Robust adaptive control of uncertain discrete-time systems\(^1\)

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Abstract

In this paper, a robust adaptive controller for a class of nonlinear uncertain discrete-time systems is developed by combining the backstepping procedures with a simple parameter estimator subject to parameter projection. It is shown that the proposed controller can ensure boundedness of all signals in the overall adaptive systems in the presence of unmodelled dynamics and disturbances. It can also guarantee that the tracking error is bounded by a function of the size of the unmodelled dynamics. In the ideal case when there are no unmodelled dynamics and disturbances, perfect tracking is ensured. © 1999 Elsevier Science Ltd. All rights reserved.

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1. Introduction

In the last few years, adaptive control of nonlinear continuous-time systems have drawn extensive attention and many significant progresses have been made (Kanel- lakopoulos, 1995a; Kokotovic, 1992; Kanellakopoulos et al., 1992; Pomet and Praly, 1992; Krstic et al., 1995; Krstic and Kokotovic, 1995). To overcome some restrictions such as matching conditions and overparameterisation that the earlier established adaptive control methods suffer from, a promising backstepping technique was developed methodologically in Kanellakopoulos et al. (1992), Krstic et al. (1995) and Kanellakopoulos (1995a) where an adaptive controller for a large class of nonlinear systems can be designed in a systematic framework. In contrast with the conventional approaches based on certainty equivalence principle, the design of the control law and the parameter update law are carried out at the same time in the backstepping design method. This can provide better transient performance. However, the results obtained on continuous-time systems are not applicable to the discrete-time systems where the increment of Lyapunov function is no longer a linear function with respect to the increments of parameter estimates.

With the increasing applications of advanced computer technologies in industries, it is much more meaningful to implement adaptive control for nonlinear discrete-time systems. So far, however, only a few results have been reported on this topic (Kung and Womach, 1983; Agarwal and Sebory, 1987; Zhang and Lang, 1989; Lin and Yong, 1992; Song and Grizzle, 1993; Kanel- lakopoulos, 1995b; Yeh and Kokotovic, 1995). However, only in Yeh and Kokotovic (1995) an adaptive controller was designed by using the backstepping technique to achieve tracking of a reference signal for a class of nonlinear discrete-time systems. It was shown that under certain geometric conditions a large class of discrete-time nonlinear systems could be transformed into the parametric-strict-feedback form and the parametric-pure-feedback form for which the backstepping design approach could be applied. By using various update laws available in Goodwin and Sin (1984) and utilizing their properties, the global boundedness and convergence were shown to be achieved without employing Lyapunov functions in the design. But the results of Yeh and Kokotovic (1995) were obtained only in the ideal case without considering nonparametric uncertainties such as unmodelled dynamics and external uncertainties.

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disturbances which are usually inevitable in practical situations. The design of robust adaptive controllers for nonlinear discrete-time systems with uncertainties remains an unresolved problem. In this paper we will address this issue.

This paper presents a design approach of robust adaptive control for a class of nonlinear discrete-time systems with both parametric and nonparametric uncertainties. In our design, the backstepping procedure incorporating a simple parameter projection update law is employed to obtain the desired controller. An intermediate constant, which can be chosen without a priori knowledge of the unmodelled dynamics, is introduced in the normalising term of the parameter estimator to allow for the nonparametric uncertainties in the estimators properties. Using a similar stability analysis method developed in Wen (1989) and Wen and Hill (1992), it is shown that the proposed adaptive controller can ensure boundedness of all the signals in the closed-loop system even in the presence of unmodelled dynamics and disturbances. It can also ensure 𝜀-small in the mean tracking error. When the disturbances and the unmodelled dynamics are removed, the ideal results obtained in Yeh and Kokotovic (1995) are still preserved. In particular, a perfect tracking is achieved.

The rest of the paper is organised as follows. Section 2 describes the class of nonlinear uncertain discrete-time systems to be controlled and Section 3 presents the design of the adaptive controllers in the presence of uncertainties. The stability of the adaptive system is analysed in Section 4. Finally, the paper is concluded in Section 5.

