Evaluation of bedload transport subject to high shear stress fluctuations

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Abstract: Many formulas available in the literature for computing sediment transport rates are often expressed in terms of time-mean variables such as time-mean bed shear stress or flow velocity, while effects of turbulence intensity, e.g., bed shear stress fluctuation, on sediment transport were seldom considered. This may be due to the fact that turbulence fluctuation is relatively limited in laboratory open-channel flows, which are often used for conducting sediment transport experiments. However, turbulence intensity could be markedly enhanced in practice. This note presents an analytical method to compute bedload transport by including effects of fluctuations in the bed
shear stress. The analytical results obtained show that the transport rate enhanced by
turbulence can be expressed as a simple function of the relative fluctuation of the bed
shear stress. The results are also verified using data that were collected recently from
specifically-designed laboratory experiments. The present analysis is applicable largely
for the condition of a flat bed that is comprised of uniform sand particles subject to
unidirectional flows.

Introduction

Turbulent flows can be characterized using time-mean flow parameters and
corresponding fluctuating intensities, so turbulence effects on sediment transport may
generally comprise those due to the time-mean flow as well as those associated with the
turbulence intensity. However, previous studies of sediment transport are largely based
on the information of the mean flow behavior such that most equations for bedload
transport rate are only associated with the time-mean bed shear stress or flow velocity.
This may be because many sediment transport experiments are conducted under
uniform flow conditions, where the variation of the near-bed turbulence intensity is
usually limited. For example, in uniform open channel flows, the fluctuation or
turbulence intensity of the streamwise velocity, expressed as the ratio of its rms to time-
mean values, varies from 12% to 32% (Nezu and Nakagawa 1993) and thus turbulence
effects on the particle motion may not be easily noticed in such laboratory experiments.

In comparison, many practical situations are associated with widely varying
turbulence intensities due to flow unsteadiness or non-uniformity. For example,
turbulence in the surf zone is enhanced due to wave breaking, and thus bed sediment
particles can be picked up and transported in obviously differing manners from those
observed in laboratory open channels. Another example is the process of local scouring around hydraulic structures such as bridge piers. For these two cases, the local turbulence structure is altered significantly such that corresponding sediment transport phenomena are much more complex. Unfortunately, relevant research is lacking in the literature.

Grass (1970) appeared the first to directly address the influence of turbulence on instability of individual bed particles. Girgis (1977) used fine sand with a mean sieve diameter of 0.143 mm for observing bedload transport on a flat bed, which was induced by turbulent and laminar water flows, respectively. An interesting finding obtained by Girgis was that for the same bed shear stress, the sediment transport rate induced by turbulent flows increased by 30% - 100% in comparison with that generated by laminar flows (also see Grass and Ayoub 1982). This result is qualitatively consistent with Yalin and Karahan’s (1979) observations on the incipient motion of sediment particles in laminar flows. For the same particle Reynolds number, which varied approximately from 0.7 to 10, Yalin and Karahan found the critical shear stress in laminar flows to be generally larger than that obtained in turbulent flows. This implies that the turbulence fluctuation effectively enhances sediment transport at the incipient condition.

Recently, Sumer et al. (2003) performed a laboratory study with controllable shear stress fluctuations to investigate turbulence effects on bedload transport. Their experimental results obtained for the two bed conditions, flat-bed and ripple-covered-bed, show that the sediment transport rate increases markedly with increasing turbulence levels. In particular, for the flat bed condition, the increased sediment transport rate was found to be closely related to the bed shear stress fluctuation.

This study attempts to perform an analysis of turbulence effects on sediment transport with stochastic considerations. It is noted that stochastic approaches have been
adopted previously by several researchers including Einstein (1950), Gessler (1970) and Paintal (1971). However, these studies actually failed to include variations of turbulence intensity because the relative fluctuation of different random variables, which is defined as the ratio of the rms to time-mean value, was always taken to be constant, for example, 0.5 for the lift force (Einstein 1950), 0.57 (Gessler 1970) and 0.5 (Paintal 1971) for the bed shear stress, and 0.36 for the near-bed velocity (Cheng and Chiew 1998).

