Brief Paper

Iterative learning control with initial rectifying action☆

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Abstract

This paper addresses the initial shift problem in iterative learning control with system relative degree. The tracking error caused by nonzero initial shift is detected when applying a conventional learning algorithm. Finite initial rectifying action is introduced in the learning algorithm and is shown effective in the improvement of tracking performance, in particular robustness with respect to variable initial shifts. The uniform convergence of the output trajectory to a desired one jointed smoothly with a specified transient trajectory from the starting position is ensured in the presence of fixed initial shift. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The research on iterative learning control (ILC) started from the midst of the 1980s and most of early work focused on the convergence analysis of learning algorithms. The fundamental characteristics were examined for systems with direct transmission term (Sugie & Ono, 1991) and relative degree (Ahn, Choi, & Kim, 1993), respectively. ILC implementations involve perturbed initial conditions, as well as state disturbances and measurement noises. A number of efforts are also observed toward the robustness analysis. A common assumption in these analyses is that the initial condition at each cycle is reset to the desired one (Ahn et al., 1993; Arimoto, Kawamura, & Miyazaki, 1984; Hauser, 1987; Sugie & Ono, 1991), or within its neighborhood (Arimoto, Naniwa, & Suzuki, 1991; Bondi, Casalino, & Gambardella, 1988; Heinzinger, Fenwick, Paden, & Miyazaki, 1992; Saab, 1994). In case of perturbed initial conditions, boundedness of the tracking error is established and the error bound is shown to be proportional to the bound on initial condition errors.

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Up to now, increasing attention has been devoted to relax the requirement (Lee & Bien, 1996; Park & Bien, 2000; Park, Bien, & Hwang, 1999; Porter & Mohamed, 1991; Wang & Cheah, 1998; Xu & Qu, 1998). In Lee and Bien (1996), for a PD-type ILC, the initial condition is required to keep the same for all cycles but different from the desired one or inside a neighborhood of any fixed point. The extended result to PID-type ILC and a general treatment can be found in Park and Bien (2000) and Park, Bien, and Hwang (1999). Benefitting from the relaxation, better tracking performance can be achieved in the face of perturbed initial conditions, where different input updates based on distinguishing the regions of the measured initial conditions result in smaller error bound. Learning algorithms in Lee and Bien (1996), Park and Bien (2000) and Park et al. (1999), however, are only applicable to systems with relative degree one. In Porter and Mohamed (1991), the problem was addressed for partially irregular LTI systems with rank-defective Markov parameters. Initial impulsive action is shown effective to totally eliminate the effect caused by fixed initial shift and enables zero-error trajectory tracking over entire tracking interval. However, the use of an impulsive action may be not practical. In Sun and Wang (1999), initial rectifying action was introduced as an alternative to avoid the impulsive action, also for partially irregular LTI systems. The initial rectifying action is finite and implementable and ensures
uniform convergence of the system output to the desired trajectory, jointed smoothly with a transient trajectory from starting position. This paper is motivated by the work (Ahn et al., 1993) on ILC with system relative degree. An initial rectifying action is utilized to address the same problem based on the properties of well-defined relative degree nonlinear systems. The tracking error caused by fixed initial shift is detected when the conventional learning algorithm (Ahn et al., 1993) is applied. Initial rectifying action is introduced in the learning algorithm and is shown effective to guarantee the converged system output to achieve a desired trajectory with a smooth transient. The robustness with respect to variable initial shifts is addressed simultaneously. A numerical example is given to illustrate the defect of the conventional algorithm and the effectiveness of the proposed method.

2. Preliminaries

Consider the class of nonlinear continuous-time systems described by
\[ \dot{x}(t) = f(x(t)) + B(x(t))u(t), \]  
\[ y(t) = g(x(t)), \]  
where
\[ x \in \mathbb{R}^n, \ u = [u_1, \ldots, u_m]^T \in \mathbb{R}^m, \]  
and
\[ y = [y_1, \ldots, y_m]^T \in \mathbb{R}^m \]
denote the state, the control input, and the output of the system, respectively, \( f(\cdot) \in \mathbb{R}^n \), \( B(\cdot) = [b_1(\cdot), \ldots, b_m(\cdot)] \in \mathbb{R}^{n \times m} \), and \( g(\cdot) = [g_1(\cdot), \ldots, g_m(\cdot)]^T \in \mathbb{R}^m \) are smooth in their domains of definition. According to Isidori (1995) and Nijmeijer and van der Schaft (1990), system (1) and (2) is said to have (vector) relative degree \( \mu(=\{\mu_1, \ldots, \mu_m\}) \) at a point \( x^0 \), if \( 1 \leq q \leq m \)

(i) \( L_{b_i}L_{y^0}^{-1}g_q(x) = 0, \ 0 \leq i \leq \mu_q - 2, \ 1 \leq p \leq m \) and for all \( x \) in a neighborhood of \( x^0 \),

(ii) \( L_{b_i}L_{y^0}^{-1}g_q(x^0) \neq 0 \), for some \( 1 \leq p \leq m \).

