A Novel Method for Analysis of Soliton Propagation in Optical Fibers

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Abstract—We have proposed a new technique to analyze the propagation behavior of solitons in an optical fiber. This technique is based on the Fourier series analysis technique (FSAT) which can consider fiber loss and third-order dispersion. Using this technique, the propagation of solitons in an optical fiber has been investigated. The results compare well with the well-known split-step Fourier method (SSFM). We found that for a given computational accuracy, in order to obtain the same results, we need fewer sampling points for FSAT compared with the SSFM.

I. INTRODUCTION

The concept of solitary waves was first introduced in 1834 by Russell [1] after he had observed that a water wave preserved its original shape (about a foot in height) over a very long distance in a Scottish canal. About two decades ago, Hasegawa and Tappert [2] showed that, theoretically, optical solitons can be formed in a dielectric fiber because the wave envelope satisfies the nonlinear Schrödinger equation. However, at that time, neither low loss fibers nor good measuring equipment was available, so their theory could not be demonstrated experimentally. In 1980, Mollenauer [3] was the first to demonstrate successfully the propagation of solitons in an optical fiber. Optical solitons have enormous potential in long haul communication systems [4] because they can be stable over a very long propagation distance and will permit wavelength multiplexing (transmission of data at two or more wavelengths simultaneously). In 1991, Nakazawa [5] demonstrated the remarkable stability of the optical soliton for 1 million km of transmission. Solitons are stable against perturbations due to local inhomogeneity and external disturbance (e.g., noises) [6], so they can be utilized as a reliable basis for high bit-rate transmission systems. In late 1991, Mollenauer [7] showed "error-free" transmission of solitons at 25 Gb/s for as long as 14,000 km. The soliton phenomenon can be used to generate a soliton laser, for signal compression and for switching purposes [8]. Recently, Nakazawa [9] demonstrated the stable transmission of optical signal at 10 Gb/s over 180 million km (in 15 min), and suggested that error-free transmission over unlimited distance is possible. In optical fibers, solitons are generally referred to as envelope solitons. The propagation of envelope solitons in a lossless fiber can be described by the well-known nonlinear Schrödinger equation [10]. The fiber loss and chromatic dispersion are the main obstacles that affect the propagation of stable soliton pulses [11]. The inverse scattering method [12] and perturbation techniques [13] are two methods available to study the propagation of solitons in an optical fiber. The inverse scattering method (ISM) gives an analytic solution to the propagation of soliton in a lossless optical fiber. If the fiber loss is to be considered, then an exact analytic solution cannot be obtained by ISM. Hence, a perturbation technique should be used to take the effect of fiber loss into account. However, the ISM and perturbation technique have their limitations. For higher order solitons, the complexity of finding the solution in the ISM will be greatly increased, so that a large amount of time will be spent on tedious calculations and substitutions for the final solution. On the other hand, the perturbation technique can give reasonable results if the normalized loss factor $\Gamma$ is smaller than 0.015. For a conventional fiber with fiber loss 0.2 dB/km and a practical input soliton pulse width of 6 ps, the value of the loss factor is calculated to be 0.04. This value of $\Gamma$ is obviously much larger than 0.015; hence, the perturbation technique cannot be applied in this practical case.

In this paper, we propose a novel numerical technique, based on Fourier series analysis, to analyze the propagation behavior of solitons in an optical fiber taking into account of both the fiber loss and the third-order dispersion. This Fourier series analysis technique (FSAT) is also compared with the commonly used split-step Fourier method (SSFM) [14]–[15].

II. ANALYSIS

The general soliton equation which includes second- and third-order dispersions, fiber nonlinearity, and loss can be expressed as follows [16]:

\[
\frac{\partial u}{\partial x} = \frac{1}{2} \frac{\partial^2 u}{\partial T^2} + B \frac{\partial^3 u}{\partial T^3} + j|u|^2 u - \Gamma u \tag{1a}
\]

where

\[
B = \frac{\beta_3}{6|\beta_3|} |\alpha| \tag{1b}
\]

\[
\Gamma = \frac{\alpha t_0^2}{2 |\beta_3|} \tag{1c}
\]

\[
\beta_i = \frac{d^i \beta}{d \omega^i} \quad (i = 2, 3) \tag{1d}
\]