2. Problem formulation

The nonlinear discrete-time system under consideration is described by

\[ \begin{align*}
    x_{1,t+1} &= x_2 + \theta^T \zeta_2(x_1) + \eta_1(t), \\
    x_{2,t+1} &= x_3 + \theta^T \zeta_2(x_1, x_2) + \eta_2(t), \\
    &\vdots \\
    x_{n-1,t+1} &= x_n + \theta^T \zeta_2(x_1, x_2, \ldots, x_{n-1}) + \eta_{n-1}(t), \\
    x_{n,t+1} &= \theta^T \zeta_2(x_1, x_2, \ldots, x_n) + \eta_n(t) + u(t), \\
    y(t) &= x_1(t),
\end{align*} \]

(1)

where \( u(t) \) and \( y(t) \) represent the system input and output, respectively, and \( \theta \) is the unknown parameter vector in \( R^p \). For each \( 1 \leq i \leq n \), \( \zeta_i(x_1, \ldots, x_i) \) are known nonlinear functions in \( C(R^i, R^p) \) satisfying \( \zeta_i(0) = 0 \), and \( \eta(t) \) are unknown functions. For simplicity of illustration, \( \zeta_i(x_1, x_2, \ldots, x_i) \) are denoted by \( \zeta_i(t) \) for each \( i = 1, 2, \ldots, n \).

The discrete-time system described by Eq. (1) has two types of uncertainties. One is the parametric uncertainty denoted by the unknown parameter vector \( \theta \). Usually, the range of \( \theta \) can be considered to be known a priori, which leads to the following assumption.

Assumption A1. \( \theta \) lies in a known compact set \( \Theta \), i.e. \( \theta \in \Theta = \{ \theta : \| \theta \| \leq k_\theta, \| \theta - \theta' \| \leq k_{\theta}, \forall \theta' \in \Theta \} \), where \( k_\theta \) is a known constant.

Another kind of uncertainty appearing in system (1) is nonparametric. It is described by the unknown functions \( \eta_i(t) \) which are due to unmodelled dynamics, external disturbances and time variations. As shown in Wen and Hill (1992), they are usually characterised by

\[ \eta_i(t) < c_n, c \sup_{0 \leq t \leq T} \| x_1, x_2, \ldots, x_i \|^2 + d, \]

for \( i = 1, 2, \ldots, n \),

where \( c_n, c > 0 \) are known constants. Taking \( c_n = \max_{1 \leq i \leq n} |\zeta_i| \) and \( d = \max_{1 \leq i \leq n} |d_i| \), we impose the following assumption about the nonparametric uncertainty.

Assumption A2.

\[ \eta_i(t) \leq c_n, c \| x_1, x_2, \ldots, x_i \|^2 + d. \]

It will be shown later that knowledge of \( c \) and \( d \) is not required to implement the adaptive controller.

The adaptive control problem is to obtain a control law for plant (1) such that all the signals in the resulting closed-loop system are bounded for arbitrary bounded reference set-point \( y_d(t) \) and initial conditions. It is also desirable that for a certain known gain \( K \), the tracking error \( |y(t) - Ky_d(t)| \) is small in some sense. To solve the problem, an additional assumption on the nonlinear functions \( z_i(t) \) is required.

Assumption A3. All the known nonlinear functions \( z_i(t) \) are Lipschitz functions, i.e.

\[ \| z_i(x_i(t)) - z_i(x_i'(t)) \| \leq k_x \| x_i(t) - x_i'(t) \|, \]

where \( x_i(t), x_i'(t) \in C(R, R^i) \) and \( k_x \) is known a priori.

3. Adaptive control design using backstepping technique

Suppose that \( M_0 \) is an intermediate positive constant such that \( \| x(0) \| \leq M_0, \| y_d(t) \| \leq M_0, \) and \( d/M_0 < \delta \) for a sufficiently small \( \delta \), where \( x(0) \) denotes the initial conditions of the system. It is noted that for a given system, such an intermediate constant can always be found for any bounded initial conditions, set-point and disturbances. Then the desired adaptive controller can be obtained by performing the backstepping procedures as in
Table 1
Robust backstepping adaptive controller

<table>
<thead>
<tr>
<th>Coordinate transformation:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_1' = x_1$</td>
</tr>
<tr>
<td>$z_2' = x_2' + \tilde{a}_i(t)z_1(t)$</td>
</tr>
<tr>
<td>$z_k' = x_k' + \tilde{a}_i(t)\tilde{z}_k(t) + \tilde{b}_i(t)\tilde{z}_k(t)$</td>
</tr>
</tbody>
</table>

where

$z(t) = \begin{bmatrix} z_1' \\ \vdots \\ z_n' \end{bmatrix}$, $F = \begin{bmatrix} 0 & I_{(n-1) \times (n-1)} \end{bmatrix} \in \mathbb{R}^{n \times n}$,

$s = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^n$, $c = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{R}^n$,

$e(t + 1) \triangleq [e_1(t + 1), e_2(t + 1), \ldots, e_n(t + 1)]^T$,

$\Psi(t + 1) \triangleq [\chi_1(t + 1), \chi_2(t + 1), \ldots, \chi_n(t + 1)]^T$,

with

$\chi_1(t + 1) = 0$.

