T_2 - T_1) P(t, T_2) \mathbb{E}_2 \left[ (m + X - \kappa)^+ \big| \mathcal{F}_t \right] \]

\[= (T_2 - T_1) P(t, T_2) \left( \frac{v}{\sqrt{2\pi}} e^{-(\kappa-m)^2/(2v^2)} + (m - \kappa) \Phi((m - \kappa)/v) \right) . \]

e) We have

\[L(T, T_1, T_2) = S(T, T_1, T_2) \]

\[= \frac{1}{T_2 - T_1} \left( \frac{P(T, T_1)}{P(T, T_2)} - 1 \right) \]

\[= \frac{1}{T_2 - T_1} \left( \frac{P(t, T_1)}{P(t, T_2)} \exp \left( \int_t^T (\zeta_s^1 - \zeta_s^2) dB_s - \frac{1}{2} \int_t^T (|\zeta_s^1|^2 - |\zeta_s^2|^2) ds \right) - 1 \right) \]

\[= \frac{1}{T_2 - T_1} \left( \frac{P(t, T_1)}{P(t, T_2)} \exp \left( \int_t^T (\zeta_s^1 - \zeta_s^2) dB_s^2 - \frac{1}{2} \int_t^T (\zeta_s^1 - \zeta_s^2)^2 ds \right) - 1 \right) \]

\[= \frac{1}{T_2 - T_1} \left( \frac{P(t, T_1)}{P(t, T_2)} \exp \left( \int_t^T (\zeta_s^1 - \zeta_s^2) dB_s^1 + \frac{1}{2} \int_t^T (\zeta_s^1 - \zeta_s^2)^2 ds \right) - 1 \right) , \]

and by Itô calculus,

\[dS(t, T_1, T_2) = \frac{1}{T_2 - T_1} d \left( \frac{P(t, T_1)}{P(t, T_2)} \right) \]

\[= \frac{1}{T_2 - T_1} \frac{P(t, T_1)}{P(t, T_2)} \left( (\zeta_t^1 - \zeta_t^2) dB_t + \frac{1}{2} (\zeta_t^1 - \zeta_t^2)^2 dt - \frac{1}{2}(|\zeta_t^1|^2 - |\zeta_t^2|^2) dt \right) \]

\[= \left( \frac{1}{T_2 - T_1} + S(t, T_1, T_2) \right) \left( (\zeta_t^1 - \zeta_t^2) dB_t + \zeta_t^2 (\zeta_t^2 - \zeta_t^1) dt \right) \]

\[= \left( \frac{1}{T_2 - T_1} + S(t, T_1, T_2) \right) \left( (\zeta_t^1 - \zeta_t^2) dB_t^1 + (|\zeta_t^2|^2 - |\zeta_t^1|^2) dt \right) \]

\[= \left( \frac{1}{T_2 - T_1} + S(t, T_1, T_2) \right) (\zeta_t^1 - \zeta_t^2) dB_t^2 , \quad t \in [0, T_1] , \]

hence \( t \mapsto \frac{1}{T_2 - T_1} + S(t, T_1, T_2) \) is a geometric Brownian motion, with

\[\frac{1}{T_2 - T_1} + S(T, T_1, T_2) \]

\[= \left( \frac{1}{T_2 - T_1} + S(t, T_1, T_2) \right) \exp \left( \int_t^T (\zeta_s^1 - \zeta_s^2) dB_s^2 - \frac{1}{2} \int_t^T (\zeta_s^1 - \zeta_s^2)^2 ds \right) , \]

\[0 \leq t \leq T \leq T_1 . \]
(T_2 - T_1) \mathbb{E}^\ast \left[ e^{-\int_t^{T_2} r_s ds} (L(T_1, T_1, T_2) - \kappa)^+ \bigg| \mathcal{F}_t \right] 
= (T_2 - T_1) \mathbb{E}^\ast \left[ e^{-\int_t^{T_1} r_s ds} P(T_1, T_2) (L(T_1, T_1, T_2) - \kappa)^+ \bigg| \mathcal{F}_t \right] 
= P(t, T_1, T_2) \mathbb{E}_{1,2} \left[ (S(T_1, T_1, T_2) - \kappa)^+ \bigg| \mathcal{F}_t \right].

The forward measure \mathbb{P}_2 is defined by

\mathbb{E}^\ast \left[ \frac{d\mathbb{P}_2}{d\mathbb{P}} \bigg| \mathcal{F}_t \right] = \frac{P(t, T_2)}{P(0, T_2)} e^{-\int_0^t r_s ds}, \quad 0 \leq t \leq T_2,

and the forward swap measure is defined by

\mathbb{E}^\ast \left[ \frac{d\mathbb{P}_{1,2}}{d\mathbb{P}} \bigg| \mathcal{F}_t \right] = \frac{P(t, T_2)}{P(0, T_2)} e^{-\int_0^t r_s ds}, \quad 0 \leq t \leq T_1,

hence \mathbb{P}_2 and \mathbb{P}_{1,2} coincide up to time \(T_1\) and \((B^2_t)_{t \in [0, T_1]}\) is a standard Brownian motion until time \(T_1\) under \mathbb{P}_2 and under \mathbb{P}_{1,2}, consequently under \mathbb{P}_{1,2} we have

\begin{align*}
L(T, T_1, T_2) &= S(T, T_1, T_2) \\
&= -\frac{1}{T_2 - T_1} + \left( \frac{1}{T_2 - T_1} + S(t, T_1, T_2) \right) e^{\int_t^T (\zeta^1_s - \zeta^2_s) dB^2_s - \frac{1}{2} \int_t^T (\zeta^1_s - \zeta^2_s)^2 ds},
\end{align*}

has same distribution as

\begin{align*}
\frac{1}{T_2 - T_1} \left( \frac{P(t, T_1)}{P(t, T_2)} e^{X - \frac{1}{2} \text{Var}[X]} - 1 \right),
\end{align*}

where \(X\) is a centered Gaussian random variable with variance

\begin{align*}
\int_t^T (\zeta^1_s - \zeta^2_s)^2 ds
\end{align*}

given \(\mathcal{F}_t\). Hence

\begin{align*}
(T_2 - T_1) \mathbb{E}^\ast \left[ e^{-\int_t^{T_2} r_s ds} (L(T_1, T_1, T_2) - \kappa)^+ \bigg| \mathcal{F}_t \right] 
&= P(t, T_1, T_2) \\
&\times \text{Bl} \left( \frac{1}{T_2 - T_1} + S(t, T_1, T_2), \frac{\int_t^T (\zeta^1_s - \zeta^2_s)^2 ds}{T_1 - t}, \kappa + \frac{1}{T_2 - T_1}, T_1 - t \right).
\end{align*}

Exercise 14.8
Notes on Stochastic Finance

a) We have
\[ L(t, T_1, T_2) = L(0, T_1, T_2) e^{\int_0^t \gamma_1(s) dW^2_s - \frac{1}{2} \int_0^t |\gamma_1(s)|^2 ds}, \quad 0 \leq t \leq T_1, \]
and \( L(t, T_2, T_3) = b. \) Note that we have \( P(t, T_2) / P(t, T_3) = 1 + \delta b \) hence \( P_2 = P_3 = P_{1,2} \) up to time \( T_1. \)

b) We use change of numéraire under the forward measure \( \mathbb{P}_2. \)

c) We have
\[
\begin{align*}
E^* \left[ e^{-\int_t^{T_2} r_s ds} (L(T_1, T_1, T_2) - \kappa)^+ \right| \mathcal{F}_t] \\
= P(t, T_2) \mathbb{E}_2 \left[ (L(T_1, T_1, T_2) - \kappa)^+ \right| \mathcal{F}_t] \\
= P(t, T_2) \mathbb{E}_2 \left[ (L(t, T_1, T_2) e^{\int_t^{T_1} \gamma_1(s) dW^2_s - \frac{1}{2} \int_t^{T_1} |\gamma_1(s)|^2 ds} - \kappa)^+ \right| \mathcal{F}_t] \\
= P(t, T_2) \mathbb{B}l(\kappa, L(t, T_1, T_2), \sigma_1(t), 0, T_1 - t),
\end{align*}
\]
where
\[
\sigma_1^2(t) = \frac{1}{T_1 - t} \int_t^{T_1} |\gamma_1(s)|^2 ds.
\]

d) We have
\[
\begin{align*}
P(t, T_1) &= \frac{P(t, T_1)}{\delta P(t, T_2) + \delta P(t, T_3)} \\
&= \frac{P(t, T_1)}{\delta P(t, T_2) / \delta P(t, T_3)} \frac{1}{1 + P(t, T_3) / P(t, T_2)} \\
&= \frac{1 + \delta b}{\delta (\delta b + 2)} (1 + \delta L(t, T_1, T_2)), \quad 0 \leq t \leq T_1,
\end{align*}
\]
and
\[
\begin{align*}
P(t, T_3) &= \frac{P(t, T_3)}{P(t, T_2) + P(t, T_3)} \\
&= \frac{1}{1 + P(t, T_2) / P(t, T_3)} \\
&= \frac{1}{\delta} \frac{1}{2 + \delta b}, \quad 0 \leq t \leq T_2.
\end{align*}
\tag{A.51}
\]
e) We have
\[
S(t, T_1, T_3) = \frac{P(t, T_1)}{P(t, T_1, T_3)} - \frac{P(t, T_3)}{P(t, T_1, T_3)} \\
= \frac{1 + \delta b}{\delta (2 + \delta b)} (1 + \delta L(t, T_1, T_2)) - \frac{1}{\delta (2 + \delta b)}
\]
\[ dS(t, T_1, T_3) = \frac{1 + \delta b}{2 + \delta b} L(t, T_1, T_2) \gamma_1(t) dW_t^2 \]

\[ = \left( S(t, T_1, T_3) - \frac{b}{2 + \delta b} \right) \gamma_1(t) dW_t^2 \]

\[ = S(t, T_1, T_3) \sigma_{1,3}(t) dW_t^2, \quad 0 \leq t \leq T_2, \]

with

\[ \sigma_{1,3}(t) = \left( 1 - \frac{b}{S(t, T_1, T_3)(2 + \delta b)} \right) \gamma_1(t) \]

\[ = \left( 1 - \frac{b}{b + (1 + \delta b)L(t, T_1, T_2)} \right) \gamma_1(t) \]

\[ = \frac{(1 + \delta b)L(t, T_1, T_2)}{b + (1 + \delta b)L(t, T_1, T_2)} \gamma_1(t) \]

\[ = \frac{(1 + \delta b)L(t, T_1, T_2)}{(2 + \delta b)S(t, T_1, T_3)} \gamma_1(t). \]

f) The process \((W^2)_{t \in \mathbb{R}_+}\) is a standard Brownian motion under \(P_2\) and

\[ P(t, T_1, T_3) \mathbb{E}_{1,3} \left[ (S(T_1, T_1, T_3) - \kappa)^+ \mid \mathcal{F}_t \right] = P(t, T_2) \text{Bl}(\kappa, S(t, T_1, T_2), \bar{\sigma}_{1,3}(t), 0, T_1 - t), \]

where \(|\bar{\sigma}_{1,3}(t)|^2\) is the approximation of the volatility

\[ \frac{1}{T_1 - t} \int_t^{T_1} |\sigma_{1,3}(s)|^2 ds = \frac{1}{T_1 - t} \int_t^{T_1} \left( \frac{(1 + \delta b)L(s, T_1, T_2)}{(2 + \delta b)S(s, T_1, T_3)} \right)^2 \gamma_1(s) ds \]

obtained by freezing the random component of \(\sigma_{1,3}(s)\) at time \(t\), i.e.