In the above equations, $u(x, T)$ is the normalized complex amplitude of the soliton pulse, $x$ represents the normalized distance along the direction of propagation, $T$ represents the
normalized time, $\beta$ is the propagation constant, $B$ is the third-order dispersion coefficient, $\Gamma$ is the normalized loss factor, $\alpha$ is the fiber loss, and $t_0$ is the $1/e$ width of the initial pulse. The first term on the right-hand side of (1a) is the nonlinear term, whereas the second and third terms are, respectively, the second- and third-order group velocity dispersions. The last term is the attenuation term corresponding to the fiber loss. At the zero dispersion wavelength, $\beta_2 \approx 0$, so that (1b) and (1c) are not suitable for normalization purposes. In this case, another normalization method is used, and similar nonlinear partial differential equation can be obtained (see part B of the Appendix). In order to solve (1a), we express $u(x, T)$ in terms of Fourier series as follows:

$$u(x, T) = \sum_{n=-N}^{N} \hat{u}_n(x) \exp(jn\epsilon T)$$  \hspace{1cm} (2)

where $\hat{u}_n(x)$ and $\epsilon$ are the Fourier amplitude coefficient and the fundamental frequency, respectively. The partial derivatives of $u(x, T)$ with respect to $x$ and $T$ can be expressed as follows:

$$\frac{\partial}{\partial x} u(x, T) = \sum_{n=-N}^{N} \frac{\partial \hat{u}_n(x)}{\partial x} \exp(jn\epsilon T)$$  \hspace{1cm} (3)

$$\frac{\partial^2}{\partial T^2} u(x, T) = \sum_{n=-N}^{N} -n^2 \alpha^2 \hat{u}_n(x) \exp(jn\epsilon T)$$  \hspace{1cm} (4)

$$\frac{\partial^3}{\partial T^3} u(x, T) = \sum_{n=-N}^{N} -jn^3 \alpha^3 \hat{u}_n(x) \exp(jn\epsilon T).$$  \hspace{1cm} (5)

Substituting (2)–(5) into (1) results in

$$\sum_{n=-N}^{N} \frac{\partial \hat{u}_n(x)}{\partial x} \exp(jn\epsilon T)$$

$$= -\frac{1}{2} j \sum_{n=-N}^{N} n^2 \epsilon^2 \hat{u}_n(x) \exp(jn\epsilon T)$$

$$+ j \sum_{n=-N}^{N} \Psi_n(x) \exp(jn\epsilon T) - jB \sum_{n=-N}^{N} n^3 \epsilon^3 \hat{u}_n(x)$$

$$\cdot \exp(jn\epsilon T) - \Gamma \sum_{n=-N}^{N} \hat{u}_n(x) \exp(jn\epsilon T)$$

$$\hat{u}_n(x) = \sum_{\mu=-N}^{N} \Psi_{\mu}(x) \exp(j\mu \epsilon T)$$

$$= \sum_{\lambda=-N}^{N} \hat{u}_{\lambda}(x) \exp(j\lambda \epsilon T)$$

$$\hat{u}_n(x) = \sum_{\mu=-N}^{N} \Psi_{\mu}(x) \exp(j\mu \epsilon T)$$

$$= \sum_{\lambda=-N}^{N} \hat{u}_{\lambda}(x) \exp(j\lambda \epsilon T)$$

where

$$\sum_{n=-N}^{N} \Psi_n(x) \exp(jn\epsilon T) = \left[ \sum_{n=-N}^{N} \hat{u}_n(x) \exp(jn\epsilon T) \right]^2$$

$$\cdot \sum_{n=-N}^{N} \hat{u}_n(x) \exp(jn\epsilon T).$$  \hspace{1cm} (6b)

It is easy to prove that the exponential coefficients $\exp(jn\epsilon T)$ in (6) are orthogonal to each other over the range of time $(-\pi/\epsilon < T < \pi/\epsilon)$. For any integer $k (-N \leq k \leq N)$, if we multiply both sides of (6a) by the conjugate of $\exp(jk\epsilon T)$ and integrate the whole equation with respect to $T$ over the interval $(-\pi/\epsilon, \pi/\epsilon)$, a set of first-order partial differential equations can be obtained as follows:

$$\frac{\partial \hat{u}_k(x)}{\partial x} = -\frac{1}{2} j n^2 \epsilon^2 \hat{u}_k(x) + j \Psi_k(x) - jBk^3 \epsilon^3 \hat{u}_k(x) - \Gamma \hat{u}_k(x)$$

with

$$k = -N, -N + 1, \cdots, N.$$  \hspace{1cm} (7)