From the definitions of $z_i'$, it is trivial to show that the relationship between the new state variables $z(t)$ and the original states $x(t) = [x_1', x_2', \ldots, x_n']^T \in \mathbb{R}^n$ can be specified by the following lemma.

**Lemma 1.** For $z(t)$ obtained by Eqs. (2)–(5), we have

$b_1 \|x(t)\| \leq \|z(t)\| \leq b_2 \|x(t)\|$, where $b_1$ and $b_2$ are constants which depend on $k_a$ and $k_o$.

The properties of estimator (18) and (19) are summarised in the following lemma, which will be used in the next section to set up the robust stability of the closed-loop system.

**Lemma 2.** Assume that

$\|x(t_0 - 1)\| \leq M_0$, $\|x(t)\| > M_0$, $\tau = t_0, t_0 + 1, \ldots, t - 1$ and $\|x(\tau_i)\| < M_1$, $\tau_i = t_0, t_0 + 1, \ldots, t - 1$ where $M_1$ is a constant such that $M_1^2 = k_1M_0^2 + k_2 > M_0^2$ where $k_1$ and $k_2$ are constants which will be determined in the later discussion.

Then

$|e_i(t_0)| \leq (k_0a_1 + a_1)M_0 + a_1$, $\forall i$, (25)

$|\tilde{e}_i(t_0) + 1| \leq k_0 + a_1$, $\forall t \geq t_0, \forall i$, (26)

where

$\tilde{e}_i(t_0) + 1 \triangleq \frac{e_i(t_0 + 1)}{1 + M_0 + \|x_i(t_0)\|^2}$, (27)
\[ a_1 = c_x \phi(k_1^{1/2} + k_2^{1/2}) + \delta. \quad (28) \]

\[ \| \hat{\theta}(t + 1) - \hat{\theta}(t) \| \leq \| \tilde{e}_i(t + 1) \|, \quad \forall i, \quad (29) \]

\[ \sum_{t = t_0}^{t + 1} |\hat{\tau}(t)|^2 \leq k_0^2 + (a_2 + a_3)(t - t_0), \quad \forall i, \quad (30) \]

where

\[ a_2 = 2(k_0(k_1^{1/2} + k_2^{1/2}) + 2c_x \phi(k_1 + k_2))c_x \phi, \quad \]

\[ a_3 = 2\delta(2\delta + k_0). \quad (31) \]

\[ \| \hat{\theta}(t + 1) \| \leq c_1 \left[ \begin{array}{c}
    e_1(t + 1) \\
    e_2(t + 1) \\
    \vdots \\
    e_{i-1}(t + 1)
  \end{array} \right] \\
  + c_2 \left[ \begin{array}{c}
    z_1^1 \\
    z_2^1 \\
    \vdots \\
    z_{i-1}^1
  \end{array} \right], \quad \forall i, \quad (32) \]

\[ \| \hat{\theta}(t_0) \| \leq (c_s + c_4a_1)M_0 + c_s a_1 + c_6, \quad \forall i, \quad (33) \]

where \( c_j \) (\( j = 1, 2, \ldots, 6 \)) are constants depending on \( k_s \) and \( k_0 \).

**Proof.** (1) From the definitions of \( e_i(t + 1) \), we have

\[ e_i(t + 1) = z_i^{t+1} - z_i^{t+1} - \hat{\theta}(t + 1) = (\theta - \hat{\theta}(t))^T z_i(t) + \eta_i(t) \]

\[ = -\hat{\theta}(t)^T z_i(t) + \eta_i(t) \quad (34) \]

with \( \hat{\theta}(t) \triangleq \hat{\theta}(t) - \theta \).

Applying Assumptions A1–A3 gives

\[ |e_i(t + 1)| \leq k_0 \| z_i(t) \| + c_x \phi \sup_{0 < t \leq t} \| x(t) \| + d \]

\[ \leq k_0 k_0 \left[ x_1^t, x_2^t, \ldots, x_4^t \right]^T + c_x \phi(k_1 M_0^2 + k_2)^{1/2} + d, \quad (35) \]

where \( M_0^2 = k_1 M_0^2 + k_2 \) is used.

