NON-COMMUTATIVE CHERN-WEIL THEORY AND THE COMBINATORICS OF WHEELING

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Abstract. This work applies the ideas of Alekseev and Meinrenken’s Non-commutative Chern-Weil Theory to describe a completely combinatorial and constructive proof of the Wheeling Theorem. In this theory, the crux of the proof is, essentially, the familiar demonstration that a characteristic class does not depend on the choice of connection made to construct it. To a large extent this work may be viewed as an exposition of the details of some elements of Alekseev and Meinrenken’s theory written for Kontsevich integral specialists. Our goal was a presentation with full combinatorial detail in the setting of Jacobi diagrams - to achieve this goal a certain level of invention proved necessary.

1. Introduction and Outline

This paper is organized around a purely combinatorial proof of what is called the Wheeling Theorem. The Wheeling Theorem constructs an algebra isomorphism between a pair of much-studied algebras, $A$ and $B$:

$$(\chi_B \circ \partial_H) : B \to A.$$ 

We will recall these spaces and maps shortly, beginning in Section 2. The formal development of the theory begins in Section 3. Our underlying aim is to use this proof to introduce a combinatorial reconstruction of some elements of Alekseev and Meinrenken’s Non-commutative Chern-Weil theory [M].

The work falls naturally into two parts:

- In the first part we will describe and prove a statement we will call “Homological Wheeling”. This is a generalization of Wheeling to the setting of a “Non-commutative Weil complex for diagrams”. That complex will be introduced in Section 5. After that, Homological Wheeling will be stated as Theorem 6.0.7. The homotopy equivalence at the heart of the proof will be constructed in Section 7.

- The remainder of the paper will be occupied with describing how the usual statement of Wheeling is recovered from Homological Wheeling when certain relations are introduced. See Section 9 for an illustration summarizing the mechanism which creates wheels. This part consists mainly of the sort of delicate “gluing legs to legs” combinatorics that Kontsevich integral specialists will find very familiar (though here we have the extra challenge of keeping track of permutations and signs). Indeed, part of the motivation for this work was that it presented an opportunity to revisit some of the combinatorial detail of such works as [GK].

Wheeling is a combinatorial version of the Duflo isomorphism of Lie theory (but doesn’t follow directly from it). Let’s take a moment to recall what it does: The
classical Poincare-Birkhoff-Witt theorem concerns $\mathfrak{g}$, a Lie algebra, $S(\mathfrak{g})^g$, the algebra of $\mathfrak{g}$-invariants in the symmetric algebra of $\mathfrak{g}$, and $U(\mathfrak{g})^g$, the algebra of $\mathfrak{g}$-invariants in the universal enveloping algebra of $\mathfrak{g}$. PBW says that the natural averaging map actually gives a vector space isomorphism between the vector spaces underlying these algebras. The averaging map, however, does not respect the product structures. Duflo’s fascinating discovery was, for a class including the semisimple Lie algebras, of a concrete isomorphism of $S(\mathfrak{g})^g$ which, when composed with the averaging map, promoted it to an algebra isomorphism $[D]$.

1.1. **The Kontsevich integral as the underlying motivation.** While it’s true that Wheeling is a combinatorial analogue of the Duflo isomorphism, its origins and role are more interesting than that fact suggests. When Bar-Natan, Garoufalidis, Rozansky and Thurston conjectured that the Wheeling map was an algebra isomorphism they were motivated, in part, by questions that arose in the developing theory of the Kontsevich integral (see [BGRT] and [BLT] and discussions therein). Recall that the Kontsevich integral is a knot invariant (amongst other things) taking values in a certain space of formal power series of trivalent graphs. The classic place to learn about this topic is [B]. Wheeling is a fundamental topic in the study of the Kontsevich integral because:

- The Kontsevich integral of the unknot turns out to be precisely the power series of graphs that is used in the construction of the Wheeling isomorphism.
- In addition to the above fact, Bar-Natan, Le, and Thurston showed that the Wheeling Theorem was a consequence of the invariance of the Kontsevich integral under a certain elementary isotopy (this is their famous ‘1+1=2’ argument). See [BLT].
- Wheeling has proved to be an indispensable tool in almost all ‘hard’ theorems that can be proved about the Kontsevich integral.

The Kontsevich integral still fascinates us for many different reasons. The promise that the Alekseev-Meinrenken Non-Commutative Chern-Weil theory might provide a new, perhaps quite powerful, point of view on the Kontsevich integral was the main motivation for this work. While no such advance can be found here, the aim of this work is to facilitate further investigations.

Let us mention one puzzle: How does this theory explain the fact that the Kontsevich integral of the unknot is the Duflo element? We speculate that this puzzle could be solved by constructing some map

$$\{\text{Embeddings } S^1 \hookrightarrow \mathbb{R}^3 \} \rightarrow \tilde{W},$$

which gave a knot invariant when basic cohomology was taken, and which covered the usual Kontsevich invariant:

$$H(\tilde{W}_{\text{basic}}) \downarrow \uparrow \quad ??? \quad \{\text{Knots}\} \rightarrow \mathbb{Z} \rightarrow \mathcal{A}$$

But first we need to learn what is $\tilde{W}$ (the “non-commutative Weil complex for diagrams”), what is $H(\tilde{W}_{\text{basic}})$ (its “basic cohomology”), and what that has to do
with the familiar space $A$. And before embarking on all that, we first need some context - Chern-Weil theory.

1.2. The origins of this work in Alekseev-Meinrenken’s Non-commutative Chern-Weil theory. This work originated in a course given by E. Meinrenken at the University of Toronto in the Spring of 2003 entitled “Group Actions on Manifolds”, in which Alekseev and Meinrenken’s work on a Non-commutative Chern-Weil theory was described. See [M] for a short review of the theory, [AM] for the original paper, and [AM05] for later developments.

Our goal is a self-contained reconstruction and exploration of their technology in the combinatorial setting of the Jacobi diagrams of Quantum Topology. We’ll begin that development in the next section. For the purpose of providing some context, however, we’ll now give a very brief sketch of the outlines of classical Chern-Weil theory. A more detailed discussion of how our combinatorial definitions fit into the classical picture will be given in Section 3.9, once those definitions have been made.

So let $G$ be a compact Lie group with Lie algebra $g$. Chern-Weil theory is a theory of characteristic classes in de Rahm cohomology of smooth principal $G$-bundles. Recall that every smooth principal $G$-bundle arises as a smooth manifold $P$ with a free smooth left $G$-action.

What is a characteristic class? The first thing to say is that it is a choice, for every principal $G$-bundle $P$, of a class $w(P)$ in the cohomology of the base space, $B = P/G$, which is functorial with respect to bundle morphisms. To be precise: if there is a morphism of principal $G$-bundles $P$ and $P'$, which means that there is a $G$-equivariant map $f : P \to P'$, then it should be true that

$$f_B^*(w(P')) = w(P),$$

where $f_B$ denotes the map induced by $f$ between the base spaces $f_B : B \to B'$.

Chern-Weil theory constructs such systems of classes in de Rahm cohomology in the following way.

1. It takes an arbitrarily chosen connection form $A \in \Omega^1(P) \otimes g$.
2. It uses it to construct closed forms in $\Omega(P)_{\text{basic}} \subset \Omega(P)$. This is the ring of basic differential forms on $P$. It is most simply introduced as the isomorphic image under $\pi^*$ of $\Omega(B)$. It is a crucial fact in the theory that this basic subring is selected as the kernel of a certain Lie algebra of differential operators arising from the generating vector fields of the $G$-action. The main task of the theory is to canonically construct forms from $A$ lying in the kernel of this Lie algebra. (See Section 3.9 for a more detailed discussion).
3. It then shows that the cohomology classes in $H_{dR}(B)$ so constructed do not depend on the choice of connection and do indeed have the required functoriality with respect to bundle morphisms.

The heart of the Chern-Weil construction is an algebraic structure called the Weil complex. The Weil complex is based on the graded algebra $S(\hat{g} \oplus \hat{g})$ (that is, the graded symmetric algebra on two copies of $\hat{g}$, the dual of $g$, one copy in grade 1 and one copy in grade 2). The Weil complex is this graded algebra equipped with a certain graded Lie algebra of graded derivations. This Lie algebra of derivations selects a key subalgebra $S(\hat{g} \oplus \hat{g})_{\text{basic}}$, the basic subalgebra.

What is the construction then? Consider some principal $G$-bundle $P$. Given the fixed data of a connection form $A \in \Omega^1(P) \otimes g$ on $P$, Chern-Weil theory constructs
a map
\[ c_A : S(\hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}}) \to \Omega(P) \]
by the correspondence
\[ g_1 g_2 \ldots g_n \mathfrak{f}_1 \ldots \mathfrak{f}_m \mapsto g_1(A) \wedge \ldots \wedge g_n(A) \wedge d\mathfrak{f}_1(A) \wedge \ldots \wedge d\mathfrak{f}_m(A). \]

It is immediate in the theory that this construction specializes to a map of differential graded algebras from the basic subalgebra of the Weil complex into the ring of basic forms on \( P \):
\[ S(\hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}})_{\text{basic}} \to \Omega(P)_{\text{basic}} = \Omega(B), \]
so that, passing to cohomology, we get a map:
\[ H^\ast(S(\hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}})_{\text{basic}}) \to H^\ast_{dR}(B). \]

The fundamental theorem of Chern-Weil theory is that this construction does indeed give a canonical system of rings of characteristic classes. We can summarize Chern-Weil theory with the following commutative diagram:

\[
\begin{array}{ccc}
H^\ast(\Omega(P)_{\text{basic}}) & \longrightarrow & H^\ast_{dR}(B) \\
H^\ast(S(\hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}})_{\text{basic}}) & \downarrow & \downarrow f_B \\
H^\ast(\Omega(P')_{\text{basic}}) & \longrightarrow & H^\ast_{dR}(B')
\end{array}
\]

So why should this theory have anything to say about the Duflo isomorphism, which seems a purely algebraic fact? The connection arises from the key computation that
\[ H(S(\hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}})_{\text{basic}}) = (S\mathfrak{g})^\mathfrak{g}. \]

In words: the basic cohomology of the Weil complex, which is the universal ring of characteristic classes (consider its place in the above diagram), is precisely the ring of \( \mathfrak{g} \)-invariants in \( S\mathfrak{g} \), a ring involved in the Duflo isomorphism. (We remark that this formulation assumes that \( \mathfrak{g} \) has a symmetric, non-degenerate, invariant inner product.)

1.3. A Weil complex for diagrams. This paper will start with a combinatorial reconstruction of the computation
\[ H(S(\hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}})_{\text{basic}}) = (S\mathfrak{g})^\mathfrak{g}. \]

Recall that when we abstract Wheeling from the Duflo isomorphism, we replace \((S\mathfrak{g})^\mathfrak{g}\) by \( \mathcal{B} \), the space of symmetric Jacobi diagrams. The familiar “Weight system” map (read \( [B] \)) connects the two:
\[ \text{Weight}_\mathfrak{g} : \mathcal{B} \to (S\mathfrak{g})^\mathfrak{g}. \]

Our paper will begin by replacing the “something” in the following equation
\[ H((\text{Something})_{\text{basic}}) = \mathcal{B}, \]
by an appropriate construction: \( \mathcal{W} \), the commutative Weil complex for diagrams. The complex \( \mathcal{W} \) is a combinatorial version of the usual Weil complex \( S(\hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}}) \). This usual Weil complex is built from two copies of \( \hat{\mathfrak{g}} \): one in grade 1 and one in grade 2. Thus \( \mathcal{W} \) will be built from diagrams with two types of legs: legs of grade 1 and
legs of grade 2 (the grade 2 legs will be distinguished by drawing them with a fat dot). The legs will be ordered and when two consecutive legs have their positions swapped, the diagram picks up a sign \((-1)^{xy}\), where \(x\) and \(y\) denote the grades of the involved legs. Observe:

\[
\begin{array}{c}
\begin{array}{c}
\text{=}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{=}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{in } \mathcal{W}^6.
\end{array}
\end{array}
\]

The details of the construction of \(\mathcal{W}\) are presented in Section 3. First, in Section 3.1, we’ll introduce the diagrams that the construction is based on: we’ll call them Weil diagrams. Then, in Section 3.3, we’ll discuss what we mean by a “formal linear differential operator” on these diagrams. After that, in Section 3.4, we’ll introduce the differential as just such a differential operator.

To fit \(\mathcal{W}\) into the equation

\[H(\mathcal{W}_{\text{basic}}) = \mathcal{B},\]

we need to understand what is \(\mathcal{W}_{\text{basic}} \subset \mathcal{W}\), its basic subcomplex. This complex is selected as the kernel of a certain grade \(-1\) operator \(\iota\). This is a pervasive algebraic arrangement in this work (i.e. selecting a basic subcomplex as the kernel of some grade \(-1\) differential operator), so we’ll introduce a definition “\(\iota\)-complex”, in Section 3.7, in order to clarify subsequent discussion. Briefly: an \(\iota\)-complex is a bigraded complex equipped with a differential and grade \(-1\) map \(\iota\) which squares to zero and graded-commutes with the differential:

\[d \circ \iota + \iota \circ d = 0.\]

In Section 3.8, we will construct \(\mathcal{W}\) as just such an \(\iota\)-complex.

The final ingredient required in the explication of the equation \(H(\mathcal{W}_{\text{basic}}) = \mathcal{B}\) is the map \(\Upsilon: \mathcal{B} \rightarrow \mathcal{W}\) which actually performs the isomorphism. This map \(\Upsilon\) occupies a key role in our presentation. It will be introduced in Section 4, where the computation that \(H(\mathcal{W}_{\text{basic}}) = \mathcal{B}\) will be concluded. Its definition on some symmetric Jacobi diagram \(w\) is quite simple: choose an ordering of the legs of \(w\), then expand each leg according to the following rule:

\[
\Upsilon: w \mapsto -\frac{1}{2}.
\]

To record that this map \(\Upsilon\) is key in our presentation, we’ll give it a name: the hair-splitting map. (Readers familiar with the history of this subject may see why.)

1.4. Some motivation for a non-commutative Weil complex. As the author learnt it from E. Meinrenken, this construction can be motivated with the following line of thought: to begin, note that there is almost a much more natural way to build these characteristic classes in \textbf{de Rahm cohomology}. The more natural way would exploit \(EG\), the classifying bundle of principal \(G\)-bundles, and its associated
base space $EG/G = BG$. Recall the key property of $EG$ that the bundle pullback operation performs a bijection, for some space $X$:

\[
\{ \text{Isomorphism classes of principal } G\text{-bundles over } X \} \leftrightarrow [X : BG].
\]

So, proceeding optimistically, we could define the ring of characteristic classes associated to some smooth principal $G$-bundle $P$ by the map:

\[
\text{Class}^*_P : H_{dR}(BG) \rightarrow H_{dR}(B),
\]

for some choice of classifying map $\text{Class}_P : B \rightarrow BG$ giving $P$ as its corresponding pullback bundle. This ring would be well-defined because the map $\text{Class}_P$ is unique up to homotopy. Furthermore, this arrangement automatically gives the required commutative diagrams:

The problem with this picture, of course, is that the space $BG$ is not in general a manifold, so there is no ring of differential forms $\Omega(BG)$ with which to work. The Weil complex can be seen as a kind of stand in for the ring of differential forms on the classifying bundle $EG$:

\[
S(g^* \oplus g^*) = \"\Omega(EG)\".
\]

But this picture is too nice to let it go so easily. It turns out that there does exist a point of view in which it is reasonable to discuss “differential forms” on $EG$: the simplicial space construction of $EG$ (see Section 2 of [MP] for a discussion of the required simplicial techniques). $EG$ can be constructed as the geometric realization of a certain simplicial manifold:

\[
EG = \left( \prod_{n=0}^{\infty} \Delta^n \times G^n \right) / \sim.
\]

It follows, with some work, from a theorem of Moore’s (see Section 4 of [MP]) that the real cohomology of such a space can, in fact, be computed as the total cohomology of an appropriate double complex built from differential forms on the pieces of the simplicial manifold (in the case at hand they are $G^n$). (Moore’s theorem is a generalization of the familiar theorem that equates singular and simplicial cohomology). To be precise, let $C^n = \oplus_{i=0}^n \Omega^i(G^{n-i})$. Then:

\[
H(EG, \mathbb{R}) \simeq H(C^*).
\]

So, with this beautiful idea in hand, we may feel that we are ready to build a Chern-Weil theory which involves the classifying spaces, as it should. There is a final hurdle to be cleared though: the natural product structure on $\oplus_{n=0}^\infty \Omega^i(G^{n-i})$ that is provided by Moore’s theorem is not graded commutative, unlike the wedge product on the usual ring of differential forms $\Omega(X)$. Well, we didn’t really need commutativity anyway, so we follow Alekseev and Meinrenken in passing to the non-commutative Weil complex. This has the same definition as the usual Weil complex, but without introducing commutativity (so that it is based on a
It has natural maps to both the usual, commutative, Weil complex, as well as the “de Rahm complex” of $EG$:

\[
\bigoplus_n \bigoplus_{i=0}^n \Omega^i(G^{n-i})
\]

The usual Chern-Weil map based on a certain “connection” on $EG$.

\[
T(\hat{g} \oplus \hat{g}) 
\]

The natural projection.

\[
S(\hat{g} \oplus \hat{g})
\]

The Alekseev-Meinrenken proof of the Duflo isomorphism arises from an algebraic study of the relationship between the two Weil complexes, $T(\hat{g} \oplus \hat{g})$ and $S(\hat{g} \oplus \hat{g})$.

1.5. The Non-commutative Weil complex for Diagrams. So in Section 5 we define $\tilde{W}$, a non-commutative Weil complex for diagrams, in the way that will be obvious at this point of the development. The interplay between $W$ and $\tilde{W}$ is at the heart of this theory. There are two natural maps between these $ι$-complexes:

\[
W \xrightarrow{\chi_W} \tilde{W} \xleftarrow{\tau} \tilde{W}.
\]

The map $\chi_W$ is a key map: the graded averaging map. Its action on some Weil diagram $w$ is to take the average of all diagrams you can get by rearranging the legs of $w$ (each multiplied by a sign appropriate to the rearrangement). The map $\tau$ is the basic projection map (corresponding to the introduction of the commutativity relations).

1.6. Homological Wheeling. When we compose $\Upsilon$, the “Hair-splitting map”, with $\chi_W$, the graded averaging map, we get a map which takes elements of $B$ to basic cohomology classes in $\tilde{W}$:

\[
B \xrightarrow{\Upsilon} W_{basic} \xrightarrow{\chi_W} \tilde{W}_{basic}.
\]

So if we take two elements $v$ and $w$ of $B$, then

\[
(\chi_W \circ \Upsilon)(v \sqcup w)
\]

represents a basic cohomology class in $\tilde{W}_{basic}$.

But there is another way to build a basic cohomology class in $\tilde{W}_{basic}$ from $v$ and $w$, using the juxtaposition product $#$ on $W$. This is a product on basic cohomology arising from the following product defined on the generating diagrams:

\[
\begin{array}{ccc}
\text{Diagram 1} & \# & \text{Diagram 2}
\end{array}
\]

\[
\Rightarrow \text{Diagram Result}
\]

Theorem 1.6.1 (Homological Wheeling). Let $v$ and $w$ be elements of $B$. Then the two elements of $\tilde{W}_{basic}$:

- $(\chi_W \circ \Upsilon)(v \sqcup w)$
- $(\chi_W \circ \Upsilon)(v) \# (\chi_W \circ \Upsilon)(w)$

represent the same basic cohomology class.
The technical fact which underlies Homological Wheeling is that the two maps from the basic subcomplex of $\tilde{W}$ to itself:

\[
\tilde{W}_{\text{basic}} \xrightarrow{\chi_W \circ \Upsilon} \tilde{W}_{\text{basic}} \xrightarrow{\chi_W \circ \tau} \tilde{W}_{\text{basic}}
\]

are chain homotopic. The homotopy is constructed in Section 7. In words: if $z \in \tilde{W}$ represents some basic cohomology class, then its graded symmetrization represents the same basic cohomology class. In the case at hand, this fact implies that:

\[
(\chi_W \circ \Upsilon) (v) \# (\chi_W \circ \Upsilon) (w)
\]

and

\[
(\chi_W \circ \Upsilon) (v \sqcup w)
\]

represent the same basic cohomology class, as required.