As a result, we can prove that making use of the orthogonality properties of the exponential terms $\exp(jn\epsilon T)$, $2N + 1$ first-order partial differential equations can be obtained as

$$\frac{\partial \hat{u}_N(x)}{\partial x} = -\frac{1}{2} j N^2 \epsilon^2 \hat{u}_N(x) + j \Psi_N(x)$$

$$- jB N^3 \epsilon^3 \hat{u}_N(x) - \Gamma \hat{u}_N(x)$$

$$\vdots$$

$$\frac{\partial \hat{u}_0(x)}{\partial x} = j \Psi_0(x) - \Gamma \hat{u}_0(x)$$

$$\vdots$$

$$\frac{\partial \hat{u}_{-N}(x)}{\partial x} = -\frac{1}{2} j N^2 \epsilon^2 \hat{u}_{-N}(x) + j \Psi_{-N}(x)$$

$$+ jB N^3 \epsilon^3 \hat{u}_{-N}(x) - \Gamma \hat{u}_{-N}(x).$$  \hspace{1cm} (8)

If we define $\sigma(n)$ as

$$\sigma(n) = \frac{n^2 \epsilon^2}{2} + BN^3 \epsilon^3$$

then for $(-N \leq n \leq N)$, (7) reduces to

$$\frac{\partial \hat{u}_n(x)}{\partial x} = -j \sigma(n) \hat{u}_n(x) + j \Psi_n(x) - \Gamma \hat{u}_n(x).$$  \hspace{1cm} (10)

On the other hand, for the nonlinear term, we have

$$\sum_{n=-N}^{N} \Psi_n(x) \exp(jn\epsilon T) = \left( \sum_{\mu=-N}^{N} \hat{u}_\mu(x) \exp(j\mu \epsilon T) \right)$$

$$\cdot \left( \sum_{\lambda=-N}^{N} \hat{u}_\lambda(x) \exp(j\lambda \epsilon T) \right)$$

where the function $\hat{u}^*$ is the complex conjugate of $\hat{u}$ and parameters $\mu, \nu$, and $\lambda$ are all integers between $-N$ and $N$. In order that (12) stands, the values of $\mu, \nu$, and $\lambda$ should satisfy the following condition:

$$\mu - \nu + \lambda = n$$  \hspace{1cm} (12)
where \( n \) is an integer \((-N \leq n \leq N)\). Under this circumstance, (11) can be further simplified as

\[
\Psi_n(x) = \sum_{\nu \mu - \nu + \lambda = n} \hat{u}_\mu(x) \hat{u}_\nu^*(x) \hat{u}_\lambda(x).
\] (13)

Substituting (13) into (10), we have

\[
\frac{\partial \hat{u}_n(x)}{\partial x} = \left[ -j \sigma(n) - \Gamma \right] \hat{u}_n(x) + j \sum_{\nu \mu - \nu + \lambda = n} \hat{u}_\mu(x) \hat{u}_\nu^*(x) \hat{u}_\lambda(x)
\] (14)

where all the functions \( \hat{u}(x) \) are complex numbers and can be separated into real and imaginary parts as follows:

\[
\hat{u}_n(x) = \hat{u}_{nR}(x) + j \hat{u}_{nI}(x)
\]

\[
\hat{u}_\mu(x) = \hat{u}_{\mu R}(x) + j \hat{u}_{\mu I}(x)
\]

\[
\hat{u}_\nu^*(x) = \hat{u}_{\nu R}(x) - j \hat{u}_{\nu I}(x)
\]

\[
\hat{u}_\lambda(x) = \hat{u}_{\lambda R}(x) + j \hat{u}_{\lambda I}(x).
\] (15)

For \(-N \leq n \leq N\), we will have \(4N+2\) first-order partial differential equations (i.e., \(2N+1\) equations for real parts and \(2N+1\) equations for imaginary parts). These are

\[
\frac{\partial \hat{u}_{nR}(x)}{\partial x} = \sigma(n) \hat{u}_{nI}(x) + \Gamma \hat{u}_{nR}(x)
\] (linear term)

\[
+ \text{Re} \left( j \sum_{\nu \mu - \nu + \lambda = n} \hat{u}_\mu(x) \hat{u}_\nu^*(x) \hat{u}_\lambda(x) \right)
\] (nonlinear term)