For simplicity, we first consider bedload transport in laminar flows. For this extreme condition, the bed shear stress fluctuation reduces to zero, and therefore the sediment transport rate is only subject to randomness related to bed particle arrangement. Then, we assume that the bedload function derived for laminar flows is applicable for computing instantaneous sediment transport in turbulent flows. Given the dimensionless transport rate for laminar flows denoted by $\phi_L$, and the probability density function of turbulent bed shear stress denoted by $f(\tau)$, the sediment transport rate for turbulent flows can be thus expressed as

$$
\phi_T = \int_{\tau_{\text{min}}}^{\tau_{\text{max}}} f(\tau) \phi_L d\tau
$$

(1)

where $\tau_{\text{min}}$ and $\tau_{\text{max}}$ represent the range of the bed shear stress variation. Eq. (1) was previously adopted by Girgis (1977) (see also Grass and Ayoub 1982), but the proposed approaches for evaluating the two functions, $f(\tau)$ and $\phi_L$, were empirical, which finally caused the verification of Eq. (1) incomplete. Other formulations similar to Eq. (1) have also been proposed by Lopez and Garcia (2001) for computing the risk of sediment erosion; and by Garcia et al. (1999) for investigating navigation-induced sediment resuspension.
In this note, Eq. (1) is analytically evaluated. First, theoretical formulations of $f(\tau)$ and $\phi_L$ are introduced so that turbulence effects on bedload transport can be theoretically explored using Eq. (1). Then, the analytical results obtained are compared with experimental data. To facilitate all analyses conducted in this study, the condition to be considered is limited to a flat bed, which is comprised of uniform sand particles subject to unidirectional flows.
**Probability density function of bed shear stress, f(τ)**

The normal or Gaussian distribution is widely used in many engineering applications because of its simplicity and good approximation for certain cases. However, this kind of distribution cannot be applied for describing variables with skewed distributions, for example, the bed shear stress considered in this study. For a unidirectional flow over the hydraulically smooth bed, it has been found that the probability density function of the bed shear stress can be represented well with the two-parameter lognormal function (Cheng and Law 2003; Cheng et al. 2003):

\[
f(\tau) = \frac{1}{\sqrt{2\pi \ln(1 + I^2)}} \frac{\tau}{\tau_{\text{rms}}} \exp\left[-\left(\frac{\ln(\frac{\tau}{\tau_{\text{mean}}} + \sqrt{1 + I^2})}{2 \ln(1 + I^2)}\right)^2\right]
\]

for \( \tau > 0 \). Here, \( I = \tau_{\text{rms}}/\tau_{\text{mean}} \) = relative fluctuation of the bed shear stress, and \( \tau_{\text{mean}} \) and \( \tau_{\text{rms}} \) = mean and rms values of \( \tau \). It can be shown that the function given by Eq. (2) reduces to the Gaussian function if \( I \) is small (Cheng and Law 2003).

**Dimensionless laminar transport rate, \( \phi_L \)**

Einstein’s (1950) probabilistic consideration on bedload transport is used herein to express transport rate in terms of the probability of erosion. The basic function proposed by Einstein was to relate the transport rate to the probability of erosion, the diameter of particle and the characteristic time, i.e.

\[
q = a \frac{p D^2}{1 - p t}
\]

(3)
where \( q \) = volumetric bedload transport rate per unit width; \( a \) = coefficient; \( p \) = probability of erosion; \( D \) = particle diameter; and \( t \) = bedload time scale. Eq. (3) was derived for the equilibrium condition of the rate of erosion being equal to the rate of deposition per unit area of bed, so it can be used, in principle, for computing the bedload transport rate for both turbulent and laminar flows. For the condition of laminar flows, \( p \) and \( t \) can be expressed respectively as (Cheng 2003):

\[
p = 1 - \exp\left[-0.278\tau^2_D D^{0.58}\right]
\]

\[
t = a_1 \frac{\nu}{\Delta g D p}
\]