Before commencing on the formalities, let’s compare Wheeling and Homological Wheeling:

<table>
<thead>
<tr>
<th>Wheeling</th>
<th>Homological Wheeling</th>
</tr>
</thead>
<tbody>
<tr>
<td>The Wheeling map $\partial_\Omega$: Gluing the Duflo element into legs in all possible ways.</td>
<td>The Hair-splitting map $\Upsilon$: Gluing forks into legs in all possible ways.</td>
</tr>
<tr>
<td>The averaging map $\chi_\mathcal{B}$.</td>
<td>The graded averaging map $\chi_W$.</td>
</tr>
<tr>
<td>Wheeling says: $(\chi_\mathcal{B} \circ \partial_\Omega) (v \sqcup w)$ and $(\chi_\mathcal{B} \circ \partial_\Omega) (v) # (\chi_\mathcal{B} \circ \partial_\Omega) (w)$ are equal in $\mathcal{A}$.</td>
<td>HW says: $(\chi_W \circ \Upsilon) (v \sqcup w)$ and $(\chi_W \circ \Upsilon) (v) # (\chi_W \circ \Upsilon) (w)$ represent the same basic cohomology class in $\tilde{W}$.</td>
</tr>
</tbody>
</table>

Homological Wheeling clarifies Wheeling in a number of different ways:

- The map $\Upsilon$ is considerably simpler and less mysterious than $\partial_\Omega$.  
- HW has a completely transparent and combinatorial proof. The underlying mechanisms of the proof are quite accessible to study and computation.

The cost of HW (at least from the combinatorial point of view) is that transferring statements in $\tilde{W}$ to statements in $\mathcal{A}$ involves a considerable amount of combinatorial work. In fact, the bulk of this paper, from Section 8 on, is taken up with such computation.

1.7. **Acknowledgements.** The author has benefitted greatly from the assistance and expertise of Dror Bar-Natan and Eckhard Meinrenken, and many others at the University of Toronto.

2. **$\mathcal{A}$, $\mathcal{B}$ and the Averaging Map.**

Apart from some comments about how we grade diagrams in this work, the material in this section is standard. The algebras $\mathcal{A}$ and $\mathcal{B}$ are built from certain graph-theoretic objects called Jacobi diagrams: $\mathcal{A}$ from ordered Jacobi diagrams and $\mathcal{B}$ from symmetric Jacobi diagrams. Here is an example of a symmetric Jacobi diagram:
What is it? It is first of all a graph. By graph we’ll mean a triple \((V,E,\psi)\) consisting of a finite set \(V\) of vertices together with a finite set \(E\) of edges and a map \(\psi : E \to V^{(2)},\) where \(V^{(2)}\) denotes the set of unordered pairs of vertices. (Note that, in particular, the edges of our graphs are unoriented, we allow multiple edges between the same pair of vertices, and we also allow edges both of whose endpoints are the same vertex.) An isomorphism between two graphs \((V_G,E_G,\psi_G)\) and \((V_H,E_H,\psi_H)\) is a pair of bijections \(\theta_V : V_G \to V_H\) and \(\theta_E : E_G \to E_H\) respecting the functions \(\psi_G\) and \(\psi_H\).

A symmetric Jacobi diagram, then, is a graph with vertices of degree 1 and 3 where, in addition, each trivalent vertex has been oriented (equipped with a cyclic ordering of its incident edges). To read the orientation of a vertex from a drawing of a symmetric Jacobi diagram (as above) simply take the counter-clockwise ordering determined by the drawing. Two symmetric Jacobi diagrams are isomorphic if there is an isomorphism of their underlying graphs which respects the orientations at the trivalent vertices.

Definition 2.0.1. The space \(\mathcal{B}\) is defined to be the rational vector space obtained by quotienting the vector space of formal finite rational linear combinations of isomorphism classes of symmetric Jacobi diagrams by its subspace generated by the anti-symmetry (AS) and Jacobi (IHX) relations:

\[
\text{AS: } \begin{array}{c} + \end{array} = 0.
\]

\[
\text{IHX: } \begin{array}{c} - \end{array} = 0.
\]

Next let’s consider the space \(\mathcal{A}\). It is built from ordered Jacobi diagrams. An ordered Jacobi diagram, such as

is the same as a symmetric Jacobi diagram but has an extra piece of structure: the set of vertices of degree 1 has been ordered.

Definition 2.0.2. The space \(\mathcal{A}\) is defined to be the rational vector space obtained by quotienting the vector space of formal finite rational linear combinations of isomorphism classes of ordered Jacobi diagrams by its subspace generated by the anti-symmetry (AS), Jacobi (IHX) and permutation (STU) relations:

\[
\text{STU: } \begin{array}{c} - \end{array} = 0.
\]

There is a natural map between these vector spaces, the averaging map

\[ \chi_B : \mathcal{B} \to \mathcal{A}. \]
This map is defined by linearly extending its value on symmetric Jacobi diagrams. Its value on some symmetric Jacobi diagram is the average of all the different ordered Jacobi diagrams that can be obtained by choosing an ordering of the legs. (Remark: in this work we require averaging maps between a variety of different spaces. To keep the notation logical we will record the domain space as a subscript on the symbol $\chi$.) The proof of the following statement can be read in [B]:

**The formal Poincare-Birkhoff-Witt theorem.** The averaging map is a vector space isomorphism.

These spaces have natural product structures. The product on $A$ (we'll refer to it as the “juxtaposition product”) looks like

$$
\begin{array}{ccc}
\ast & \# & = \\
\ast & \ast & \\
\end{array}
$$

The product on $B$ (the “disjoint union” product) looks like

$$
\begin{array}{ccc}
\ast & \⊔ & = \\
\ast & \ast & \\
\end{array}
$$

Famously, while the averaging map is a vector space isomorphism it is not an isomorphism of algebras. (That is, the statement $\chi_B(a \⊔ b) = \chi_B(a)\#\chi_B(b)$ is not, in general, true.) It was Duflo’s amazing discovery (in the Lie algebra setting [D], the combinatorial version here due to Bar-Natan, Garoufalidis, Rozansky and Thurston [BGRT, BLT]) that if the averaging map is preceded by a certain vector space isomorphism of $B$

$$
\partial_\Omega : B \to B
$$

then one does obtain an isomorphism of algebras:

$$(\chi_B \circ \partial_\Omega)(a \⊔ b) = (\chi_B \circ \partial_\Omega)(a)\#(\chi_B \circ \partial_\Omega)(b).$$

Our next task, then, is to recall the isomorphism $\partial_\Omega$.

**2.1. Operating with diagrams on diagrams.** Let $D_1$ and $D_2$ be two symmetric Jacobi diagrams. The notation $\partial_{D_1}(D_2)$ will denote the result of operating with $D_1$ on $D_2$. That is, $\partial_{D_1}(D_2)$ is defined to be the sum of all the possible symmetric Jacobi diagrams that can be obtained by gluing all of the legs of $D_1$ to some of the legs of $D_2$.

Let’s say that again with more combinatorial precision. Let $D_1$ have $n$ legs, and number them in some way. Let $D_2$ have $m$ legs, and number them in some way. For some injection

$$
\sigma : \{1, \ldots, n\} \to \{1, \ldots, m\}
$$

let $D_\sigma$ denote the diagram that results when the legs of $D_1$ are joined to the legs of $D_2$ according to the map $\sigma$. For example, if

$$
D_1 = 1 \quad 2 \quad \text{and} \quad D_2 = \begin{array}{ccc}
2 & 1 \\
3 & 4 \\
\end{array}
$$

then $D_{(13)} = \text{ (diagram) }$. 
In these terms the diagram operation can be defined:
\[
\partial_{D_1}(D_2) = \sum_{\text{injections } \sigma} D_\sigma .
\]
(Note that if \( n > m \) then the result is zero.) We can extend this operation to the case when \( D_1 \) and \( D_2 \) are finite linear combinations of symmetric Jacobi diagrams in an obvious way.

The wheeling isomorphism \( \partial_\Omega \) is obtained by operating with a certain remarkable power series of diagrams, \( \Omega \). Before introducing this power series we must say a few words about gradings.

2.2. Gradings and \( \Omega \). It is traditional to grade the spaces \( A \) and \( B \) by half the number of vertices in the diagram. This grading has no use in this paper, so, to avoid complication, we will ignore it.

On the other hand the leg-grade of a diagram will play a crucial role.

**Definition 2.2.1.** Define the leg-grade of a symmetric Jacobi diagram to be twice the number of legs of the diagram.

The reason for the weight of 2-per-leg will become clear very soon. If \( B^n \) denotes the subspace of leg-grade \( n \) then we have a direct-sum decomposition
\[
B = \bigoplus_{n=0}^{\infty} B^n .
\]
When we refer to a power series \( \Gamma \) of symmetric Jacobi diagrams we are referring to an element of the direct product vector space
\[
\Gamma \in \prod_{n=0}^{\infty} B^n .
\]
Note that if \( \Gamma \) is such a power series, then we get a perfectly well-defined map
\[
\partial_\Gamma : B \to B
\]
because all but finitely many of the terms of \( \Gamma \) contribute zero.

Now to introduce \( \Omega \). For this purpose we will employ a certain notation that gets used in disparate works in the literature. If we draw a diagram, orient an edge of the diagram at some point, and label that point with a power series in some formal variable \( a \), then that diagram refers to the power series of symmetric Jacobi diagrams that one obtains by replacing powers of \( a \) with legs. That is,
\[
\begin{align*}
&\begin{cases}
c_0 + c_1 a + c_2 a^2 + c_3 a^3 + \ldots
\end{cases} \\
\text{denotes} &\quad
\begin{cases}
c_0 \quad + \quad c_1 \quad + \quad c_2 \quad + \quad c_3 \quad + \ldots
\end{cases}
\end{align*}
\]
Definition 2.2.2. The Wheels element, \( \Omega \), is the formal power series of symmetric Jacobi diagrams defined by the expression

\[
\Omega = \exp \left( \frac{1}{2} \bigcirc \ln \left( \frac{\sinh \left( \frac{a}{2} \right)}{\frac{a}{2}} \right) \right) \in \prod_{n=0}^{\infty} B^n.
\]

This element next appears in computations towards the end of this paper, in Theorem 11.0.3.

3. The commutative Weil complex for diagrams

The purpose of this section is the introduction of a certain cochain complex \( W_{\text{basic}} \). The next section will describe a certain map of cochain complexes (with the spaces \( B^i \) assembled into a complex with zero differential),

\[
\Upsilon^i_{\text{basic}} : B^i \to W^i_{\text{basic}},
\]

which gives isomorphisms in cohomology:

\[
H^i(\Upsilon_{\text{basic}}) : B^i \simeq H^i(B) \rightarrow H^i(W_{\text{basic}}).
\]

The complex \( W_{\text{basic}} \) will be obtained as a subcomplex of a larger complex \( W \). This will be the first instance of a procedure that will pervade this work; to organize the discussion we’ll employ the term “\( \iota \)-complex”. For example, \( W \) will be an example of an \( \iota \)-complex. An \( \iota \)-complex is just a triple \((W, W_\iota, \iota)\) consisting of a pair of cochain complexes \( W \) and \( W_\iota \) together with a collection of grade \(-1\) maps \( \iota^i : W^i \to W^i_{\iota} \) which graded-commute with the differentials. The corresponding basic complex, \( W_{\text{basic}} \), is obtained by taking the kernels of the maps \( \iota^i \).

The spaces \( \{W^i\} \) and \( \{W^i_{\iota}\} \) are built from something we’ll call a Weil diagram, as we will shortly detail.

Our goal is a clean presentation of the combinatorial constructions. These constructions may appear quite mysterious at first glance, however, so to provide some context to the theory, we’ll close the section by constructing a “characteristic class-valued weight system”. Given the data of a compact Lie group \( G \), a smooth principal \( G \)-bundle \( P \), and a connection form \( A \) on \( P \), we’ll construct a map of complexes:

\[
\text{Weight}_{(G,P,A)} : W \to \Omega(P)
\]

which yields a map

\[
H \left( \text{Weight}_{(G,P,A)} \right) : H(W_{\text{basic}}) \to H(\Omega(P)_{\text{basic}}) \simeq H_{dR}(B),
\]

where \( B = P \setminus G \) is the base of the principal bundle.

3.1. Weil diagrams. The distinguishing feature of Weil diagrams is that their legs are graded. Every leg of a Weil diagram is either of grade 1 or of grade 2. Here is our first Weil diagram:
The grade 2 legs are distinguished by drawing them with a fat dot. The leg-grade of a Weil diagram is the total grade of the legs. Thus, the above example has leg-grade 6.

(Otherwise, the definition of Weil diagram is unsurprising: it is a graph with vertices of degree 1 and 3, the degree 3 vertices are oriented, its set of degree 1 vertices $U$ is ordered, and its degree 1 vertices are weighted by a grading $U \rightarrow \{1, 2\}$, as just mentioned.)

Occasionally we need something slightly more general. A “Weil diagram with a special degree 1 vertex” is defined just as a Weil diagram except that exactly one of the degree 1 vertices is neither included in the total ordering nor assigned a leg-grading. Such diagrams are depicted with the special degree 1 vertex floating freely. Observe:

3.2. Permuting the legs of Weil diagrams. When constructing spaces from these diagrams, we’ll always employ the familiar AS and IHX relations. The numerous spaces we employ will differ from each other according to how they treat the legs of the diagrams. As there are really quite a number of different sets of relations applied to legs in this work we’ll remind the reader of which space we are in by drawing the arrow on the orienting line in a different style depending on which space the diagram is in.

To begin: consider relations which say that we are allowed to move the legs around freely, as long as when we transpose an adjacent pair of legs we introduce the sign $(-1)^{y-x}$, where $x$ and $y$ are the grades of the two legs involved:

\[
\begin{align*}
\text{Perm}_1 : & \quad \begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{image}}
\end{array} = 0 \\
\text{Perm}_2 : & \quad \begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{image}}
\end{array} = 0 \\
\text{Perm}_3 : & \quad \begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{image}}
\end{array} = 0
\end{align*}
\]

Definition 3.2.1. Let $i$ be a non-negative integer. The vector space $W^i$ is defined to be the quotient of the vector space of formal finite $\mathbb{Q}$-linear combinations of isomorphism classes of Weil diagrams of leg-grade $i$ by the subspace generated by AS, IHX and Perm relations.

We are now going to assemble these vector spaces into a complex. Before we do that we will make some general comments about formal linear differential operators.
3.2.2. A clarification concerning $\mathcal{W}^0$. We regard the empty diagram as a Weil diagram. Thus, for example,

$$\frac{1}{2} - \frac{7}{6} \quad \text{is an element of } \mathcal{W}^0.$$

3.3. Formal linear differential operators. A formal linear differential operator of grade $j$ is a sequence $\{D^0, D^1, \ldots\}$ of linear maps $D^i : \mathcal{W}^i \to \mathcal{W}^{i+j}$ defined, as will be described presently, by a pair of substitution rules. The substitution rules are specified by the following data: $X$, a Weil diagram of leg-grade $j + 1$ with a special degree 1 vertex, and $Y$, a Weil diagram of leg-grade $j + 2$ with a special degree 1 vertex.

Given $w$, a Weil diagram of leg-grade $i$, the evaluation of $D^i(w)$ begins by placing a copy of the word “$D(“ on the far left-hand end of the base vector and a copy of the symbol “$)“ on the far right-hand end of the base vector:

\[
\text{The word “$D(“ is then pushed towards the “$)“ by the repeated application of the two substitution rules:}
\]

When the word “$D(“ reaches the symbol “$)“ then that diagram is eliminated and the procedure terminates. The closing bracket sometimes appears earlier along the orienting line. In the typical situation, when the closing bracket appears on the far-right of the orienting line, we’ll omit the brackets altogether. The reader should check that this operation is well-defined (that is, that it sends all the Perm relations to combinations of Perm relations).

(More generally, $X$ and $Y$ may be finite formal $\mathbb{Q}$-linear combinations of the relevant diagrams. The operator in this case is defined by linearly extending the substitution rules.)

3.4. Example: the differential. Let us illustrate all this by an important specific example. Define the differential $d^i : \mathcal{W}^i \to \mathcal{W}^{i+1}$ to be the formal linear differential
Figure 1. An example of the calculation of $d$.

Thus:

$$d^6\left(\begin{array}{c}
\end{array}\right) = \begin{array}{c}
\end{array} - \begin{array}{c}
\end{array}.$$

operator of degree +1 corresponding to the following substitution rules:

$$d \leftrightarrow -d \leftrightarrow 0 + d.$$  

Figure 1 gives an example of the calculation of the value of $d$.

Before showing that this map actually is a differential (i.e. that $d \circ d = 0$), we’ll record an extremely useful lemma.

3.5. The Lie algebra of formal linear differential operators. The graded commutator of two formal linear differential operators is another formal linear differential operator. To see this, consider a pair of formal linear differential operators
$F$ and $G$, of grades $|F|$ and $|G|$. Their graded commutator is defined by

$$[F,G] = F \circ G - (-1)^{|F||G|} G \circ F.$$  

Consider the terms that arise when this map is applied to the first leg of a diagram.

The term $F \circ G$ leads to:

$$\text{Diagram} \quad \Rightarrow \quad \text{Diagram} \quad + \quad (-1)^{|G|} \quad \text{Diagram} \quad \Rightarrow \quad \text{Diagram} \quad + \quad (-1)^{|F|(l+|G|)} \quad \text{Diagram} \quad + \quad (-1)^{|F|+|G|} \quad \text{Diagram} \quad + \quad (-1)^{|F||G|} \quad \text{Diagram}.$$  

Similarly, the term $(-1)^{|F||G|} G \circ F$ leads to:

$$\text{Diagram} \quad \Rightarrow \quad \text{Diagram} \quad + \quad (-1)^{|F||G|} \quad \text{Diagram} \quad + \quad (-1)^{|G|l} \quad \text{Diagram} \quad + \quad (-1)^{|F|(l+|G|)} \quad \text{Diagram} \quad + \quad (-1)^{|F|+|G|} \quad \text{Diagram} \quad + \quad (-1)^{|F||G|} \quad \text{Diagram}.$$  

Taking the difference of these two expressions gives the following lemma.

**Lemma 3.5.1.** Let $F$ and $G$ denote formal linear differential operators. Then the map $[F,G]$ is the formal linear differential operator of grade $|F| + |G|$ associated to the substitution rule

$$\text{Diagram} \quad \Rightarrow \quad \text{Diagram} \quad + \quad (-1)^{|F|+|G|} \quad \text{Diagram} \quad,$$

where $l$ denotes the grade of the leg in the box.
3.6. The complex $W$. The system of maps

$$\mathcal{W} : 0 \rightarrow \mathcal{W}^0 \xrightarrow{d^0} \mathcal{W}^1 \xrightarrow{d^1} \mathcal{W}^2 \rightarrow \ldots$$

forms a differential complex. This is because $d \circ d = \frac{1}{2} [d, d]$, and Lemma 3.5.1 implies that $[d, d] = 0$.

Proposition 3.6.1.

$$H^i(\mathcal{W}) = \begin{cases} \mathcal{W}^0 & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $t$ be the formal linear differential operator of grade $-1$ defined by the substitution rules

\begin{align*}
\tau & \Rightarrow + \\
\tau & \Rightarrow 0 \\
\tau & \Rightarrow -
\end{align*}

Observe that it follows from Lemma 3.5.1 that if $D$ is a Weil diagram then $[t, d](D) = (\# \text{ of legs of } D) D$.

So for every $i > 0$ define a map $s^i : \mathcal{W}^i \rightarrow \mathcal{W}^{i-1}$ by

$$s^i(D) = \frac{1}{(\# \text{ of legs of } D)} \tau^i(D).$$

This is the required contracting homotopy. When $i \geq 1$:

$$d^{i-1} \circ s^i + s^{i+1} \circ d^i = \text{id}.$$

□

So the complex $W$ has trivial cohomology. There is, however, a subcomplex of $W$, the basic subcomplex $W_{\text{basic}}$, whose cohomology spaces are canonically isomorphic to the $\{B^i\}$:

$$H^i(\Upsilon_{\text{basic}}) : B^i \xrightarrow{\cong} H^i(W_{\text{basic}}).$$

To cut out the subcomplex $W_{\text{basic}} \subset W$, we need a simple structure we'll call an $i$-complex.

3.7. $i$-complexes. An $i$-complex is a pair of complexes together with a grade $-1$ map $\iota$ between them. Our first $i$-complex, to be defined presently, looks like this:

$$0 \rightarrow \mathcal{W}^0 \xrightarrow{d^0} \mathcal{W}^1 \xrightarrow{d^1} \mathcal{W}^2 \xrightarrow{d^2} \ldots$$

To be precise: the equations $d^i \circ d^{i+1} = 0$, $d^i \circ d_i^{i+1} = 0$ and $d^i \circ \iota^{i+1} = -\iota^i \circ d_i^{i-1}$ hold in the above system. (This last equation has a $(-1)$ because $d$ is grade 1 and $i$ is grade $-1$.)
A map \( f : (K, K_\iota, \iota K) \to (L, L_\iota, \iota L) \) between a pair of \( \iota \)-complexes is just a pair of sequences of maps \( \{ f^i : K^i \to L^i \}, \{ f^i_\iota : K^i_\iota \to L^i_\iota \} \) commuting with the \( d \)'s and the \( \iota \)'s. (Commuting up to sign as determined by the understanding that \( d \) and \( d_\iota \) are regarded as grade 1 maps, \( f \) and \( f_\iota \) as grade 0 maps, and \( \iota \) as a grade \(-1\) map.)

**Definition 3.7.1.** The basic subcomplex \( K_\text{basic} \) associated to an \( \iota \)-complex \((K, K_\iota, \iota K)\) is defined by setting \( K^i_\text{basic} = \ker(\iota^i) \) and by defining the differential to be the restriction of the differential on \( K \). (The rule that \( \iota \) commutes with \( d \) implies that this actually is a subcomplex.)