Since \( \| x(t_0 - 1) \| \leq M_0 \), it follows immediately that

\[ |e_i(t_0)| \leq k_0 k_0 M_0 + c_x \phi(k_1 M_0^2 + k_2)^{1/2} + d \]

\[ \leq (k_0 k_0 + a_1)M_0 + a_2. \quad (36) \]

From Eqs. (27) and (35), we have

\[ |\hat{\theta}(t + 1)| = \frac{|e_i(t + 1)|}{(1 + M_0 + \| z_i(t) \|^2)^{1/2}} \]

\[ \leq k_0 + \frac{c_x \phi k_1 M_0^2 + k_2)^{1/2} + d}{(1 + M_0 + \| z_i(t) \|^2)^{1/2}} \]

\[ \leq k_0 + \frac{d}{(1 + M_0^2)^{1/2}} + \frac{c_x \phi(k_1 M_0^2 + k_2)^{1/2}}{(1 + M_0^2)^{1/2}} \leq k_0 + \delta + c_x \phi(k_1^{1/2} + k_2^{1/2}) \leq k_0 + a_1. \quad (37) \]

(2) Let \( \hat{\theta}_{\mu}(t) \) denote a parameter estimate before applying the projector \( \phi \), i.e.

\[ \hat{\theta}_{\mu}(t + 1) - \hat{\theta}(t) = \frac{x_1(t) e_i(t + 1)}{1 + M_0 + \| z_i(t) \|^2} \cdot \]

Then

\[ \| \hat{\theta}(t + 1) - \hat{\theta}(t) \| \leq \| \hat{\theta}_{\mu}(t + 1) - \hat{\theta}(t) \| \]

\[ = \frac{\| x_i(t) e_i(t + 1) \|}{(1 + M_0 + \| z_i(t) \|^2)^{1/2}} \leq |\hat{e}_i(t + 1)|, \quad \forall i. \quad (38) \]

(3) Introducing \( \nu(t + 1) = \hat{\theta}(t + 1) \hat{\theta}(t + 1) \), we get

\[ \nu(t + 1) - \nu(t) \leq \hat{\theta}(t + 1) \hat{\theta}(t + 1) \]

\[ \leq \hat{\theta}_{\mu}(t + 1) \hat{\theta}_{\mu}(t + 1) - \hat{\theta}(t) \hat{\theta}(t) \]

\[ = [\hat{\theta}_{\mu}(t + 1) - \hat{\theta}(t)]^T [\hat{\theta}_{\mu}(t + 1) - \hat{\theta}(t) + 2 \hat{\theta}(t)] \]

\[ = \frac{\| x_i(t) \|^2 e_i(t + 1)^2}{(1 + M_0 + \| z_i(t) \|^2)^2} + \frac{2x_i(t) \hat{\theta}(t) e_i(t + 1)}{1 + M_0 + \| z_i(t) \|^2} \]

\[ \leq \frac{e_i(t + 1)^2}{1 + M_0 + \| z_i(t) \|^2} + \frac{2x_i(t) \hat{\theta}(t) e_i(t + 1)}{1 + M_0 + \| z_i(t) \|^2}. \quad (39) \]

From Eq. (34), we have

\[ 2x_i(t) \hat{\theta}(t) e_i(t + 1) \]

\[ = 2(\eta_i(t) - e_i(t + 1)) e_i(t + 1) \]

\[ \leq -2e_i(t + 1)^2 + 2(\eta_i(t)) (k_0 \| z_i(t) \| + |\eta_i(t)|) \]

\[ \leq -2e_i(t + 1)^2 + 2k_0 \| z_i(t) \| |\eta_i(t)| + 2|\eta_i(t)|^2. \quad (40) \]

Substituting Eq. (41) into Eq. (40) yields

\[ \nu(t + 1) - \nu(t) \leq \frac{-e_i(t + 1)^2}{1 + M_0 + \| z_i(t) \|^2} + 2 \frac{k_0 \| z_i(t) \| |\eta_i(t)|}{1 + M_0 + \| z_i(t) \|^2} \]

\[ + \frac{2}{1 + M_0 + \| z_i(t) \|^2} |\eta_i(t)|^2. \]
\[\leq - |\tilde{c}(t + 1)|^2 + 2k_0c_0^2(k_1M_0 + k_2)^{1/2} + d \left( 1 + M_0 + \|\tilde{x}(t)\|^2 \right) \]
\[+ 4c_0^2\tilde{c}^2(k_1M_0^2 + k_2^2) + \frac{d^2}{1 + M_0 + \|\tilde{x}(t)\|^2} \]
\[\leq - \tilde{c}(t + 1)^2 + 2k_0c_0^2(k_1M_0^2 + k_2)^{1/2} + \frac{4c_0^2\tilde{c}^2(k_1M_0^2 + k_2^2) + 4d^2}{1 + M_0 + \|\tilde{x}(t)\|^2} \]
\[+ \frac{4c_0^2\tilde{c}^2(k_1M_0^2 + k_2^2)}{1 + M_0^2}, \quad (42) \]

Therefore,
\[\tilde{c}(t + 1)^2 \leq \tilde{v}(t) - \tilde{v}(t + 1) + a_2 + a_3. \quad (43)\]

Summing both sides of Eq. (43) gives
\[\sum_{t = t_0 + 1}^{t} |\tilde{c}(t)|^2 \leq \|\tilde{c}(t_0)\|^2 - \|\tilde{c}(t)\|^2 + (a_{12} + a_{13})(t - t_0), \quad (44)\]

which results in Eq. (30) by applying Assumption A.3.