\[ \bar{\sigma}_{1,3}^2(t) = \frac{1}{T_1 - t} \left( \frac{(1 + \delta b)L(t, T_1, T_2)}{(2 + \delta b)S(t, T_1, T_3)} \right)^2 \int_t^{T_1} |\gamma_1(s)|^2 ds. \]

Exercise 14.9

a) We have

\[ \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} (P(T, S) - \kappa)^+ \mid \mathcal{F}_t \right] = V_T = V_0 + \int_0^T dV_t \]

\[ = P(0, T) \mathbb{E}_T \left[ (P(T, S) - \kappa)^+ \right] + \int_0^t \xi_s^T dP(s, T) + \int_0^t \xi_s^S dP(s, S). \]

b) We have
\[ d\tilde{V}_t = d \left( e^{-\int_0^t r_s ds} V_t \right) \]
\[ = -r_t e^{-\int_0^t r_s ds} V_t dt + e^{-\int_0^t r_s ds} dV_t \]
\[ = -r_t e^{-\int_0^t r_s ds} (\xi_t T P(t, T) + \xi_t^S P(t, S)) dt + e^{-\int_0^t r_s ds} \xi_t^T dP(t, T) + e^{-\int_0^t r_s ds} \xi_t^S dP(t, S) \]
\[ = \xi_t^T d\tilde{P}(t, T) + \xi_t^S d\tilde{P}(t, S). \]

c) By Itô’s formula we have
\[ \mathbb{E}_T \left[ (P(T, S) - \kappa)^+ | F_t \right] = C(X_T, 0, 0) \]
\[ = C(X_0, T, v(0, T)) + \int_0^t \frac{\partial C}{\partial x} (X_s, T - s, v(s, T)) dX_s \]
\[ = \mathbb{E}_T \left[ (P(T, S) - \kappa)^+ \right] + \int_0^t \frac{\partial C}{\partial x} (X_s, T - s, v(s, T)) dX_s, \]
since the process
\[ t \mapsto \mathbb{E}_T \left[ (P(T, S) - \kappa)^+ | F_t \right] \]
is a martingale under \( \tilde{P} \).

d) We have
\[ d\tilde{V}_t = d(V_t / P(t, T)) \]
\[ = d \mathbb{E}_T \left[ (P(T, S) - \kappa)^+ | F_t \right] \]
\[ = \frac{\partial C}{\partial x} (X_t, T - t, v(t, T)) dX_t \]
\[ = \frac{P(t, S)}{P(t, T)} \frac{\partial C}{\partial x} (X_t, T - t, v(t, T))(\sigma_t^S - \sigma_t^T) dB_t^T. \]

e) We have
\[ dV_t = d(P(t, T)\tilde{V}_t) \]
\[ = P(t, T)d\tilde{V}_t + \tilde{V}_tdP(t, T) + d\tilde{V}_t \cdot dP(t, T) \]
\[ = P(t, S)\frac{\partial C}{\partial x} (X_t, T - t, v(t, T))(\sigma_t^S - \sigma_t^T) dB_t^T + \tilde{V}_tdP(t, T) \]
\[ + P(t, S)\frac{\partial C}{\partial x} (X_t, T - t, v(t, T))(\sigma_t^S - \sigma_t^T) \sigma_t^T dt \]
\[ = P(t, S)\frac{\partial C}{\partial x} (X_t, T - t, v(t, T))(\sigma_t^S - \sigma_t^T) dB_t + \tilde{V}_tdP(t, T). \]

f) We have
h) We have

\[
d\tilde{V}_t = d\left(e^{-\int_0^t r_s ds} V_t\right) \\
= -r_t e^{-\int_0^t r_s ds} V_t dt + e^{-\int_0^t r_s ds} dV_t \\
= \dot{P}(t, S) \frac{\partial C}{\partial x}(X_t, T - t, v(t, T)) (\sigma_t^S - \sigma_t^T) dB_t + \dot{V}_t d\dot{P}(t, T).
\]

g) We have

\[
d\tilde{V}_t = \dot{P}(t, S) \frac{\partial C}{\partial x}(X_t, T - t, v(t, T)) (\sigma_t^S - \sigma_t^T) dB_t + \dot{V}_t d\dot{P}(t, T) \\
= \frac{\partial C}{\partial x}(X_t, T - t, v(t, T)) d\dot{P}(t, S) \\
- \frac{P(t, S)}{P(t, T)} \frac{\partial C}{\partial x}(X_t, T - t, v(t, T)) d\dot{P}(t, T) + \dot{V}_t d\dot{P}(t, T) \\
= \left( \dot{V}_t - \frac{P(t, S)}{P(t, T)} \frac{\partial C}{\partial x}(X_t, T - t, v(t, T)) \right) d\dot{P}(t, T) \\
+ \frac{\partial C}{\partial x}(X_t, T - t, v(t, T)) d\dot{P}(t, S),
\]

hence the hedging strategy \((\xi_t^T, \xi_t^S)_{t \in [0, T]}\) of the bond option is given by

\[
\xi_t^T = \dot{V}_t - \frac{P(t, S)}{P(t, T)} \frac{\partial C}{\partial x}(X_t, T - t, v(t, T)) \\
= C(X_t, T - t, v(t, T)) - \frac{P(t, S)}{P(t, T)} \frac{\partial C}{\partial x}(X_t, T - t, v(t, T)),
\]

and

\[
\xi_t^S = \frac{\partial C}{\partial x}(X_t, T - t, v(t, T)),
\]

\(t \in [0, T].\)

h) We have

\[
\frac{\partial C}{\partial x}(x, \tau, v) \\
= \frac{\partial}{\partial x} \left[ x \Phi \left( \frac{v \sqrt{\tau}}{2} + \frac{1}{v \sqrt{\tau}} \log \frac{x}{\kappa} \right) \right] - \kappa \Phi \left( \frac{-v \sqrt{\tau}}{2} + \frac{1}{v \sqrt{\tau}} \log \frac{x}{\kappa} \right) \\
= x \frac{\partial}{\partial x} \Phi \left( \frac{v \sqrt{\tau}}{2} + \frac{1}{v \sqrt{\tau}} \log \frac{x}{\kappa} \right) - \kappa \frac{\partial}{\partial x} \Phi \left( \frac{-v \sqrt{\tau}}{2} + \frac{1}{v \sqrt{\tau}} \log \frac{x}{\kappa} \right) \\
+ \Phi \left( \frac{v \sqrt{\tau}}{2} + \frac{1}{v \sqrt{\tau}} \log \frac{x}{\kappa} \right) \\
= x e^{-\frac{1}{2} \left( \frac{v \sqrt{\tau}}{2} + \frac{1}{v \sqrt{\tau}} \log \frac{x}{\kappa} \right)^2} \left( \frac{1}{v \sqrt{\tau} x} \right) - \kappa e^{-\frac{1}{2} \left( \frac{-v \sqrt{\tau}}{2} + \frac{1}{v \sqrt{\tau}} \log \frac{x}{\kappa} \right)^2} \left( \frac{1}{v \sqrt{\tau} x} \right)
\]
As a consequence we get

\[ \xi_t^T = C(X_t, T - t, v(t, T)) - \frac{P(t, S)}{P(t, T)} \frac{\partial C}{\partial x}(X_t, T - t, v(t, T)) \]

\[ = \frac{P(t, S)}{P(t, T)} \Phi \left( \frac{(T - t)v^2(t, T)/2 + \log X_t}{\sqrt{T - t}v(t, T)} \right) \]

\[ - \kappa \Phi \left( - \frac{v(t, T)}{2} + \frac{1}{v(t, T)} \log \frac{P(t, S)}{\kappa P(t, T)} \right) \]

\[ - \frac{P(t, S)}{P(t, T)} \Phi \left( \frac{\log(X_t/\kappa) + (T - t)v^2(t, T)/2}{\sqrt{T - t}v(t, T)} \right) \]

\[ = - \kappa \Phi \left( \frac{\log(X_t/\kappa) - (T - t)v^2(t, T)/2}{v(t, T)\sqrt{T - t}} \right), \]

and

\[ \xi_t^S = \frac{\partial C}{\partial x}(X_t, T - t, v(t, T)) = \Phi \left( \frac{\log(X_t/\kappa) + (T - t)v^2(t, T)/2}{v(t, T)\sqrt{T - t}} \right), \]

\( t \in [0, T] \), and the hedging strategy is given by

\[ V_T = \mathbb{E}^* \left[ e^{-\int_0^T r_s ds} (P(T, S) - \kappa)^+ \right] \mathcal{F}_t \]

\[ = V_0 + \int_0^t \xi_t^S dP(s, T) + \int_0^t \xi_t^S dP(s, S) \]

\[ = V_0 - \kappa \int_0^t \Phi \left( \frac{\log(X_t/\kappa) - (T - t)v^2(t, T)/2}{v(t, T)\sqrt{T - t}} \right) dP(s, T) \]

\[ + \int_0^t \Phi \left( \frac{\log(X_t/\kappa) + (T - t)v^2(t, T)/2}{v(t, T)\sqrt{T - t}} \right) dP(s, S). \]

Consequently the bond option can be hedged by shortselling a bond with maturity \( T \) for the amount

\[ \kappa \Phi \left( \frac{\log(X_t/\kappa) - (T - t)v^2(t, T)/2}{v(t, T)\sqrt{T - t}} \right), \]

and by holding a bond with maturity \( S \) for the amount

\[ \Phi \left( \frac{\log(X_t/\kappa) + (T - t)v^2(t, T)/2}{v(t, T)\sqrt{T - t}} \right). \]
Exercise 14.10

a) The LIBOR rate $L(t, T, S)$ is a driftless geometric Brownian motion with deterministic volatility process $\sigma(t)$ under the forward measure $\hat{P}_S$.