...\[
\frac{\partial \hat{u}_{0R}(x)}{\partial x} = \sigma(0) \hat{u}_{0I}(x) + \Gamma \hat{u}_{0R}(x)
\] (linear term)

\[
+ \text{Re} \left( j \sum_{\nu \mu - \nu + \lambda = 0} \hat{u}_\mu(x) \hat{u}_\nu^*(x) \hat{u}_\lambda(x) \right)
\] (nonlinear term)

\[
\frac{\partial \hat{u}_{-N}(x)}{\partial x} = \sigma(-N) \hat{u}_{-NI}(x) + \Gamma \hat{u}_{-NR}(x)
\] (linear term)

\[
+ \text{Re} \left( j \sum_{\nu \mu - \nu + \lambda = -N} \hat{u}_\mu(x) \hat{u}_\nu^*(x) \hat{u}_\lambda(x) \right)
\] (nonlinear term)

where

\[
\text{Im} \left( j \sum_{\nu \mu - \nu + \lambda = -N} \hat{u}_\mu(x) \hat{u}_\nu^*(x) \hat{u}_\lambda(x) \right)
\]

\[
= \sum_{\nu \mu - \nu + \lambda = -N} \left( [\hat{u}_\mu R(x) \hat{u}_\nu R(x) + \hat{u}_\mu I(x) \hat{u}_\nu I(x)] \right)
\]

\[
+ (\hat{u}_\mu R(x) \hat{u}_\nu I(x) - \hat{u}_\mu I(x) \hat{u}_\nu R(x)) \hat{u}_\lambda(x). \] (17a)

In order to solve the above equations, initial conditions are required. Considering an initial pulse \( u(0, T) \), its Fourier series
can be written as

\[ u(0, T) = \sum_{n=-N}^{N} \hat{u}_n(0) \exp(jnnT). \]  

(18)

Multiplying both sides of (18) by the exponential term \( \exp(-jmmT) \), where \( m \) is an integer \(-N \leq m \leq N\), gives

\[ u(0, T) \exp(-jmmT) = \sum_{n=-N}^{N} \hat{u}_n(0) \exp(j(n-m)T). \]  

(19)

Integrating both sides of the above equation from \(-\pi/\varepsilon\) to \(\pi/\varepsilon\) results in

\[
\int_{-\pi/\varepsilon}^{\pi/\varepsilon} u(0, T) \exp(-jmmT) dT
= \sum_{n=-N}^{N} \hat{u}_n(0) \int_{-\pi/\varepsilon}^{\pi/\varepsilon} \exp[j\varepsilon T(n-m)] dT
= \begin{cases} 
0 & \text{for } m \neq n \\
\frac{2\pi}{\varepsilon} \hat{u}_m(0) & \text{for } m = n.
\end{cases}
\]

(20)

From (20), we can generate \(2N+1\) initial conditions as follows:

\[ \hat{u}_m(0) = \frac{\varepsilon}{2\pi} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} u(0, T) \exp(-jmmT) dT. \]  

(21)

In general, both \(u(0, T)\) and \(\hat{u}_m(0)\) are complex functions; hence, they can be expressed as follows:

\[ \hat{u}_m(0) = \hat{u}_{mR}(0) + j\hat{u}_{mI}(0) \]  

(22)

\[ u(0, T) = u_R(0, T) + ju_I(0, T) \]  

(23)

where \(\hat{u}_{mR}(0)\) is the real part and is given by

\[
\hat{u}_{mR}(0) = \text{Re} \left( \frac{\varepsilon}{2\pi} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} u(0, T) \exp(-jmmT) dT \right)
= \frac{\varepsilon}{2\pi} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} \left[ u_R(0, T) \cos(mmT) + u_I(0, T) \sin(mmT) \right] dT
\]

(24)

and \(\hat{u}_{mI}(0)\) is the imaginary part given by

\[
\hat{u}_{mI}(0) = \text{Im} \left( \frac{\varepsilon}{2\pi} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} u(0, T) \exp(-jmmT) dT \right)
= \frac{\varepsilon}{2\pi} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} \left[ u_I(0, T) \cos(mmT) - u_R(0, T) \sin(mmT) \right] dT.
\]

(25)

Consequently, we will have \(4N+2\) initial conditions altogether.