A map \( f : (K, K_\iota, \iota K) \to (L, L_\iota, \iota L) \) between \( \iota \)-complexes restricts to a chain map \( f_\text{basic} \) between their basic subcomplexes.

### 3.8. The \( \iota \)-complex \((W, W_\iota, \iota)\)

The space \( W^i \) is defined in exactly the same way as the space \( W^0 \), except that it is based on Weil diagrams with precisely one special univalent vertex (in diagrams the special vertex will be labelled by the symbol \( \iota \)). For example, the following diagram is a generator of \( W^6_\iota \):

The map \( \iota \) is the formal linear differential operator of grade \(-1\) defined by the substitution rules:

**Proposition 3.8.1.** With these definitions, \((W, W_\iota, \iota)\) forms an \( \iota \)-complex.

**Proof.** The only thing to verify is that \([\iota, d] = 0\). Lemma 3.5.1 tells us that \([\iota, d] \) is the formal linear differential operator associated to the following substitution rule (the leg in the box can be of either type):
This operator is precisely zero on $W$. To see why this is true we’ll examine its action on a generic Weil diagram. To begin:

\[
\begin{align*}
\delta[\iota, d] & = \iota + \iota = 0.
\end{align*}
\]

The box above represents the sum over all ways of placing the end of the $\iota$-labelled edge on one of the edges going through the box. Then we can just use the AS and IHX relations to “sweep” the box up and off the diagram. Figure 2 shows how it is done in this example. □

So the triple $(W, W_\iota, \iota)$ forms an $\iota$-complex. Thus we can pass to the basic subcomplex $W_{\text{basic}}$. The computation of the corresponding basic cohomology is the subject of Section 4.

3.9. A lengthy aside: the characteristic class-valued weight system. The most commonly asked question that arises when this formalism is described is: what is the $\iota$ for? In this section we’ll give some explanation of the $\iota$ by describing how to build characteristic classes from the complex $W$. This section is not part of the
Consider a compact Lie group $G$ and a smooth manifold $P$ with a free, smooth left $G$-action. It turns out that the space of orbits $P/G$, which we'll denote $B$, may be uniquely equipped with the structure of a smooth manifold so that $P$, $B$ and the projection map $\pi : P \to B$ form a smooth principal $G$-bundle. Indeed, it turns out that every smooth principal $G$-bundle arises in this fashion.

The Chern-Weil theory of characteristic classes begins by examining the corresponding pull-back map of differential forms

$$\pi^* : \Omega(B) \to \Omega(P)$$

and asking: is this map an injection? If so, can we characterize the image of this map? Yes, and yes, are the answers.

The characterization requires the generating vector fields of the $G$-action on $P$. Let $\mathfrak{g}$ denote the Lie algebra of $G$. Given a vector $\xi \in \mathfrak{g}$, the vector of the corresponding generating vector field $\xi_P$ at a point $p$ of $P$ is given by the derivative:

$$\left( \xi_P f \right)(p) = \frac{d}{dt} f \left( \exp \left( -t\xi \right) \cdot p \right) \bigg|_{t=0}.$$ 

There are two graded differential operators on $\Omega(P)$ that are naturally associated to a generating vector field $\xi_P$:

- $\mathcal{L}_\xi : \Omega^\bullet(P) \to \Omega^\bullet(P)$, the Lie derivative along $\xi_P$,
- $\iota_{\xi} : \Omega^\bullet(P) \to \Omega^{\bullet-1}(P)$, the corresponding contraction operator. Given $\tau$, a 1-form on $P$, $\iota_{\xi}$ simply evaluates $\tau$ on the vector field $\xi_P$. In other words,

$$\left( \iota_{\xi} \tau \right)(p) = \tau_p(\xi_p).$$

**Fact.** The graded differential operators $\mathcal{L}_\xi$, $\iota_{\xi}$ and $d$ satisfy the following graded commutation relations:

$$\begin{align*}
[\mathcal{L}_\xi, \mathcal{L}_{\xi'}] &= \mathcal{L}_{[\xi, \xi']} \quad [\mathcal{L}_\xi, \iota_{\xi'}] = \iota_{[\xi, \xi']} \quad [\mathcal{L}_\xi, d] = 0 \\
[\iota_{\xi}, d] &= \mathcal{L}_\xi \\
[\iota_{\xi}, \iota_{\xi'}] &= 0
\end{align*}$$

Consider, now, some differential form $\omega$ on $B$, and consider its pull-back $\pi^*(\omega)$ to $P$. Our first observation is that any such pull-back has the property that it is annihilated by every $\mathcal{L}_\xi$ and also by every $\iota_{\xi}$. Before noting the simple reasons for this fact, let’s encode it in a definition:

**Definition 3.9.1.** A differential form on $P$ is said to be basic if it is annihilated by every $\mathcal{L}_\xi$ and also by every $\iota_{\xi}$. Let $\Omega(P)_{\text{basic}}$ denote the ring of basic forms.

**Proposition 3.9.2.** The pull-back map $\pi^*$ maps into the ring of basic forms:

$$\pi^*(\Omega(B)) \subset \Omega(P)_{\text{basic}}.$$ 

**Proof.** (1st point: $\mathcal{L}_\xi (\pi^*(\omega)) = 0$.) A direct implementation of the definitions gives:

$$\begin{align*}
\left( \mathcal{L}_\xi (\pi^*(\omega)) \right)_p &= \frac{d}{dt} \left( (\exp(-t\xi)^* \circ \pi^*)(\omega)_p \right) \\
&= \frac{d}{dt} \left( (\pi \circ \exp(-t\xi))^*(\omega)_p \right) \\
&= \frac{d}{dt} (\pi^*(\omega)_p), \\
&= 0,
\end{align*}$$
where to get the penultimate line we have used the fact that the projection map is invariant. (2nd point: \( \iota_\xi (\pi^*(\omega)) = 0. \) This is for the simple reason that the projection map \( \pi_\star \) sends any vector from a generating vector field to zero. □

Actually, quite a bit more is true. The following improvement follows from the local triviality of a principal bundle.

**Fact.** The pull back map gives an isomorphism of differential graded algebras

\[
\pi^*: \Omega(B) \rightarrow \Omega(P)_{\text{basic}}.
\]

The next step in Chern-Weil theory is to consider connection forms on \( P \), as we will presently recall. Chern-Weil theory asks: how can we build basic forms on \( P \) from the components of a connection form?

A **connection form** is an element \( A \in \mathfrak{g} \otimes \Omega^1(P) \) satisfying the two properties that:

\[ (* ) \quad L_\xi (A) = \text{ad}_\xi (A), \text{ where } \text{ad}_\xi (A) \text{ denotes the adjoint action of } \xi \text{ on the first factor of } \mathfrak{g} \otimes \Omega^1(P). \]

\[ (** ) \quad \iota_\xi (A) = \xi. \]

Connection forms are equivalent to connections on \( P \). To recall what a connection is: Letting \( V_p P \) (the “vertical subspace”) denote the subspace of \( T_p P \) spanned by the generating vector fields at \( p \), a **connection** is a smooth, \( G \)-equivariant decomposition \( T_p P = V_p P \oplus H_p P \) (where \( H_p P \) is called the “horizontal subspace”). To get the corresponding connection form from a connection: observe that the decomposition

\[
T_p P = V_p P \oplus H_p P \simeq \mathfrak{g} \oplus H_p P
\]
gives, at every point \( p \) of \( P \), a linear map \( T_p P \rightarrow \mathfrak{g} \). That the element of \( \mathfrak{g} \otimes \Omega^1(P) \) that this map corresponds to satisfies the two properties above is a very instructive exercise.

### 3.9.3. The construction of the weight system.

With these preliminaries in hand we can now turn to constructing the chain map

\[
\text{Weight}^{(G,P,A)} : \mathcal{W} \rightarrow \Omega(P),
\]

depending on the initial data of a compact Lie group \( G \), a principal \( G \)-bundle \( P \), and a connection form \( A \) on \( P \). We will then observe that this map restricts to a map between the basic subcomplexes

\[
\text{Weight}^{(G,P,A)}_{\text{basic}} : \mathcal{W}_{\text{basic}} \rightarrow \Omega(P)_{\text{basic}}
\]

so that we end up with a map

\[
H \left( \text{Weight}^{(G,P,A)}_{\text{basic}} \right) : H(\mathcal{W}_{\text{basic}}) \rightarrow H(\Omega(P)_{\text{basic}}) \simeq H(B).
\]

We will refer to this map as the **characteristic class-valued weight system**. The computation of \( H(\mathcal{W}_{\text{basic}}) \) is the subject of the next section.

The “weight system” \( \text{Weight}^{(G,P,A)} \) will be constructed as a state-sum in the familiar way. For completeness, we’ll describe the method in detail here. To begin, let’s fix some notation to refer to certain tensors associated to the initial data.

- **(The inner product) The Lie algebra \( \mathfrak{g} \), being the Lie algebra of a compact Lie group, comes equipped with an inner product which is:**
  1. Symmetric, so that \( \langle v, w \rangle = \langle w, v \rangle \);
  2. Invariant, so that \( \langle [u, v], w \rangle + \langle v, [u, w] \rangle = 0 \);
(3) Non-degenerate, so that the map from $\mathfrak{g}$ to $\hat{\mathfrak{g}}$ given by $v \mapsto \langle v, \cdot \rangle$ is an isomorphism. Let $\varpi : \hat{\mathfrak{g}} \to \mathfrak{g}$ denote the inverse map.

- (The Casimir element) The identity map from $\mathfrak{g}$ to itself can be viewed as an element of $\hat{\mathfrak{g}} \otimes \mathfrak{g}$. (In coordinates: introduce a basis $\{t_a\}_{a=1}^{\dim \mathfrak{g}}$. Let $\hat{t}_a$ denote the dual vector defined by the rule $\hat{t}_a(t_b) = \delta_{ab}$. Then the identity map is $\sum_a \hat{t}_a \otimes t_a$.) If we take the tensor representing the identity map and apply $\varpi$ to the first factor (what physicists call “lowering the index”) then we get a tensor in $\mathfrak{g} \otimes \mathfrak{g}$ which we’ll refer to as the Casimir element. Write this element as a sum

$$\sum_a s_a \otimes t_a.$$ 

The following identity is often useful:

$$\sum_a (s_a, v)t_a = v, \quad v \in \mathfrak{g}. \quad (†)$$

- (The connection form) The connection form $A$ is an element of $\mathfrak{g} \otimes \Omega^1(P)$. Write it as a sum:

$$A = \sum_i r_i \otimes \omega_i.$$ 

We can proceed to construct the state-sum. So consider some Weil diagram. To begin, chop the diagram into pieces by cutting every edge at its midpoint. For example:

The next step is to decorate the chopped-up diagram with tensors. Every piece of the diagram is decorated with a tensor according to what kind of piece it is. The rules are:

- (Trivalent vertices) Recall that we denoted the Casimir element $\sum_a s_a \otimes t_a$. Every trivalent vertex is decorated with a tensor from $\mathfrak{g}^\otimes 3$ in the following way:

$$\sum_{a,b} [t_a, t_b] s_a s_b.$$ 

- (Grade 1 legs) Recall that the connection form was denoted $A = \sum_i r_i \otimes \omega_i$. Every grade 1 leg is decorated as follows:

$$\sum_i r_i \omega_i.$$ 

- (Grade 2 legs) Every grade 2 leg is decorated as follows:

$$\sum_i r_i d\omega_i.$$
In the example we are following the decoration is:

\[
\sum_{a,\ldots,r}
\]

To finish the construction: use the inner product to pair up every pair of elements that face each other where you broke the edges; then take the wedge product of the differential forms that appear along the orienting line at the bottom:

\[
(2)
\sum_{a,\ldots,r}
\]

Thus, for this example, we get:

\[
\text{Weight}^{(G,P,A)}
\]

Proposition 3.9.4. The map \(\text{Weight}^{(G,P,A)} : W \to \Omega(P)\) is well-defined. In other words, it respects the anti-symmetry, Jacobi, and leg permutation relations, and is a chain map.

The map respects the anti-symmetry and Jacobi relations for familiar reasons. That the map respects the leg permutation relations obviously follows from the grading of the differential forms involved in the state-sum.
Proposition 3.9.5. The map $\text{Weight}^{(G,P,A)}$ restricts to a map

$$\text{Weight}^{(G,P,A)}_{\text{basic}} : W_{\text{basic}} \rightarrow \Omega(P)_{\text{basic}}$$

between the basic subcomplexes.

Our explanation of this fact uses a small extension of the state-sum construction given earlier. The extension applies to Weil diagrams with a special degree 1 vertex labelled by some element of the Lie algebra $\xi \in \mathfrak{g}$. Given such a diagram, we construct a corresponding differential form just as we did earlier, but with an extra rule that says what to do with the special vertex. The extra rule is just to use the inner product to pair the label $\xi$ with whatever it faces where you broke the edge. For example:

Proof. Recall that a differential form $\omega \in \Omega(P)$ is said to be basic if $L_\xi(\omega) = 0$ and $\iota_\xi(\omega) = 0$ for all $\xi \in \mathfrak{g}$. Our job is to show that if an element $v \in W$ satisfies $\iota(v) = 0$, then $\text{Weight}^{(G,P,A)}(v)$ is basic in $\Omega(P)$. We’ll split this into two observations.

Observation 1: Any form in the image of $\text{Weight}^{(G,P,A)}$ is annihilated by every Lie derivative $L_\xi$. To observe this fact, we must first understand how a Lie derivative $L_\xi$ operates on a differential form constructed by such a state sum.

To begin, recall that any Lie derivative $L_\xi$ acts as a grade 0 differential operator on the forms along the bottom of the diagram. Thus, for example, if we applied $L_\xi$ to the example of Line 2, the result would be (only drawing the bottom of Line 2):
To proceed, then, we must ask how \( L_\xi \) operates on the legs. We calculate:

\[
\sum_i r_i = \sum_i [\xi, r_i] = \sum_{a,b,i} -\langle \xi, S_a \rangle \frac{1}{\langle r_i, S_b \rangle},
\]

where the first equality above used the assumed property (⋆) of connection forms, and the second equality used Equation (†) twice. Similarly, because \( [L_\xi, d] = 0 \) (see Line 1), there is an equality:

\[
\sum_i r_i = \sum_{a,b,i} -\langle \xi, S_a \rangle \frac{1}{\langle r_i, S_b \rangle}.
\]

These facts can be interpreted graphically by saying that \( L_\xi \) operates on differential forms in the image of \( \text{Weight}^{(G,P,A)} \) by the following two substitution rules (where the diagrams in these rules represent the corresponding differential forms in \( \Omega(P) \)):

\[
L_\xi \mapsto \frac{1}{\xi} L_\xi + \frac{1}{\xi} L_\xi.
\]

We can now see that \( L_\xi \) of such a form will always be zero by the usual “sweeping” argument (compare Figure 2).

**Observation 2:** If \( \iota(v) = 0 \), for some \( v \in W \), then, for every \( \xi \in g \),

\[
\iota_\xi \left( \text{Weight}^{(G,P,A)}(v) \right) = 0.
\]

This is a special case of a more general statement. The more general statement uses a certain map

\[
\text{Weight}_{\iota=\xi}^{(G,P,A)} : \mathcal{W}_i \to \Omega(P).
\]

Recall that the diagrams which generate \( \mathcal{W}_i \) are just like the usual Weil diagrams but have a special \( \iota \)-labelled leg. The map \( \text{Weight}_{\iota=\xi}^{(G,P,A)} \) is defined in exactly the same way as \( \text{Weight}^{(G,P,A)} \) but with an extra rule that says that when constructing the state-sum the special leg is first relabelled \( \xi \in g \).

The more general statement is:

(3) \[
\iota_\xi \circ \text{Weight}^{(G,P,A)} = \text{Weight}_{\iota=\xi}^{(G,P,A)} \circ \iota.
\]
To explain this statement we must consider how the differential operator $\iota_\xi$ acts on a connection form. According to property (***) of differential forms:

$$\iota_\xi \left( \sum_i r_i \otimes \omega_i \right) = \sum_i r_i \otimes \iota_\xi (\omega_i) = \xi.$$ 

Furthermore, using the relation $[d, \iota_\xi] = L_\xi$ (see Line 1), and property (*) of connection forms, we have that:

$$\iota_\xi \left( \sum_i r_i \otimes d\omega_i \right) = \sum_i r_i \otimes \iota_\xi d\omega_i = \sum_i r_i \otimes (L_\xi - d\iota_\xi)\omega_i = \sum_i [\xi, r_i] \otimes \omega_i.$$ 

The graphical interpretation of these facts is that the differential operator $\iota_\xi$ acts on the differential forms in the image of Weight\((G,P,A)\) according to the following substitution rules:

\[\begin{aligned}
\iota_\xi \xrightarrow{\xi} & \xi, \\
\iota_\xi \xrightarrow{-\xi} & -\xi + \iota_\xi \\
\iota_\xi \xrightarrow{\xi} & -\xi \iota_\xi.
\end{aligned}\]

This explains Equation 3. □

4. INTRODUCING THE MAP $\Upsilon$. 

In this section we’ll see how to map $B$, the familiar space of symmetric Jacobi diagrams, into $W$, the just-introduced commutative Weil complex for diagrams, in order to get isomorphisms in basic cohomology:

$$H^i(\Upsilon_{\text{basic}}) : B_i \simeq H^i(B_{\text{basic}}) \xrightarrow{\simeq} H^i(W_{\text{basic}}).$$

The most important concept in this section is the (straightforward) definition of the map of $\iota$-complexes $\Upsilon$. The fact that it gives isomorphisms in cohomology will not be needed elsewhere in this work, but serves to organize the discussion here. We’ll introduce $\Upsilon$ as the composition of two other maps of $\iota$-complexes:

$$(B, 0, 0) \xrightarrow{\phi} (W_F, W_{\mathfrak{F}}) \xrightarrow{B_F} (W, W_{\iota}).$$

Formally speaking, this section will be rather routine. We’ll introduce the $\iota$-complex $(W_F, W_{\mathfrak{F}}, \iota)$. Then we’ll introduce the maps $\phi$ and $B_F$. Along the way we’ll explain that they both induce isomorphisms in basic cohomology.

4.1. A change of basis: The $\iota$-complex $(W_F, W_{\mathfrak{F}}, \iota)$. So, to begin, we’ll introduce the $\iota$-complex $(W_F, W_{\mathfrak{F}}, \iota)$. This should be regarded as the $\iota$-complex $(W, W_{\iota})$ viewed under a different basis. We’ll set this up formally in a moment, but the idea is to replace the usual grade 2 legs by “curvature” legs, which will be labelled by an $F$.

\[\begin{aligned}
\iota_\xi & \xrightarrow{\xi} \xi,
\iota_\xi & \xrightarrow{-\xi} -\xi + \iota_\xi,
\iota_\xi & \xrightarrow{\xi} -\xi \iota_\xi.
\end{aligned}\]
This is done to simplify the action of $\iota$ (at the cost of a more complicated differential), and so permit the calculation of the basic cohomology.

Now let’s set this up formally. This complex employs Weil diagrams with two types of leg: grade 2 legs labelled by the symbol $F$, and grade 1 legs with no decoration. These legs obey the usual permutation rules appropriate to this grading. For example:

\[
\begin{align*}
\text{F} & \equiv - \\
\text{F} & \equiv - \\
\text{F} & \equiv -
\end{align*}
\]

The spaces $W^i_F$ and $W^i_{\bar{F}}$ are defined in the obvious way. Indeed, the $\iota$-complexes $(W, W_\iota, \iota)$ and $(W_F, W_{\bar{F}}$, $\iota)$ differ from each other only in the definition of the maps $\iota$ and $d$.

In this case, the map $\iota$ is defined to be the formal linear differential operator defined by the substitution rules

\[
\begin{align*}
\text{F} \mapsto 0 + \\
\text{F} \mapsto 1
\end{align*}
\]

This simplification in $\iota$ comes at the cost of a more complicated differential:

\[
\begin{align*}
d \text{F} & \equiv - \\
d \text{F} & \equiv \left( \text{F} + \frac{1}{2} \right) -
\end{align*}
\]

With these definitions, $(W_F, W_{\bar{F}}, \iota)$ forms an $\iota$-complex. (This is a quick calculation based on Lemma 3.5.1 together with a sweeping argument.)