Here \( \mu_q \) is the minimum order of time derivative of the \( q \)th output to which a directly transmission is established from at least one component of the control input \( u \), as follows:
\[ y_q^{(j)} = L_{y^0}^{j-1}g_q(x), \quad 0 \leq j \leq \mu_q - 1, \]
\[ y_q^{(\mu_q)} = L_{y^0}^{\mu_q - 1}g_q(x) \]
\[ + [L_{b_i}L_{y^0}^{\mu_q - 1}g_q(x), \ldots, L_{b_m}L_{y^0}^{\mu_q - 1}g_q(x)]u. \]  

Throughout this paper, the vector norm is defined as \( \|a\| = \max_{1 \leq i \leq n} |a_i| \) for an \( n \)-dimensional vector \( a = [a_1, \ldots, a_n]^T \) and the matrix norm as the induced norm by the vector norm, i.e., for a matrix \( A = [a_{ij}] \in \mathbb{R}^{m \times n} \), \( \|A\| = \max_{1 \leq i \leq n} \sum_{j=1}^{m} |a_{ij}| \). The \( \lambda \)-norm for a vector-valued function \( b(t) \in \mathbb{R}^m \) is defined as \( \|b(\cdot)\| = \sup_{t \in [0, T]} \{e^{-\lambda t}\|b(t)\|\} \), \( \lambda > 0 \). The system undertaken is assumed to perform identical tasks repeatedly over a finite time interval, i.e., \( t \in [0, T] \).

Also the following properties are assumed.

(A1) The system dynamics described by (1) and (2) are invertible.

(A2) The system has relative degree \( \mu(=\{\mu_1, \ldots, \mu_m\}) \) for all \( x(t), \ t \in [0, T] \).

(A3) The functions \( f(\cdot), B(\cdot), g(\cdot), L_{y^0}^{\mu_q - 1}g_q(\cdot), \ 1 \leq q \leq m, \) and \( L_{b_i}L_{y^0}^{\mu_q - 1}g_q(\cdot), \ 1 \leq p \leq m, 1 \leq q \leq m, \) are locally Lipschitz in \( x(t) \) for \( t \in [0, T] \), and \( L_{b_i}, L_{b_m}, L_{y^0}^{\mu_q}, L_{b_m}L_{y^0}^{\mu_q - 1} \) and \( L_{b_i}L_{y^0}^{\mu_q} \) denote the Lipschitz constants, respectively.

(A4) The operator \( B(\cdot) \) is bounded for all \( x(t), \ t \in [0, T] \).

(A5) For an appropriate \( x_0 \), all operations start within a neighborhood of \( x_0 \) in the sense that \( \|x_0 - x(t)\|_e \leq c_0 \) for a positive constant \( c_0 \), where \( x_0(0) \) is the initial condition at the kth cycle.

3. Analysis

3.1. Conventional ILC and its tracking error

To deal with the system of relative degree \( \mu \), conventional D-type learning algorithm takes the form of (Ahn et al., 1993)
\[ u_{k+1}(t) = u_k(t) + \Gamma(y_k(t))(y_d^{(\nu)}(t) - y_k^{(\mu)}(t)), \]  
where \( k \) refers to the number of operation cycle and \( y_d^{(\nu)} = [y_1^{(\nu)}, \ldots, y_m^{(\nu)}]^T \). \( \Gamma(\cdot) \in \mathbb{R}^{n \times m} \) is the bounded learning gain and should be chosen such that
\[ ||I - \Gamma(g(x))D(x)|| \leq \rho < 1 \]  
with
\[ D(x) = \begin{bmatrix} L_{b_1}L_{y^0}^{\mu_q - 1}g_1(x), \ldots, L_{b_m}L_{y^0}^{\mu_q - 1}g_1(x) \\ \vdots \\ L_{b_1}L_{y^0}^{\mu_q - 1}g_m(x), \ldots, L_{b_m}L_{y^0}^{\mu_q - 1}g_m(x) \end{bmatrix}. \]