Using the Fourier series technique and making use of the orthogonality of the exponential terms, we can transform the nonlinear soliton equation (1a) into another form of expression that is easier to solve by numerical procedures. After transformation, a system \((2N+1)\) of first-order partial differential equations is obtained. Since all the variables involved are complex, before any numerical method can be applied, we need to further divide the set of partial differential equations into real and imaginary parts. As a result, we come up with two sets of equations (16) and (17) where all the parameters involved are real. From here, we will need to apply numerical procedures to solve the \(4N+2\) first-order partial differential equations obtained above. A quite stable method, called the Runge–Kutta method [17], is employed to deal with these problems. With the initial conditions obtained from (24), the Runge–Kutta method can be used to solve the \(4N+2\) equations above.

The Runge–Kutta method is efficient in terms of total time unless the number of sampling points is very large. The fourth-order Runge–Kutta method is employed to solve the partial differential equations obtained in (16) and (17) because it has the advantages of simplicity and is an efficient method for many routine calculations. The fourth-order Runge–Kutta method requires four evaluations per step, and is more accurate than the lower order Runge–Kutta methods which take fewer evaluations per step. There is no fifth-order Runge–Kutta method which involves five equations per step. The reason is that the complexity of the higher order derivatives increases rapidly with order, so that in the case of fifth order, the number of terms to be matched exceeds the number of parameters available. In order to apply this fourth-order Runge–Kutta method to solve the system of partial differential equations above, we need to know the \(2N+1\) initial conditions. Based on these initial conditions, we can use the Runge–Kutta method to calculate the propagation behavior of a soliton pulse. If we define \("h"\) as the step size of each iteration, \("i"\) as an integer between \(-N\) and \(N\), and \("x_j"\) as the \(j\)th mesh point of \(x\) (i.e., \(0 + jh\) in our case), then the general fourth-order Runge–Kutta method can be expressed as below:

\[ u_i(x_{j+1}) = u_i(x_j) + \frac{1}{6} \left( K_1,i + 2K_{2,i} + K_{3,i} + K_{4,i} \right) \]  

(26)

where

\[ K_{1,i} = h f_1[x_j, u_{-N}(x_j), u_{-N+1}(x_j), \ldots, u_N(x_j)] \]
\[ K_{2,i} = h f_1 \left[ x_j + \frac{h}{2}, u_{-N+1}(x_j) + \frac{1}{2} K_{1,N} \right] \]
\[ K_{3,i} = h f_1 \left[ x_j + \frac{h}{2}, u_{-N}(x_j) + \frac{1}{2} K_{2,N} \right] \]
\[ K_{4,i} = h f_1 \left[ x_j + \frac{h}{2}, u_{-N+1}(x_j) + \frac{1}{2} K_{3,N} \right] \]

The basic idea of the above expression is to calculate a new set of values for \(u_i(x + h)\) each time the distance parameter \(x\) moves a step forward of size \(h\). After a new set of values of \(u_i(x + h)\) has been obtained, we can then move another step forward to obtain another set of values \(u_i(x + 2h)\). Similarly, following this procedure, we can obtain the
propagation behavior of the inputting signal. The Runge–Kutta method is employed because it is stable (small errors do not get amplified) and it can deal with nonlinear equations. There is another method, called the predictor–corrector method, which has the major advantage over the Runge–Kutta method of requiring only two computations of \( u(x) \) per step (instead of four evaluations in the Runge–Kutta method). Unfortunately, the predictor–corrector method as originally developed is unstable; small roundoff or truncation errors can easily get amplified as they propagate.

III. RESULTS

Since our following results are compared with the SSFM, a brief description of this technique is given in part A of the Appendix. Fig. 1 shows the three-dimensional propagation of the fundamental soliton pulse over one soliton period of an optical fiber. In the figure, \( Z \) is defined as \( z/z_0 \) where \( z \) is the soliton propagation distance along the fiber and \( z_0 \) is the soliton period [18]. In the calculations, we used (1a) without third-order dispersion and fiber loss. Unlike pulse propagation in the linear fiber regime, both the soliton power and soliton pulse width remain unchanged as the soliton propagates along the fiber. This is because the self-phase modulation (SPM) and group velocity dispersion (GVD) effects have totally cancelled each other, and this happens only in a lossless case. Fig. 2 shows a comparison between the results obtained by the FSAT (solid line) and the SSFM (solid circles) after propagating a soliton pulse for ten soliton periods. A good agreement between the two techniques is obtained.