Let’s now consider how this complex fits into the construction of $\Upsilon$. We’ll focus first on the following factor:

\[
(B, 0, 0) \xrightarrow{\phi} (W_F, W_{\bar{F}}, \iota) \xrightarrow{B_{\text{F} \mapsto \bullet}} (W, W_\iota, \iota).
\]

**Definition 4.1.1.** Define maps $B_{\text{F} \mapsto \bullet} : W^i_F \rightarrow W^i$ and $B_{\bar{F} \mapsto \bullet, \iota} : W^i_{\bar{F}} \rightarrow W^i_\iota$ by expanding $F$-labelled legs as follows:

\[
\begin{align*}
\text{F} & \mapsto - \\
- \frac{1}{2}
\end{align*}
\]
For example:

\[
B_{\mathcal{F} \to \mathcal{F}}^\dagger \left( \begin{array}{cc}
\mathcal{F} & \mathcal{F} \\
\mathcal{F} & \mathcal{F}
\end{array} \right) = \begin{array}{ccc}
\mathcal{F} & -\frac{1}{2} & \mathcal{F} \\
-\frac{1}{2} & \mathcal{F} & + \frac{1}{4} & \mathcal{F}
\end{array}
\]

**Proposition 4.1.2.** The ‘change of basis’ map \( B_{\mathcal{F} \to \mathcal{F}} \) is a map of \( \iota \)-complexes.

**Proof.** Let’s begin by showing that the equation \( d \circ B_{\mathcal{F} \to \mathcal{F}} = B_{\mathcal{F} \to \mathcal{F}} \circ d \) holds. First consider how \( d \) operates on the expansion of an \( \mathcal{F} \)-labelled leg:

The required equation holds if the second term above is zero. The following argument provides that fact:

\[
\begin{align*}
(IHX) & = (Perm) & = (AS) = -2
\end{align*}
\]
The case of the action of $d$ on the grade 1 legs is treated straightforwardly. The equation $\iota \circ B \rightarrow \bullet = B \rightarrow \bullet \circ \iota$ is also straightforward. □

**Proposition 4.1.3.** The maps $B^i \rightarrow \bullet$ and $B^i_\rightarrow \bullet$ are vector space isomorphisms, and so yield isomorphisms in basic cohomology:

$$H^i((B \rightarrow \bullet)_{\text{basic}}) : H^i((W \rightarrow \bullet)_{\text{basic}}) \xrightarrow{\cong} H^i(W_{\text{basic}}).$$

**Proof.** The inverse $B^i_\rightarrow \bullet : W_i \rightarrow W_i \rightarrow \bullet$ is given by replacing legs as follows:

4.2. **The calculation of the basic cohomology of** $(W, W_i, \iota)$. We now turn to the following factor:

$$(B, 0, 0) \xrightarrow{\phi} (W \rightarrow \bullet, W_i, \iota) \xrightarrow{B \rightarrow \bullet} (W, W_i, \iota).$$

We begin by assembling the spaces $B^i$ into an $\iota$-complex in the following way:

Note the obvious fact that

$$H^i(B_{\text{basic}}) = \begin{cases} B^i & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

Define a map $\phi : B^i \rightarrow W_i \rightarrow \bullet$ on some symmetric Jacobi diagram $D$ by simply choosing some ordering of its legs and labelling each of them with an $F$. For example:

**Proposition 4.2.1.** The map $\phi$ is a map of $\iota$-complexes.

**Proof.** The only thing we need to show is that if $D$ is a symmetric Jacobi diagram then $\iota(\phi(D)) = 0$ and $d(\phi(D)) = 0$. The first equation is clear (given the defined operation of $\iota$ on curvature legs). The following example will show why the second
equation holds:

\[
\begin{align*}
\cdots & \xrightarrow{d} F F F F F \\
\sim & \quad F F F F F + F F F F F + F F F F F + F F F F F \\
= & \quad F F F F F + F F F F F + F F F F F + F F F F F \\
= & \quad 0.
\end{align*}
\]

\begin{proposition}
\label{prop:iso}
The map \( \phi \) yields isomorphisms

\[ H^i(\phi_{\text{basic}}) : H^i(\mathcal{B}_{\text{basic}}) \xrightarrow{\sim} H^i((W_F)_{\text{basic}}). \]

\end{proposition}

\begin{proof}
(In this proof the symbol \( \iota \) refers to the \( \iota \) map in the complex \( W_F \).) It suffices to show that:

\[ \begin{cases} 
\phi^j : \mathcal{B}^j \xrightarrow{\sim} \ker(\iota^j) & \text{if } j \text{ is even,} \\
\ker(\iota^j) = 0 & \text{if } j \text{ is odd.}
\end{cases} \]

To determine \( \ker(\iota^j) \): observe that there is a direct-sum decomposition

\[ W_F^i = \bigoplus_{j=0} W_F^{i,j} \]

where \( W_F^{i,j} \) is the subspace generated by Weil diagrams with exactly \( j \) grade 1 legs. For example,

\[ \begin{array}{c}
\text{is a generator of } W_F^{6,2}. \text{ This direct-sum decomposition exists because there are no relations involving diagrams with varying numbers of such legs.}
\end{array} \]

The proposition follows immediately from the claim that \( \ker(\iota^i) = W_F^{i,0} \). To prove this claim: let \( \iota^i : W_F^{i-1} \to W_F^i \) be the map which takes the special \( i \)-labelled leg and places it at the far left hand end of the orienting line. For example:

\[ \begin{array}{c}
\text{Observe that:}
\end{array} \]

\[ (\iota^i \circ \iota^j)(D) = jD \quad \text{if } D \in W_F^{i,j}, \]

from which the claim follows.
\end{proof}
4.3. Summary. If we define maps \( Y^i : B^i \to W^i \) by the formula

\[ Y^i = B^i_F \circ \phi^i, \]

then Propositions 4.1.3 and 4.2.2 give us the following theorem.

**Theorem 4.3.1.** The maps

\[ H^i(Y_{\text{basic}}) : B^i \simeq H^i(B_{\text{basic}}) \to H^i(W_{\text{basic}}) \]

are isomorphisms.

5. The non-commutative Weil complex.

In this section we embed the Weil complex in a larger \( \iota \)-complex, the non-commutative Weil complex:

\[ \chi_W : (W, W, \iota) \to (\tilde{W}, \tilde{W}, \iota). \]

This larger space is built in the same way as \( W \) but without introducing the permutation relations; the embedding \( \chi_W \) is the graded averaging map. The key technical theorem says, in essence, that every cocycle \( z \) in \( \tilde{W}_{\text{basic}} \) has its corresponding basic cohomology class \([z]\) represented by its graded symmetrisation \((\chi_W \circ \tau)(z)\).

**Definition 5.0.2.** Define vector spaces \( \tilde{W}^i \) in exactly the same way as the vector spaces \( W^i \) but without introducing the permutation relations.

For the purposes of clarity, we’ll always draw generators of \( \tilde{W} \), which we’ll refer to as non-commutative Weil diagrams, with a certain arrow head on their orienting lines. Observe:

\[ \neq \quad \neq \quad \neq \quad \text{in } \tilde{W}^6. \]

Define \( d, \tilde{W}, \) and \( \iota \) in the obvious way. These definitions form an \( \iota \)-complex (the calculation that \([d, \iota] = 0\) is exactly the same here as it is in the commutative case, Lemma 3.8.1).

The basic subcomplex here seems much harder to construct explicitly than it is in the commutative case. It is a startling and central fact, however, that the two \( \iota \)-complexes have canonically isomorphic basic cohomology spaces. We’ll show this by constructing a chain equivalence of \( \iota \)-complexes:

\[ W \xrightarrow{\chi_W} \tilde{W}. \]

**Definition 5.0.3.** Define the graded averaging maps

\[ \chi_W^i : W^i \to \tilde{W}^i \quad \text{and} \quad \chi_{W, \iota}^i : W^i \to \tilde{W}^i. \]
by declaring the value of $\chi_W$ (and similarly for $\chi_{W,i}$) on some commutative Weil diagram to be the average of the (signed) orderings of its legs. For example:

$$\chi_W^4 \left( \begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2} \\
\text{Diagram 3} \\
\text{Diagram 4}
\end{array} \right) = \frac{1}{6!} \left( \begin{array}{c}
\text{Diagram 5} \\
\text{Diagram 6} \\
\text{Diagram 7} \\
\text{Diagram 8}
\end{array} \right).$$

**Definition 5.0.4.** Define the forget-the-ordering maps

$$\tau^i : \widetilde{W}^i \rightarrow W^i \quad \text{and} \quad \tau_i^i : \widetilde{W}^i_i \rightarrow W^i_i$$

by simply obtaining a commutative Weil diagram from a given non-commutative Weil diagram by forgetting the ordering of the legs. For example:

$$\tau^4 \left( \begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2} \\
\text{Diagram 3} \\
\text{Diagram 4}
\end{array} \right) \in W^4.$$

Note that $\chi_W : W \rightarrow \widetilde{W}$ and $\tau : \widetilde{W} \rightarrow W$ are both maps of $i$-complexes. (This is immediate for $\tau$; for $\chi_W$ it may take a few minutes reflection.) We shall presently declare these two maps of $i$-complexes to be “$i$-chain homotopic”. This fact is the technical heart of this work. Let us first spell out what “$i$-chain homotopic” means.

An $i$-chain homotopy $s$ between two maps of $i$-complexes

$$(\mathcal{K}, \mathcal{K}_i, i) \xrightarrow{f} (\mathcal{L}, \mathcal{L}_i, i)$$

is a pair of sequences of maps

$$s^i : \mathcal{K}^i \rightarrow \mathcal{L}^{i-1} \quad \text{and} \quad s^i_i : \mathcal{K}^i_i \rightarrow \mathcal{L}^{i-1}_i$$

such that

1. $d^i_{\mathcal{L}_i} \circ s^i + s^{i+1}_i \circ d^i_{\mathcal{K}_i} = f^i - g^i$,  
2. $d^i_{\mathcal{L}_i} \circ s^i_i + s^{i+1}_i \circ d^i_{\mathcal{K}_i} = f^i_i - g^i_i$,  
3. $s^{i-1}_i \circ \iota^i + \iota^{i-1} \circ s^i = 0$.

(Here $i$ runs from 0 on. In the above equations it is understood that $s^j = \iota^j = d^{j-1} = 0$ if $j \leq 0$.) It will not surprise the reader to learn that if two maps of $i$-complexes are $i$-chain homotopic, then the induced maps on basic cohomology

$$H^\bullet(\mathcal{K}_{\text{basic}}) \xrightarrow{H(f)} H^\bullet(\mathcal{L}_{\text{basic}})$$

are equal.
Theorem 5.0.5 (The key technical theorem). There exists an \( \iota \)-chain homotopy\

\[
s : \tilde{W} \to \tilde{W}
\]

between the two maps of \( \iota \)-complexes \( id_{\tilde{W}} \) and \( \chi_W \circ \tau \).

The construction of this \( \iota \)-chain homotopy is the subject of Section 7.

6. Homological wheeling

In this section we will carefully state the Homological Wheeling (HW) theorem, and point out why it is an immediate consequence of the existence of the \( \iota \)-chain homotopy described in Theorem 5.0.5. First we'll make some brief comments concerning the ingredients of HW.

In Section 4.2 we showed how to take symmetric Jacobi diagrams (i.e. generators of \( \mathcal{B} \)) and construct basic cocycles in \( W \). Recall this map \( \Upsilon \): order the legs of the symmetric Jacobi diagram, label each leg with an \( F \), then expand these legs into the usual basis. For example:

\[
\Upsilon^4 \left( \begin{array}{c}
\end{array} \right) = \begin{array}{c}
\end{array} - \frac{1}{2} \begin{array}{c}
\end{array} + \frac{1}{4} \in W^4.
\]

If we continue on and compose \( \Upsilon \) with the graded averaging map \( \chi_W \), then we have constructed basic cocycles in \( \tilde{W} \). HW concerns these cocycles. Formalizing this composition:

**Definition 6.0.6.** Let \( H^i : B^i \to H^i(\tilde{W}_{\text{basic}}) \) denote the linear map given by the formula

\[
H^i = H^i((\chi \circ \Upsilon)_{\text{basic}}).
\]

The other ingredient in HW is the natural graded product on the \( \iota \)-complex \( \tilde{W} \). For two non-commutative diagrams \( D_1 \) and \( D_2 \), \( D_1 \# D_2 \) is defined by placing \( D_2 \) to the right of \( D_1 \) on the orienting line. For example:

\[
\begin{array}{c}
\end{array} \quad \# \quad \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array}.
\]

Observe that \( d \) and \( \iota \) satisfy the graded Leibniz rule with respective to this product, so that this product descends to a product on the basic cohomology.

**Homological Wheeling.** Let \( v \) be an element of \( B^i \) and \( w \) be an element of \( B^j \). Then:

\[
H^{i+j}(v \sqcup w) = H^i(v) \# H^j(w) \in H^{i+j}(\tilde{W}_{\text{basic}}).
\]

It is worth re-expressing this theorem in more concrete terms.
Theorem 6.0.7 (A re-expression of Homological Wheeling). Let \( v \) be an element of \( B^i \) and \( w \) be an element of \( B^j \). Then there exists an element \( x_{v,w} \) of \( \tilde{W}^{i+j-1} \) basic, (that is, an element of \( \tilde{W}^{i+j} \) satisfying \( \iota(x_{v,w}) = 0 \), with the property that

\[
(\chi_W \circ \Upsilon)^i(v) \# (\chi_W \circ \Upsilon)^j(w) = (\chi_W \circ \Upsilon)^{i+j}(v \sqcup w) + d(x_{v,w}) \in \tilde{W}^{i+j}.
\]

Let us take a moment to illustrate this theorem. If we take \( v = \) and \( w = \), then \( (\chi \circ \Upsilon)^i(v) \# (\chi \circ \Upsilon)^j(w) \) is a combination of diagrams like

![Diagram 1](image1)

and \( (\chi \circ \Upsilon)^{i+j}(v \sqcup w) \) is a combination of diagrams like

![Diagram 2](image2)

6.1. Why the existence of the \( \iota \)-chain homotopy \( s \) (Theorem 5.0.5) implies Homological Wheeling. Consider the left-hand side of the equation stated in Theorem 6.0.7:

\[
(\chi_W \circ \Upsilon)(v) \# (\chi_W \circ \Upsilon)(w)
\]

If we insert this element into the equation

\[
id_{\tilde{W}} - \chi_W \circ \tau = s \circ d + d \circ s
\]

then we learn that it is equal to

\[
(\chi_W \circ \tau)((\chi_W \circ \Upsilon)(v) \# (\chi_W \circ \Upsilon)(w)) + d(s((\chi_W \circ \Upsilon)(v) \# (\chi_W \circ \Upsilon)(w))),
\]

where we have used the fact that \( (\chi_W \circ \Upsilon)(v) \# (\chi_W \circ \Upsilon)(w) \) is a cocycle in \( \tilde{W} \).

It takes but a moment to agree that

\[
(\chi_W \circ \tau)((\chi_W \circ \Upsilon)(v) \# (\chi_W \circ \Upsilon)(w)) = (\chi_W \circ \Upsilon)(v \sqcup w),
\]

and, setting

\[
x_{v,w} = s((\chi_W \circ \Upsilon)(v) \# (\chi_W \circ \Upsilon)(w)),
\]

we have the required right-hand side (noting that \( \iota(x_{v,w}) = 0 \) because \( s \) commutes with \( \iota \) and \( (\chi_W \circ \Upsilon)(v) \# (\chi_W \circ \Upsilon)(w) \) is a basic element of \( \tilde{W} \)).

\[\square\]

7. The construction of the \( \iota \)-chain homotopy \( s \).

The construction of the chain homotopy \( s \) and the verification of its properties is a 100% combinatorial exercize. The combinatorial work takes place in two \( \iota \)-complexes \( T \) and \( T_{dR} \).
7.1. **The $\iota$-complex $\mathcal{T}$.** The $\iota$-complex $\mathcal{T}$ is based on diagrams of the following sort, which will be referred to as $\mathcal{T}$-diagrams:

In a $\mathcal{T}$-diagram two legs never lie on the same vertical line. For example, the following is not allowed:

Thus the set of legs of a $\mathcal{T}$-diagram is ordered. The rules for permuting the order of legs depend on which orienting lines they lie on. If they lie on different orienting lines, or both lie on the bottom (the *commutative*) orienting line, then they can be permuted, up to sign, in the usual way. If, however, they both lie on the top (the *non-commutative*) line, then they cannot be permuted. Thus:

We’ll apply formal linear differential operators to these spaces. To apply a differential operator one thinks of a vertical line sweeping along the pair of orienting lines. Here is an example of the calculation of the differential:

Thus:
\[ d_T^\epsilon \cdot \begin{pmatrix} \text{Diagram} \end{pmatrix} = \text{Diagram} \begin{pmatrix} \text{Diagram} \end{pmatrix} - \text{Diagram} \begin{pmatrix} \text{Diagram} \end{pmatrix} + \text{Diagram} \begin{pmatrix} \text{Diagram} \end{pmatrix}. \]

The map \( \epsilon \) is defined similarly, and with these definitions \((T, T_\epsilon, \iota_T)\) forms an \( \iota \)-complex.

Constructions in \( T \) can be translated to \( \widetilde{W} \) by means of a certain map of \( \iota \)-complexes

\[ \omega : (T, T_\epsilon, \iota_T) \rightarrow (\widetilde{W}, \widetilde{W}_\iota, \iota_{\widetilde{W}}). \]

Definition 7.1.1. The value of the map \( \omega \) on some \( T \)-diagram is constructed in two steps.

1. Permute (with the appropriate signs) the legs of the diagram so that all the legs lying on the commutative orienting line lie to the right of all the legs lying on the non-commutative orienting line. For example:

\[ \begin{array}{cc}
\begin{array}{c}
\text{Diagram}
\end{array} & \Rightarrow \\
\begin{array}{c}
\text{Diagram}
\end{array} & - \\
\begin{array}{c}
\text{Diagram}
\end{array} & +
\end{array} \]

2. Now take the average of all (signed) permutations of the legs on the commutative orienting line and adjoin the result to the right-hand end of the non-commutative orienting line. Continuing the example:

\[ \begin{array}{cc}
\begin{array}{c}
\text{Diagram}
\end{array} & \Rightarrow \\
\begin{array}{c}
\text{Diagram}
\end{array} & - \\
\begin{array}{c}
\text{Diagram}
\end{array} & +
\end{array} \]

There are two natural maps of \( \iota \)-complexes from the \( \iota \)-complex \( \widetilde{W} \) to the \( \iota \)-complex \( T \). They are \( i_n : W \rightarrow T \), the map which puts all legs of the original
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diagram on the top (non-commutative) line, as in:

\[
\begin{array}{c}
\begin{array}{cc}
\tau \circ \chi = \omega \circ i_c
\end{array}
\end{array}
\]

and \( i_c : \overline{W} \rightarrow T \), the map which puts all legs of the original diagram on the bottom (commutative) line:

\[
\begin{array}{c}
\begin{array}{cc}
\tau \circ \chi = \omega \circ i_n
\end{array}
\end{array}
\]

Observe that

\[
\chi \circ \tau = \omega \circ i_c
\]

and that \( \text{id}_{\overline{W}} = \omega \circ i_n \)

**Theorem 7.1.2.** There exists an \( \iota \)-chain homotopy

\[
s : \overline{W} \rightarrow T
\]

between the two maps \( i_n \) and \( i_c \).

The construction of this \( \iota \)-chain homotopy is the subject of the next two sections. The key theorem, Theorem 5.0.5, is a quick consequence of it. The required \( \iota \)-homotopy is defined by the formula

\[
s = \omega \circ s_T.
\]

**Proof of Theorem 5.0.5.** We must check that this map satisfies the properties of an \( \iota \)-chain homotopy between the maps \( \chi \circ \tau \) and \( \text{id}_{\overline{W}} \). To begin, note that \( s \) commutes with \( \iota \). This is because \( s_T \) commutes with \( \iota \) (as asserted by Theorem 7.1.2) and because \( \omega \) is a map of \( \iota \)-complexes. We must also check:

\[
\text{id}_{\overline{W}} - \chi \circ \tau = \omega \circ (i_n - i_c)
\]

\[
= \omega \circ (d \circ s_T + s_T \circ d),
\]

\[
= d \circ \omega \circ s_T + \omega \circ s_T \circ d,
\]

\[
= d \circ s + s \circ d.
\]

To obtain the penultimate line above we used the fact that \( \omega \) is a map of \( \iota \)-complexes.

\[
\square
\]

7.2. **The \( \iota \)-complex \( T_{dR} \).** It remains for us to prove Theorem 7.1.2: to construct an \( \iota \)-chain homotopy \( s_T \) between the map \( i_n \) which puts all legs on the non-commutative line, and the map \( i_c \) which puts all legs on the commutative line. This construction will employ a subsidiary \( \iota \)-complex: \( T_{dR} \).