To show the final result, the following inequality is useful.
\[|\tilde{c}(t + 1)^2\tilde{x}_{t+1}(t + 1) - \tilde{c}(t)^2\tilde{x}_{t+1}(t + 1)| \]
\[\leq c_{1,i} \begin{bmatrix} e_{k-i+1}(t + 1) \\ e_{k-i+2}(t + 1) \\ \vdots \\ e_{k}(t + 1) \end{bmatrix} + c_{2,i} \begin{bmatrix} \tilde{z}_{k-i+1}(t + 1) \\ \tilde{z}_{k-i+2}(t + 1) \\ \vdots \\ \tilde{z}_{k}(t + 1) \end{bmatrix} \]
\[+ c_{3,i} \begin{bmatrix} \tilde{z}_{k+1}^{i+1} \\ \tilde{z}_{k+1}^{i+3} \\ \vdots \\ \tilde{z}_{k+1} \end{bmatrix} \begin{bmatrix} e_{k-i+1}(t + 1) \\ e_{k-i+2}(t + 1) \\ \vdots \\ \tilde{c}(t + 1) \end{bmatrix} \], \quad (45)

where \(c_{1,i}, c_{2,i}, c_{3,i}\) are constants depending on \(k_i\) and \(k_0\) only.

An inductive strategy is adopted to examine the validation of this inequality. Firstly, consider \(i = 1\). From the definitions of \(\tilde{x}_{t+1}(t)\) and \(e_{k}(t + 1)\), Assumptions A.1 and A.3, and Eq. (29), we have
\[|\tilde{c}(t + 1)^2\tilde{x}_{t+1}(t + 1) - \tilde{c}(t)^2\tilde{x}_{t+1}(t + 1)| \]
\[\leq |\tilde{c}(t + 1)^2\tilde{x}_{t+1}(t + 1) - \tilde{c}(t)^2\tilde{x}_{t+1}(t + 1)| \]
\[+ |\tilde{c}(t + 1)^2\tilde{x}_{t+1}(t + 1) - \tilde{c}(t)^2\tilde{x}_{t+1}(t + 1)| \]
\[\leq k_0k_4|e_{k}(t + 1) + k_0|z_{k+1}^i|\tilde{c}(t + 1)|, \quad (46)\]

which obviously supports the inequality (45). In particular, if \(k = 1\), we obtain
\[|\tilde{x}_{t+1}(t + 1)| \leq k_0k_4|e_{k}(t + 1) + k_0|z_{k+1}^i|\tilde{c}(t + 1)|, \quad (47)\]

where \(\tilde{x}_{t+1}(t + 1) = 0\) is used. This actually verifies Eq. (32) for \(i = 1\). Then consider \(i = 2\), we have
\[|\tilde{c}(t + 1)^2\tilde{x}_{t+1}(t + 1) - \tilde{c}(t)^2\tilde{x}_{t+1}(t + 1)| \]
\[\leq |\tilde{c}(t + 1)^2\tilde{x}_{t+1}(t + 1) - \tilde{c}(t)^2\tilde{x}_{t+1}(t + 1)| \]
\[+ |\tilde{c}(t + 1)^2\tilde{x}_{t+1}(t + 1) - \tilde{c}(t)^2\tilde{x}_{t+1}(t + 1)| \]
\[\leq k_0k_4\left(\begin{bmatrix} z_{k+1}^1 - z_k^1 \\ z_{k+1}^2 - z_k^2 \end{bmatrix} \right) \]
\[+ (|\tilde{c}(t + 1)^2\tilde{x}_{t+1}(t + 1) - \tilde{c}(t)^2\tilde{x}_{t+1}(t + 1)|) \]
\[+ c \left( \begin{bmatrix} z_{k+1}^1 \\ z_{k+1}^2 \end{bmatrix} \right) \|\tilde{c}(t + 1) - \tilde{c}(t)\|, \quad (48)\]

where \(c = \max\{k_0, k_0k_4\}\).