Theorem 1. Given a desired trajectory \( y_d(t), \ t \in [0, T] \), let updating law (5) be applied to system (1) and (2) under assumptions (A1)–(A5). If at the beginning of each cycle \( x_k(0) = x_0 \), i.e., \( c_0 = 0 \), (6) holds and the trajectory \( y^*(t) \) is realizable, where \( y^*(t) = [y_1^*(t), \ldots, y_m^*(t)]^T \) and for \( 1 \leq q \leq m, \)
\[ y_q^*(t) = y_{q,d}(t) - \sum_{j=0}^{\mu_q - 1} \frac{t^j}{j!} (y_{q,d}^{(j)}(0) - y_{q,d}^{(j)}(0)), \]
\[ t \in [0, T], \]  
which is said to have (vector) relative degree \( \mu(=\{\mu_1, \ldots, \mu_m\}) \) for all \( x(t), \ t \in [0, T] \).
then the system output $y_k(t)$ converges uniformly to $y^*(t)$ on $[0, T]$ as $k \to \infty$.

**Remark 3.1.** Note that the system is called invertible if it is both right- and left-invertible. Clearly, for an appropriate $x_0$ and a realizable trajectory $y^*(t)$, assumption (A1) guarantees that there exists a unique control input $u^*(t)$ that will generate the trajectory (Nijmeijer and van der Schaft, 1990). Namely, $y^*(t) = g(x^*(t))$, $x^*(t) = f(x^*(t)) + B(x^*(t))u^*(t)$, $x^*(0) = x_0$, where $x^*(t)$ is the corresponding state.

**Proof of Theorem 1.** For simplicity, the time $t$ is dropped in the following proof where confusion will not occur. Define

$$\hat{g}(x) = [L^{i_1}_1 g_1(x), \ldots, L^{i_n}_n g_m(x)]^T.$$

From (7), $y_{q,d}^{(m)} - y_{q,k}^{(m)} = y_q^*(x_k) - y_q^*(x_0)$, $1 \leq q \leq m$, which leads to

$$u^* - u_{k+1} = (I - \Gamma(g(x_0)))D(x_s)(u^* - u_k) - \Gamma(g(x_0))$$

Taking norms and applying the bounds and the Lipschitz conditions give rise to $\|\hat{g}(x^*) - \hat{g}(x_0)\| \leq l_1\|\Delta x^*\|$, $\|D(x^*) - D(x_s)\| \leq l_2\|\Delta x^*\|$ and

$$\|\Delta u^*_{k+1}\| \leq \rho\|\Delta u^*_k\| + c_1\|\Delta x^*_k\|,$$

where $\Delta u^*_k = u^* - u_k$, $\Delta x^*_k = x^* - x_0$, $l_1 = \max\{l_1^1, \ldots, l_1^m\}$, $l_2 = \max\{l_2^1, \ldots, l_2^m\}$, $c_1 = c_1r(l + l_2c_u)$, $c_T$ is the norm bound for $\Gamma(\cdot)$ and $c_u = \sup_{x \in [0, T]}\|u^*(t)\|$. In order to evaluate the state error on the right-hand side of (8), we integrate both sides of the state equations to obtain

$$\Delta x^*_k(t) = \int_0^t \left[ f(x^*(s)) - f(x_k(s)) \right] ds + (B(x^*(s)) - B(x_k(s)))u^*(s) + B(x_k(s))\Delta u^*_k(s) ds.$$

Taking norms and using their properties produce

$$\|\Delta x^*_k(t)\| \leq \int_0^t \left( c_2\|\Delta x^*_k(s)\| + c_b\|\Delta u^*_k(s)\| \right) ds,$$

where $c_b$ is the norm bound for $B(\cdot)$ and $c_2 = l_f + l_2c_u$. Multiplying both sides of (9) by $e^{-\lambda t}$ ($\lambda > 0$) results in

$$\|\Delta x^*_k\| \leq \frac{1 - e^{-\lambda T}}{\lambda} (c_2\|\Delta x^*_k\| + c_b\|\Delta u^*_k\|).$$

A $\lambda > 0$ is chosen such that $1 - c_2(1 - e^{-\lambda T})/\lambda > 0$, which implies

$$\|\Delta x^*_k\| \leq \frac{c_b(1 - e^{-\lambda T})/\lambda}{1 - c_2(1 - e^{-\lambda T})/\lambda} \|\Delta u^*_k\|.$$

Then multiplying both sides of (8) by $e^{-\lambda t}$ and using (10) yield

$$\|\Delta u^*_k\| \leq \left( \frac{\rho + c_1c_b(1 - e^{-\lambda T})/\lambda}{1 - c_2(1 - e^{-\lambda T})/\lambda} \right) \|\Delta u^*_k\|.$$