The \( \iota \)-complex \( T_{dR} \) is based on diagrams which have a third orienting line. No legs lie on this third line; instead, this third line can be labelled by hollow discs,
which are of grade 0, and filled-in discs, which are of grade 1. These circles can be
moved about with the permutation rules appropriate to that grading. For example:

\[
\begin{align*}
\text{[Diagram]} & \quad = \quad \text{[Diagram]} \\
\text{[Diagram]} & \quad = \quad \text{[Diagram]} \\
\text{[Diagram]} & \quad = \quad \text{[Diagram]}
\end{align*}
\]

It is a consequence of the permutation rules that a diagram with more than one
filled-in circle is precisely zero:

\[
\begin{align*}
\text{[Diagram]} & \quad = \quad -\text{[Diagram]}
\end{align*}
\]

It will be convenient for the discussion to come to define the differential as a sum
of two grade 1 operators

\[
(4) \quad d_{T_{air}} = d_T + d_\bullet.
\]

The operator \(d_T\) is defined by the usual substitution rules operating on the first
two lines, such as

\[
\begin{align*}
\text{[Diagram]} & \quad \sim \quad \text{[Diagram]} \\
\text{[Diagram]} & \quad \sim \quad \text{[Diagram]}
\end{align*}
\]

and rules that say that \(d_T\) (graded) commutes through any labels on the third line:

\[
\begin{align*}
\text{[Diagram]} & \quad \sim \quad \text{[Diagram]} \quad \text{and} \quad \text{[Diagram]} & \quad \sim \quad -\text{[Diagram]}
\end{align*}
\]

On the other hand, the operator \(d_\bullet\) is defined by rules that say that it (graded)
commutes through any legs on the first two lines, for example

\[
\begin{align*}
\text{[Diagram]} & \quad \sim \quad -\text{[Diagram]}
\end{align*}
\]
together with the following rules which apply to the third line:

\[
\begin{align*}
&\mapsto + \\
&\mapsto 0 \\
&\mapsto 
\end{align*}
\]

The map \(\iota\) is defined by the usual rules applied to the first two lines, such as:

\[
\begin{align*}
&\mapsto + \\
&\mapsto 
\end{align*}
\]

together with rules which say that \(\iota\) (graded) commutes through anything on the third line:

\[
\begin{align*}
&\mapsto \quad \text{and} \quad \mapsto 
\end{align*}
\]

We invite the reader to check that, with these definitions, \(T_{dR}\) does indeed form an \(\iota\)-complex.

7.3. The construction of the \(\iota\)-chain homotopy \(s_T\). What is this curious complex \(T_{dR}\) all about then? The empty disc can be thought of as a formal parameter (it may be useful to mentally replace the open discs with the symbol ‘\(t\)’), and the filled disc is its differential (‘\(dt\)’).

We’ll begin the construction by introducing the elements of the following diagram of maps of \(\iota\)-complexes:

\[
\begin{array}{c}
\overline{W} \\
\downarrow \theta \\
\overline{T}_{dR} \\
\downarrow \begin{array}{c}
\iota_n \\
\iota_c \\
\iota_c
\end{array} \\
T
\end{array}
\]

In words: we are going to \(\iota\)-map \(\overline{W}\) into \(T_{dR}\) in such a way that when we then set the formal variable to 0 (this will be \(\text{Ev}_{\theta \to 0}\)) then all legs are pushed to the commutative line (in other words, we get \(\iota_n\)), while when we set the formal variable to 1 (i.e. \(\text{Ev}_{\theta \to 1}\)) then all legs are pushed to the non-commutative line (i.e. \(\iota_c\)). Later, we’ll exploit a formal version of the fundamental theorem of calculus.
to get an expression for the difference $i_n - i_c$ that is the subject of this construction.

So define a map of $\iota$-complexes

$$\theta : \tilde{W} \to T_{dR}.$$ 

by replacing legs according to the following rules:

- $\mapsto \mapsto + -$,
- $\mapsto \mapsto - +$.

It is not hard to motivate this seemingly complicated map. The maps of degree 1 legs are a formal representation of the combination:

$$t \ast \left( \text{Place the leg on the non-commutative line.} \right) + (1 - t) \ast \left( \text{Place the leg on the commutative line.} \right)$$

The degree 2 legs then map in the only way possible that will result in a map of complexes.

Proposition 7.3.1. This defines a map of $\iota$-complexes.

Checking that this is true is an instructive exercise. Now let’s meet the combinatorial maps which represent the act of “setting the parameter to 0” and “setting the parameter to 1”.

Definition 7.3.2. Define a map of $\iota$-complexes

$$\text{Ev}_{\circ \to 0} : T_{dR} \to T$$

by declaring its value on some diagram to be zero if the third line of the diagram is labelled with anything, otherwise declaring its value to be the diagram that remains when that unlabelled third line is removed.

For example:

$$\text{Ev}_{\circ \to 0} \begin{pmatrix} \quad + \quad \end{pmatrix} = \quad .$$

Definition 7.3.3. Define a map of $\iota$-complexes

$$\text{Ev}_{\circ \to 1} : T_{dR} \to T$$
by declaring its value on a diagram to be 0 if the third line of that diagram has any
grade 1 (i.e. filled) circles on it, otherwise declaring its value to be the diagram that
remains when the third line is removed.

For example:

\[
\begin{array}{c}
\text{Ev}_{0 \rightarrow 1} \\
\text{Ev}_{0 \rightarrow 1} \\
\end{array}
\]

\[
\begin{array}{c}
+ \quad + \\
\end{array}
\]

\[
= \quad .
\]

The reader can check that, as desired:

(5) \[ i_c = \text{Ev}_{0 \rightarrow 0} \circ \theta, \]

(6) \[ i_n = \text{Ev}_{0 \rightarrow 1} \circ \theta. \]

Now what we want to study is the difference \[ i_n - i_c, \]

which we have just shown is equal to:

\[ i_n - i_c = (\text{Ev}_{0 \rightarrow 1} - \text{Ev}_{0 \rightarrow 0}) \circ \theta. \]

Intuitively, then, we want to take a difference of what we get when we evaluate
some expression at 1 with what we get when we evaluate it at 0. A formal version
of the fundamental theorem of calculus will will do that for us. First we need to
learn to integrate:

**Definition 7.3.4.** Define linear maps \( \int \text{T}_{\text{dR}} \rightarrow \text{T}^i \) (and also \( \iota \) versions) by

the following procedure:

1. The value of \( \int \text{T}_{\text{dR}} \) on some diagram is zero unless the diagram has precisely
   one grade 1 (i.e. filled-in) disc on its third line.
2. If a diagram has precisely one such disc, then to calculate the value of \( \int \text{T}_{\text{dR}} \)
   on the diagram, begin by permuting (with the appropriate signs) the disc to
   the far-left end of the third line.
3. Then remove the third line and multiply the resulting diagram by \( \frac{1}{n+1} \), where
   \( n \) is the number of grade 0 circles on that line.

Here is an example of the calculation of \( \int \text{T}_{\text{dR}} \):

\[
\int \text{T}_{\text{dR}} \\
\text{Ev}_{0 \rightarrow 1} \\
\text{Ev}_{0 \rightarrow 1} \\
\end{array}
\]

\[
+ \quad + \\
\]

\[
= \quad .
\]

These grade \(-1\) maps do not quite commute with all the involved differential
operators; observe that the following lemma is not stated in terms of the full dif-
ferential on \( \text{T}_{\text{dR}} \).
Lemma 7.3.5.

\[ \int_{T_{dR}} \circ \nu_{T_{dR}} + \nu_T \circ \int_{T_{dR}} = 0. \]  
\[ \int_{T_{dR}} \circ d_T + d_T \circ \int_{T_{dR}} = 0. \]

Here is the key lemma:

Lemma 7.3.6 ("The fundamental theorem of the calculus.").

\[
\operatorname{Ev}_0 - \operatorname{Ev}_0 = \int_{T_{dR}} \circ d^\bullet.
\]

Proof. We’ll simply check that this equation is true on the different classes of generators. If the \( T_{dR} \) diagram has a grade 1 circle on its third line then both sides of the equation are zero. If, on the other hand, the \( T_{dR} \) diagram has no labels on its third line at all then again, both sides are zero. It remains to consider \( T_{dR} \) diagrams with no grade 1 circle and some positive number \( n \) of grade 0 circles on its third line. Let’s do an example to observe why the equation is true in this case.

First the left-hand side:

\[
(\operatorname{Ev}_0 - \operatorname{Ev}_0) \begin{pmatrix} \end{pmatrix} = \begin{pmatrix} \end{pmatrix}.
\]

Now consider the right-hand side:

\[
d^\bullet \begin{pmatrix} \end{pmatrix} = \begin{pmatrix} \end{pmatrix} + \begin{pmatrix} \end{pmatrix} = 3 \begin{pmatrix} \end{pmatrix}.
\]
Thus,
\[
\left( \int_{T_{\text{an}}} d_\bullet \right) \left( \begin{array}{c}
\bullet \\
\circ
\end{array} \right) = 3 \int_{T_{\text{an}}} \left( \begin{array}{c}
\bullet \\
\circ
\end{array} \right) = (3) \left( \frac{1}{3} \right).
\]

The reader should not have any difficulties constructing a general argument from this example.

\[\square\]

Proof of Theorem 7.1.2. Define the required homotopy \( s_T \) by the formula

\[ s_T = \int_{T_{\text{an}}} \circ \theta. \]

Observe:

\[ i_n - i_c = \left( \text{Ev}_{\circ \rightarrow 1} - \text{Ev}_{\circ \rightarrow 0} \right) \circ \theta, \]  
(Eqns. 6 and 6),

\[ = \int_{T_{\text{an}}} \circ d_\bullet \circ \theta, \]  
(Lemma 7.3.6),

\[ = \int_{T_{\text{an}}} \circ (d_{T_{\text{an}}} - d_T) \circ \theta, \]  
(Eqn. 4),

\[ = \left( \int_{T_{\text{an}}} \circ d_{T_{\text{an}}} \circ \theta \right) - \left( \int_{T_{\text{an}}} \circ d_T \circ \theta \right), \]  
(Prop. 7.3.1),

\[ = \left( \int_{T_{\text{an}}} \circ \theta \circ d_{\tilde{W}} \right) - \left( \int_{T_{\text{an}}} \circ d_T \circ \theta \right), \]  
(Eqn. 7),

\[ = s_T \circ d_{\tilde{W}} + d_T \circ s_T. \]

\[\square\]

8. How to Obtain Wheeling from Homological Wheeling.

Before we begin this discussion we’ll introduce some vocabulary.

8.1. \((d, \iota)-\text{pairs}\). For the remainder of this work we’ll be working with a number of systems without a natural \(\mathbb{Z}\)-grading, so it will be necessary to adjust our language a little. Define a \((d, \iota)\)-pair to be a pair of vector spaces with maps,

\[ d \left( \begin{array}{c}
\mathcal{F} \\
\iota
\end{array} \right) \left( \begin{array}{c}
\mathcal{F}_i \\
\bullet
\end{array} \right) \left( \begin{array}{c}
n \\
\circ
\end{array} \right) \]

such that

(1) \( d^2 = 0 \),

(2) \( d^n = 0 \), and

(3) \( d \circ \iota = -\iota \circ d \).

An \(\iota\)-complex, such as \(\tilde{W}\), may be viewed as a \((d, \iota)\)-pair in the following way:

\[ \oplus d^i \left( \oplus_{i=0}^{\infty} \tilde{W}^i \right) \left( \oplus_{i=0}^{\infty} \tilde{W}^i_i \right) \left( \oplus d^i \right). \]

Write \(\tilde{W} = \oplus_{i=0}^{\infty} \tilde{W}^i\) and \(\tilde{W}_i = \oplus_{i=0}^{\infty} \tilde{W}^i_i\).
8.2. Getting Wheeling from HW: an outline. Recall the statement of Homological Wheeling. Given elements $v \in B_i$ and $w \in B_j$, there exists an element $x_{v,w}$ of $\tilde{W}^{i+j-1}$ such that $\iota(x_{v,w}) = 0$ and such that

$$(\chi_W \circ \Upsilon)^i(v) \# (\chi_W \circ \Upsilon)^j(w) = (\chi_W \circ \Upsilon)^{i+j}(v \sqcup w) + d(x_{v,w}) \in \tilde{W}^{i+j}.$$ 

This statement holds in $\tilde{W}$, whereas we want a statement in $A$, the space of ordered Jacobi diagrams. Well, the non-commutative space $\tilde{W}$ has no relations which concern the legs of diagrams. We can introduce whatever relations we can think of, and see what consequences may be derived. For example, in attempting to derive a statement in $A$, we could introduce the STU relations:

That is what we will do (more carefully) in the next section. It is an interesting question whether any interesting statements can be obtained by introducing different classes of relations at this point.

In outline, to get the usual Wheeling Theorem in $A$, we will take the following route:

\[
\begin{align*}
\tilde{W} &\xrightarrow{\pi} \hat{W} & \hat{W} &\xrightarrow{B} \hat{W}_F & \hat{W}_F &\xrightarrow{\lambda} \hat{W}_\lambda = A \oplus \ldots.
\end{align*}
\]

If the reader gets lost in the details that follow, they may want to return to this map.

8.3. The $(d, \iota)$-pair $(\tilde{W}, \tilde{W}_\iota, \iota)$. The maps $d$ and $\iota$ play a crucial role in the statement of HW. But simply introducing the STU relations leaves these maps ill-defined; indeed, observe that when you apply $\iota$ to an STU relation you do not get a relation back. So we must introduce a number of other relations at the same time as we introduce STU, to ensure that the maps $d$ and $\iota$ descend to the quotient spaces.

Define $\tilde{W}$ to be the vector space generated by Weil diagrams modulo AS, IHX and the following three classes of relations:

Similarly define the vector space $\tilde{W}_\iota$. Note that diagrams generating this space will have their arrow heads drawn in the style shown.
Proposition 8.3.1. The maps $d$ and $\iota$ descend to the spaces $\hat{W}$ and $\hat{W}_\iota$. Thus, projection gives a map of $(d, \iota)$-pairs:

$$\pi : (\hat{W}, \hat{W}_\iota, \iota) \rightarrow (\hat{W}, \hat{W}_\iota, \iota).$$

The calculations to demonstrate that $d$ and $\iota$ send relations to relations constitute an amusing exercise.

8.4. A change of basis: The $(d, \iota)$-pair $\hat{W}_F$. We’ll now change back to the “curvatures” basis. Here it has the salutary effect of making the two kinds of legs invisible to each other. To be precise: define $\hat{W}_F$ to be the vector space generated by $F$-Weil diagrams, such as,

![Diagram of Weil diagrams]

modulo AS, IHX, and the following three classes of relations:

$$\begin{align*}
F F F & \rightarrow 0, \\
F F & \rightarrow 0, \\
\pm F F F & \rightarrow 0.
\end{align*}$$

Similarly define $\hat{W}_F\iota$. If we equip this pair of spaces with maps $d$ and $\iota$ using the usual substitution rules for $F$-Weil diagrams then we have a $(d, \iota)$-pair (see Section 4). Define maps $B_{\ast \rightarrow F} : \hat{W} \rightarrow \hat{W}_F$ and $B_{\ast \rightarrow F, \iota} : \hat{W}_\iota \rightarrow \hat{W}_{F\iota}$ by expanding grade 2 legs as follows:

![Diagram of maps]

These maps give a map of $(d, \iota)$-pairs

$$B_{\ast \rightarrow F} : \hat{W} \rightarrow \hat{W}_F.$$

8.5. Symmetrizing the grade 1 legs. The final step on our journey to $\mathcal{A}$ is to (graded) symmetrize the grade 1 legs. Intuitively speaking, we are just taking a special choice of basis in $\hat{W}_F$. Formally speaking, we’ll define new vector spaces $\hat{W}_\lambda$ and $\hat{W}_{\lambda, \iota}$ and equip them with “averaging” maps into $\hat{W}_F$ and $\hat{W}_{F\iota}$.

Then we’ll notice that this final space $\hat{W}_\lambda$ admits a direct-sum decomposition

$$\hat{W}_\lambda = \bigoplus_{i=0}^{\infty} \hat{W}_{\lambda}^i.$$
According to the number of grade 1 legs in a diagram. We'll locate \( \mathcal{A} \) in the grade 0 position of this decomposition:

\[
\mathcal{A} \simeq \hat{W}^0_{\lambda_1}.
\]

Finally, we'll notice that

\[
\ker \iota = \hat{W}^0_{\lambda_1} \simeq \mathcal{A}.
\]

8.5.1. The vector spaces \( \hat{W}_{\lambda_1} \) and \( \hat{W}_{\lambda_2} \). Define \( \hat{W}_{\lambda_1} \) to be the vector space generated by\( \mathbf{F} \)-Weil diagrams modulo AS, IHX and the following three classes of relations:

\[
\begin{align*}
\mathbf{F} \mathbf{F} - \mathbf{F} \mathbf{F} &= 0, \\
\mathbf{F} \mathbf{F} - \mathbf{F} &= 0, \\
\mathbf{F} \mathbf{F} + \mathbf{F} &= 0.
\end{align*}
\]

Define \( \hat{W}_{\lambda_2} \) similarly. In these spaces, then, the grade 1 legs may be moved about freely (up to sign). For example:

\[
\begin{align*}
\begin{array}{c}
\mathbf{F} \\
\mathbf{F}
\end{array}
\quad &=
\begin{array}{c}
\mathbf{F} \\
\mathbf{F}
\end{array}, \\
\begin{array}{c}
\mathbf{F} \\
\mathbf{F}
\end{array}
\quad &=
\begin{array}{c}
\mathbf{F} \\
\mathbf{F}
\end{array}.
\end{align*}
\]

We will not introduce a differential into these spaces. We will only define the map \( \iota : \hat{W}_{\lambda_1} \to \hat{W}_{\lambda_2} \). This is defined as a formal linear differential operator, in the usual way. The following theorem will be established in some detail in Section 8.8:

**Theorem 8.5.2.** There exist vector space isomorphisms

\[
\chi_{\lambda_1} : \hat{W}_{\lambda_1} \to \hat{W}_{\lambda_1} \quad \text{and} \quad \chi_{\lambda_2} : \hat{W}_{\lambda_2} \to \hat{W}_{\lambda_2}
\]

commuting with the action of \( \iota \):

\[
\begin{array}{c}
\hat{W}_{\lambda_1} \\
\hat{W}_{\lambda_2}
\end{array}
\quad \overset{\iota}{\longrightarrow}
\begin{array}{c}
\hat{W}_{\lambda_2} \\
\hat{W}_{\lambda_1}
\end{array}
\]

\[
\begin{array}{c}
\chi_{\lambda_1} \\
\chi_{\lambda_2}
\end{array}
\quad \overset{\simeq}{\longrightarrow}
\begin{array}{c}
\chi_{\lambda_1} \\
\chi_{\lambda_2}
\end{array}
\]

\[
(9)
\]

The map \( \chi_{\lambda_1} : \hat{W}_{\lambda_1} \to \hat{W}_{\lambda_2} \) is defined by declaring the value of the map on some diagram to be the average of the terms that can be obtained by doing (signed) permutations of the degree 1 legs of the diagram. \( \chi_{\lambda_2} \) is defined similarly. For
example:
\[
\chi_{\lambda}\left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
F
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
F
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}\right) = \frac{1}{6}\left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
F
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
F
\end{array}
\end{array}
\end{array}
\end{array}\right) - \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
F
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
F
\end{array}
\end{array}
\end{array}
\end{array}\right)
+ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
F
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
F
\end{array}
\end{array}
\end{array}
\end{array}
+ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
F
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
F
\end{array}
\end{array}
\end{array}
\end{array}\right).
\]

**Definition 8.5.3.** The inverse to \(\chi_{\lambda}\) is denoted
\[
\lambda : \hat{W}_{\lambda} \rightarrow \hat{W}_{\lambda}.
\]

Similarly we have \(\lambda_{\iota}\). This map \(\lambda\) is a crucial map which will be constructed explicitly in Section 8.7. After constructing \(\lambda\) we’ll go through the details of Theorem 8.5.2.

8.5.4. **Decomposing according to the number of grade 1 legs.** Observe that there are no relations in \(\hat{W}_{\lambda}\) which involve diagrams with differing numbers of the degree 1 legs. Thus the space \(\hat{W}_{\lambda}\) is graded by that quantity. Let \(\hat{W}_{\lambda}^{i}\) denote the subspace of \(\hat{W}_{\lambda}\) generated by diagrams which have exactly \(i\) grade 1 legs. We have a direct-sum decomposition
\[
\hat{W}_{\lambda} = \bigoplus_{i=0}^{\infty} \hat{W}_{\lambda}^{i}.
\]

Observe now that there is a canonical isomorphism (“label every leg with an \(F\)”) from \(A\), the familiar space of ordered Jacobi diagrams, to \(\hat{W}_{\lambda}^{0}\).
\[
\phi_{A} : A \cong \hat{W}_{\lambda}^{0}.
\]

For example:
\[
\phi_{A}\left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
F
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
F
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}\right) = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
F
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
F
\end{array}
\end{array}
\end{array}
\end{array}\right).
\]

Summarizing, we write:
\[
\hat{W}_{\lambda} = A \oplus \left(\bigoplus_{i=1}^{\infty} \hat{W}_{\lambda}^{i}\right).
\]
8.5.5. The kernel of \( \iota \). Here we will establish the kernel of the map \( \iota : \hat{W}_{\lambda} \to \hat{W}_{\lambda} \):

\[
\ker \iota = \hat{W}^0_{\lambda} \subset \hat{W}_{\lambda}.
\]

This is a consequence of the map \( \hat{i} : \hat{W}_{\lambda} \to \hat{W}_{\lambda} \) that is defined by placing the special \( \iota \)-labelled leg on the far left-hand end of the orienting line. For example:

\[
\hat{i}^7 \left( \begin{array}{c}
\mu \\
F \\
F
\end{array} \right) = \left( \begin{array}{c}
\text{Number of grade 1 legs of } v \\
F \\
F
\end{array} \right) v.
\]

Because, for some generator \( v \in \hat{W}_{\lambda} \),

\[
(\hat{i} \circ \iota) (v) = \left( \begin{array}{c}
\text{Number of grade 1 legs of } v \\
F \\
F
\end{array} \right) v,
\]

it follows that an element of \( \hat{W}_{\lambda} \) lies in the kernel of \( \iota \) if and only if it lies in \( \hat{W}^0_{\lambda} \).