Explanation: The LIBOR rate $L(t, T, S)$ can be written as the forward price $L(t, T, S) = \hat{X}_t = X_t / N_t$ where $X_t = (P(t, T) - P(t, S)) / (S - T)$ and $N_t = P(t, S)$. Since both discounted bond prices $e^{-\int_0^t r_s ds} P(t, T)$ and $e^{-\int_0^t r_s ds} P(t, S)$ are martingales under $P^*$, the same is true of $X_t$. Hence $L(t, T, S) = X_t / N_t$ becomes a martingale under the forward measure $\hat{P}_S$ by Proposition 2.1, and computing its dynamics under $\hat{P}_S$ amounts to removing any “$dt$” term in the original SDE defining $L(t, T, S)$, i.e. we find

$$dL(t, T, S) = \sigma(t)L(t, T, S)d\hat{W}_t, \quad 0 \leq t \leq T,$$

hence

$$L(t, T, S) = L(0, T, S) \exp \left( \int_0^t \sigma(s)d\hat{W}_s - \int_0^t \sigma^2(s)ds/2 \right),$$

where $(\hat{W}_t)_{t \in \mathbb{R}^+}$ is a standard Brownian motion.

b) Choosing the annuity numéraire $N_t = P(t, S)$, we have

$$\mathbb{E}^* \left[ e^{-\int_t^S r_s ds} \phi(L(T, T, S)) | \mathcal{F}_t \right] = \mathbb{E}^* \left[ e^{-\int_t^S r_s ds} N_S \phi(L(T, T, S)) | \mathcal{F}_t \right] = N_t \mathbb{E} \left[ \phi(L(T, T, S)) | \mathcal{F}_t \right] = P(t, S) \mathbb{E} [\phi(L(T, T, S)) | \mathcal{F}_t].$$

c) Given the solution

$$L(T, T, S) = L(0, T, S) \exp \left( \int_0^T \sigma(s)d\hat{W}_s - \int_0^T \sigma^2(s)ds/2 \right)$$

$$= L(t, T, S) \exp \left( \int_t^T \sigma(s)d\hat{W}_s - \int_t^T \sigma^2(s)ds/2 \right),$$

we find

$$P(t, S) \mathbb{E} \left[ \phi(L(T, T, S)) | \mathcal{F}_t \right]$$

$$= P(t, S) \mathbb{E} \left[ \phi \left( L(t, T, S) e^{\int_t^T \sigma(s)d\hat{W}_s - \int_t^T \sigma^2(s)ds/2} \right) | \mathcal{F}_t \right]$$

$$= P(t, S) \int_{-\infty}^{\infty} \phi(L(t, T, S) e^x - \eta^2/2) e^{-x^2/(2\eta^2)} \frac{dx}{\sqrt{2\pi\eta^2}},$$

where $\eta = \sqrt{\int_0^T \sigma^2(s)ds}$. 

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because \( \int_t^T \sigma(s)d\hat{W}_s \) is a centered Gaussian variable with variance \( \eta^2 := \int_t^T \sigma^2(s)ds \), independent of \( \mathcal{F}_t \) under the forward measure \( \hat{\mathbf{P}} \).

**Exercise 14.11**

a) Choosing the annuity numéraire \( N_t = P(t, T_i, T_j) \), we have

\[
\mathbb{E}^* \left[ e^{\int_{T_i}^{T_j} r_s ds} P(T_i, T_j) \phi(S(T_i, T_j)) \right] = N_t \hat{E}_{i,j} \left[ \frac{P(T_i, T_j)}{N_{T_i}} \phi(S(T_i, T_j)) \right] = P(t, T_i, T_j) \hat{E}_{i,j} [\phi(S(T_i, T_j)) | \mathcal{F}_t].
\]

b) Since \( S(t, T_i, T_j) \) is a forward price for the numéraire \( P(t, T_i, T_j) \), it is a martingale under the forward swap measure \( \hat{\mathbf{P}}_{i,j} \) and we have

\[
S(t, T_i, T_j) = \sigma S(t, T_i, T_j) d\hat{W}_{i,j}, \quad 0 \leq t \leq T_i,
\]

where \( \hat{W}_{i,j} \) is a standard Brownian motion under the forward swap measure \( \hat{\mathbf{P}}_{i,j} \).

c) Given the solution

\[
S(T_i, T_i, T_j) = S(0, T_i, T_j) e^{-\sigma^2 T_i/2} = S(t, T_i, T_j) e^{-\sigma^2 (T_i - t)/2}
\]

of (14.33), we find

\[
P(t, T_i, T_j) \hat{E}_{i,j} [\phi(S(T_i, T_i, T_j)) | \mathcal{F}_t]
= P(t, T_i, T_j) \hat{E}_{i,j} \left[ \phi \left( S(t, T_i, T_j) e^{-\sigma^2 (T_i - t)/2} \right) \right] \mathcal{F}_t
= P(t, T_i, T_j) \int_{-\infty}^{\infty} \phi \left( S(t, T_i, T_j) e^{-\sigma^2 (T_i - t)/2} \right) e^{-x^2/(2(T_i-t))} \frac{dx}{\sqrt{2\pi(T_i-t)}},
\]

because \( \hat{W}_{i,j} \) is a centered Gaussian variable with variance \( T_i - t \), independent of \( \mathcal{F}_t \) under the forward measure \( \hat{\mathbf{P}}_{i,j} \).

d) We find

\[
P(t, T_i, T_j) \hat{E}_{i,j} [(\kappa - S(T_i, T_i, T_j))^+ | \mathcal{F}_t]
= P(t, T_i, T_j) \hat{E}_{i,j} \left[ (\kappa - S(t, T_i, T_i) e^{-\sigma^2 (T_i - t)/2} \phi(W_{T_i} - W_t)) \right] \mathcal{F}_t
= P(t, T_i, T_j) (\kappa \Phi(-d_-(T_i - t)) - \hat{X}_t \Phi(-d_+(T_i - t)))
= P(t, T_i, T_j) \kappa \Phi(-d_-(T_i - t)) - P(t, T_i, T_j) S(t, T_i, T_j) \Phi(-d_+(T_i - t))
= P(t, T_i, T_j) \kappa \Phi(-d_-(T_i - t)) - (P(t, T_i) - P(t, T_j)) \Phi(-d_+(T_i - t)) / (T_j - T_i),
\]
where \( e^m = S(t, T_i, T_j) e^{-\sigma^2(T-t)/2}, \) \( \nu^2 = (T-t)\sigma^2, \) and

\[
d_+(T_i - t) = \log(S(t, T_i, T_j) / \kappa) \frac{\sigma \sqrt{T_i - t}}{\sigma \sqrt{T_i - t}} + \frac{\sigma \sqrt{T_i - t}}{2},
\]

and

\[
d_-(T_i - t) = \log(S(t, T_i, T_j) / \kappa) \frac{\sigma \sqrt{T_i - t}}{\sigma \sqrt{T_i - t}} - \frac{\sigma \sqrt{T_i - t}}{2},
\]

because \( S(t, T_i, T_j) \) is a driftless geometric Brownian motion with volatility \( \sigma \) under the forward measure \( \hat{\mathbb{P}}_{i,j} \).

**Exercise 14.12**

a) Apply the Itô formula to \( d(P(t, T_1) / P(t, T_2)) \).

b) We have

\[
L_{T_1} = L_t \exp \left( \int_t^{T_1} \sigma(s) d\hat{B}_s - \frac{1}{2} \int_t^{T_1} |\sigma(s)|^2 ds \right).
\]

c) We have

\[
P(t, T_2) \mathbb{E} \left[ (L_{T_1} - \kappa)^+ \bigg| \mathcal{F}_t \right] = P(t, T_2) \mathbb{E} \left( L_t e^{J_t^1 \sigma(s) d\hat{B}_s - \frac{1}{2} \int_t^{T_1} |\sigma(s)|^2 ds} - \kappa e^{J_t^1 \sigma(s) d\hat{B}_s - \frac{1}{2} \int_t^{T_1} |\sigma(s)|^2 ds} \right) + \mathcal{F}_t
\]

\[
= P(t, T_2) \text{Bl} \left( \kappa, v(t, T_1) / \sqrt{T_1 - t}, 0, T_1 - t \right)
\]

\[
= P(t, T_2) \left( L_t \Phi \left( \frac{\log(x/K)}{v(t, T_1)} + \frac{v(t, T_1)}{2} \right) - \kappa \Phi \left( \frac{\log(x/K)}{v(t, T_1)} - \frac{v(t, T_1)}{2} \right) \right).
\]

d) Integrate the self-financing condition (14.38) between 0 and \( t \).

e) We have

\[
d\tilde{V}_t = d \left( e^{-\int_0^t r_s ds V_t} \right)
\]

\[
= -r_t e^{-\int_0^t r_s ds V_t} dt + e^{-\int_0^t r_s ds} dV_t
\]

\[
= -r_t e^{-\int_0^t r_s ds} \xi_1(t) P(t, T_1) - r_t e^{-\int_0^t r_s ds} \xi_2(t) P(t, T_2) dt
\]

\[
+ e^{-\int_0^t r_s ds} \xi_1(t) dP(t, T_1) + e^{-\int_0^t r_s ds} \xi_2(t) dP(t, T_2)
\]

\[
= \xi_1(t) d\tilde{P}(t, T_1) + \xi_2(t) d\tilde{P}(t, T_2), \quad 0 \leq t \leq T_1.
\]

since

\[
\frac{d\tilde{P}(t, T_1)}{\tilde{P}(t, T_1)} = \xi_1(t) dt, \quad \frac{d\tilde{P}(t, T_2)}{\tilde{P}(t, T_2)} = \xi_2(t) dt.
\]
We have
\[ (L_{T_1} - \kappa)^+ \hat{F}_t \]
and apply the Itô formula and the fact that \((L_t)_{t \in \mathbb{R}_+}\) are both martingales under \(\hat{\mathbb{P}}\). Next we use the fact that
\[ \hat{V}_t = \hat{\mathbb{E}} \left[ (L_{T_1} - \kappa)^+ \bigg| \mathcal{F}_t \right], \]
and apply the result of Question (f).

Apply the Itô rule to \(\hat{V}_t = P(t, T_2)\hat{V}_t\) using Relation (14.36) and the result of Question (f).