(4)

(5)

where \( \tau^*_L = \tau_L / (\rho g D) \) = dimensionless bed shear stress for laminar flows; \( \tau_L \) = bed shear stress exerted by laminar flows; \( \rho \) = fluid density; \( g \) = gravitational acceleration; \( \Delta = (\rho_s - \rho) / \rho \); \( \rho_s \) = particle density; \( D^* = D (\Delta g / \nu^2)^{1/3} \) = dimensionless particle diameter; \( \nu \) = kinematic viscosity of fluid; and \( a_1 \) = coefficient.

Using Eq. (5), Eq. (3) can be rewritten in the dimensionless form:

\[
\phi_L = a_2 \frac{\nu^2}{1 - p} D^{1.5}
\]

(6)

where \( \phi_L = q / \sqrt[3]{\Delta g D} \) = dimensionless bedload transport rate for laminar flows, and \( a_2 \) = coefficient. Furthermore, for a flat sand bed, we may assume that the shear stress is small so that the following approximation can be taken:

\[
\frac{\nu^2}{1 - p} \approx 0.0773 \tau^2_D D^{1.6}
\]

(7)

which can be derived based on Eq. (4) using the power series expansion. Substituting Eq. (7) into Eq. (6) yields

\[
\phi_L = 0.773 \tau^4_D D^{2.66}
\]

(8)
where the coefficient (0.773) was obtained by fitting the power function given by Eq. (8) to the experimental data collected by Girgis (1977).

**Bedload transport rate subject to various shear stress fluctuations**

Substituting Eqs. (8) and (2) into Eq. (1) yields

\[
\phi_t = \int_{0}^{\infty} 0.773 \tau_*^2 D_*^{2.66} \frac{1}{\sqrt{2\pi \ln(1 + I^2)}} \exp \left[ - \left( \frac{\ln \frac{\tau_*}{\bar{\tau}_*} + \ln \sqrt{1 + I^2}}{2\ln(1 + I^2)} \right)^2 \right] d\tau_* \tag{9}
\]

where \( \tau_* = \tau/(\rho g \Delta D) = \) dimensionless instantaneous bed shear stress, and \( \bar{\tau}_* = \tau_{\text{mean}}/(\rho g \Delta D) = \) dimensionless time-mean bed shear stress or the Shields parameter. Eq. (9) can be used to compute the transport rate for a flat bed comprised of uniform sediment particles, which is subject to flows with various turbulence levels. It indicates that the dimensionless transport rate generally depends on the Shields parameter, \( \bar{\tau}_* \), relative shear stress fluctuation, \( I \), and dimensionless particle diameter, \( D_* \). Examples computed using Eq. (9) are given in Figs. 1 and 2. Fig. 1 demonstrates increases in the transport rate with the turbulence intensity for \( D_* = 10 \) and \( \bar{\tau}_* = 0.05 - 0.15 \), and Fig. 2 shows similar results for \( D_* = 4 - 50 \) and \( \bar{\tau}_* = 0.08 \). Both figures show that the increased transport rate is considerable if \( I > 0.3 \).

**Comparison of Eq. (9) with experimental data**

The experimental data used for comparison are those given by Sumer et al. (2003), who conducted their experiments using a tilting flume, 10 m long and 0.3 m wide. The test section was chosen at 5.6 m from the entrance of the flume. For each
experiment, the bed shear stress was first measured using a 1-D hotfilm probe for a flow over a rigid smooth bed. The statistics of the shear stress so measured for each test was further extended for the corresponding sediment bed condition. The latter was prepared using fine sand with a diameter of 0.22 mm, which was hydraulically smooth as the rigid bed. Sediment discharge was then recorded for the same flow condition but prior to initiation of bedforms. The assumption of the shear stress remaining unchanged for the two hydraulically smooth beds is acceptable according to Kurose and Komori (2001), who reported that there were only very slight increases (less than 2%) in the shear velocity for a hydraulically smooth bed with moving bed particles.