8.6. Wheeling - the key lemmas. Now that we have introduced the spaces involved, we can turn to the explanation of how Wheeling arises from HW. Recall the statement of HW: Given elements \( v \in B^i \) and \( w \in B^j \), there exists an element \( x_{v,w} \) of \( \tilde{W}^{i+j-1} \) such that \( \iota(x_{v,w}) = 0 \) and such that

\[
(\chi_{\hat{W}} \circ \Upsilon)^i (v) \# (\chi_{\hat{W}} \circ \Upsilon)^j (w) = (\chi_{\hat{W}} \circ \Upsilon)^{i+j} (v \sqcup w) + d(x_{v,w}) \in \tilde{W}^{i+j}.
\]

We'll now consider taking the summands of this equation and inserting them into the composition of linear maps

\[
\hat{W} \xrightarrow{\rho} \hat{W} B_{\bullet \to F} \xrightarrow{\lambda} \hat{W}_{\lambda},
\]

producing a statement in \( \hat{W}_{\lambda} \).

To begin, note that each of the three summands \((\chi_{\hat{W}} \circ \Upsilon)^i (v) \# (\chi_{\hat{W}} \circ \Upsilon)^j (w), (\chi_{\hat{W}} \circ \Upsilon)^{i+j} (v \sqcup w) \) and \( d(x_{v,w}) \) lie in the kernel of \( \iota_{\hat{W}} \). Thus, each of their images under the map \((\lambda \circ B_{\bullet \to F} \circ \pi)\) lies in the kernel of \( \iota_{\hat{W}} \), which we know is isomorphic to \( A \):

\[
\phi_A : A \to \hat{W}^0_{\lambda} \subset \hat{W}_{\lambda}.
\]

So we move on to asking: if we pull these elements back to \( A \), what do we get? The following, crucial, lemma says that we just get The Wheeling Map composed with the averaging map from \( B \) to \( A \) (just as appears in the Wheeling Theorem).

Lemma 8.6.1. For some \( x \in B \),

\[
(\phi_A^{-1} \circ \lambda \circ B_{\bullet \to F} \circ \pi) ((\chi_{\hat{W}} \circ \Upsilon) (x)) = (\chi_B \circ \partial_{\hat{A}}) (x).
\]

The proof of this lemma will occupy almost all the remainder of this paper, starting with Section 9. The next lemma is an quick corollary of it, as will be explained at the end of this subsection:

Lemma 8.6.2. The element

\[
(\phi_A^{-1} \circ \lambda \circ B_{\bullet \to F} \circ \pi) ((\chi_{\hat{W}} \circ \Upsilon) (v) \# (\chi_{\hat{W}} \circ \Upsilon) (w))
\]

is equal to

\[
(\chi_B \circ \partial_{\hat{A}}) (v) \# (\chi_B \circ \partial_{\hat{A}}) (w).
\]
With these lemmas under our belt, the only thing which stands in the way of the Wheeling Theorem is the error term:
\[ d(x,v,w), \quad \text{where } \iota(x,v,w) = 0. \]
This is dispatched by the following lemma.

**Lemma 8.6.3.** If \( z \in \tilde{W} \) is such that \( \iota_{\tilde{W}}(z) = 0 \), then
\[
(B_{\bullet \rightarrow F} \circ \pi)(d(z)) = 0 \in \hat{W}_F.
\]

**Proof.** We'll start by establishing a small claim. We claim that if an element \( w \in \hat{W}_F \) is in the kernel of \( \iota_{\hat{W}_F} \), then it can be expressed as a linear combination of diagrams each of whose legs is an \( F \)-labelled leg. The deduction is that:
- \( \lambda(w) \) is in the kernel of \( \iota_{\hat{W}_F} \) (because \( \lambda \) commutes with \( \iota \), by Theorem 8.5.2, or directly using the construction of \( \lambda \) given in the next section).
- Thus \( \lambda(w) \) lies in \( \hat{W}_F \) (by Equation 10).
- Thus \( \lambda(w) \) is a linear combination of diagrams all of whose legs are \( F \)-labelled.
- Thus \( w = \chi_{\tilde{W}_F}(\lambda(w)) \) must also consist of entirely-\( F \)-labelled diagrams. This establishes the claim.

Then:
\[
(B_{\bullet \rightarrow F} \circ \pi)(d(z)) = d((B_{\bullet \rightarrow F} \circ \pi)(z)) = d\left( \text{A combination of diagrams in } \hat{W}_F \text{ where every leg of which is labelled by an } F. \right) = 0.
\]
The first equality follows because the maps \( B_{\bullet \rightarrow F} \) and \( \pi \) are both maps of \((d, \iota)\)-pairs. The second equality follows from the claim that started this proof (noting that \( \iota((B_{\bullet \rightarrow F} \circ \pi)(z)) = 0 \)). The third equality we leave as an exercise. (Compare with the proof of Proposition 4.2.1.)

(We remark that the error term could also have been dealt with by introducing a \( \mathbb{Z}_2 \)-grading and tracking that through the composition of maps.)

**Proof of Lemma 8.6.2 using Lemma 8.6.1.** We wish to compute the expression
\[
(\phi^{-1}_A \circ \lambda \circ B_{\bullet \rightarrow F} \circ \pi)(\chi_W \circ T)(v)\#(\chi_W \circ T)(w)).
\]
Briefly considering the constructions of the maps \( B_{\bullet \rightarrow F} \) and \( \pi \), we realize that we can write this expression:
\[
(\phi^{-1}_A \circ \lambda)((B_{\bullet \rightarrow F} \circ \pi \circ \chi_W \circ T)(v)\#(B_{\bullet \rightarrow F} \circ \pi \circ \chi_W \circ T)(w)).
\]
Now, just as in the claim used in the proof of Lemma 8.6.3, we observe that because the factors \((B_{\bullet \rightarrow F} \circ \pi \circ \chi_W \circ T)(v) \) and \((B_{\bullet \rightarrow F} \circ \pi \circ \chi_W \circ T)(w) \) are both in the kernel of \( \iota_{\hat{W}_F} \), they can be expressed using diagrams whose legs are entirely \( F \)-legs. The map \( \lambda \) does not do anything to such diagrams. Thus, the expression can be written:
\[
(\phi^{-1}_A \circ \lambda \circ B_{\bullet \rightarrow F} \circ \pi \circ \chi_W \circ T)(v)\#(\phi^{-1}_A \circ \lambda \circ B_{\bullet \rightarrow F} \circ \pi \circ \chi_W \circ T)(w).
\]
This completes the proof.

\( \square \)
8.7. **The map $\lambda$.** We now turn to a detailed construction of the maps

$$\lambda : \tilde{\mathcal{W}}_F \to \tilde{\mathcal{W}}_\lambda$$

and

$$\lambda_\ast : \tilde{\mathcal{W}}_{F \ast} \to \tilde{\mathcal{W}}_{\lambda_\ast}$$

which invert the averaging maps:

$$\lambda \circ \chi_{\wedge} = \text{id}_{\tilde{\mathcal{W}}_\lambda}$$

and

$$\lambda_\ast \circ \chi_{\wedge_\ast} = \text{id}_{\tilde{\mathcal{W}}_{\lambda_\ast}}.$$ 

The definitions are quite natural and easily remembered: “glue chords (with signs) into the grade 1 legs in all possible ways”. Getting the signs and coefficients right can be a delicate affair, however, so it proves worthwhile to introduce some machinery to set up the definition of $\lambda$ now in order to clarify the computations later. We’ll describe two complementary approaches to $\lambda$: the first is a formal combinatorial definition; the second is a more visual method.

8.7.1. **The combinatorial definition of $\lambda$.** Consider a diagram $w$ from $\tilde{\mathcal{W}}_F$. Let $L_{\perp}(w)$ denote the set of grade 1 legs of $w$. To begin, we’ll explain how to operate with a two-element subset $S$ of $L_{\perp}(w)$ on $w$. The resulting term, which will be denoted $D_S(w)$, is obtained from $w$ by:

1. Doing signed permutations to move one of the legs in $S$ until it is adjacent to the other leg.
2. Then gluing those two legs together and multiplying the result by $\frac{1}{2}$.

For example, consider the following diagram $w$, with the elements of its set $L_{\perp}(w)$ enumerated:

To illustrate this operation we will calculate $D_{\{2,4\}}(w)$. First we do signed permutations to make leg 2 adjacent to leg 4:

Then we just glue those two legs together, then multiply by $\frac{1}{2}$. Thus:

$$D_{\{2,4\}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ F & F & F & F \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 1 & 3 & 24 & 24 \\ F & F & F & F \end{pmatrix}.$$ 

Now we’ll extend this to an action of a *pairing* of the legs of $w$ on $w$. The desired map $\lambda$ will then be defined to be a sum over one term for every pairing.

**Definition 8.7.2.** A pairing of the legs of a diagram $w$ is defined to be a set of mutually disjoint 2-element subsets of $L_{\perp}(w)$. The empty set will be regarded as a
pairing. Let \( \mathcal{P}(w) \) denote the set of pairings of the diagram \( w \). Define the action of a pairing \( \phi = \{S_1, \ldots, S_n\} \) on the diagram \( w \), denoted \( \mathcal{D}_\phi(w) \), to be:

\[
\mathcal{D}_\phi(w) = (\mathcal{D}_{S_1} \circ \ldots \circ \mathcal{D}_{S_n})(w).
\]

For example:

\[
\mathcal{D}_{\{(2, 4), (1, 3)\}} \left( \begin{array}{ccc}
1 & 2 & 3 \\
F & F & F
\end{array} \right) = \mathcal{D}_{\{(2, 4)\}} \left( \mathcal{D}_{\{(1, 3)\}} \left( \begin{array}{ccc}
1 & 2 & 3 \\
F & F & F
\end{array} \right) \right),
\]

\[
= -\frac{1}{2} \mathcal{D}_{\{(2, 4)\}} \left( \begin{array}{c}
2 \\
F \\
F
\end{array} \right),
\]

\[
= -\left(\frac{1}{2}\right)^2 \mathcal{D}_{\{(2, 4)\}},
\]

For this to give a well-defined operation on diagrams it is necessary that a different choice of ordering of the elements of \( \phi \) will yield the same result. It is sufficient to show that the following equation will always hold

\[
(\mathcal{D}_{\{a, b\}} \circ \mathcal{D}_{\{c, d\}})(w) = (\mathcal{D}_{\{c, d\}} \circ \mathcal{D}_{\{a, b\}})(w),
\]

where \( a, b, c \) and \( d \) are different legs of some diagram \( w \). To show this equation we just have to observe that it is true for all possible arrangements of the positions of these legs. Let’s just check one possibility, where the two pairs are ‘linked’:

\[
\begin{array}{cccc}
a & c & b & d \\
\hline
\end{array}
\]

Assume that the total grade of the legs between \( a \) and \( c \) in the above diagram is \( p \), between \( c \) and \( b \) is \( q \), and between \( b \) and \( d \) is \( r \). Then the result of applying \( \mathcal{D}_{\{a, b\}} \circ \mathcal{D}_{\{c, d\}} \) to the above diagram is readily calculated to be

\[
(-1)^{q+1+r}(-1)^{p+q} \frac{1}{4}
\]

On the other hand, the result of applying \( (\mathcal{D}_{\{c, d\}} \circ \mathcal{D}_{\{a, b\}}) \) is

\[
(-1)^{p+q+1}(-1)^{q+r} \frac{1}{4}
\]

which is clearly the same thing. Nothing surprising happens in the other cases.
**Definition 8.7.3.** Define a linear map \( \lambda : \hat{\mathcal{W}}_F \to \hat{\mathcal{W}}_\lambda \) by declaring the value of \( \lambda \) on some diagram \( w \) to be

\[
\lambda(w) = \sum_{\psi \in \mathcal{P}(w)} D_\psi(w).
\]

Define a map \( \lambda_\iota \) similarly.

To ensure that we agree on the precise definition of this important map, here is the result of a computation:

\[
\lambda \begin{pmatrix} F & F & F \end{pmatrix} = F F F + \frac{1}{2} F F F - \frac{1}{2} F F F + \frac{1}{2} F F F - \frac{1}{4} F F F + \frac{1}{4} F F F + \frac{1}{4} F F F - \frac{1}{4} F F F.
\]

We’ll check that \( \lambda \) respects the relations that define \( \hat{\mathcal{W}}_F \) shortly, in Section 8.7.5. But first we’ll describe a more visual/graphical approach to its definition.

**8.7.4. A visual approach to the map \( \lambda \).** We are defining the map \( \lambda \) as a sum

\[
\lambda(w) = \sum_{\psi \in \mathcal{P}(w)} D_\psi(w)
\]

of one term for every pairing of the grade 1 legs of \( w \). The term \( D_\psi(w) \) can be constructed in the following way. To illustrate this point of view we’ll use:

\[
w = \begin{pmatrix} F & F \end{pmatrix}
\text{ and } \psi = \{\{1, 3\}, \{2, 4\}, \{5, 7\}\}.
\]
To construct $D_\varphi$: Begin by introducing a second orienting line underneath the diagram, with a gap separating the two orienting lines, in the following way:

Next: For every pair of legs in the pairing $\varphi$, add an arc, using a full line, between the corresponding legs of the diagram (such that the introduced arc has no self-intersections and stays within the gap between the two orienting lines):

Finally, carry all the remaining legs straight down onto the bottom orienting line, using a full line for the grade 1 legs and a dashed line for the grade 2 legs:

Let $x$ denote the number of intersections between full lines displayed within the gap. The term $D_\varphi(w)$ is this diagram (with the original orienting line forgotten and the dashed lines filled in) multiplied by $(-1)^x$ and a factor of $\left(\frac{1}{2}\right)$ for every introduced arc.

Thus, in the example at hand:

$$D_\varphi(\{1,3\}, \{2,4\}, \{5,7\}) = (-1)^2 \left(\frac{1}{2}\right)^3$$

This procedure constructs the same $D_\varphi$ as the earlier definition.

This visual approach can be useful because it lets us see facts about $\lambda$ as “homotopies” of the arcs of the glued diagrams.

8.7.5. The map $\lambda$ respects relations.

Proposition 8.7.6. The map $\lambda$ is well-defined.
**Proof.** We must check that \( \lambda \) maps relations to relations. The map \( \lambda \) only 'sees' the grade 1 legs, so our problem is to show that \( \lambda \) maps the expressions in the following class:

\[
\begin{align*}
\lambda \text{ only 'sees' the grade 1 legs, so our problem is to show that } & \lambda \text{ maps the expressions in the following class:} \\
\text{to zero in } & \hat{W}_\lambda.
\end{align*}
\]

So consider a particular case of this expression, and let \( D_1, D_2 \) and \( D_3 \) denote the three involved diagrams. For example,

\[
\begin{align*}
D_1 & + D_2 & - D_3
\end{align*}
\]

Identify the sets of grade 1 legs \( L_\perp(D_1) \) and \( L_\perp(D_2) \) in the natural way. (So in the above example the 2nd leg of \( D_1 \) is identified with the third leg of \( D_2 \)) Thus we can write:

\[
\lambda(D_1 + D_2) = \sum_{\varphi \in \mathcal{P}(D_1)} (D_\varphi(D_1) + D_\varphi(D_2)).
\]

There are three possibilities for the pairing \( \varphi \). Call the two grade 1 legs that change position in the expression the *active* legs. The other grade 1 legs are called the *inactive* legs. The grade 2 legs can be ignored completely.

**Possibility #1:** Neither of the active legs appear in \( \varphi \). In this case it is clear that

\[
D_\varphi(D_1) + D_\varphi(D_2) = 0,
\]

using a leg permutation relation in \( \hat{W}_\lambda \).

**Possibility #2:** At least one of the active legs is paired by \( \varphi \) with an inactive leg. In this case, we again have that

\[
D_\varphi(D_1) + D_\varphi(D_2) = 0.
\]

To see this, write

\[
D_\varphi = D_{S_1} \circ \ldots \circ D_{(a,i)}
\]

where \( a \) denotes the mentioned active leg, and \( i \) the corresponding inactive leg.

Now calculate:

\[
D_{(a,i)}(D_1) = D_{(a,i)} \left( \begin{array}{c}
\text{i} \\
\text{a}
\end{array} \right) = \frac{1}{2}(-1)^x,
\]

where \( x \) denotes the total grade of the legs between \( i \) and \( a \). Also:

\[
D_{(a,i)}(D_2) = D_{(a,i)} \left( \begin{array}{c}
\text{i} \\
\text{a}
\end{array} \right) = \frac{1}{2}(-1)^{x+1}.
\]
Possibility #3: The two active legs are paired with each other. Let \( \mathcal{P}^a(D_1) \) denote the set of pairings of \( D_1 \) which pair the two active legs together. There is a bijection \( r : \mathcal{P}^a(D_1) \to \mathcal{P}(D_3) \).

The map is just to remove the pair \( \{a_1, a_2\} \) from the pairing. A quick calculation gives the equation
\[
\mathcal{D}_\psi(D_1) + \mathcal{D}_\psi(D_2) = \mathcal{D}_{r(\psi)}(D_3).
\]

8.7.7. The map \( \lambda \) does indeed invert \( \chi_\wedge \).

**Proposition 8.7.8.**

\[
\lambda \circ \chi_\wedge = \text{id}_{\hat{\mathcal{W}}_\wedge} \quad \text{and} \quad \lambda_\rightharpoonup \circ \chi_\wedge = \text{id}_{\hat{\mathcal{W}}_\wedge}.\]

**Proof.** Consider some diagram \( w \) with \( n \) degree 1 legs. Write the signed average of the grade 1 legs \( \chi_\wedge(w) = \sum_{\sigma \in \text{Perm}_n} w_\sigma \), where \( \text{Perm}_n \) denotes the set of permutations of the set \( \{1, \ldots, n\} \). For example:

If \( w = \quad \begin{array}{c} \begin{array}{c} \text{F} \end{array} \end{array} \) then \( w_{(1234)} = -\frac{1}{24} \).

For each \( \sigma \), identify the set of grade 1 legs \( L_\wedge(w_\sigma) \) with \( L_\wedge(w) \) using the order of the legs as they appear along the orienting line from left-to-right. Given a pairing \( \psi \in \mathcal{P}(w) \), we claim that
\[
\mathcal{D}_\psi \left( \sum_{\sigma \in \text{Perm}_n} w_\sigma \right) = 0
\]
unless \( \psi = \phi \).

To begin the establishment of this claim, choose some decomposition of the operation
\[
\mathcal{D}_\psi = \mathcal{D}_{S_1} \circ \ldots \circ \mathcal{D}_{\{l_1, l_2\}}.
\]
Assuming w.l.o.g. that \( l_1 < l_2 \), let \( \text{Perm}_n^{(l_1, l_2)} \subset \text{Perm}_n \) denote the subset of permutations \( \sigma \) with the property that if \( \sigma(x) = l_1 \) and \( \sigma(y) = l_2 \) then \( x < y \). Graphically speaking, \( \text{Perm}_n^{(l_1, l_2)} \) consists of those permutations where the straight lines ending at positions \( l_1 \) and \( l_2 \) do not cross:

Similarly, define \( \text{Perm}_n^{X}(l_1, l_2) \subset \text{Perm}_n \) to be the subset of permutations where they do cross. There is clearly a bijection \( \vartheta : \text{Perm}_n^{(l_1, l_2)} \xrightarrow{\simeq} \text{Perm}_n^{X}(l_1, l_2) \):

\[
\vartheta \left( \begin{array}{c} \begin{array}{c} \text{F} \end{array} \end{array} \right) = \begin{array}{c} \begin{array}{c} \text{F} \end{array} \end{array}.
\]
Now calculate:
\[ D_\varphi \left( \sum_{\sigma \in \text{Perm}_n} w_\sigma \right) = (D_{S_1} \circ \ldots \circ D_{\{l_1,l_2\}}) \left( \sum_{\sigma \in \text{Perm}_n} w_\sigma \right) \]
\[ = (D_{S_1} \circ \ldots \circ D_{\{l_1,l_2\}}) \left( \sum_{\sigma \in \text{Perm}_n(l_1,l_2)} (w_\sigma + w_\vartheta(\sigma)) \right) \]

The claim now follows from the observation that
\[ D_{\{l_1,l_2\}}(w_\sigma) = -D_{\{l_1,l_2\}}(w_\vartheta(\sigma)). \]

\( \square \)

8.8. Proof of Theorem 8.5.2. To prove this theorem we’ll:

1. Prove that Diagram 9 commutes.
2. Prove that \( \chi_\Lambda \) and \( \chi_{\Lambda \Lambda} \) are surjective.
3. Observe that because \( \lambda \circ \chi_\Lambda = \text{id}_{\widehat{W}_\Lambda} \) and \( \lambda_\Lambda \circ \chi_{\Lambda \Lambda} = \text{id}_{\widehat{W}_{\Lambda \Lambda}} \), it follows that \( \chi_\Lambda \) and \( \chi_{\Lambda \Lambda} \) are also injective.