We have
\[
d\hat{V}_t = \frac{\partial C}{\partial x} (L_t, v(t, T_1)) P(t, T_1) (\zeta_1(t) - \zeta_2(t)) dB_t + \hat{V}_t dP(t, T_2)
\]
\[= \frac{\partial C}{\partial x} (L_t, v(t, T_1)) P(t, T_1) (\zeta_1(t) - \zeta_2(t)) dB_t + \hat{V}_t \zeta_2(t) P(t, T_2) dB_t \]
\[= (1 + L_t) \frac{\partial C}{\partial x} (L_t, v(t, T_1)) P(t, T_2) (\zeta_1(t) - \zeta_2(t)) dB_t + \hat{V}_t \zeta_2(t) P(t, T_2) dB_t \]
\[= L_t \zeta_1(t) \frac{\partial C}{\partial x} (L_t, v(t, T_1)) P(t, T_2) dB_t - L_t \frac{\partial C}{\partial x} (L_t, v(t, T_1)) P(t, T_2) \zeta_2(t) dB_t \]
\[+ \frac{\partial C}{\partial x} (L_t, v(t, T_1)) P(t, T_2) (\zeta_1(t) - \zeta_2(t)) dB_t + \hat{V}_t \zeta_2(t) P(t, T_2) dB_t \]
\[= L_t \zeta_1(t) \frac{\partial C}{\partial x} (L_t, v(t, T_1)) P(t, T_2) dB_t + \left( \hat{V}_t - L_t \frac{\partial C}{\partial x} (L_t, v(t, T_1)) \right) P(t, T_2) \zeta_2(t) dB_t \]
\[+ \frac{\partial C}{\partial x} (L_t, v(t, T_1)) P(t, T_2) (\zeta_1(t) - \zeta_2(t)) dB_t, \]

hence
\[
d\tilde{V}_t = L_t \zeta_1(t) \frac{\partial C}{\partial x} (L_t, v(t, T_1)) \hat{P}(t, T_2) dB_t \]
\[+ \left( \hat{V}_t - L_t \frac{\partial C}{\partial x} (L_t, v(t, T_1)) \right) \hat{P}(t, T_2) \zeta_2(t) dB_t \]
\[+ \frac{\partial C}{\partial x} (L_t, v(t, T_1)) \hat{P}(t, T_2) (\zeta_1(t) - \zeta_2(t)) dB_t \]
\[= (P(t, T_1) - \hat{P}(t, T_2)) \zeta_1(t) \frac{\partial C}{\partial x} (L_t, v(t, T_1)) dB_t \]
\[+ \left( \hat{V}_t - L_t \frac{\partial C}{\partial x} (L_t, v(t, T_1)) \right) d\hat{P}(t, T_2) \]
\[+ \frac{\partial C}{\partial x} (L_t, v(t, T_1)) \hat{P}(t, T_2) (\zeta_1(t) - \zeta_2(t)) dB_t \]
\[= (\zeta_1(t) \hat{P}(t, T_1) - \zeta_2(t) \hat{P}(t, T_2)) \frac{\partial C}{\partial x} (L_t, v(t, T_1)) dB_t \]
We have
\[ + \left( \dot{V}_t - L_t \frac{\partial C}{\partial x} (L_t, v(t, T_1)) \right) d\tilde{P}(t, T_2) \]
\[ = \frac{\partial C}{\partial x} (L_t, v(t, T_1)) d\tilde{P}(t, T_2) + \left( \dot{V}_t - L_t \frac{\partial C}{\partial x} (L_t, v(t, T_1)) \right) d\tilde{P}(t, T_2). \]

Exercise 14.13

a) We have
\[ S(T_i, T_i, T_j) = S(t, T_i, T_j) \exp \left( \int_t^{T_i} \sigma_{i,j}(s) dB_s^{i,j} - \frac{1}{2} \int_t^{T_i} |\sigma_{i,j}|^2(s) ds \right). \]

b) We have
\[ P(t, T_i, T_j) \mathbb{E}_{i,j} \left[ (S(T_i, T_i, T_j) - \kappa)^+ \right \vert F_t \]
\[ = P(t, T_i, T_j) \mathbb{E}_{i,j} \left[ \left( S(t, T_i, T_j) e^{\int_t^{T_i} \sigma_{i,j}(s) dB_s^{i,j} - \frac{1}{2} \int_t^{T_i} |\sigma_{i,j}|^2(s) ds - \kappa \right)^+ \right \vert F_t \]
\[ = P(t, T_i, T_j) \text{Bl} \left( \kappa, v(t, T_i) / \sqrt{T_i - t}, 0, T_i - t \right) \]
\[ = P(t, T_i, T_j) \times \left( S(t, T_i, T_j) \Phi \left( \frac{\log(x/K)}{v(t, T_i)} \right) + v(t, T_i) \right) - \kappa \Phi \left( \frac{\log(x/K)}{v(t, T_i)} - \frac{v(t, T_i)}{2} \right) \]
\[ \right. \]
where
\[ v^2(t, T_i) = \int_t^{T_i} |\sigma_{i,j}|^2(s) ds. \]

c) Integrate the self-financing condition (14.43) between 0 and t.

d) We have
\[ d\tilde{V}_t = d \left( e^{-\int_0^t r_s ds} V_t \right) \]
\[ = -r_t e^{-\int_0^t r_s ds} V_t dt + e^{-\int_0^t r_s ds} dV_t \]
\[ = -r_t e^{-\int_0^t r_s ds} \sum_{k=i}^j \xi^{(k)}_t P(t, T_k) dt + e^{-\int_0^t r_s ds} \sum_{k=i}^j \xi^{(k)}_t dP(t, T_k) \]
\[ = \sum_{k=i}^j \xi^{(k)}_t d\tilde{P}(t, T_k), \quad 0 \leq t \leq T_i. \]
since
\[ \frac{d\tilde{P}(t, T_k)}{P(t, T_k)} = \zeta_k(t) dt, \quad k = i, \ldots, j. \]
e) We apply the Itô formula and the fact that

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From the expression (14.42) of the swap rate volatilities we have

\[\mathbb{E}_{i,j} \left[ (S(T_i, T_j) - \kappa)^+ \bigg| \mathcal{F}_t \right] \]

and \((S_t)_{t \in \mathbb{R}_+}\) are both martingales under \(\mathbb{P}_{i,j}\).

f) Use the fact that

\[\hat{V}_t = \mathbb{E}_{i,j} \left[ (S(T_i, T_j) - \kappa)^+ \bigg| \mathcal{F}_t \right],\]

and apply the result of Question (e).

g) Apply the Itô rule to \(V_t = P(t, T_i, T_j)\hat{V}_t\) using Relation (14.40) and the result of Question (f).

h) From the expression (14.42) of the swap rate volatilities we have

\[dV_t = S_t \frac{\partial C}{\partial x} (S_t, v(t, T_i)) \times \left( \sum_{k=i}^{j-1} (T_{k+1} - T_k)P(t, T_{k+1})(\zeta_i(t) - \zeta_{k+1}(t)) + P(t, T_j)(\zeta_i(t) - \zeta_j(t)) \right) dB_t \]

\[+ \hat{V}_t dP(t, T_i, T_j) \]

\[= S_t \frac{\partial C}{\partial x} (S_t, v(t, T_i)) \times \left( \sum_{k=i}^{j-1} (T_{k+1} - T_k)P(t, T_{k+1})(\zeta_i(t) - \zeta_{k+1}(t)) + P(t, T_j)(\zeta_i(t) - \zeta_j(t)) \right) dB_t \]

\[+ \hat{V}_t \sum_{k=i}^{j-1} (T_{k+1} - T_k)\zeta_{k+1}(t)P(t, T_{k+1})dB_t \]

\[= S_t \frac{\partial C}{\partial x} (S_t, v(t, T_i)) \sum_{k=i}^{j-1} (T_{k+1} - T_k)P(t, T_{k+1})(\zeta_i(t) - \zeta_{k+1}(t))dB_t \]

\[+ \frac{\partial C}{\partial x} (S_t, v(t, T_i)) P(t, T_j)(\zeta_i(t) - \zeta_j(t))dB_t \]

\[+ \hat{V}_t \sum_{k=i}^{j-1} (T_{k+1} - T_k)\zeta_{k+1}(t)P(t, T_{k+1})dB_t \]

\[= S_t \zeta_i(t) \frac{\partial C}{\partial x} (S_t, v(t, T_i)) \sum_{k=i}^{j-1} (T_{k+1} - T_k)P(t, T_{k+1})dB_t \]

\[+ S_t \frac{\partial C}{\partial x} (S_t, v(t, T_i)) \sum_{k=i}^{j-1} (T_{k+1} - T_k)P(t, T_{k+1})\zeta_{k+1}(t)dB_t \]

\[+ \frac{\partial C}{\partial x} (S_t, v(t, T_i)) P(t, T_j)(\zeta_i(t) - \zeta_j(t))dB_t \]
\[ + \hat{V}_i \sum_{k=1}^{j-1} (T_{k+1} - T_k) \zeta_{k+1}(t) P(t, T_{k+1}) dB_t \]

\[ = S_t \zeta_i(t) \frac{\partial C}{\partial x} (S_t, v(t, T_i)) \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) dB_t \]

\[ + \left( \hat{V}_i - S_t \frac{\partial C}{\partial x} (S_t, v(t, T_i)) \right) \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) \zeta_{k+1}(t) dB_t \]

\[ + \frac{\partial C}{\partial x} (S_t, v(t, T_i)) P(t, T_j) (\zeta_i(t) - \zeta_j(t)) dB_t. \]

i) We have

\[ d\hat{V}_i = S_t \zeta_i(t) \frac{\partial C}{\partial x} (S_t, v(t, T_i)) \sum_{k=i}^{j-1} (T_{k+1} - T_k) \bar{P}(t, T_{k+1}) dB_t \]

\[ + \left( \hat{V}_i - S_t \frac{\partial C}{\partial x} (S_t, v(t, T_i)) \right) \sum_{k=i}^{j-1} (T_{k+1} - T_k) \bar{P}(t, T_{k+1}) \zeta_{k+1}(t) dB_t \]

\[ + \frac{\partial C}{\partial x} (S_t, v(t, T_i)) \bar{P}(t, T_j) (\zeta_i(t) - \zeta_j(t)) dB_t \]

\[ = (\bar{P}(t, T_i) - \bar{P}(t, T_j)) \zeta_i(t) \frac{\partial C}{\partial x} (S_t, v(t, T_i)) dB_t \]

\[ + \left( \hat{V}_i - S_t \frac{\partial C}{\partial x} (S_t, v(t, T_i)) \right) d\bar{P}(t, T_i, T_j) \]

