Boundary Integral Equation for 
Axisymmetric Potential Problem 
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The boundary integral equation for the three-dimensional potential problem is given in Chapter 6 of the book “A Beginner’s Course in Boundary Element Methods” as

$$\lambda(\xi, \eta, \zeta) \phi(\xi, \eta, \zeta) = \int_S (\phi(x, y, z) \frac{\partial}{\partial n}[\Phi_{3D}(x, y, z; \xi, \eta, \zeta)] - \Phi_{3D}(x, y, z; \xi, \eta, \zeta) \frac{\partial}{\partial n}[\phi(x, y, z)]) \, ds(x, y, z),$$  \hspace{1cm} (1)

where $\phi$ satisfies the three-dimensional Laplace’s equation in the region $R$ bounded by a closed surface $S$, $\lambda(\xi, \eta, \zeta)$ is defined by

$$\lambda(\xi, \eta, \zeta) = \begin{cases} 
0 & \text{if } (\xi, \eta, \zeta) \notin R \cup S, \\
1/2 & \text{if } (\xi, \eta, \zeta) \text{ lies on a smooth part of } S, \\
1 & \text{if } (\xi, \eta, \zeta) \in R,
\end{cases}$$  \hspace{1cm} (2)

and $\Phi_{3D}$ is the fundamental solution given by

$$\Phi_{3D}(x, y, z; \xi, \eta, \zeta) = -\frac{1}{4\pi \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}}.$$  \hspace{1cm} (3)

Let us now consider the axisymmetric case in which the surface $S$ of the solution domain can be generated by rotating a curve $\Gamma$ about the $z$-axis by an angle of $360^\circ$. For example, if $S$ is the sphere $x^2 + y^2 + (z-2)^2 = 1$ (sphere of center $(0, 0, 2)$ and radius 1) then we can generate the surface $S$ by rotating the semi-circle $x^2 + (z-2)^2 = 1, x \geq 0$, on the $Oxz$ plane by an angle of $360^\circ$ about the $z$-axis. For such a surface $S$ and for $\phi$ which does not change with the polar coordinate $\theta$ but only with $r$ and $z$, that is, for an axisymmetric problem, the boundary integral over $S$ in (1) can be reduced to an integral over the curve $\Gamma$ as explained below. (In cylindrical polar coordinates, points can be described using $(r, \theta, z)$ instead of $(x, y, z)$, where $x = r \cos \theta$ and $y = r \sin \theta$.)

Firstly, let us define

$$\phi^*(r, \theta, z) = \phi(r \cos \theta, y \sin \theta, z)$$
$$p^*(r, \theta, z) = \left. \frac{\partial}{\partial n}[\phi(x, y, z)] \right|_{(x, y, z) = (r \cos \theta, y \sin \theta, z), \quad (n_x, n_y) = (r \cos \theta - n_y \sin \theta, r \sin \theta + n_y \cos \theta)}$$
where \( n_r \) and \( n_{\theta} \) are respectively the \( r \) and \( \theta \) components of the outward unit normal vector to the surface \( S \) as explained below. In Cartesian coordinates, the normal vector is given by \([n_x,n_y,n_z]\).

For an axisymmetric problem, \( \phi^* \) is independent of \( \theta \) and we can write \( \phi^*(r,z) \). We will show now that, for axisymmetric problem, \( p^* \) also depends only on \( r \) and \( z \).

We have:

\[
\frac{\partial}{\partial n}[\phi(x,y,z)] = n_x \frac{\partial \phi}{\partial x} + n_y \frac{\partial \phi}{\partial y} + n_z \frac{\partial \phi}{\partial z}
\]

\[
= n_x \left( \frac{\partial \phi^*}{\partial r} + \frac{\partial \phi^*}{\partial \theta} \frac{\partial \theta}{\partial x} \right) + n_y \left( \frac{\partial \phi^*}{\partial r} + \frac{\partial \phi^*}{\partial \theta} \frac{\partial \theta}{\partial y} \right) + n_z \frac{\partial \phi^*}{\partial z}.
\]

If we introduce a local (polar) coordinate system with base vectors \( e_r, e_{\theta}, e_z \), then the unit normal vector is given by \( n_r e_r + n_{\theta} e_{\theta} + n_z e_z \). On a fixed plane \( z = c \) (constant), if the body is axisymmetric, the components \( n_r, n_{\theta} \) and \( n_z \) do not change with \( \theta \), but \( n_x \) and \( n_y \) change with \( \theta \). It may be shown that

\[
n_x = n_r \cos \theta - n_{\theta} \sin \theta
\]

\[
n_y = n_r \sin \theta + n_{\theta} \cos \theta.
\]