Since $0 < \rho < 1$, it is possible to find a sufficiently large $\lambda$ such that $\rho + \frac{c_1c_b(1 - e^{-\lambda T})/\lambda}{1 - c_2(1 - e^{-\lambda T})/\lambda} < 1$. Then, (11) is a contraction in $\|\Delta u^*_k\|$. As the iterations increase, we obtain $\|\Delta u^*_k\| \to 0$ so that $u_k \to u^*$ uniformly on $[0, T]$ as $k \to \infty$. It follows from (10) that $x_k \to x^*$ uniformly on $[0, T]$ as $k \to \infty$. Furthermore, by the assumption on $g(\cdot)$ in (A3), $y_k \to y^*$ uniformly on $[0, T]$ as $k \to \infty$. This completes the proof. $\square$

**Remark 3.2.** The above theorem shows that the converged output follows $y^*(t)$ which deviates from $y_q(t)$ by $\sum_{j=0}^{m-1} (t^j/j!)(y^*_q(0)) - y^*_q(t))$, $1 \leq q \leq m$, $t \in [0, T]$. Thus, uniform convergence of (5) is guaranteed only when $y^*_q(0) = y^*_q(0)$, $0 \leq j \leq m - 1$, $1 \leq q \leq m$. At the $k$th cycle, the output trajectory $y_{q,k}(t)$, $1 \leq q \leq m$, depends on $y^*_q(t)$. However, (7) indicates that the limit trajectory $y^*_q(t)$ only relies on $y^*_q(0)$ as $k$ tends to infinity. That is due to the aligned initial condition, $x_0(t) = x_0$, which implies $y^*_q(0) = y^*_q(0)$, $0 \leq j \leq m$.

### 3.2. Initial rectifying action

To overcome the deviated convergence shown in Theorem 1, an initial rectifying action is introduced into (5) in the form of

$$u_{k+1}(t) = u_0(t) + \Gamma(y_k(t))(y^*_q(t) - y^*_q(0)) - \Gamma(y_k(t))$$

$$= u_0(t) + \sum_{j=0}^{m-1} \left( \frac{t^j}{j!} \int_0^t \theta_{\mu,h}(s) ds \right) (y^*_q(0) - y^*_q(t)),$$

where $\theta_{\mu,h} : [0, T] \to R$, $1 \leq q \leq m$, satisfies

$$\int_0^h \theta_{\mu,h}(s) ds = 1,$$

$$\left. \left( \frac{t^j}{j!} \int_0^t \theta_{\mu,h}(s) ds \right) \right|_{t=0} = \begin{cases} 1, & \mu_q = j, \\ 0, & \mu_q \neq j. \end{cases}$$

One candidate of such function is given as

$$\theta_{\mu,h}(t) = \begin{cases} \frac{1}{h^{\mu_q + 1}} \frac{(2\mu_q + 1)!}{\mu_q^{\mu_q}} e^{(h - t)^{\mu_q}}, & t \in [0, h], \\ 0, & t \in (h, T]. \end{cases}$$
Theorem 2. Given a desired trajectory $y_d(t)$, $t \in [0, T]$, let updating law (12) be applied to system (1) and (2) under assumptions (A1)–(A5). If the learning gain $\Gamma(\cdot)$ is chosen such that (6) holds and the trajectory $y^*(t)$ is realizable, where $y^*(t) = [y^*_1(t), \ldots, y^*_m(t)]$ and for $1 \leq q \leq m$,

$$y^*_q(t) = y_{q,d}(t) - \sum_{j=0}^{\mu_q-1} \int_{t-j}^{t} \theta_{\mu_q,k}(s) \, ds,$$

(14) then the asymptotic bound of the error $y^*(t) - y_k(t)$ is proportional to $c_{x_0}$ on $[0, T]$ as $k \to \infty$.