\[ + \frac{\partial C}{\partial x} (S_t, v(t, T_i)) \bar{P}(t, T_j) (\zeta_i(t) - \zeta_j(t)) dB_t \]

\[ = (\zeta_i(t) \bar{P}(t, T_i) - \zeta_j(t) \bar{P}(t, T_j)) \frac{\partial C}{\partial x} (S_t, v(t, T_i)) dB_t \]

\[ + \left( \hat{V}_i - S_t \frac{\partial C}{\partial x} (S_t, v(t, T_i)) \right) d\bar{P}(t, T_i, T_j) \]

\[ = \frac{\partial C}{\partial x} (S_t, v(t, T_i)) d(\bar{P}(t, T_i) - \bar{P}(t, T_j)) \]

\[ + \left( \hat{V}_i - S_t \frac{\partial C}{\partial x} (S_t, v(t, T_i)) \right) d\bar{P}(t, T_i, T_j).\]

j) We have

\[ \frac{\partial C}{\partial x} (x, \tau, v) = \frac{\partial}{\partial x} \left( x \Phi \left( \frac{v}{2} + \frac{1}{v} \log \frac{x}{\kappa} \right) - \kappa \Phi \left( -\frac{v}{2} + \frac{1}{v} \log \frac{x}{\kappa} \right) \right) \]

\[ = x \frac{\partial}{\partial x} \Phi \left( \frac{v}{2} + \frac{1}{v} \log \frac{x}{\kappa} \right) - \kappa \frac{\partial}{\partial x} \Phi \left( -\frac{v}{2} + \frac{1}{v} \log \frac{x}{\kappa} \right) + \Phi \left( \frac{v}{2} + \frac{1}{v} \log \frac{x}{\kappa} \right) \]

\[ = x \frac{\partial}{\partial x} \Phi \left( \frac{v}{2} + \frac{1}{v} \log \frac{x}{\kappa} \right) - \kappa \frac{\partial}{\partial x} \Phi \left( -\frac{v}{2} + \frac{1}{v} \log \frac{x}{\kappa} \right) + \Phi \left( \frac{v}{2} + \frac{1}{v} \log \frac{x}{\kappa} \right) \]
\[
\frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2} - \frac{v}{2} \log(x/\kappa)} e^{-\frac{(v/2 + v^{-1} \log(x/\kappa))^2}{2\kappa}} + \Phi\left(\frac{\log(x/\kappa)}{v} + \frac{v}{2}\right) = \Phi\left(\frac{\log(x/\kappa)}{v} + \frac{v}{2}\right).
\]

k) We have
\[
d\tilde{V}_t = \frac{\partial C}{\partial x}[S_t, v(t, T_i)]d(\tilde{P}(t, T_i) - \tilde{P}(t, T_j)) + \left(\hat{V}_t - S_t \frac{\partial C}{\partial x}[S_t, v(t, T_i)]\right)d\tilde{P}(t, T_i, T_j)
\]
\[
= \Phi\left(\frac{\log(S_t/K)}{v(t, T_i)} - \frac{v(t, T_i)}{2}\right)d(\tilde{P}(t, T_i) - \tilde{P}(t, T_j)) - \kappa\Phi\left(\frac{\log(S_t/K)}{v(t, T_i)} - \frac{v(t, T_i)}{2}\right)d\tilde{P}(t, T_i, T_j).
\]

l) We compare the results of Questions (d) and (k).

**Chapter 15**

Exercise 15.1

a) When \(t \in [0, T_1]\) the equation reads
\[
dS_t = -\eta \lambda S_t^- dt = -\eta \lambda S_t^- dt,
\]
which is solved as \(S_t = S_0 e^{-\eta \lambda t}, \quad 0 \leq t < T_1\). Next, at the first jump time \(t = T_1\) we have
\[
\Delta S_t := S_t - S_t^- = \eta S_t^- dN_t = \eta S_t^-,
\]
which yields \(S_t = (1 + \eta)S_t^-\), hence \(S_{T_1} = (1 + \eta)S_{T_1^-} = S_0(1 + \eta) e^{-\eta \lambda T_1}\). Repeating this procedure over the \(N_t\) jump times contained in the interval \([0, t]\) we get
\[
S_t = S_0(1 + \eta)^{N_t} e^{-\lambda t}, \quad t \in \mathbb{R}_+.
\]

b) When \(t \in [0, T_1)\) the equation reads
\[
dS_t = -\eta \lambda S_t^- dt = -\eta \lambda S_t^- dt,
\]
which is solved as \(S_t = S_0 e^{-\eta \lambda t}, \quad 0 \leq t < T_1\). Next, at the first jump time \(t = T_1\) we have
\[
dS_t = S_t - S_t^- = dN_t = 1.
\]
which yields $S_t = 1 + S_{t-}$, hence $S_{T_1} = 1 + S_{T_1-} = 1 + S_0 e^{-\eta\lambda T_1}$, and for $t \in [T_1, T_2)$ we will find

$$S_t = (1 + S_0 e^{-\eta\lambda T_1}) e^{-\eta\lambda (t-T_1)}, \quad t \in [T_1, T_2).$$

More generally, the equation can be solved by letting $Y_t := e^{\eta\lambda t} S_t$ and noting that $(Y_t)_{t \in \mathbb{R}_+}$ satisfies $dY_t = e^{\lambda t} dN_t$, which has the solution

$$Y_t = Y_0 + \int_0^t e^{\eta\lambda s} dN_s, \quad t \in \mathbb{R}_+,$$

hence in general we have

$$S_t = e^{-\eta\lambda t} S_0 + \int_0^t e^{-\eta\lambda (t-s)} dN_s, \quad t \in \mathbb{R}_+,$$

Exercise 15.2

a) We have

$$X_t = \begin{cases} X_0 e^{\alpha t}, & 0 \leq t < T_1, \\ (X_0 e^{\alpha T_1} + \sigma) e^{(t-T_1)\alpha} = X_0 e^{\alpha t} + \sigma e^{(t-T_1)\alpha}, & T_1 \leq t < T_2, \\ (X_0 e^{\alpha T_1} + \sigma) (e^{(T_2-T_1)\alpha} + \sigma) e^{(t-T_2)\alpha} = X_0 e^{\alpha t} + \sigma e^{(t-T_1)\alpha} + \sigma e^{(t-T_2)\alpha}, & T_2 \leq t < T_3, \end{cases}$$

and more generally the solution $(X_t)_{t \in \mathbb{R}_+}$ can be written as

$$X_t = X_0 e^{\alpha t} + \sigma \sum_{k=1}^{N_t} e^{(t-T_k)\alpha} = X_0 e^{\alpha t} + \sigma \int_0^t e^{(t-s)\alpha} dN_s, \quad t \in \mathbb{R}_+. \tag{A.52}$$

b) Letting $f(t) := \mathbb{E}[X_t]$ and taking expectation on both sides of the stochastic differential equation $dX_t = \alpha X_t dt + \sigma dN_t$ we find

$$df(t) = \alpha f(t) dt + \sigma \lambda dt,$$

or

$$f'(t) = \alpha f(t) + \sigma \lambda.$$

Letting $g(t) = f(t) e^{-\alpha t}$ we check that

$$g'(t) = \sigma \lambda e^{-\alpha t},$$

hence
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\[ g(t) = g(0) + \int_0^t g'(s) ds = g(0) + \sigma \lambda \int_0^t e^{-\alpha s} ds = f(0) + \sigma \frac{\lambda}{\alpha} (1 - e^{-\alpha t}), \]

and

\[ f(t) = \mathbb{E}[X_t] = g(t) e^{\alpha t} = f(0) e^{\alpha t} + \sigma \frac{\lambda}{\alpha} (e^{\alpha t} - 1) = X_0 e^{\alpha t} + \sigma \frac{\lambda}{\alpha} (e^{\alpha t} - 1), \quad t \in \mathbb{R}_+. \]

We could also take the expectation on both sides of (A.52) and directly find

\[ f(t) = \mathbb{E}[X_t] = X_0 e^{\alpha t} + \sigma \lambda \int_0^t e^{(t-s)\alpha} ds = X_0 e^{\alpha t} + \sigma \frac{\lambda}{\alpha} (e^{\alpha t} - 1), \quad t \in \mathbb{R}_+. \]

Exercise 15.3

a) We have \( X_t = X_0 \prod_{k=1}^{N_t} (1 + \sigma) = X_0 (1 + \sigma)^{N_t} = (1 + \sigma)^{N_t}, \quad t \in \mathbb{R}_+. \)

b) By stochastic calculus and using the relation \( dX_t = \sigma X_t dN_t \) we have

\[
\begin{align*}
\frac{dS_t}{dt} &= d \left( S_0 X_t + r X_t \int_0^t X_s^{-1} ds \right) = S_0 dX_t + d \left( X_t \int_0^t X_s^{-1} ds \right) \\
&= S_0 dX_t + \lambda X_t \int_0^t X_s^{-1} ds + \left( \int_0^t X_s^{-1} ds \right) dX_t - r dX_t \cdot \frac{d}{dt} \left( \int_0^t X_s^{-1} ds \right) \\
&= S_0 dX_t + \lambda X_t \int_0^t X_s^{-1} ds + \left( \int_0^t X_s^{-1} ds \right) dX_t - r dX_t \cdot \left( \int_0^t X_s^{-1} ds \right) \\
&= S_0 dX_t + r dX_t + r \left( \int_0^t X_s^{-1} ds \right) dX_t = r dt + \left( S_0 + \int_0^t X_s^{-1} ds \right) dX_t \\
&= r dt + \sigma \left( S_0 X_{t^+} + X_t - \int_0^t X_s^{-1} ds \right) dN_t = r dt + \sigma S_t dN_t.
\end{align*}
\]

c) We have

\[
\mathbb{E}[X_t / X_s] = \mathbb{E}[(1 + \sigma)^{N_t - N_s}] = \sum_{k=0}^{\infty} (1 + \sigma)^k \mathbb{P}(N_t - N_s = k) \]

\[
= e^{-\lambda(t-s)} \sum_{k=0}^{\infty} (1 + \sigma)^k \frac{(\lambda(t-s))^k}{k!} = e^{-\lambda(t-s)} \sum_{k=0}^{\infty} \frac{(\lambda(1 + \sigma)(t-s))^k}{k!} \]