For all experiments, the water depth at the test section was maintained at 20 cm, the Shields parameter was maintained as 0.085 while I varied ranging from 0.21 to 1.01. To generate significant fluctuations in the bed shear stress at the test section, three turbulence generators were employed to produce external disturbances in the flow. They were a pipe placed laterally at the mid-depth of the flow, a short series of grids superimposed on part of the channel (referred to as short-grid), and a long series of grids covering the entire channel (referred to as long-grid). Both pipe and grids were considered to be typical structures for shedding eddies. Such eddies, when moving downstream, would serve as external turbulence with various scales to affect the bed shear stress and thus sediment transport at the test section.

The experimental data are compared herein with analytical results that are computed using Eq. (9) by numerical integration (or computed indirectly using Eqs. (8) and (15), as shown later). Fig. 3 shows good agreement achieved between the measurements and computations. It should be mentioned that there are not constants that have been tuned for this comparison. Also plotted in the figure is the transport rate computed by assuming that the bed shear stress follows the Gaussian distribution. It can
be seen that the Gaussian distribution can be used only for the case of very low stress fluctuations, for example, $I < 0.4$.

**Variation of relative transport rate with turbulence intensity**

The relative bedload transport rate is defined as the ratio of the rate in the turbulent flow to that in the laminar flow for the same time-mean bed shear stress. This can be computed by dividing Eq. (9) by Eq. (8), which gives

$$
\frac{\phi_T}{\phi_L} = \frac{\int_0^\infty \frac{1}{\sqrt{2\pi \ln(1+I^2)}} \exp \left[ -\frac{\left( \frac{\tau}{\tau_*} + \ln \sqrt{1+I^2} \right)^2}{2 \ln(1+I^2)} \right] d\tau_*}{\tau_*^4}
$$

(10)

From Eq. (10), it follows that the relative transport rate, $\phi_T/\phi_L$, is equal to the forth moment of the relative shear stress, $(\tau/\tau_{mean})^4$. With the coefficients of kurtosis (K) and skewness (S) for the lognormal function being given as (Cheng and Law 2003)

$$
K = \left( \frac{\tau - \tau_{mean}}{\tau_{mean}} \right)^4 \frac{(\tau - \tau_{mean})^2}{(\tau_{mean}^2)} = I^8 + 6I^6 + 15I^4 + 16I^2 + 3
$$

(11)

$$
S = \left( \frac{\tau - \tau_{mean}}{\tau_{mean}} \right)^3 \frac{(\tau - \tau_{mean})^{1.5}}{(\tau_{mean}^2)^{1.5}} = I^3 + 3I
$$

(12)

one can get

$$
\left( \frac{\tau}{\tau_{mean}} \right)^4 = 4 \left( \frac{\tau}{\tau_{mean}} \right)^3 - 6 \left( \frac{\tau}{\tau_{mean}} \right)^2 + 3 + (I^8 + 6I^6 + 15I^4 + 16I^2 + 3)I^4
$$

(13)

$$
\left( \frac{\tau}{\tau_{mean}} \right)^3 = 3 \left( \frac{\tau}{\tau_{mean}} \right)^2 - 2 + (I^3 + 3I)I^3
$$

(14)
Substituting Eq. (14) into Eq. (13) and noting that $(\tau / \tau_{\text{mean}})^2 = I^2 + 1$ yields

$$\frac{\phi_T}{\phi_L} = \left( \frac{\tau}{\tau_{\text{mean}}} \right)^4 = (1 + I^2)^4$$  \hspace{1cm} (15)

Eq. (15) indicates that the relative transport rate is solely dependent on the bed shear stress fluctuation for the same bed condition. For comparison, if the Gaussian function, of which the skewness is zero and the kurtosis equals three, is used for describing the probability distribution of the bed shear stress, then the relative transport rate, $\phi_T/\phi_L$, can be expressed as

$$\frac{\phi_T}{\phi_L} = 1 + 6I^2 + 3I^4$$  \hspace{1cm} (16)

Eqs. (15) and (16) are plotted in Fig. 4 with the experimental data reported by Sumer et al. (2003), showing that Eq. (15) generally provides a good prediction of the transport rate enhanced by turbulence, while Eq. (16) is close to the measurements only for small I-values.