Fig. 3 shows both the 3-D and 2-D plots of a fundamental soliton propagation in a lossy fiber when \( B = 0 \) and \( \Gamma = 0.037 \) (i.e., an input pulse with \( t_{\text{FWHM}} = 1.76z_0 = 10 \) ps and a conventional fiber with \( \alpha = 0.2 \) dB/km and \( \beta_2 = -20 \) ps²/km). As the pulse propagates along the fiber, its amplitude decreases gradually due to the fiber loss. The pulse with the highest peak power is the input signal, and the one with the smallest peak power is the signal received at \( Z = 10 \) (i.e., after it has propagated ten soliton periods). As can be observed from this figure, the pulse width broadens significantly after it propagates over the distance of ten soliton periods. As the soliton peak power decreases along the fiber due to the loss, the nonlinear effect is no longer strong enough to cancel the group velocity dispersion effects. Thus, unlike the lossless case, the dispersion effect is dominant. The numerical results obtained are also compared with those obtained by the SSFM. This is shown in Fig. 4 for \( Z = 5 \) and 10, respectively, which shows a good agreement between the two methods. Fig. 5 shows both the 3-D and 2-D plots of soliton pulse propagation along the optical fiber for two soliton periods when \( B = 0 \) and \( \Gamma = 0.37 \) (i.e., an input pulse with \( t_{\text{FWHM}} = 10 \) ps and a dispersion-shifted fiber with \( \alpha = 0.2 \) dB/km and \( \beta_2 = -2 \) ps²/km). As can be observed from the figures, the soliton pulse amplitude decreases rapidly along the fiber and to about half of its input power after propagating half a soliton period. The pulse width broadening is less significant in this case, and the peak power of the input signal decreases so rapidly that the signal vanishes before any observable pulse width broadening can be seen. Fig. 6 shows a comparison between FSAT and the commonly used SSFM, and again a good agreement is obtained.

For silica fibers, the zero dispersion wavelength (ZDWL) occurs at about 1.3 \( \mu \)m. It can be shifted to 1.55 \( \mu \)m by appropriate design modifications in order to take advantage of the minimum fiber loss occurring at this wavelength. If the pulse wavelength is close to the ZDWL, \( \beta_2 \) becomes very small and \( \beta_3 \) provides the dominant contribution to the chromatic dispersion effects. For ultrashort pulses with \( t_0 < 0.1 \) ps, it is often necessary to include the \( \beta_3 \) term because the ratio of the spectral width \( \Delta \omega \) to the carrier frequency \( \omega_0 \) is no longer small enough to make the third-order term negligible. For a dispersion-shifted fiber \( \beta_2 \approx -2 \) ps²/km; if we assume \( t_0 = 0.08 \) ps and \( \beta_3 = 0.1 \) ps³/km, then from
Fig. 3. The 3-D and 2-D plots of the fundamental soliton propagation over ten soliton periods in a conventional optical fiber ($B = 0, \Gamma = 0.037$).

Fig. 5. The 3-D and 2-D plots of the fundamental soliton propagation over two soliton periods in a dispersion-shifted optical fiber ($B = 0, \Gamma = 0.37$).

Fig. 4. The soliton waveforms obtained after a propagation distance of ten soliton periods in a lossy fiber by (a) the Fourier series analysis technique, and (b) the split-step Fourier method. The initial pulse and conditions are the same as Fig. 3.

Fig. 6. The soliton waveforms obtained after a propagation distance of two soliton periods in a dispersion-shifted fiber by (a) the Fourier series analysis technique, and (b) the split-step Fourier method. The initial pulse and conditions are the same as Fig. 5.

(1b), $B = 0.1$. Fig. 7 shows the 3-D and 2-D plots for the propagation of soliton in a lossy fiber with third-order dispersion ($\beta_3 = 0.1 \text{ ps}^3/\text{km}$ and $\alpha = 0.2 \text{ dB/km}$). As shown in the figure, the introduction of the third-order dispersion term distorts the pulse shape, making the soliton asymmetric and forming a small ripple at the trailing edge of the pulse. The
peak power position is observed to be shifted towards the positive time axis. This shift in the peak power is due to the group velocity $v_g$ of the pulse changes ($T = (t - z/v_g)/t_0$). Without the third-order dispersion term, the input soliton pulse is expected to be constant in shape as it propagates along a lossless fiber (as shown in Fig. 1). A comparison with the SSFM is given in Fig. 8, and shows a very good agreement. If the initial pulse width $t_0$ is further decreased to 0.01 ps, then the effect of the third-order dispersion term will be more significant, as shown in Fig. 9 for $\beta_3 = 0.1$ ps$^3$/km and $\alpha = 0.2$ dB/km. A continuous pulse broadening is observed, and oscillatory tails are formed at the trailing edge as the pulse propagates along the fiber. The comparison with the SSFM is given in Fig. 10, and shows a good agreement.