\[
= e^{-\lambda(t-s)} e^{\lambda(1+\sigma)(t-s)} = e^{\lambda \sigma (t-s)}, \quad 0 \leq s \leq t.
\]
\begin{align*}
\text{Exercise 15.4} \\
\text{a) Since } \mathbb{E}[N_t] = \lambda t, \text{ the expectation } \mathbb{E}[N_t - 2\lambda t] = -\lambda t \text{ is a decreasing function of } t \in \mathbb{R}_+, \text{ and } (N_t - 2\lambda t)_{t \in \mathbb{R}_+} \text{ is a supermartingale.} \\
\text{b) We have } S_t = S_0 e^{r t - \lambda \sigma t} (1 + \sigma)^N_t, \quad t \in \mathbb{R}_+. \\
\text{c) The stochastic differential equation} \\
dS_t = r S_t dt + \sigma S_t (dN_t - \lambda dt) \\
\text{contains a martingale component } (dN_t - \lambda dt) \text{ and a positive drift } r S_t dt, \text{ therefore } (S_t)_{t \in \mathbb{R}_+} \text{ is a submartingale.} \\
\text{d) Given that } \sigma > 0 \text{ we have } ((1 + \sigma)^k - 1)^+ = (1 + \sigma)^k - 1 \text{ and} \\
e^{-r T} \mathbb{E}^*[(S_T - K)^+] = e^{-r T} \mathbb{E}^*[(S_0 e^{(r - \sigma \lambda) T} (1 + \sigma)^N_T - K)^+] \\
= e^{-r T} \mathbb{E}^*[(S_0 e^{(r - \sigma \lambda) T} (1 + \sigma)^N_T - S_0 e^{(r - \lambda \sigma) T})^+] \\
= S_0 e^{-\sigma \lambda T} \mathbb{E}^*[((1 + \sigma)^N_T - 1)^+] \\
= S_0 e^{-\sigma \lambda T} \sum_{k=0}^{\infty} ((1 + \sigma)^k - 1)^+ \mathbb{P}(N_T = k) \\
= S_0 e^{-\sigma \lambda T} \sum_{k=0}^{\infty} ((1 + \sigma)^k - 1) \mathbb{P}(N_T = k) \\
= S_0 e^{-\sigma \lambda T} \sum_{k=0}^{\infty} (1 + \sigma)^k \mathbb{P}(N_T = k) - S_0 e^{-\sigma \lambda T} \sum_{k=0}^{\infty} \mathbb{P}(N_T = k) \\
= S_0 e^{-\sigma \lambda T - \lambda T} \sum_{k=0}^{\infty} \frac{\lambda (1 + \sigma)^T^k}{k!} - S_0 e^{-\sigma \lambda T} \\
= S_0 (1 - e^{-\sigma \lambda T}),
\end{align*}
where we applied the exponential identity
\[ e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \]

to \( x := \lambda(1 + \sigma)T \).

Exercise 15.5

a) For all \( k = 1, 2, \ldots, N_t \) we have
\[ X_{T_k} - X_{T_{k-1}} = a + \sigma X_{T_{k-1}}, \]
hence
\[ X_{T_k} = a + (1 + \sigma) X_{T_{k-1}}, \]
and continuing by induction we find
\[ X_{T_k} = a + a(1 + \sigma) + \cdots + a(1 + \sigma)^{k-1} + X_0(1 + \sigma)^k = a \frac{(1 + \sigma)^k - 1}{\sigma} + X_0(1 + \sigma)^k, \]
which shows that
\[ X_t = X_{T_{N_t}} = X_0(1 + \sigma)^{N_t} + a \frac{(1 + \sigma)^{N_t} - 1}{\sigma} = (1 + \sigma)^{N_t} \left(X_0 + \frac{a}{\sigma}\right) - \frac{a}{\sigma}, \quad t \in \mathbb{R}^+. \]
This result can also be obtained by noting that
\[ X_{T_k} + \frac{a}{\sigma} = (1 + \sigma) \left(X_{T_{k-1}} + \frac{a}{\sigma}\right), \quad k = 1, 2, \ldots, N_t. \]

b) We have
\[ \mathbb{E}[(1 + \sigma)^{N_t}] = e^{-\lambda t} \sum_{n=0}^{\infty} (1 + \sigma)^k \frac{(\lambda t)^k}{k!} = e^{\sigma \lambda t}, \quad t \in \mathbb{R}^+, \]
hence
\[ \mathbb{E}[X_t] = X_0 e^{\lambda t} + a \frac{e^{\lambda t} - 1}{\sigma} = e^{\lambda t} \left(X_0 + \frac{a}{\sigma}\right) - \frac{a}{\sigma}, \quad t \in \mathbb{R}^+. \]

Exercise 15.6 We have \( S_t = S_0 e^{rt} \prod_{k=1}^{N_t} (1 + \eta Z_k), \quad t \in \mathbb{R}^+. \)

Exercise 15.7 We have
\[ \text{Var}[Y_T] = \mathbb{E} \left[ \left( \sum_{k=1}^{N_T} Z_k - \mathbb{E}[Y_T] \right)^2 \right] \]
Exercise 15.8

\[
\sum_{n=0}^{\infty} \mathbb{E} \left[ \left( \sum_{k=1}^{N_T} Z_k - \lambda t \mathbb{E}[Z_1] \right)^2 \right] \mathbb{P}(N_T = k)
\]
\[
e^{-\lambda t} \sum_{n=0}^{\infty} \frac{\lambda^{n+1}}{n!} \mathbb{E} \left[ \left( \sum_{k=1}^{n} Z_k - \lambda t \mathbb{E}[Z_1] \right)^2 \right]
\]
\[
e^{-\lambda t} \sum_{n=0}^{\infty} \frac{\lambda^{n+1}}{n!} \left[ \left( \sum_{k=1}^{n} Z_k \right)^2 - 2\lambda t \mathbb{E}[Z_1] \sum_{k=1}^{n} Z_k + \lambda^2 t^2 (\mathbb{E}[Z_1])^2 \right]
\]
\[
e^{-\lambda t} \sum_{n=0}^{\infty} \frac{\lambda^{n+1}}{n!} \times \mathbb{E} \left[ 2 \sum_{1 \leq k < l \leq n} Z_k Z_l + \sum_{k=1}^{n} |Z_k|^2 - 2\lambda t \mathbb{E}[Z_1] \sum_{k=1}^{n} Z_k + \lambda^2 t^2 (\mathbb{E}[Z_1])^2 \right]
\]
\[
e^{-\lambda t} \sum_{n=0}^{\infty} \frac{\lambda^{n+1}}{n!} \times (n(n-1)(\mathbb{E}[Z_1])^2 + n \mathbb{E}[|Z_1|^2] - 2n\lambda t(\mathbb{E}[Z_1])^2 + \lambda^2 t^2 (\mathbb{E}[Z_1])^2)
\]
\[
e^{-\lambda t} (\mathbb{E}[Z_1])^2 \sum_{n=2}^{\infty} \frac{\lambda^{n+1}}{(n-2)!} + e^{-\lambda t} \mathbb{E}[|Z_1|^2] \sum_{n=1}^{\infty} \frac{\lambda^{n+1}}{(n-1)!}
\]
\[-2e^{-\lambda t}\lambda t (\mathbb{E}[Z_1])^2 \sum_{n=1}^{\infty} \frac{\lambda^{n+1}}{(n-1)!} + \lambda^2 t^2 (\mathbb{E}[Z_1])^2)
\]
\[= \lambda t \mathbb{E}[|Z_1|^2],
\]

or, using the moment generating function of Proposition 15.6,

\[
\text{Var}[Y_T] = \mathbb{E}[|Y_T|^2] - (\mathbb{E}[Y_T])^2
\]
\[
= \frac{\partial^2}{\partial \alpha^2} \mathbb{E}[e^{\alpha Y_T}]\big|_{\alpha=0} - \lambda^2 t^2 (\mathbb{E}[Z_1])^2
\]
\[
= \lambda t \int_{-\infty}^{\infty} |y|^2 \mu(dy) = \lambda t \mathbb{E}[|Z_1|^2].
\]

Exercise 15.8

a) Applying the Itô formula (15.18) to the function \( f(x) = e^x \) and to the process \( X_t = \mu t + \sigma W_t + Y_t \), we find

\[
dS_t = \left( \mu + \frac{1}{2} \sigma^2 \right) S_t dt + \sigma S_t dW_t + (S_t - S_{t-}) dN_t
\]
\[
= \left( \mu + \frac{1}{2} \sigma^2 \right) S_t dt + \sigma S_t dW_t + (S_0 e^{\mu t + \sigma W_t + Y_t} - S_0 e^{\mu t + \sigma W_t + Y_{t-}}) dN_t
\]
\[= \left( \mu + \frac{1}{2} \sigma^2 \right) S_t dt + \sigma S_t dW_t + (S_0 e^{\mu t + \sigma W_t + Y_t - Z_{N_t}} - e^{\mu t + \sigma W_t + Y_t -}) dN_t \]

\[= \left( \mu + \frac{1}{2} \sigma^2 \right) S_t dt + \sigma S_t dW_t + S_t (e^{Z_{N_t}} - 1) dN_t, \]

hence the jumps of \( S_t \) are given by the sequence \( (e^{Z_k} - 1)_{k \geq 1} \).

b) The discounted process \( e^{-rt} S_t \) satisfies

\[ d(e^{-rt} S_t) = e^{-rt} \left( \mu - r + \frac{1}{2} \sigma^2 \right) S_t dt + \sigma e^{-rt} S_t dW_t + e^{-rt} S_t (e^{Z_{N_t}} - 1) dN_t. \]

Hence by the Girsanov Theorem 15.12, choosing \( u, \tilde{\lambda}, \tilde{\nu} \) such that

\[ \mu - r + \frac{1}{2} \sigma^2 = \sigma u - \tilde{\lambda} \mathbb{E}_\tilde{\nu}[e^{Z1} - 1], \]

shows that

\[ d(e^{-rt} S_t) = \sigma e^{-rt} S_t (dW_t + u dt) + e^{-rt} S_t (e^{Z_{N_t}} - 1) dN_t - \tilde{\lambda} \mathbb{E}_\tilde{\nu}[e^{Z1} - 1] dt \]

is a martingale under \( (\mathbb{P}_u, \tilde{\lambda}, \tilde{\nu}) \).