Proof. As in the proof of Theorem 1, for an appropriate $x_0$, let $u_0$ be the control input which generates the trajectory $y^*$ and $x^*$ is the corresponding state. From (14), $y^*_q(t) - y^*_q(0)$ is the 0th cycle initial error $y^*_q(0)$ and $y^*_q(t)$ is the corresponding state. From (14), $y^*_q(t) - y^*_q(0)$ is the 0th cycle initial error $y^*_q(0)$ and $y^*_q(t)$ is the corresponding state. From (14), $y^*_q(t) - y^*_q(0)$ is the 0th cycle initial error $y^*_q(0)$ and $y^*_q(t)$ is the corresponding state. From (14), $y^*_q(t) - y^*_q(0)$ is the 0th cycle initial error $y^*_q(0)$ and $y^*_q(t)$ is the corresponding state. From (14), $y^*_q(t) - y^*_q(0)$ is the 0th cycle initial error $y^*_q(0)$ and $y^*_q(t)$ is the corresponding state. From (14), $y^*_q(t) - y^*_q(0)$ is the 0th cycle initial error $y^*_q(0)$ and $y^*_q(t)$ is the corresponding state. From (14), $y^*_q(t) - y^*_q(0)$ is the 0th cycle initial error $y^*_q(0)$ and $y^*_q(t)$ is the corresponding state. From (14), $y^*_q(t) - y^*_q(0)$ is the 0th cycle initial error $y^*_q(0)$ and $y^*_q(t)$ is the corresponding state. From (14), $y^*_q(t) - y^*_q(0)$ is the 0th cycle initial error $y^*_q(0)$ and $y^*_q(t)$ is the corresponding state. From (14), $y^*_q(t) - y^*_q(0)$ is the 0th cycle initial error $y^*_q(0)$ and $y^*_q(t)$ is the corresponding state. From (14), $y^*_q(t) - y^*_q(0)$ is the 0th cycle initial error $y^*_q(0)$ and $y^*_q(t)$ is the corresponding state.
4. Example

Consider the following nonlinear continuous-time system

\[ \begin{align*}
    \dot{x}_1(t) &= x_2(t), \\
    \dot{x}_2(t) &= \sin(x_3(t)) + u(t), \\
    \dot{x}_3(t) &= \cos(x_1(t) + x_2(t))u(t), \\
    y(t) &= x_3(t).
\end{align*} \]

The system has relative degree two. Let the desired trajectory be given to be \( y_d(t) = 6t^5 - 15t^4 + 10t^3, \ t \in [0, 1] \). Simulations are conducted for two ILCs: (I) using conventional learning algorithm \( (5) \) with \( \Gamma(y_k(t)) = 0.7; \) and (II) using learning algorithm with initial rectifying action \( (12), h = 0.3 \) and \( \Gamma(y_k(t)) = 0.7 \). For both ILCs, the initial control input is chosen as \( u_k(0) = 0, \ t \in [0, 1] \). For this case, convergence performance of the learning algorithms is examined in the presence of fixed initial shift where the initial condition at each iteration is reset to \( x_k(0) = [x_{1,k}(0), x_{2,k}(0), x_{3,k}(0)]^T = [0.1, 0.2, 0.3]^T \), and thus \( y_k(0) = 0.1, \dot{y}_k(0) = 0.2 \). According to Theorem 1, the converged trajectory by applying \( (5) \) is \( \lim_{k \to \infty} y_k(t) = y_d(t) - (y_d(0) - y_0(0)) - t(y_d(0) - \dot{y}_0(0)) \).

Fig. 1(a) shows the output trajectory at the tenth iteration where the output trajectory tracks the desired one with the lasting tracking error. The ILC with initial rectifying action \( (12) \) can be used to eliminate the error and it is given as

\[ \begin{align*}
    u_{k+1}(t) &= u_k(t) + \Gamma(y_k(t))(\dot{y}_d(t) - \dot{y}_k(t)) \\
    &+ \Gamma(y_k(t))[\dot{\theta}_{2,k}(t)(y_d(0) - y_0(0)) \\
    &+ (t\dot{\theta}_{2,k}(t) + 2\theta_{2,k}(t)(\dot{y}_d(0) - \dot{y}_0(0)))],
\end{align*} \]

where

\[ \dot{\theta}_{2,k}(t) = \begin{cases} 
    \frac{30}{h^2}f(h - t)^2, & t \in [0, h], \\
    0, & t \in (h, T].
\end{cases} \]

From Theorem 2, the converged trajectory is \( \lim_{k \to \infty} y_k(t) = y_d(t) - \int_t^h \theta_{2,k}(s) \, ds(y_d(0) - y_0(0)) - t \int_t^h \theta_{2,k}(s) \, ds \).

5. Conclusion

For trajectory tracking of a class of nonlinear systems with well-defined relative degree, the defect in tracking performance exists due to initial shift when the conventional
learning algorithm is applied. The learning algorithm with the proposed initial action is able to rectify this defect in tracking performance and achieves uniform convergence of the output trajectories to the desired one with specified smooth transience. The results also show that the robustness performance of the conventional learning algorithm with respect to variable initial shifts can be improved by the initial rectifying action.

References


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