Fig. 11 is produced for the case where $\beta_2 = 0$ ps$^2$/km (at ZDWL), $\beta_3 = 0.1$ ps$^3$/km, and $t_0 = 1$ ps. The pulse amplitude is observed to decrease exponentially along the fiber, and the amplitude of the oscillatory tail formed at the trailing edge of the pulse is also observed to decrease significantly along the fiber. A general peak amplitude shift towards the positive time axis is also observed. The numerical results show that a significant pulse distortion due to the third-order dispersion can occur at the zero dispersion wavelength, which will then limit the performance of an optical fiber communication system.

Fig. 12 shows a comparison with the SSFM, and a good agreement is obtained.
IV. COMPUTATIONAL CONSIDERATIONS

In order to compare the CPU time of both SSFM and FSAT, we fixed the value of ε to 0.314 so that sampling points are always taken between the time interval of \((-10 < T < 10)\). Also, the accuracy for computation in the z direction is fixed to \(10^{-6}\) for both models. The minimum number of sampling points required to reproduce a fundamental soliton after propagating one soliton period is found for both models, as shown in Fig. 13. Using the FSAT, 31 sampling points are found to be enough to calculate the propagation of a fundamental soliton in a lossless fiber over one soliton period. However, using the SSFM, the same graph can only be obtained by taking a minimum of 201 sampling points. Using the IBM3090 mainframe computer, the CPU time used for each model has been compared and is listed in Table I. Even though the fast Fourier transform (FFT) used in the SSFM is very efficient, the FSAT is found to be faster for 31 sampling points. This is because the FSAT works entirely in the frequency domain along the z direction, and only needs to transform back to the time domain at the end of the transmission. This is done by simply using (18). Since the accuracy for computation in the z direction is fixed to \(10^{-6}\) for both models, the step size in the z direction should be kept reasonably small for the SSFM. Therefore, the computational load of the FFT to transform sampling points between the time and frequency domains is very heavy, and this is the reason that more computational time is required. We have also taken the effect of fiber loss into consideration, and have found that the numerical result obtained by 31-point FSAT is the same as that obtained by the SSFM using 201 sampling points. Further comparison of both models for a longer distance has been carried out, and similar results were obtained as listed in Table I. With a fixed accuracy along the fiber length, the CPU time required for the SSFM is almost proportional to the propagation distance, whereas the CPU time required for the FSAT is shown to be more efficient as the propagation distance increases. We have also considered the effects of both fiber loss and the third-order dispersion, and have found that in order to get the same results, the minimum numbers of sampling points required to solve the equation using FSAT and SSFM techniques were 41 and 201, respectively. This again confirms that fewer sampling points are required in FSAT as compared with the SSFM. However, there is a drawback with the FSAT; as the number of sampling points increases, the number of first-order partial differential equations in the Fourier domain increases too, and hence more computing time will be required for the fourth-order RKM to solve the problem. The reason that we have used the RKM is because of its stability and accuracy. When the computational time is a matter of concern, the use of lower order RKM or the Adams method may be helpful.

There are several advantages of the proposed technique. 1) the number of sampling points required by the FSAT is less than that required by the SSFM in order to obtain smooth and accurate results. 2) The nonlinear partial differential equation is not necessary to be analytically solvable before or after splitting. 3) The technique is very efficient, in particular when the number of sampling points required is not too large. 4) The FSAT can handle whatever the SSFM can handle with ease of implementation and understanding. 5) It works entirely in
the frequency domain, and the cumulative errors due to the heavy use of FFT in the SSFM can be avoided. b) Using the FSAT, there is no need to employ other numerical methods to solve the higher order nonlinear terms (e.g., finite difference method). Using SSFM, we may well need to do so if one or both split equations cannot be solved analytically. Obviously, the additional numerical methods will introduce extra errors, more CPU time, and greater difficulties in the control of the accuracy.