Proposition 8.8.1. The map \( \iota \) commutes with \( \chi_\Lambda \) and \( \chi_{\Lambda \Lambda} \). (i.e. Diagram 9 commutes.)

Proof. Consider some generator of \( \widehat{W}_\Lambda \) with \( n \) grade 1 legs. We must show that \( \iota(\chi_\Lambda(D)) = \chi_{\Lambda \Lambda}(\iota(D)) \). To do this we’ll enumerate the terms that arise when each side of this equation is calculated, and then we’ll observe that the two calculated expressions are the same. For the purposes of discussion we’ll focus on the diagram

\[ (\iota(\chi_\Lambda(D))). \]

First, note that \( \chi_\Lambda(D) \) has one term for every word in the symbols \( \{1, \ldots, n\} \). The corresponding term is \( (\pm 1^n n!) \) times the diagram which is obtained by permuting the legs of the original diagram according to the word. The sign of the term is just the sign of the corresponding permutation. For example:

\[ [1 \ 3 \ 2] \text{ corresponds to the term } -\frac{1}{3!} \]

Observe, now, that \( \iota(\chi_\Lambda(D)) \) has one term for every pair consisting of a word in the symbols \( \{1, \ldots, n\} \) together with a choice of one of the symbols in the word. The corresponding term is \( (\pm 1^n n!) \) times the diagram that is obtained by first permuting the legs of the original diagram according to the word, and then making the leg corresponding to the chosen letter the \( \iota \)-labelled leg. The sign of this term is the sign of the permutation that is required to go from the standard word \( [1 \ldots n] \) to the word that is obtained from the given word by moving the chosen symbol to the
first place in the word. For example:

\[
\begin{bmatrix} 1 & 3 & 2 \end{bmatrix}
\]

corresponds to the term \( \frac{1}{3!} \cdot \frac{\chi}{F} \).

(Note that the sign of this term is +1 because that is the sign of the permutation that is required to go from \([1 2 3]\) to the word \([3 1 2]\) (which is obtained from \([1 3 2]\) by moving the 3 to first place).)

\( (\chi_{\lambda, n}(\iota(D))) \) This expression has one term for every pair consisting of a choice of one of the symbols \( \{1, \ldots, n\} \) together with a word in the remaining symbols. The corresponding term is \( \pm \frac{1}{(n-1)!} \) times the diagram that is obtained from the original diagram by first making the chosen leg the \( \iota \)-labelled leg, and then permuting the remaining legs according to the given word. The sign of this term is the sign of the permutation that is required to go from the standard word to the word that is obtained by placing the chosen symbol at the head of the given word. For example:

\[
(3, [1 2])
\]

corresponds to the term \( \frac{1}{2!} \cdot \frac{\chi}{F} \).

There is an \( n \)-to-1 map between the two sets of terms, e.g.:

\[
\begin{align*}
[3 & 1 2] \\
[1 & 3 2] &\rightarrow (3, [1 2]) \\
[1 & 2 3]
\end{align*}
\]

Observing that each term in the domain of this map is exactly \( \frac{1}{n} \) times its image, we are done.

\( \square \)

**Proposition 8.8.2.** *The maps \( \chi_{\lambda} \) and \( \chi_{\lambda^\dagger} \) are surjective.*

**Proof.** This argument will be familiar. Take \( w \), an arbitrary \( F \)-Weil diagram. Let \( w_{\lambda} \) be the same diagram, but regarded as a generator of \( \widetilde{W}_{\lambda} \). Then:

\[
w - \chi_{\lambda}(w_{\lambda}) = \left( \text{A sum of diagrams with less grade 1 legs than } w \right).
\]

By induction, \( w \) lies in the image of \( \chi_{\lambda} \).

\( \square \)

9. **How wheels appear: An illustration.**

The remaining sections of this paper will present a detailed proof of the following theorem (which was stated earlier as Lemma 8.6.1).
Theorem 9.0.3. Let $v$ be an element of $\mathcal{B}$.

\[(\phi_A^{-1} \circ \lambda \circ B_{\rightarrow F} \circ \pi \circ \chi_W \circ T)(v) = (\chi_B \circ \partial_B)(v).\]

Figure 3 illustrates this crucial theorem by taking a symmetric Jacobi diagram and following it as it maps through this sequence. At each stage in the figure we have drawn a diagram that is typical of the diagrams in the expression at that point. Observe that the last diagram is equal to the image under $\phi_A$ of
the original diagram with two wheels glued into its legs and its remaining legs given some ordering. These are precisely the sort of diagrams the map $\chi_B \circ \partial_1$ will produce.

It seems a challenging combinatorial problem to compute the composition:

$$(\lambda \circ B \circ \pi \circ \chi_W \circ \Upsilon)(v).$$

To compute this composition we’ll begin by expressing it in terms of a certain operation product. The next section, Section 10, will introduce the relevant formalism, of “diagrams operating on diagrams”. Section 11.1 develops the key expression; the result is recorded in Theorem 11.0.2.

Finally, in Sections 12 and 13 we’ll perform a direct and detailed combinatorial evaluation of that operator expression.

10. Operating with diagrams on diagrams.

10.1. Operator Weil diagrams. We’ll now begin to operate on Weil diagrams with other Weil diagrams. To introduce this formalism we’ll build a vector space $\hat{W}_F[[a, b]]$. (The constructions to follow adapt in an unambiguous way to build spaces like $\hat{W}_{\lambda}[[a, b]]$, etc.) Intuitively: we are taking the vector space $\hat{W}_F$, adjoining a formal grade 2 variable $a$ and a formal grade 1 variable $b$ and their corresponding differential operators, and then taking power series with respect to those introduced symbols.

Formally: this space will be built from diagrams which may have the usual legs for $\hat{W}_F$, namely

\begin{center}
\begin{tikzpicture}
  \draw[->] (0,0) -- (1,0) node[midway, above] {F};
  \draw[->] (1,0) -- (2,0) node[midway, above] {F};
  \node at (0.5,0.2) {grade 2 legs};
  \node at (1.5,0.2) {and grade 1 legs};
\end{tikzpicture}
\end{center}

but which may have, in addition, parameter legs

\begin{center}
\begin{tikzpicture}
  \draw[->] (0,0) -- (1,0) node[midway, above] {a};
  \draw[->] (1,0) -- (2,0) node[midway, above] {b};
  \node at (0.5,0.2) {and};
\end{tikzpicture}
\end{center}

and their corresponding operator legs

\begin{center}
\begin{tikzpicture}
  \draw[->] (0,0) -- (1,0) node[midway, above] {\partial_a};
  \draw[->] (1,0) -- (2,0) node[midway, above] {\partial_b};
  \node at (0.5,0.2) {and};
\end{tikzpicture}
\end{center}

We require that the operator legs appear in a group at the far right-hand end of the diagram. The parameter legs may appear amongst the usual legs in any order.
Here is an example of such an operator Weil diagram:

Let the parameter-grade of a diagram be the total grade of its parameter legs, where $a$-labelled legs count for 2 and $b$-labelled legs count for 1. The diagram above has parameter-grade 3. Similarly define the quantity operator-grade; the diagram above has operator-grade 4. If a diagram has parameter-grade $i$ and operator-grade $j$ then we say that it is an $(i, j)$-operator Weil diagram. The pair $(i, j)$ will be referred to as the type of the diagram. Thus, the operator Weil diagram above has type $(3, 4)$.

**Definition 10.1.1.** Define the vector space $\hat{\mathcal{W}}_F[[a, b]^{(i,j)}]$ to be the space of formal $\mathbb{Q}$-linear combinations of operator Weil diagrams of type $(i, j)$, subject to the same relations that the space $\hat{\mathcal{W}}_F$ uses, together with relations that say that the parameter and operator legs can be moved about freely (up to the appropriate sign), as long as the operator legs all stay at the far-right hand end of the orienting line.

For example, the following equations hold in $\hat{\mathcal{W}}_F[[a, b]^{(3,2)}]$:

We will work with power series of operator Weil diagrams. Here is what we mean by that:

**Definition 10.1.2.** Define the space of formal power series of operator Weil diagrams in the following way:

\[ \hat{\mathcal{W}}_F[[a, b]] = \prod_{(i,j) \in \mathbb{N}_0 \times \mathbb{N}_0} \hat{\mathcal{W}}_F[[a, b]^{(i,j)}], \]

where $\mathbb{N}_0$ denotes the set of non-negative integers.

We suggest taking a moment to decode this: a formal power series of operator diagrams is a choice, for every pair $(i, j)$ of non-negative integers, of a vector from $\hat{\mathcal{W}}_F[[a, b]^{(i,j)}]$. 
10.2. **The operator pairing.** We will now introduce a bilinear pairing on these power series:

\[ \upharpoonright : \hat{\mathcal{W}}_F[[a,b]] \times' \hat{\mathcal{W}}_F[[a,b]] \to \hat{\mathcal{W}}_F[[a,b]]. \]

The notation \( \times' \) is to record the fact that the pairing is only defined (only “converges”) on certain pairs of power series:

\[ \hat{\mathcal{W}}_F[[a,b]] \times' \hat{\mathcal{W}}_F[[a,b]] \subset \hat{\mathcal{W}}_F[[a,b]] \times \hat{\mathcal{W}}_F[[a,b]]. \]

The discussion below requires the projection map

\[ \pi^{(i,j)} : \hat{\mathcal{W}}_F[[a,b]] \to \hat{\mathcal{W}}_F[a,b]^{(i,j)}. \]

10.2.1. **How to operate with a diagram.** The purpose of operator Weil diagrams, of course, is to have them operate on each other. We’ll first define how individual diagrams operate on each other, and then extend that action to power series. Consider, then, two operator Weil diagrams:

![Diagram](image1)

To operate with the first on the second, you begin by placing the two diagrams adjacent to each other on the orienting line. Then you proceed to push the operator legs to the far-right hand side of the resulting diagram by using substitution rules which declare that the operator legs act as graded differential operators. To be precise: if the operator leg encounters a parameter leg corresponding to the same parameter, then it operates on that leg:

![Diagram](image2)

or

![Diagram](image3)

If, on the other hand, the operator leg encounters any leg it is not matched to, then the operator is just pushed past the leg, incurring the appropriate sign \((-1)^{g_1g_2}\), where \(g_1\) and \(g_2\) are the leg-grades of the two legs involved). Figure 4 illustrates such a computation. The reader might like to check that they get the final result, which is contained in Figure 5.

10.2.2. **Checking relations.** Having defined the operation \( \upharpoonright \) on the level of diagrams, let us proceed to verify that it respects the relations that we introduce amongst those diagrams.

**Proposition 10.2.3.** Let \( i, j, k \) and \( l \) be elements of \( \mathbb{N}_0 \). The linear extension of the above definition of \( \upharpoonright \) gives a well-defined bilinear map

\[ \upharpoonright : \hat{\mathcal{W}}_F[a,b]^{(i,j)} \times' \hat{\mathcal{W}}_F[a,b]^{(k,l)} \to \hat{\mathcal{W}}_F[[a,b]]. \]
Figure 4. An illustration of one diagram operating on another.

Proof. We require that the relations that define $\hat{W}_F[a,b]^{(i,j)}$ and $\hat{W}_F[a,b]^{(k,l)}$ are sent to zero in $\hat{W}_F[[a,b]]$. The sort of thing that we must show is that

$$ F \begin{array}{c} \partial a \\ \partial b \\ \partial b \\ \partial b \end{array} + \begin{array}{c} b \\ b \\ b \\ b \\ \partial a \\ \partial a \\ \partial a \\ \partial a \end{array} = 0. $$

In this discussion we'll use a box to represent the part of the diagram that varies in a relation vector. For example, the equation above will be represented schematically as follows:

There are four classes of relations that must be checked:
Figure 5. The final result of the diagram operation started in Figure 4.

(1) Amongst the operator legs of $\hat{W}_F[a,b]^{(i,j)}$. For example:

(2) Amongst the non-operator legs of $\hat{W}_F[a,b]^{(i,j)}$. For example:

(3) Amongst the operator legs of $\hat{W}_F[a,b]^{(i,j)}$. For example:
(4) Amongst the other legs of $\mathcal{W}_F[a, b]^{(k,l)}$. For example:

$$
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[draw] (a) {} -- (1,0) node[draw] (b) {} -- (2,0) node[draw] (c) {};
\draw (a) -- (0,1) node[draw] (d) {} -- (b);\end{tikzpicture}
\end{array}
\quad \vdash \quad
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[draw] (a) {} -- (1,0) node[draw] (b) {} -- (2,0) node[draw] (c) {};
\draw (a) -- (0,1) node[draw] (d) {} -- (b);\end{tikzpicture}
\end{array}
= 0.
\end{array}
$$

Classes 2 and 3 are obviously sent to zero because they play no part in the calculation. We will restrict ourselves to checking Class 4 as these relations will play a role in the subsequent calculation. Class 1 is also straightforward.

To demonstrate why such relations are respected we’ll consider a specific example. We must show that

$$
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[draw] (a) {} -- (1,0) node[draw] (b) {} -- (2,0) node[draw] (c) {};
\draw (a) -- (0,1) node[draw] (d) {} -- (b);\end{tikzpicture}
\end{array}
\quad \vdash \quad
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[draw] (a) {} -- (1,0) node[draw] (b) {} -- (2,0) node[draw] (c) {};
\draw (a) -- (0,1) node[draw] (d) {} -- (b);\end{tikzpicture}
\end{array}
= 0,
\end{array}
$$

where the box can represent any one of ten possible relations. It might be one of the usual relations for $\mathcal{W}_F$:

$$
\begin{array}{c}
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[draw] (a) {} -- (1,0) node[draw] (b) {} -- (2,0) node[draw] (c) {};
\draw (a) -- (0,1) node[draw] (d) {} -- (b);\end{tikzpicture}
\end{array}
= \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[draw] (a) {} -- (1,0) node[draw] (b) {} -- (2,0) node[draw] (c) {};
\draw (a) -- (0,1) node[draw] (d) {} -- (b);\end{tikzpicture}
\end{array}
- \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[draw] (a) {} -- (1,0) node[draw] (b) {} -- (2,0) node[draw] (c) {};
\draw (a) -- (0,1) node[draw] (d) {} -- (b);\end{tikzpicture}
\end{array}
- \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[draw] (a) {} -- (1,0) node[draw] (b) {} -- (2,0) node[draw] (c) {};
\draw (a) -- (0,1) node[draw] (d) {} -- (b);\end{tikzpicture}
\end{array},
\end{array}
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[draw] (a) {} -- (1,0) node[draw] (b) {} -- (2,0) node[draw] (c) {};
\draw (a) -- (0,1) node[draw] (d) {} -- (b);\end{tikzpicture}
\end{array}
= \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[draw] (a) {} -- (1,0) node[draw] (b) {} -- (2,0) node[draw] (c) {};
\draw (a) -- (0,1) node[draw] (d) {} -- (b);\end{tikzpicture}
\end{array}
+ \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[draw] (a) {} -- (1,0) node[draw] (b) {} -- (2,0) node[draw] (c) {};
\draw (a) -- (0,1) node[draw] (d) {} -- (b);\end{tikzpicture}
\end{array}
- \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[draw] (a) {} -- (1,0) node[draw] (b) {} -- (2,0) node[draw] (c) {};
\draw (a) -- (0,1) node[draw] (d) {} -- (b);\end{tikzpicture}
\end{array},
\end{array}
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[draw] (a) {} -- (1,0) node[draw] (b) {} -- (2,0) node[draw] (c) {};
\draw (a) -- (0,1) node[draw] (d) {} -- (b);\end{tikzpicture}
\end{array}
= \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[draw] (a) {} -- (1,0) node[draw] (b) {} -- (2,0) node[draw] (c) {};
\draw (a) -- (0,1) node[draw] (d) {} -- (b);\end{tikzpicture}
\end{array}
- \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[draw] (a) {} -- (1,0) node[draw] (b) {} -- (2,0) node[draw] (c) {};
\draw (a) -- (0,1) node[draw] (d) {} -- (b);\end{tikzpicture}
\end{array},
\end{array}
\end{array}
$$

On the other hand it might be one of the relations involving one of the introduced parameters:

$$
\begin{array}{c}
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[draw] (a) {} -- (1,0) node[draw] (b) {} -- (2,0) node[draw] (c) {};
\draw (a) -- (0,1) node[draw] (d) {} -- (b);\end{tikzpicture}
\end{array}
= \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[draw] (a) {} -- (1,0) node[draw] (b) {} -- (2,0) node[draw] (c) {};
\draw (a) -- (0,1) node[draw] (d) {} -- (b);\end{tikzpicture}
\end{array}
- (\text{\text{-1)}|x|}^\mu \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[draw] (a) {} -- (1,0) node[draw] (b) {} -- (2,0) node[draw] (c) {};
\draw (a) -- (0,1) node[draw] (d) {} -- (b);\end{tikzpicture}
\end{array},
\end{array}
\end{array}
$$

where $x$ can be either $a$ or $b$, where $|x|$ is the grade of that leg, and where there can be any type of leg in the shaded box with $\mu$ representing its grade. To proceed, note that we can perform the calculation on all the diagrams involved in the relation at
the same time. We begin:

Now we employ the following formulas, which the reader can check are true for each of the ten possible replacements for the box (where $\mu$ is the total grade of the legs in the box):

Continuing with the example from earlier we obtain:

This is clearly zero in $\widehat{\mathcal{W}}_F[[a, b]]$ (because it is a combination of relations).

10.2.4. The extension to power series. Recall that we are working with “formal power series” of Weil operator diagrams, which are a choice, for every pair $(i, j)$ of non-negative integers, of a vector from $\widehat{\mathcal{W}}_F[a, b]^{(i, j)}$:

In this section we will extend the operation product $\sqcup$ to these power series in the obvious way: to multiply two power series, do it term-by-term, then add up the
results. Because there is the possibility of infinite sums coming out of this, we must be careful making statements in generality about this product.

For a power series $v$ write

$$v = \sum_{(i,j) \in \mathbb{N}_0 \times \mathbb{N}_0} v^{(i,j)},$$

where $v^{(i,j)} = \pi^{(i,j)}(v)$, the type $(i,j)$ piece of $v$.

**Definition 10.2.5.** Let $v$ and $w$ be power series from $\widetilde{W}_F[[a,b]]$. If for every pair $(i,j) \in \mathbb{N}_0 \times \mathbb{N}_0$ it is true that

$$\pi^{(i,j)} \left( v^{(k,l)} \vdash w^{(m,n)} \right) = 0$$

for all but finitely many pairs $((k,l), (m,n))$, then we say that the product $v \vdash w$ is **convergent**, in which case it is defined to be

$$\sum_{(k,l), (m,n) \in \mathbb{N}_0 \times \mathbb{N}_0} \left( v^{(k,l)} \vdash w^{(m,n)} \right) \in \widetilde{W}_F[[a,b]].$$

It is not difficult to construct non-convergent products. Here is one:

$$\exp\# \left( \begin{array}{c}
\begin{array}{c}
\partial_a
\end{array}
\end{array} \right) \vdash \exp\# \left( \begin{array}{c}
\begin{array}{c}
a
\end{array}
\end{array} \right).$$

10.3. **A convenient graphical method for doing diagram operations.** In certain of the computations to come later in this paper, we’ll need to be able to give a direct construction of all the terms that contribute to a diagram operation, together with an easy way to determine the signs of those contributions.

For this purpose we’ll now introduce a convenient visual method for doing a diagram operation. In this method, the “operation” is defined to be a sum of one term for every gluing of the two diagrams:

**Definition 10.3.1.** Consider two operator Weil diagrams $v$ and $w$. Let $Op(v)$ denote the set of operator legs of $v$, and let $Par(w)$ denote the set of parameter legs of $w$. A gluing of $v$ onto $w$ is an injection of a (possibly empty) subset of $Op(v)$ into $Par(w)$ that respects labels (so a $\partial_a$-labelled leg of $v$ will only be mapped to an $a$-labelled leg of $w$, and similarly for the $b$-labels). Let $G(v,w)$ denote the set of gluings of $v$ onto $w$.