It follows that (for axisymmetric problem)

\[
p^* = (n_r \cos \theta - n_{\theta} \sin \theta)[\cos \theta \frac{\partial \phi^*}{\partial r} - \sin \theta \frac{\partial \phi^*}{\partial \theta}] + (n_r \sin \theta + n_{\theta} \cos \theta)[\sin \theta \frac{\partial \phi^*}{\partial r} + \cos \theta \frac{\partial \phi^*}{\partial \theta}] + n_z \frac{\partial \phi^*}{\partial z}
\]

\[
= n_r \frac{\partial \phi^*}{\partial r} + n_{\theta} \frac{\partial \phi^*}{\partial \theta} + n_z \frac{\partial \phi^*}{\partial z}.
\]

Since \( \phi^*, n_r \) and \( n_z \) are independent of \( \theta \), we find that \( p^* = n_r \frac{\partial \phi^*}{\partial r} + n_z \frac{\partial \phi^*}{\partial z} \) is also independent of \( \theta \).

For convenience, we will now drop the asterisk \( * \) and write \( \phi^*(r,z) \) as merely \( \phi(r,z) \) and \( p^*(r,z) \) as \( p(r,z) \).

For \( S \) which is symmetrical about the \( z \)-axis, the infinitesimal area \( ds(x,y,z) \) in (1) can be written as

\[
ds(x,y,z) = r \, dl \, d\theta.
\]
where \( d\ell \) is the length of an infinitesimal portion of the curve \( \Gamma \).

Consider now (1) for axisymmetric potential problem. For a point \((\xi, \eta, \zeta) = (r_0 \cos \theta_0, r_0 \sin \theta_0, z_0)\) on the \( Oxz \) plane (where \( y = 0 \) or \( \theta_0 = 0 \)), we can rewrite (1) as

\[
\lambda(r_0, z_0) \phi(r_0, z_0) = \int_S \left( \phi(r, z) \frac{\partial}{\partial n} \Phi_{3D}(r \cos \theta, r \sin \theta, z; r_0, 0, z_0) \right) - \Phi_{3D}(r \cos \theta, r \sin \theta, z; r_0, 0, z_0) p(r, z) \, r \, d\ell \, d\theta, \tag{5}
\]

where \( \lambda(r_0, z_0) = 1/2 \) if \((r_0, z_0)\) lies on a smooth part of \( \Gamma \) and \( \lambda(r_0, z_0) = 1 \) if \((r_0, z_0)\) lies in the interior of the solution domain on the \( Oxz \) plane.

We need to integrate with respect to \( \theta \) from 0 to \( 2\pi \) as the complete surface \( S \) is obtained by rotating \( \Gamma \) by an angle of \( 360^\circ \). Note that \( \theta \) appears only in the function \( \Phi_{3D} \) and not in \( \phi(r, z) \). The integration involving the coordinates \( r \) and \( z \) (that is, with respect to \( \ell \)) is over the curve \( \Gamma \). Thus, we can rewrite (5) as

\[
\lambda(r_0, z_0) T(r_0, z_0) = \int_\Gamma \left( T(r, z) \Psi_{axis}(r, z; r_0, z_0; n_r, n_z) - \Phi_{axis}(r, z; r_0, z_0) p(r, z) \right) r \, d\ell (r, z), \tag{6}
\]

where

\[
\Phi_{axis}(r, z; r_0, z_0) = \int_0^{2\pi} \Phi_{3D}(r \cos \theta, r \sin \theta, z; r_0, 0, z_0) \, d\theta = -\frac{1}{4\pi} \sqrt{r^2 + r_0^2 + (z - z_0)^2 - 2rr_0 \cos \theta} \, d\theta
\]

\[
= -\frac{1}{\pi} \sqrt{\frac{r^2 + r_0^2 + (z - z_0)^2}{r_0^2 + (z - z_0)^2}} \int_0^{2\pi} \frac{1}{4 \sqrt{1 - \frac{2rr_0(1+\cos \theta)}{r_0^2 + (z - z_0)^2 + 2rr_0}}} \, d\theta
\]