Exercise 15.9

a) We have

\[ S_t = S_0 e^{\mu t} \prod_{k=1}^{N_t} (1 + Y_k) = S_0 \exp \left( \mu t + \sum_{k=1}^{N_t} X_k \right), \quad t \in \mathbb{R}_+. \]

b) We have

\[ e^{-rt} S_t = S_0 \exp \left( (\mu - r) t + \sum_{k=1}^{N_t} X_k \right), \quad t \in \mathbb{R}_+, \]

which is a martingale if

\[ 0 = \mu - r + \lambda \mathbb{E}[Y_k] = \mu - r + \lambda \mathbb{E}[e^{X_k} - 1] = \mu - r + \lambda (e^{\sigma^2/2} - 1). \]

c) We have

\[ e^{-(T-t)r} \mathbb{E}[(S_T - \kappa)^+ | S_t] \]

\[= e^{-(T-t)r} \mathbb{E} \left[ \left( S_0 \exp \left( \mu T + \sum_{k=1}^{N_T} X_k \right) - \kappa \right)^+ | S_t \right]. \]
Exercise 15.10

a) We have

\[ d(e^{\alpha t} S_t) = \sigma e^{\alpha t} (dN_t - \beta dt), \]

hence

\[ e^{\alpha t} S_t = S_0 + \sigma \int_0^t e^{\alpha s} (dN_s - \beta d), \]

and \( \mu = r + \lambda (1 - e^{\sigma^2/2}). \)
and
\[ S_t = S_0 e^{-\alpha t} + \sigma \int_0^t e^{-(t-s)\alpha} (dN_s - \beta ds), \quad t \in \mathbb{R}_+. \tag{A.53} \]

b) We have
\[
 f(t) = \mathbb{E}[S_t] = S_0 e^{-\alpha t} + \sigma \mathbb{E} \left[ \int_0^t e^{-(t-s)\alpha} dN_s \right] - \beta \sigma \int_0^t e^{-(t-s)\alpha} ds
 = S_0 e^{-\alpha t} + \lambda \sigma \int_0^t e^{-(t-s)\alpha} ds - \beta \sigma \int_0^t e^{-(t-s)\alpha} ds
 = S_0 e^{-\alpha t} + \sigma \left( \lambda - \beta \right) \frac{1 - e^{-\alpha t}}{\alpha}
 = \sigma \frac{\lambda - \beta}{\alpha} + \left( S_0 + \sigma \frac{\beta - \lambda}{\alpha} \right) e^{-\alpha t}, \quad t \in \mathbb{R}_+.
\]

c) By rewriting (A.53) as
\[
 S_t = S_0 - \alpha S_0 \int_0^t e^{-(t-s)\alpha} ds + \sigma \int_0^t e^{-(t-s)\alpha} (dN_s - \beta ds)
 = S_0 + \sigma \int_0^t e^{-(t-s)\alpha} (dN_s - (\beta + \alpha S_0 / \sigma) ds)
 = S_0 + \sigma \int_0^t e^{-(t-s)\alpha} (\lambda - \beta - \alpha S_0 / \sigma) ds + \sigma \int_0^t e^{-(t-s)\alpha} (dN_s - \lambda ds),
\]
t \in \mathbb{R}_+, we check that the process \((S_t)_{t \in \mathbb{R}_+}\) is a submartingale, provided that \(\lambda - \beta - \alpha S_0 / \sigma \geq 0\), i.e. \(S_0 + \sigma (\beta - \lambda) / \alpha \leq 0\). We also check that this condition makes the expectation \(f(t) = \mathbb{E}[S_t]\) decreasing in Question (b).

d) Since, given that \(N_T = n\) the jump times \((T_1, T_2, \ldots, T_n)\) of the Poisson process \((N_t)_{t \in \mathbb{R}_+}\) are independent uniformly distributed random variables over \([0, T]\), hence we can write
\[
 \mathbb{E}[\phi(S_T)] = \mathbb{E} \left[ \phi \left( S_0 e^{-\alpha T} + \sigma \int_0^T e^{-(T-s)\alpha} (dN_s - \beta ds) \right) \right]
 = \sum_{n=0}^{\infty} \mathbb{P}(N_T = n) \mathbb{E} \left[ \phi \left( S_0 e^{-\alpha T} + \sigma \int_0^T e^{-(T-s)\alpha} (dN_s - \beta ds) \right) \bigg| N_T = n \right]
 = e^{-\lambda T} \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!} \mathbb{E} \left[ \phi \left( S_0 e^{-\alpha T} + \sigma \sum_{k=1}^n e^{-(T-T_k)\alpha} - \sigma \beta \int_0^T e^{-(T-s)\alpha} ds \right) \bigg| N_T = n \right]
 = e^{-\lambda T} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_0^T \cdots \int_0^T \phi \left( S_0 e^{-\alpha T} + \sigma \sum_{k=1}^n e^{-(T-s_k)\alpha} - \sigma \beta \frac{1 - e^{-\alpha T}}{\alpha} \right) ds_1 \cdots ds_k,
\]

Exercise 15.11
a) From the decomposition $Y_t - \lambda t(t + \mathbb{E}[Z_1]) = Y_t - \lambda \mathbb{E}[Z_1]t - \lambda t^2$ as the sum of a martingale and a decreasing function, we conclude that $t \mapsto Y_t - \lambda t(t + \mathbb{E}[Z_1])$ is a supermartingale.

b) Writing
\[ dS_t = \mu S_t dt + \sigma S_t dY_t \]
\[ = r S_t dt + \sigma S_t \left( dY_t - \frac{r - \mu}{\sigma} dt \right) \]
\[ = r S_t dt + \sigma S_t \left( dY_t - \tilde{\lambda} \mathbb{E}[Z_1] dt \right), \quad 0 \leq t \leq T, \]
we conclude that $(S_t)_{t \in [0, T]}$ is a martingale under $\mathbb{P}_{\tilde{\lambda}}$ provided that
\[ \frac{\mu - r}{\sigma} = -\tilde{\lambda} \mathbb{E}[Z_1] dt, \]
\[ i.e. \quad \tilde{\lambda} = \frac{r - \mu}{\sigma \mathbb{E}[Z_1]}. \]

We note that $\tilde{\lambda} < 0$ if $\mu < r$, hence in this case there is no risk-neutral probability measure and the market admits arbitrage opportunities as the risky asset always overperforms the risk-free rate $r$.

c) We have
\[ e^{-(T-t)r} \mathbb{E}_{\tilde{\lambda}}[S_T - \kappa | \mathcal{F}_t] = e^{rt} \mathbb{E}_{\tilde{\lambda}}[e^{-rT} S_T | \mathcal{F}_t] - K e^{-(T-t)r} \]
\[ = S_t - K e^{-(T-t)r}, \]
since $(S_t)_{t \in [0, T]}$ is a martingale under $\mathbb{P}_{\tilde{\lambda}}$.

Exercise 15.12

a) We have
\[ S_t = S_0 e^{\mu t} \prod_{k=1}^{N_t} (1 + Z_k), \quad t \in \mathbb{R}_+. \]

b) We have
\[ e^{-rt} S_t = S_0 \exp \left( (\mu - r)t + \sum_{k=1}^{N_t} X_k \right), \quad t \in \mathbb{R}_+, \]
which is a martingale if
\[ 0 = \mu - r + \lambda \mathbb{E}[Z_k] = \mu - r + \lambda \int_{-\infty}^{\infty} z \mu(dz). \]

c) We have
\[ e^{-(T-t)r} \mathbb{E}[(S_T - \kappa)^+ | S_t] \]
\[ = e^{-(T-t)r} \mathbb{E} \left[ \left( S_0 e^{\mu T} \prod_{k=1}^{N_T} Z_k - \kappa \right)^+ | S_t \right] \]
\[ = e^{-(T-t)r} \sum_{n=0}^{\infty} \mathbb{E} \left[ \left( S_t e^{\mu(T-t)} \prod_{k=N_t+1}^{N_T} Z_k - \kappa \right)^+ | S_t \right] \mathbb{P}(N_T - N_t = n) \]
\[ = e^{-(r+\lambda)(T-t)} \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n}{n!} \]
\[ \times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( S_t e^{\mu(T-t)} \prod_{k=1}^{n} z_k - \kappa \right)^+ \mu(dz_1) \cdots \mu(dz_n). \]

**Exercise 15.13**

a) The discounted price process \((e^{-rt}S_t)_{t \in [0,T]}\) is a martingale, hence it is also a submartingale and a supermartingale.

b) The discounted price process \((e^{-rt}S_t)_{t \in [0,T]}\) is a supermartingale.

c) The discounted price process \((e^{-rt}S_t)_{t \in [0,T]}\) is a submartingale.

d) Under the probability measure \(\tilde{\mathbb{P}}_\lambda\), the discounted price process \((e^{-rt}S_t)_{t \in [0,T]}\) is a martingale, hence it is also a submartingale and a supermartingale.

**Chapter 16**

**Exercise 16.1**

a) We have \(\mathbb{E}[N_t - \alpha t] = \mathbb{E}[N_t] - \alpha t = \lambda t - \alpha t\), hence \(N_t - \alpha t\) is a martingale if and only if \(\alpha = \lambda\). Given that
\[ d(e^{-rt}S_t) = \eta e^{-rt}S_t (dN_t - \alpha dt), \]
we conclude that the discounted price process \(e^{-rt}S_t\) is a martingale if and only if \(\alpha = \lambda\).

b) Since we are pricing under the risk-neutral probability measure we take \(\alpha = \lambda\). Next, we note that
\[ S_T = S_0 e^{(r-\eta\lambda)T} (1 + \eta)^{N_T} = S_t e^{(r-\eta\lambda)(T-t)} (1 + \eta)^{N_T - N_t}, \quad 0 \leq t \leq T, \]
hence the price at time \(t\) of the option is
e^{-r(T-t)} \mathbb{E}[[S_T]^2 \mid \mathcal{F}_t]
= e^{-r(T-t)} \mathbb{E}[[S_t]^2 e^{2(r-\eta \lambda)(T-t)} (1 + \eta)^{2(N_T-N_t)} \mid \mathcal{F}_t]
= |S_t|^2 e^{(r-2\eta \lambda)(T-t)} \mathbb{E}[(1 + \eta)^{2(N_T-N_t)} \mid \mathcal{F}_t]
= |S_t|^2 e^{(r-2\eta \lambda)(T-t)} \mathbb{E}[(1 + \eta)^{2(N_T-N_t)}]
= |S_t|^2 e^{(r-2\eta \lambda)(T-t)} \sum_{n=0}^{\infty} (1 + \eta)^{2n} \mathbb{P}(N_T - N_t = n)
= |S_t|^2 e^{(r-2\eta \lambda - \lambda)(T-t) + (1+\eta)^2 \lambda(T-t)}
= |S_t|^2 e^{(r+\eta^2 \lambda)(T-t)}, \quad 0 \leq t \leq T.