Substituting Eq. (46) into Eq. (48) and using the definition of \(e_{k}(t + 1)\) gives
\[|\tilde{c}(t + 1)^2\tilde{x}_{t+1}(t + 1) - \tilde{c}(t)^2\tilde{x}_{t+1}(t + 1)| \]
\[\leq c_{1,2} \left| e_{k-i+1}(t + 1) \right| + c_{2,2} \left| \tilde{z}_{k-i+1}(t + 1) \right| \]
\[+ c_{3,2} \left| \tilde{z}_{k+1}^i \right| \|\tilde{c}(t + 1)\|, \quad (49)\]

where \(c_{1,2}, c_{2,2}, c_{3,2}\) are constants combining \(k_0\) and \(k_i\). Thus Eq. (45) holds for \(i = 2\).
Now assume Eq. (45) holds for all \(1 \leq p \leq i - 1\), i.e.
\[
\|\hat{\theta}(t^+) - \hat{\theta}(t^-)\| \leq c_{1,p}
\begin{bmatrix}
\begin{array}{c}
\varepsilon_{k-p-1}(t^+) \\
\varepsilon_{k-p}(t^+) \\
\vdots \\
\varepsilon_{k+1}(t^+)
\end{array}
\end{bmatrix}
+ c_{2,p}
\begin{bmatrix}
\begin{array}{c}
\varepsilon_{k-p+1}(t^-) \\
\varepsilon_{k-p+2}(t^-) \\
\vdots \\
\varepsilon_{k+1}(t^-)
\end{array}
\end{bmatrix}
+ c_{3,p}
\begin{bmatrix}
\begin{array}{c}
\varepsilon_{k-p+2}(t^+) \\
\varepsilon_{k-p+3}(t^+) \\
\vdots \\
\varepsilon_{k+2}(t^+)
\end{array}
\end{bmatrix}
\]
(50)
where \(c_{1,p}, c_{2,p}\) and \(c_{3,p}\) are constants depending upon \(k_s\) and \(k_p\). Then we show that Eq. (45) also holds for \(p = i\).

From the definitions of \(\hat{z}_{i,p}(t)\), it follows that
\[
\|\hat{\theta}(t^+) - \hat{\theta}(t^-)\| \leq \|\hat{\theta}(t^+) - \hat{\theta}(t^-)\| \\
+ \|\hat{\theta}(t^+)\| \|\hat{\theta}(t^-)\| \\
\leq k_o \|z_i\| \left(\sum_{i=1}^{i-1} \|\hat{z}_{i-1}(t)\| + \|\hat{z}_{i-1}(t)\|\right) \\
- \sum_{i=1}^{i-1} \|\hat{z}_{i-1}(t)\| \|\hat{\theta}(t^+)\| \\
- \sum_{i=1}^{i-1} \|\hat{z}_{i-1}(t)\| \|\hat{\theta}(t^-)\| \\
+ \|\hat{z}_{i-1}(t)\| \|\hat{\theta}(t^+)\| + \|\hat{z}_{i-1}(t)\| \|\hat{\theta}(t^-)\|.
\]
(51)

Using Assumption A.3 and Eq. (29), and noting that \(\hat{z}_{i-1}(t)\) is a function of \(\hat{z}_{i-1}^+, \hat{z}_{i-1}^-, \ldots, \hat{z}_{i-1}^+\), we have
\[
\|\hat{\theta}(t^+) - \hat{\theta}(t^-)\| \leq k_o k_s \\
+ K \sum_{i=1}^{i-1} \|\hat{z}_{i-1}(t)\| \\
+ K' \|\hat{\theta}(t^-)\| \\
+ K'' \|\hat{\theta}(t^+)\|.
\]
(52)
where \(K'\) and \(K''\) are constants depending on \(k_o\) and \(k_s\) only.

Substituting Eqs. (14) and (50) into Eq. (52) gives
\[
\|\hat{\theta}(t^+) - \hat{\theta}(t^-)\| \leq c_{1,i} \\
+ c_{2,i} \\
+ c_{3,i} \\
+ c_{4,i}.
\]
(53)

where \(c_{m,p}\) are constants combining \(k_o, k_s\) depend on \(k_o, k_s\) only. So far we have proved inequality (45).