V. CONCLUSION

In this paper, various soliton propagation characteristics have been analyzed using the Fourier series analysis technique.
From the initial condition of $u(x, \omega)$ at $x = 0$, the equation above can be solved as follows:

$$u(x, \omega) = u(0, \omega) \exp \left( -\frac{\omega^2}{2} x + B\omega^3 x \right).$$  \hspace{1cm} (A5)

**Solving the Nonlinear Part:** The nonlinear part can be solved analytically in the time domain. First, consider the equation below:

$$\frac{\partial |u|^2}{\partial x} = \frac{\partial uu^*}{\partial x} = u \frac{\partial u^*}{\partial x} + u^* \frac{\partial u}{\partial x}. \hspace{1cm} (A6)$$

Taking the conjugate on both sides of (A2), we obtain

$$\frac{\partial u^*}{\partial x} = -j|u|^2 u^* - \Gamma^* u^*. \hspace{1cm} (A7)$$

As a result, (A6) can be simplified as follows:

$$\frac{\partial |u|^2}{\partial x} = -2\Gamma |u|^2. \hspace{1cm} (A8)$$

From the initial condition of $u(0, T)$, the equation can be solved as below:

$$|u(x, T)|^2 = |u(0, T)|^2 \exp(-2\Gamma x). \hspace{1cm} (A9)$$

Substituting (A9) into (A2) and using the initial condition of $u(0, T)$ the following general solution can be obtained:

$$u(x, T) = u(0, T) \exp \left[ -\frac{j|u(0, T)|^2}{2\Gamma} \left( \exp(-2\Gamma x) - 1 - \Gamma x \right) \right]. \hspace{1cm} (A10)$$

If the fiber loss is ignored, then the loss factor becomes zero; thus, we have the solution as follows:

$$u(x, T) = u(0, T) \exp[j|u(0, T)|^2 x]. \hspace{1cm} (A11)$$

The fast Fourier transform (FFT) is an algorithm that can compute the discrete Fourier transform much more rapidly than other available algorithms, and this is used heavily in the SSFM.

**B. Soliton Equation for Zero Dispersion Case (i.e., $\beta_2 = 0$)**

At the zero dispersion wavelength, the second-order dispersion term is zero. Thus, we have the following soliton propagation equation [16]:

$$\frac{\partial A}{\partial z} = \frac{1}{6} \frac{\partial^2 A}{\partial \omega^3} \frac{\partial^2 A}{\partial T^3} + j\gamma |A|^2 A - \frac{\alpha}{2} A \hspace{1cm} (B1)$$

where $A$ is the soliton amplitude and $\gamma$ is the nonlinear coefficient. This equation can be expressed in terms of a normalized wave amplitude $\bar{u}$, a normalized time $\bar{T}$, and a normalized distance $\bar{x}$ by using the following transformations:

$$\bar{u} = \frac{N A}{\sqrt{P_N}} \hspace{1cm} (B2)$$

$$\bar{T} = \frac{T}{t_0} \hspace{1cm} (B3)$$

$$\bar{x} = \frac{z}{z_1^*} \hspace{1cm} (B4)$$

with

$$P_N = \frac{|\beta_3| N^2}{\gamma t_0^2} \hspace{1cm} (B5)$$

$$z_1^* = \frac{t_0^3}{|\beta_3|} \hspace{1cm} (B6)$$

where $P_N$ is the peak power of the soliton pulse, $N$ is related to the order of soliton, $t_0$ is the (1/e) input pulse width, and $z_1^*$ is the normalization factor of distance. Substituting the above equations (B2)–(B6) into (B1), the following nonlinear propagation equation, which describes the propagation of solitons at the zero dispersion wavelength, can be obtained:

$$\frac{\partial \bar{u}}{\partial \bar{T}} = \frac{1}{6} \frac{\partial^2 \bar{u}}{\partial \bar{T}^3} + j|\bar{u}|^2 \bar{u} - \Gamma^* \bar{u} \hspace{1cm} (B7)$$
where the loss factor $\Gamma'$ is defined as

$$\Gamma' = \frac{\alpha}{2 |\beta_s|}.$$  \hspace{1cm} (B8)

The second term of (B7) refers to the third-order dispersion, the third term refers to the nonlinear SPM, and the last term corresponds to the fiber loss.

REFERENCES


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