Exercise 16.2

a) Regardless of the choice of a particular risk-neutral probability measure
$\mathbb{P}_{u, \tilde{\lambda}, \tilde{\nu}}$, we have

e^{-r(T-t)} \mathbb{E}_{u, \tilde{\lambda}, \tilde{\nu}}[S_T - K \mid \mathcal{F}_t] = e^{rt} \mathbb{E}_{u, \tilde{\lambda}, \tilde{\nu}}[e^{-rT} S_T \mid \mathcal{F}_t] - K e^{-r(T-t)}
= e^{rt} e^{-rt} S_t - K e^{-r(T-t)}
= S_t - K e^{-r(T-t)}
= f(t, S_t),

for

\[ f(t, x) = x - K e^{-r(T-t)}, \quad t, x > 0. \]

b) Clearly, holding one unit of the risky asset and shorting a (possibly fractional) quantity
$K e^{-rT}$ of the risk-free asset will hedge the payoff $S_T - K$, and this (static) hedging strategy is self-financing because it is constant in time.

c) Since $\frac{\partial f}{\partial x}(t, x) = 1$ we have

\[ \xi_t = \frac{\sigma^2 \frac{\partial f}{\partial x}(t, S_t^-) + \frac{a\tilde{\lambda}}{S_t^-} (f(t, S_t^- (1 + a)) - f(t, S_t^-))}{\sigma^2 + a^2 \tilde{\lambda}} \]
\[ \xi_t = \frac{\sigma^2 + \frac{a\tilde{\lambda}}{S_t^-} (S_t^- (1 + a) - S_t^-)}{\sigma^2 + a^2 \tilde{\lambda}} \]
\[ = 1, \quad 0 \leq t \leq T, \]

which coincides with the result of Question (b).

Exercise 16.3

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a) We have
\[ S_t = S_0 \exp \left( \mu t + \sigma B_t - \frac{1}{2} \sigma^2 t \right) (1 + \eta)^{N_t}. \]

b) We have
\[ \tilde{S}_t = S_0 \exp \left( (\mu - r) t + \sigma B_t - \frac{1}{2} \sigma^2 t \right) (1 + \eta)^{N_t}, \]
and
\[ d\tilde{S}_t = (\mu - r + \lambda \eta) \tilde{S}_t dt + \eta \tilde{S}_t \left( dN_t - \lambda dt \right) + \sigma \tilde{S}_t dW_t, \]

hence we need to take
\[ \mu - r + \lambda \eta = 0, \]
since the compensated Poisson process \((N_t - \lambda t)_{t \in \mathbb{R}^+}\) is a martingale.

c) We have
\[
e^{-r(T-t)} \mathbb{E}^* \left[ (S_T - \kappa)^+ | S_t \right] \\
e^{-r(T-t)} \mathbb{E}^* \left[ \left( S_0 \exp \left( \mu T + \sigma B_T - \frac{1}{2} \sigma^2 T \right) (1 + \eta)^{N_T - \kappa} \right)^+ | S_t \right] \\
e^{-r(T-t)} \mathbb{E}^* \left[ \left( S_t e^{\mu(T-t)} + \sigma(B_T - B_t) - \sigma^2(T-t)/2 (1 + \eta)^{n} - \kappa \right)^+ | S_t \right] \\
e^{-r(\lambda + \mu)(T-t)} \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^{n}}{n!} \\
\times \mathbb{E}^* \left[ \left( S_t e^{(r-\lambda \eta)(T-t)} + \sigma(B_T - B_t) - \sigma^2(T-t)/2 (1 + \eta)^{n} - \kappa \right)^+ | S_t \right] \\
e^{-\lambda(T-t)} \sum_{n=0}^{\infty} \frac{\text{Bi}(S_t e^{-\lambda \eta(T-t)} (1 + \eta)^n, r, \sigma^2, T - t, \kappa)}{n!} \frac{(\lambda(T-t))^{n}}{n!} \\
e^{-\lambda(T-t)} \sum_{n=0}^{\infty} \left( S_t e^{-\lambda \eta(T-t)} (1 + \eta)^n \Phi(d_+) - \kappa e^{-r(T-t)} \Phi(d_-) \right) \frac{(\lambda(T-t))^{n}}{n!}, \]
with
\[
d_+ = \frac{\log(S_t e^{-\lambda \eta(T-t)} (1 + \eta)^n / \kappa) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \\
= \frac{\log(S_t (1 + \eta)^n / \kappa) + (r - \lambda \eta + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}, \]

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and
\[
\begin{align*}
\log(S_t e^{-\lambda T} (1 + \eta^n) / \kappa) &+ (r - \sigma^2 / 2)(T - t) \\
&= \frac{\log(S_t (1 + \eta^n) / \kappa) + (r - \lambda \eta - \sigma^2 / 2)(T - t)}{\sigma \sqrt{T - t}}.
\end{align*}
\]

Exercise 16.4  

a) The discounted process \( \tilde{S}_t = e^{-rt} S_t \) satisfies the equation  
\[
d \tilde{S}_t = Y_{N_t} \tilde{S}_t - dN_t,
\]
and it is a martingale since the compound Poisson process \( Y_{N_t} dN_t \) is centered with independent increments as \( \mathbb{E}[Y_1] = 0 \).

b) We have  
\[
S_T = S_0 e^{rT} \prod_{k=1}^{N_T} (1 + Y_k),
\]

hence  
\[
e^{-rT} \mathbb{E}[(S_T - \kappa)^+] = e^{-rT} \mathbb{E}\left[\left( S_0 e^{rT} \prod_{k=1}^{N_T} (1 + Y_k) - \kappa \right)^+ \right]
\]
\[
= e^{-rT} \sum_{n=0}^{\infty} \mathbb{E}\left[\left( S_0 e^{rT} \prod_{k=1}^{n} (1 + Y_k) - \kappa \right)^+ \right] \mathbb{P}(N_T = n)
\]
\[
= e^{-rT - \lambda T} \sum_{k=0}^{\infty} \frac{(\lambda T)^n}{2^n n!} \int_{-1}^{1} \cdots \int_{-1}^{1} \left( S_0 e^{rT} \prod_{k=1}^{n} (1 + y_k) - \kappa \right)^+ dy_1 \cdots dy_n.
\]

Exercise 16.5  

a) We find \( \alpha = \lambda \) where \( \lambda \) is the intensity of the Poisson process \( (N_t)_{t \in \mathbb{R}_+} \).

b) We have  
\[
e^{-r(T-t)} \mathbb{E}[S_T - \kappa \mid \mathcal{F}_t] = e^{rt} \mathbb{E}[e^{-rT} S_T \mid \mathcal{F}_t] - e^{-r(T-t)} \mathbb{E}[\kappa \mid \mathcal{F}_t]
\]
\[
= e^{rt} \mathbb{E}[e^{-rT} S_t \mid \mathcal{F}_t] - e^{-r(T-t)} \kappa
\]
\[
= S_t - e^{-r(T-t)} \kappa,
\]
since the process \( (e^{-rt} S_t)_{t \in \mathbb{R}_+} \) is a martingale.
Exercise 16.6

a) We have

\[ dV_t = df(t, S_t) = r\eta e^{rt} dt + \xi_t dS_t \]
\[ = r\eta e^{rt} dt + \xi_t(rS_t dt + \alpha S_t (dN_t - \lambda dt)) \]
\[ = rV_t dt + \alpha \xi_t S_t (dN_t - \lambda dt) \]
\[ = r f(t, S_t) dt + \alpha \xi_t S_t (dN_t - \lambda dt). \]

b) We apply the Itô formula with jumps and the martingale property of \( t \mapsto e^{rt} f(t, S_t) \) to get

\[ df(t, S_t) = r f(t, S_t) dt \]
\[ + (f(t, S_t-(1+\alpha)) - f(t, S_t))dN_t - \lambda (f(t, S_t(1+\alpha)) - f(t, S_t)) dt, \]

and we identify the terms in the above formula with those appearing in (16.18).

Background on Probability Theory

Exercise A.1 We have

\[ \mathbb{E}[X] = \sum_{k=0}^{\infty} k \mathbb{P}(X = k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \]
\[ = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} = \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \lambda. \]

Exercise A.2 We have

\[ \mathbb{P}(e^X > c) = \mathbb{P}(X > \log c) = \int_{\log c}^{\infty} e^{-y^2/(2\eta^2)} \frac{dy}{\sqrt{2\pi} \eta^2} \]
\[ = \int_{(\log c)/\eta}^{\infty} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}} = 1 - \Phi((\log c)/\eta) = \Phi(-(\log c)/\eta). \]

Exercise A.3

a) If \( \mu = 0 \) we have

\[ \mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) dx = \frac{1}{\sqrt{2\pi} \sigma^2} \int_{-\infty}^{\infty} x e^{-x^2/(2\sigma^2)} dx \]
\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{-y^2/2} dy = \frac{1}{\sqrt{2\pi}} \lim_{A \to +\infty} \int_{-A}^{A} y e^{-y^2/2} dy = 0, \]
by symmetry of the function \( y \mapsto y e^{-y^2/2} \). Note that we have

\[
\int_{-\infty}^{\infty} |y| e^{-y^2/2} \, dy = \lim_{A \to +\infty} \int_{-A}^{A} |y| e^{-y^2/2} \, dy = 2 \lim_{A \to +\infty} \int_{0}^{A} y e^{-y^2/2} \, dy
\]

\[
= -2 \lim_{A \to +\infty} \left[ e^{-y^2/2} \right]_0^A = 2 \lim_{A \to +\infty} (1 - e^{-A^2/2}) = 2 < \infty,
\]

hence the function \( y \mapsto y e^{-y^2/2} \) is integrable on \( \mathbb{R} \) and the above computation of \( \mathbb{E}[X] \) is valid. Next, for all \( \mu \in \mathbb{R} \) we have

\[
\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) \, dx = \frac{1}{\sqrt{2\pi \sigma^2}} \int_{-\infty}^{\infty} x e^{-(x-\mu)^2/(2\sigma^2)} \, dx
\]

\[
= \frac{1}{\sqrt{2\pi \sigma^2}} \int_{-\infty}^{\infty} (y + \mu) e^{-y^2/(2\sigma^2)} \, dy
\]

\[
= \frac{1}{\sqrt{2\pi \sigma^2}} \int_{-\infty}^{\infty} y e^{-y^2/(2\sigma^2)} \, dy + \frac{\mu}{\sqrt{2\pi \sigma^2}} \int_{-\infty}^{\infty} e^{-y^2/(2\sigma^2)} \, dy
\]

\[
= \frac{\mu}{\sqrt{2\pi \sigma^2}} \int_{-\infty}^{\infty} e^{-y^2/(2\sigma^2)} \, dy = \mu \int_{-\infty}^{\infty} f(y) \, dy = \mu \mathbb{P}(X \in \mathbb{R}) = \mu.
\]