Using Eq. (45), it follows immediately from the definition of \(\hat{z}_{i-1}(t)\) that:
\[
\|\hat{\theta}(t^+) - \hat{\theta}(t^-)\| \leq c_{1,i} \\
+ c_{2,i} \\
+ c_{3,i} \\
+ c_{4,i}.
\]
(54)

where \(c_{m,p}\) are constants combining \(k_o, k_s\) depend on \(k_o, k_s\) only. Since \(\hat{\theta}(t^+ - 1) = 0\) and \(\hat{\theta}(t^+ - 1) \leq k_o k_s \|\hat{\theta}(t^+ - 1)\| + k_o \|\hat{\theta}(t^+ - 1)\|\), it can be shown from Eq. (54) that
\[
\|\hat{\theta}(t^+)\| \leq c_{1,i} \\
+ c_{2,i} \\
+ c_{3,i} \\
+ c_{4,i}.
\]
(55)

where \(c_{1,i}, c_{2,i}\) are constants combining \(k_o, k_s\).

Taking \(c_1 = \max_{1 \leq i \leq n} c_{1,i}\) and \(c_2 = \max_{1 \leq i \leq n} c_{2,i}\), Eq. (32) follows.

Using Eq. (25) and inequality
\[
\|\hat{\theta}(t_0)\| \leq (1 + M_o + \|\hat{\theta}(t_0 - 1)\|)\|\hat{\theta}(t_0)\| \\
\leq (1 + M_o + k_o \|\hat{\theta}(t_0 - 1)\|)\|\hat{\theta}(t_0)\|,
\]
we get (33). □
It is noted that the presented update law has the same properties as those given in Yeh and Kokotovic (1995) if the nonparametric uncertainty is removed, i.e. $\varepsilon$ and $\delta$ are identically zeros. Moreover, the constants $a_1$ and $a_2$ in Lemma 2 are functions of $\varepsilon$ and can be made sufficiently small by reducing $\varepsilon$.

4. Stability analysis

In this section we show that there exists a small constant $\varepsilon^*$ such that for each $\varepsilon \in [0, \varepsilon^*]$, all the signals in closed-loop system (17) are bounded for any bounded initial conditions, bounded tracking reference signal and external disturbances. The stability analysis method used in Wen (1989) is adapted to derive the conclusion.

The stability of the closed-loop system can be established by the following theorem.

**Theorem 1.** Consider the adaptive system consisting of plant (1), update law (11) and controller (16). Under Assumptions A.1–A.3, there exists constant $\varepsilon^*$ such that for each $\varepsilon \in [0, \varepsilon^*]$, $\|z(t)\|$ is bounded for all bounded initial conditions, setpoints and disturbances. In addition, the tracking error satisfies that

$$
\sum_{t=t_0}^{t} |y(t) - y_m(t)| \leq \beta_1 + \beta_2 \delta(t - t_0),
$$

(56)

where $\beta_1$ and $\beta_2$ are constants, and $0(\varepsilon)$ is a function such that $\lim_{\varepsilon \to 0} O(\varepsilon, \delta) = 0$.

**Proof.** By the same idea as in Wen and Hill (1992), the time interval is divided into two subsequences

$$
N_1 = \{ t \in Z_+ | \|x(t)\| > M_0 \},
$$

(57)

$$
N_2 = \{ t \in Z_+ | \|x(t)\| \leq M_0 \},
$$

(58)

where $Z_+$ denotes all positive integers.

Clearly, it is sufficient to show that $\|z(t)\|$ is bounded for $t \in N_1$ to obtain the boundedness of $z(t)$ in the whole time interval $[0, \infty)$. To this end, we choose time instant $t_0$ such that $t_0 - 1 \in N_2$ and $[t_0, t - 1] \in N_1$. The inductive strategy is used to prove the result. Suppose that $M$ is a positive constant such that $M_1 < M/b_o$. It is known from Lemma 1 that $\|z(t_0 - 1)\| \leq b_o \|x(t_0 - 1)\| \leq b_o M_1 \leq M$. Now assume that $\|z(t)\| < M$ for $t = t_0, t_0 + 1, \ldots, t - 1$. Then we show that $\|z(t)\| < M$.

The solution of system (23) is

$$
z(t) = \Phi(t, t_0)z(t_0) + \sum_{t = t_0}^{t - 1} \Phi(t, \tau + 1)
\times [by_m(\tau + n) + \Psi(\tau + 1) + \delta(t + 1)]
$$

(59)

$$
= \Phi(t, t_0)[Fz(t_0 - 1) + by_m(t_0 + n - 1) + \Psi(t_0) + \delta(t_0)]
\sum_{t = t_0}^{t - 1} \Phi(t, \tau + 1)[by_m(\tau + n) + \Psi(\tau + 1) + \delta(t + 1)].
$$

(60)

where $\Phi(t, \tau)$ is the transition function of the system $z(t + 1) = Fz(t)$, i.e.

$$
\Phi(t, \tau) = \begin{cases}
F^{\tau - \tau'}, & 0 \leq t - \tau < n, \\
0, & t - \tau \geq n.
\end{cases}
$$

Since $F$ is a strictly stable matrix, the transition matrix $\Phi(t, \tau)$ satisfies

$$
\|\Phi(t, \tau)\| \leq Cq^{\tau - \tau'},
$$

where $C$ and $q$ are constants, and $q < 1$.