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\[
\Phi_{\text{axis}}(r, z; r_0, z_0) = -\frac{1}{\pi \sqrt{r^2 + r_0^2 + (z - z_0)^2 + 2rr_0}} \int_0^{\pi} \frac{1}{2\sqrt{1 - \frac{2rr_0(1+\cos(2t))}{r^2 + r_0^2 + (z - z_0)^2 + 2rr_0}}} dt
\]

\[
= -\frac{1}{\pi \sqrt{r^2 + r_0^2 + (z - z_0)^2 + 2rr_0}} \int_0^{\pi/2} \frac{1}{2\sqrt{1 - \frac{4rr_0 \cos^2(t)}{r^2 + r_0^2 + (z - z_0)^2 + 2rr_0}}} dt
\]

\[
= -\frac{1}{\pi \sqrt{r^2 + r_0^2 + (z - z_0)^2 + 2rr_0}} \int_0^{\pi/2} \frac{1}{2\sqrt{1 - \frac{4rr_0 \sin^2(t)}{r^2 + r_0^2 + (z - z_0)^2 + 2rr_0}}} dt.
\]

If we define the function \(K(m)\) as

\[
K(m) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - m \sin^2(t)}} dt,
\]

then we can write

\[
\Phi_{\text{axis}}(r, z; r_0, z_0) = -\frac{1}{\pi \sqrt{r^2 + r_0^2 + (z - z_0)^2 + 2rr_0}} K\left(\frac{4rr_0}{r^2 + r_0^2 + (z - z_0)^2 + 2rr_0}\right). \quad (7)
\]

In mathematics, \(K\) is a special function and is called the complete elliptic integral of the first kind. There is a simple approximate but accurate formula in Abramowitz and Stegun’s Handbook of Mathematical Functions for evaluating \(K(m)\). Some mathematical softwares may have inbuilt functions for calculating \(K(m)\). Note that \(0 \leq \frac{4rr_0}{r^2 + r_0^2 + (z - z_0)^2 + 2rr_0} \leq 1\) and \(K(m)\) is undefined for \(m = 1\).
Also, for axisymmetric body, we have:

\[
\Psi_{axis}(r, z; r_0, z_0; n_r, n_z) = 2\pi \int_0^{2\pi} \frac{\partial}{\partial n} \left[ \Phi_{3D}(r \cos \theta, r \sin \theta, z; r_0, 0, z_0) \right] d\theta
\]

\[
= \int_0^{2\pi} \left( n_r \frac{\partial}{\partial r} \left[ \Phi_{3D}(r \cos \theta, r \sin \theta, z; r_0, 0, z_0) \right] + \frac{1}{r} n_\theta \frac{\partial}{\partial \theta} \left[ \Phi_{3D}(r \cos \theta, r \sin \theta, z; r_0, 0, z_0) \right] + n_z \frac{\partial}{\partial z} \left[ \Phi_{3D}(r \cos \theta, r \sin \theta, z; r_0, 0, z_0) \right] \right) d\theta
\]

(9)

For the axisymmetric body, can you see why \( n_\theta = 0 \)?

There is this relationship:

\[
\frac{d}{dm}(K(m)) = \frac{1}{2m} (E(m) - K(m)),
\]

(10)

where

\[
E(m) = \int_0^{\pi/2} \sqrt{1 - m \sin^2 \theta} d\theta.
\]

(11)

Note that \( E(m) \) is known as the complete elliptic integral of the second kind. Details on computing \( E(m) \) are available in Abramowitz and Stegun.

From (9) and (10), it can be shown that

\[
\Psi_{axis}(r, z; r_0, z_0; n_r, n_z) = -\frac{1}{\pi \sqrt{r^2 + r_0^2 + (z - z_0)^2 + 2rr_0}}
\]

\[
\times \left[ \frac{n_r}{2r^2} \left( \frac{r^2 - r_0^2 + (z - z_0)^2}{4rr_0} \right) \left( \frac{4rr_0}{r^2 + r_0^2 + (z - z_0)^2 + 2rr_0} \right) \right]
\]

\[
+ \frac{n_z}{r^2 + r_0^2 + (z - z_0)^2 + 2rr_0} \frac{z_0 - z}{4rr_0} \right] \}
\]

(12)
If we use (6) to devise a boundary element method for solving the axisymmetric potential problem, we have to discretize the curve $\Gamma$ on the $rz$ space (that is, the $Oxz$ plane). The curve $\Gamma$ can be discretized into straight line elements. The $k$-th typical element which has endpoints $(r^{(k)}, z^{(k)})$ and $(r^{(k+1)}, z^{(k+1)})$