**Summary and conclusions**

Bedload transport comprises a number of probabilistic events of the bed particle movement, which is closely related to statistical characteristics of flow and bed particle configuration. In this study, both kinds of randomness caused by turbulent flows and bed geometry are considered. However, relevant analyses are conducted only for the condition of flat bed comprised of uniform sand particles.

First, the laminar bedload function is assumed applicable for evaluation of the instantaneous transport rate in turbulent flows. Using the lognormal function for
describing the probability density function of the bed shear stress for unidirectional flows, bedload transport rate in turbulent flows is then formulated as a probabilistic average of the laminar bedload function for the entire range of bed shear stress. The analytical results obtained show that the transport rate enhanced by turbulence can be expressed as a simple function of the relative fluctuation of the bed shear stress. The predicted transport rates using the present approach compare well with experimental data. The computed results also indicate that the prediction using the shear stress distribution following the Gaussian function is applicable only for low turbulence levels.

**Notation**

- \( a \) coefficient.
- \( a_1 \) coefficient.
- \( a_2 \) coefficient.
- \( D \) diameter of particles.
- \( D* \) dimensionless particle diameter \([= D(\Delta g/\nu^2)^{1/3}]\).
- \( g \) gravitational acceleration.
- \( f(\tau) \) probability density function of turbulent bed shear stress.
- \( I \) relative bed shear stress fluctuation \((= \tau_{rms}/\tau_{mean})\).
- \( K \) Kurtosis coefficient.
- \( p \) probability of erosion.
- \( q \) volumetric bedload transport rate per unit width.
- \( S \) skewness coefficient.
- \( t \) bedload time scale.
- \( \nu \) kinematic viscosity of fluid.
\( \tau_L \) bed shear stress exerted by the laminar flow.

\( \tau_{\text{max}} \) maximum bed shear stress.

\( \tau_{\text{mean}} \) time-mean bed shear stress.

\( \tau_{\text{min}} \) minimum bed shear stress.

\( \tau_{\text{rms}} \) rms value of bed shear stress.

\( \tau^* \) dimensionless instantaneous shear stress \([= \tau/(\rho g \Delta D)]\).

\( \tau^*_{\text{L}} \) \( \tau_L/(\rho g \Delta D) \).

\( \bar{\tau}^* \) dimensionless time-mean shear stress or the Shields parameter \([= \tau_{\text{mean}}/(\rho g \Delta D)]\).

\( \phi_T \) dimensionless bedload transport rate for turbulent flows.

\( \phi_L \) dimensionless bedload transport rate for laminar flows.

\( \Delta \) \((\rho_s - \rho)/\rho\).

\( \rho \) fluid density.

\( \rho_s \) particle density.

**References**


Fig. 1. Variations of bedload transport rate with bed shear stress fluctuation for $\tau_s = 0.05 – 0.15$ and $D_s = 10$. 
Fig. 2. Variations of bedload transport rate with bed shear stress fluctuation for $D^* = 4$ – 50 and $\tau_s = 0.08$. 
Fig. 3. Comparison of computed bedload transported rates with experimental data reported by Sumer et al. (2003). The solid curve was computed using Eq. (9) or Eq. (15) and the dashed line obtained based on the shear stress following Gaussian distribution.
Fig. 4. Comparison of computed relative transport rates with experimental data reported by Sumer et al. (2003). The solid curve was computed using Eq. (15) and the dashed line obtained with Eq. (16).