Using Eqs. (25), (26), (32), (33) and (59) gives

$$
\|z(t)\| \leq C \sum_{t = t_0}^{t - 1} Cq^{t - \tau - 1}[C_0 + C_1 + C_2 a_1 + C_3]
$$

(61)

where $C_0, C_1, C_2$ and $C_3$ are constants depending only upon $k_0$ and $k_o$. Let

$$
k_2 = C_0/b_o^2,
$$

(63)

$$
k_1 = \frac{1}{b_o^2} \max \left\{ 1, \frac{C_1 + C_2 \varepsilon^*}{1 - C_3 \varepsilon^*} \right\},
$$

(64)

where $\varepsilon^*$ is a constant satisfying $C_3 \varepsilon^* < 1$.

Then it follows from Eq. (62) that

$$
\|z(t)\| \leq b_o^2 (k_1 M_1 + k_2) = b_o^2 M_1^2 \leq M^2.
$$

(65)

Therefore, taking $\varepsilon^* = \max\{\tilde{\varepsilon}^*, \varepsilon^*\}$ confirms the first part of theorem.

Since the boundedness of all the states in the closed-loop system has been established, it follows immediately from the definitions of $\tilde{e}(t + 1)$ and Eq. (30) that

$$
\sum_{t = t_0}^{t} \|e(t)\| \leq \beta_3 + \beta_4 \delta(t - t_0),
$$

(66)

where $\beta_3$, $\beta_4$ are constants.

Using Eq. (32), we have

$$
\sum_{t = t_0}^{t} \|\Psi(t + 1)\| \leq \beta_5 + \beta_6 \delta(t - t_0),
$$

(67)

where $\beta_5$ and $\beta_6$ are constants. Applying Eqs. (66) and (67) to system (17), (56) follows. □
Remark 3.1. It is noted that if there is no nonparametric uncertainty, i.e. \( \varepsilon = 0\) and \( \delta = 0\), the stability of the adaptive system can also be ensured by Theorem 1. In such case, \( \| \hat{y}(t+1) - \hat{y}(t) \| \to 0\), \( |e(t+1)| \to 0\), and \( |y(t) - y_d(t) | \to 0\), which means perfect tracking is achieved.

Remark 3.2. The adaptive controller in Table 1 was obtained by employing an updating law in each backstepping step. This gives rise to overparameterisation. To avoid this problem, an identifier-based indirect adaptive controller can be designed by following the same steps as in Table 1 but postponing the determination of the update law. Replacing all \( \hat{y}_i \) in Table 1 with a common \( \hat{y} \), the adaptive controller without overparameterisation is thus obtained by

\[
\begin{align*}
 u(t) &= y_m(t + n) - \hat{y}(t)^T z_d(t) - \sum_{k=1}^{n-1} \hat{y}(t)^T z_{k,d}(t), \\
 \hat{y}(t + 1) &= \omega \left( \hat{y}(t) + \frac{\Phi(t)^T \zeta(t + 1)}{1 + M_0 + \text{trace}(\Phi(k) \Phi(k)^T)} \right),
\end{align*}
\]

where

\[
\zeta(t + 1) = z(t + 1) - Fz(t) - by_m(t + n) - \Psi(t + 1),
\]

\[
\Phi(t) = [z_1(t), z_2(t), \ldots, z_n(t)]^T,
\]

\[
\eta(t + 1) = [\eta_1(t), \eta_2(t), \ldots, \eta_n(t)]^T.
\]

For this adaptive controller, we can also obtain the same robust stability results as stated in Lemma 1 and Theorem 1.

5. Conclusion

This paper studies the problem of adaptive control for a class of nonlinear discrete-time systems with unmodelled dynamics and external disturbances. Since the modelling errors are considered, the class of the nonlinear discrete-time systems for which the adaptive control can be applied has been enlarged. By combining the backstepping technique with parameter projection, a design scheme of robust adaptive controller is obtained. With this scheme, the boundedness of the whole adaptive closed-loop system is guaranteed for any bounded initial conditions, set-point signals and external disturbances. A small mean tracking error is also achieved. It is also clear that the stability and convergence results obtained in the ideal case are still preserved if there are no unmodelled dynamics and disturbances. Particularly in this ideal case, perfect tracking of a reference trajectory is ensured.

References


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