Below we will explain how to associate a term

$$t(v, w, \sigma)$$

to a gluing $\sigma \in G(v, w)$, and then we’ll define the operation of $v$ on $w$ to be:

$$v \vdash w = \sum_{\sigma \in G(v, w)} t(v, w, \sigma).$$

This construction will obviously agree with the usual definition of the operation given in Section 10.2.1.
In the discussion to follow, we’ll consider the example of the diagrams

\[ v = \begin{array}{c}
\partial_b \\
\partial_b \\
\partial_a \\
\partial_b \\
\partial_b \\
\end{array} \quad \text{and} \quad w = \begin{array}{c}
\partial_b \\
\partial_b \\
a \\
\partial_b \\
\partial_b \\
\end{array} \]

The operator legs of \( v \) are numbered from left to right, and so are the parameter legs of \( w \), as displayed by the above diagrams. Below, we’ll construct the term \( t(v, w, \sigma) \) corresponding to some gluing \( \sigma \in G(v, w) \) will be constructed by the following procedure. To begin, place the operator legs of \( v \) up the left-hand side of a grid, and the non-operator legs of \( w \) across the top of the grid, above an orienting line:

The next step is to join up legs along the grid according to the gluing. Join up grade 1 legs using a full line, and grade 2 legs using a dashed line (this is so that we will be able to easily read off the sign of the term at the end of the construction). Continuing the example:
Join any remaining legs on the top of the grid to the orienting line. For grade 1 legs use a full line and for grade 2 legs use a dashed line:

Finally, carry any remaining operator legs lying along the left-hand side of the grid to the far side of the grid, and then place them on the orienting line using nested right-angles:

Let \( x \) denote the number of intersections between full lines displayed within the box. The term \( t(v, w, \sigma) \) is just the diagram that has been constructed (with the dashed lines now filled in), with a sign \( (-1)^x \) out the front.

In the example at hand \( x = 4 \), so:

\[
\begin{align*}
\left( v, w, \begin{pmatrix} 245 \\ 132 \end{pmatrix} \right) & = (-1)^4 \\
\end{align*}
\]

10.4. **Associativity.** As an illustration of this graphical method we’ll now use it to give a (probably more detailed than necessary) proof that the operation product is associative:

**Proposition 10.4.1.** Let \( u, v \) and \( w \) denote operator Weil diagrams. Then

\[
\begin{align*}
u \updownarrow (v \updownarrow w) & = (u \updownarrow v) \updownarrow w.
\end{align*}
\]
Proof. We’ll illustrate the discussion with the example of:

\[ u = \begin{array}{c}
\quad \\
\quad \\
\end{array} \quad , \quad v = \begin{array}{c}
\quad \\
\quad \\
\end{array} \quad \text{and} \quad w = \begin{array}{c}
\quad \\
\quad \\
\end{array} . \]

To begin, assemble the three diagrams around the edges of a “step-ladder” grid, in the following way:

To prove associativity we’ll:

- Show how to express \( u \uplus (v \uplus w) \) as a sum of diagrams built from this step-ladder grid.
- Show how to express \((u \uplus v) \uplus w\) in the same way.
- Observe that the terms of the two sums correspond.

So focus first on the product \( u \uplus (v \uplus w) \). If we follow the definitions directly, we learn that this product has one term for every pair of gluings (recalling that a gluing is a parameter-respecting injection):

\[
\rho : \text{Op}(v) \supset K \rightarrow \text{Par}(w),
\]
\[
\rho' : \text{Op}(u) \supset K' \rightarrow \text{Par}(v) \cup (\text{Par}(w) \setminus \text{Im}(\rho)) .
\]

Let the set of such pairs be denoted by \( G_1 \). The term \( t_1(\rho, \rho') \) corresponding to some pair \((\rho, \rho') \in G_1\) is constructed by wiring up the top box of the step-ladder using \( \rho \), and then wiring up the bottom two boxes using \( \rho' \). The sign of the term, as usual, is a product of a \((-1)\) for every intersection between full lines displayed by the diagram.
For example, to construct the contribution $t_1 \left( \left( \frac{1'}{2'}, \frac{1'}{2'3'} \right), \left( \frac{1}{2} \right) \right)$, we start by wiring up the top box using the gluing $\left( \frac{1'}{2'} \right)$, which gives

Then we wire up the bottom two boxes using $\left( \frac{1}{2'} \right)$, which gives

It is clear, as a direct application of the definitions, that

$$u \vdash (v \vdash w) = \sum_{(\rho, \rho') \in G_1} t_1(\rho, \rho').$$

Next we’ll consider the other bracketting: $(u \vdash v) \vdash w$. In this case the appropriate indexing set is $G_2$, the set of pairs of gluings

$\varsigma : \text{Op}(u) \supset L \rightarrow \text{Par}(v)$,

$\varsigma' : (\text{Op}(u) \setminus L \cup \text{Op}(v)) \supset L' \rightarrow \text{Par}(w)$.

The corresponding expression is:

$$(u \vdash v) \vdash w = \sum_{(\varsigma, \varsigma') \in G_2} t_2(\varsigma, \varsigma').$$

To finish, note that there is an obvious correspondence $C : G_1 \simeq G_2$ between the terms of the two expressions: $t_1(\rho, \rho') = t_2(C(\rho, \rho'))$.
10.5. **Associativity and power series.** We recommend this section for the second reading; it consists of some unsurprising details about associativity and convergence.

In the computations that are the core of this work, we’ll need to re-bracket certain products of power series. We’ve just shown that we can re-bracket products of the generators; to re-bracket products of power series proves to be a more delicate affair (because of convergence issues). To avoid getting bogged down by the logic of our definitions, we’ll introduce a simple finiteness condition (Condition (§), below).

When Condition (§) holds, for a triple $u$, $v$ and $w$ of power series from $\hat{W}_F[[a,b]]$, then it will be true that:

$$(u \vdash v) \vdash w = u \vdash (v \vdash w).$$

We’ll state the condition as a lemma:

**Lemma 10.5.1.** Let $u$, $v$ and $w$ be power series from $\hat{W}_F[[a,b]]$. Assume that:

- The product $u \vdash v$ converges.
- The product $v \vdash w$ converges.
- For all $(i, j)$, there are only finitely many triples $((k, l), (m, n), (p, q))$ with the property that

$$\pi(i,j) \left( u^{(k,l)} \vdash v^{(m,n)} \vdash w^{(p,q)} \right) \neq 0 \quad (\text{§}).$$

Then the products $(u \vdash v) \vdash w$ and $u \vdash (v \vdash w)$ converge, and:

$$(u \vdash v) \vdash w = u \vdash (v \vdash w).$$

There is also a version for the other bracketting. It will be a straightforward matter to check that this condition holds in any situation that we perform a re-bracketting.

**Proof.** First, note that $(u \vdash v) \vdash w$ obviously converges. (Otherwise Condition (§) would be violated.) Second, note that because we can re-bracket the generators, Condition (§) implies its re-bracketed version: that for each $(i, j)$ there are only finitely many triples $((k, l), (m, n), (p, q))$ such that

$$\pi(i,j) \left( u^{(k,l)} \vdash v^{(m,n)} \vdash w^{(p,q)} \right) \neq 0 \quad (\text{§}').$$

Thus $u \vdash (v \vdash w)$ converges as well. Thirdly, note that Condition (§) implies that the expression

$$\sum_{(k,l), (m,n), (p,q)} u^{(k,l)} \vdash v^{(m,n)} \vdash w^{(p,q)}$$

makes sense. It is almost tautological that it is equal to $(u \vdash v) \vdash w$. And similarly, $(\text{§}')$ implies that

$$u \vdash (v \vdash w) = \sum_{(k,l), (m,n), (p,q)} u^{(k,l)} \vdash v^{(m,n)} \vdash w^{(p,q)}.$$ 

Associativity of products of generators gives the required equality. □

The major reason we need such detail is that there are products $(u \vdash v) \vdash w$ which don’t satisfy Condition (§) but which nevertheless converge. This makes general statements difficult.
11. Expressing the Composition as an Operator Product.

The computation that is the subject of Theorem 9.0.3 is based on an expression of the value of the composition

$$B \xrightarrow{\chi_W} \bar{W} \xrightarrow{\pi} \bar{W} \xrightarrow{B_F} \hat{W}_F \xrightarrow{\lambda} \hat{W}_\lambda$$

on some symmetric Jacobi diagram in terms of the operation product $\triangleright$. The statement of the expression uses a linear map $B_{b \triangleright \partial_a} : B \rightarrow \mathcal{W}_\lambda[[a, b]]$ which acts on a Jacobi diagram $v$ by first choosing an ordering of the legs of $v$ and then labelling every leg with a $\partial_h$. For example:

$$B_{b \triangleright \partial_a}(v) = \text{Diagram}.$$

The purpose of this section is to prove the following theorem:

**Theorem 11.0.2.** Let $v$ be an element of $B$, the space of symmetric Jacobi diagrams. Then the element $(\lambda \circ B_F \circ \pi \circ \chi_W \circ \gamma)(v)$ is equal to

$$\left[ B_{b \triangleright \partial_a}(v) \triangleright \left( \exp_{\#} \left( \frac{1}{2} \begin{array}{c} \exp_{\#} \left( \frac{1}{2} \begin{array}{c} a \\ \end{array} \right) \\ \end{array} \right) \right) \right]_{a, b, \partial_a, \partial_b=0}$$

where $\mathcal{X}$ is equal to

$$\exp_{\#} \left( \frac{1}{2} \begin{array}{c} a \\ \end{array} \right) \triangleright \left( \exp_{\#} \left( \frac{1}{2} \begin{array}{c} a \\ \end{array} \right) \right).$$

In Section 12 we commence the computation of $\mathcal{X}$ by performing the $\lambda$ operation in the second factor above. Section 13 takes that result and performs the remaining operation product.

To state the final result we’ll employ a certain notation for the $a$-labelled legs. Note that the $a$-labelled legs commute with every type of leg, and so can be moved around freely. It proves useful, then, to avoid drawing them in explicitly. We’ll record the $a$-labelled legs by (locally) orienting the edge they are incident to, and labelling that edge with some power series in $a$. For example:

$$2a^l = \text{Diagram}.$$

Section 13 completes the computation that:
Theorem 11.0.3.

\[ [\mathcal{X}]_{b, \partial_b = 0} = \exp \left( \frac{1}{2} \ln \left( \frac{\sinh \left( \frac{3}{2} \right)}{\frac{3}{2}} \right) \right) \]

Substituting this computation into Theorem 11.0.2 completes the proof of Theorem 9.0.3, that for \( v \in \mathcal{B} \),

\[ (\phi_A^{-1} \circ \lambda \circ B \circ \pi \circ \chi_W \circ \Upsilon) (v) = (\chi_B \circ \partial_B) (v). \]

11.1. Using operator diagrams to average. We'll build up to Theorem 11.0.2 piece-by-piece. The construction begins with the following piece:

\[ B \xrightarrow{\pi} W \xrightarrow{\chi_W} \tilde{W} \xrightarrow{\pi} \tilde{W} \xrightarrow{B \circ \lambda} \hat{W}. \]

To present this piece, we'll turn a diagram \( v \in W \) into an operator diagram, and then have that diagram operate on an exponential of parameters.

**Definition 11.1.1.** Define a linear map

\[ B_{\bullet \to \partial_a, \perp \to \partial_b} : W \rightarrow \tilde{W}[a, b] \]

by replacing legs according to the rules

\[ \xrightarrow{\partial_a} \quad \text{and} \quad \xrightarrow{\partial_b}. \]

For example:

\[ B_{\bullet \to \partial_a, \perp \to \partial_b} \left( \begin{array}{c}
\text{diagram}
\end{array} \right) = \begin{array}{c}
\text{operator diagram}
\end{array}. \]

The key proposition follows. It says that we can take the signed average of a diagram by changing it into an operator, applying the resulting operator to an exponential of formal parameters, then setting all parameters to zero. The map which sets all the parameters to zero, denoted \([\cdot]_{a, b, \partial_a, \partial_b = 0}\) below, is precisely \( \text{pr}^{(0,0)} \), the projection of the \((0,0)\) factor out of the power series.

**Proposition 11.1.2.** Let \( v \in W \). Then

\[ (\pi \circ \chi_W)(v) = \left[ B_{\bullet \to \partial_a, \perp \to \partial_b} (v) \dagger \left( \exp_{\#} \left( \begin{array}{c}
\text{operator diagram}
\end{array} \right) \right) \right]_{a, b, \partial_a, \partial_b = 0}. \]

**Proof.** Both sides are linear maps, so it suffices to check this formula on generators. So take some symmetric Weil diagram \( v \), and assume that it has \( p \) grade 2 legs and \( q \) grade 1 legs. We'll evaluate the value that the right-hand side takes on \( v \) and observe that it is precisely the signed average of \( v \), as required.
For convenience, write
\[ v_{\text{op}} \] for \( B \circlearrowleft \rightarrow_{a \mapsto \partial_a} (v) \).

Consider, then, the exponential that \( v_{\text{op}} \) is to be applied to. We’ll index its terms by the set of words (including the empty word) that can be built from the symbols \( A \) and \( B \). Given some such word \( w \), let \( f_w \) denote the corresponding diagram. For example:

\[ f_{BABBA} = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array} \]

To begin, then, we can expand the right-hand-side of equation 11 to get:

\[
(12) \left[ v_{\text{op}} \vdash \left( \sum_{\text{Words made from the symbols } A \text{ and } B} \frac{1}{|w|!} f_w \right) \right]_{a, b, \partial_a, \partial_b = 0}.
\]

Because \( v_{\text{op}} \) has exactly \( p \) legs labelled by \( \partial_a \) and exactly \( q \) legs labelled by \( \partial_b \), the only terms that will survive when all the parameters are set to zero will arise from the terms \( f_w \) corresponding to words \( w \) built from exactly \( p \) copies of \( A \) and exactly \( q \) copies of \( B \). Restricting expression 12 to these terms we write:

\[
(13) \frac{1}{(p + q)!} \sum_{\text{Words } w \text{ built from } p \text{ copies of } A \text{ and } q \text{ copies of } B} [v_{\text{op}} \vdash f_w]_{a, b, \partial_a, \partial_b = 0}.
\]

Fix, then, a word \( w \) built from \( p \) copies of \( A \) and \( q \) copies of \( B \), and let’s proceed to calculate \([v_{\text{op}} \vdash f_w]_{a, b, \partial_a, \partial_b = 0}\). For example, if

\[ v = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array} \]

and \( w = BABAA \), then we wish to calculate

\[
\left[ \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array} \right] \vdash \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array} \right]_{a, b, \partial_a, \partial_b = 0}.
\]

To do such a computation directly, it is worth employing the graphical method for doing diagram operations that was described in Section 10.3.
Let us take a moment to recall this method. We begin by placing the operator legs up the left-hand side of a grid, and the legs of $f_w$ along the top of the grid:

According to this method, we get precisely one contribution $t(v_{op}, f_w, \sigma)$ to the operation for every permutation

$$\sigma : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$$

which respects the parameters. In general, if we let $\text{Perm}_n$ denote the set of all permutations on $\{1, \ldots, n\}$ and let $\text{Perm}_{p+q}(v, w) \subset \text{Perm}_{p+q}$ denote the set of permutations respecting the parameters, then we can write

$$\left[v_{op} \vdash f_w\right]_{a, b, \partial_a, \partial_b=0} = \sum_{\sigma \in \text{Perm}_{p+q}(v, w)} t(v_{op}, f_w, \sigma),$$

where $t(v_{op}, f_w, \sigma)$ is determined by the usual graphical method. For example, the term corresponding to the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix}$ is determined by joining up legs in the following way:

Recall that the sign of the contribution is determined by counting the number of intersections between full lines displayed within the box. Thus:

$$t \left( v_{op}, f_w, \begin{pmatrix} 12345 \\ 24351 \end{pmatrix} \right) = (-1)^3.$$

Here is what we need to notice: the diagram $t(v_{op}, f_w, \sigma)$ put out by this operation is (up to some to-be-determined sign) precisely what we get by permuting the legs of the original diagram using the permutation $\sigma$. To state this observation precisely, let’s introduce some notation: Given a permutation $\sigma \in \text{Perm}_{p+q}$, let $\psi^\sigma$ denote the
diagram one gets by permuting (without introducing signs) the legs of \(v\) according to \(\sigma\). For example, if

\[
v = \\
\text{then } v^{(1\ 2\ 3\ 4\ 5)} = 
\]

The key observation is contained in the following lemma.

**Lemma 11.1.3.** Let \(v\) be some diagram from \(W\) with \(p\) grade 2 legs and \(q\) grade 1 legs. Let \(w\) be some word which uses \(p\) copies of the symbol \(A\) and \(q\) copies of the symbol \(B\). Let \(\sigma\) be some permutation from \(\text{Perm}_{p+q}(v,w)\). Then:

\[
t(v_{\text{op}}, f_w, \sigma) = \varphi(\sigma)v^\sigma,
\]

where \(\varphi(\sigma)\) is the product of a \((-1)\) for every pair of grade 1 legs of \(v\) which reverse their order in \(v^\sigma\).

**Proof of the Lemma:** The fact that \(t(v_{\text{op}}, f_w, \sigma)\) is some sign multiplied by \(v^\sigma\) is quite clear. The subtlety is in the sign. This is quite clear too, once we regard the problem from the point of view of the visual method of doing the gluing. Here is the general picture of a gluing:

The legs of \(v\) appear in their original order up the left-hand side of the grid, then continue through the box (and the arcs at the top) along indeterminate paths, then finish along the orienting line at the bottom in the order determined by the permutation \(\sigma\). The sign we are trying to determine is \((-1)^i\) raised to the number of intersections between full lines displayed inside the box. It is clear (for simple topological reasons) that the number of intersections between full lines is given by the stated formula. *End of the proof of the Lemma.*

Now we can put it all together. According to Equation 13 and Equation 14, the right-hand side of the Equation 11 can be written

\[
\frac{1}{(p+q)!} \sum_{\text{Words } w \text{ built from } \text{\(p\) copies of } A \text{ and } \text{\(q\) copies of } B} \sum_{\sigma \in \text{Perm}_{p+q}(v,w)} t(v_{\text{op}}, f_w, \sigma)
\]
Substituting the result of the lemma into this, we get:

\[
\frac{1}{(p+q)!} \sum_{\text{Words } w \text{ built from } A \text{ and } q \text{ copies of } B.} \sum_{\sigma \in \text{Perm}_{p+q}(v,w)} \varphi(\sigma)v^\sigma.
\]

This, of course, is just \(\pi \circ \chi_W\), the required graded averaging map. \(\square\)

It is a simple step to extend this proposition to give a formula for the following piece of the composition

\[
\begin{align*}
B \xrightarrow{\Upsilon} W \xrightarrow{\chi_W} \tilde{W} \xrightarrow{\pi} \hat{W} \xrightarrow{\lambda} \hat{W}_\Lambda.
\end{align*}
\]

**Corollary 11.1.4.** Let \(v \in W\). Then \((B \circ \pi \circ \chi_W)(v)\) is given by the expression

\[
\left[ B_{\perp \to \partial_b}(v) \vdash \left( \exp \left( \frac{1}{2} \sum_{k=1}^{n-|S|} \frac{1}{n!} B_{\perp \to \partial_b} \right) \right) \right]_{a,b,\partial_a,\partial_b=0}.
\]

**11.2. Hair-splitting with operator diagrams.** We now turn our focus to the first step in the composition:

\[
\begin{align*}
B \xrightarrow{\Upsilon} W \xrightarrow{\chi_W} \tilde{W} \xrightarrow{\pi} \hat{W} \xrightarrow{\lambda} \hat{W}_\Lambda.
\end{align*}
\]

**Proposition 11.2.1.** Let \(v \in B\). Then the equation

\[
B_{\perp \to \partial_b}(v) \vdash \exp \left( -\frac{1}{2} \sum_{k=1}^{n-|S|} \frac{1}{n!} B_{\perp \to \partial_b} \right) = \exp \left( -\frac{1}{2} \sum_{k=1}^{n-|S|} \frac{1}{n!} B_{\perp \to \partial_b} \right) \vdash B_{\perp \to \partial_b}(\Upsilon(v))
\]

holds in \(\hat{W}_F[[a,b]]\).

**Proof.** As the two sides of this equation are both linear maps, it suffices to show that the equation holds for generators. So let \(v\) be a symmetric Jacobi diagram. We begin by expanding the left-hand side of Equation 11.2.1 in the following way:

\[
\sum_{n=0}^{\infty} \left( -\frac{1}{2} \right)^n \frac{1}{n!} B_{\perp \to \partial_b}(v) \vdash \left( \sum_{k=1}^{n-|S|} \frac{1}{n!} B_{\perp \to \partial_b} \right).
\]

Now consider the diagram operation in the above sum. Recalling how to do diagram operations (see Section 10.3), we get (letting \(L\) denote the set of the legs of \(v\)):

\[
\sum_{n=0}^{\infty} \left( -\frac{1}{2} \right)^n \frac{1}{n!} \sum_{\text{Injections } S \to \text{Perm}(1,\ldots,n)} B_{\perp \to \partial_b} \vdash v_S,
\]
where, for example,

\[
\frac{1}{2} \begin{pmatrix} 2 & 3 & 4 \\ 1 \end{pmatrix}
\]

then

\[
v_{(1,3,4)} = \begin{pmatrix} \partial_b & \partial_b & \partial_b & \partial_b & \partial_b \\ \partial_b & \partial_b & \partial_b & \partial_b & \partial_b \end{pmatrix}
\]

Now for every subset \( S \subset \mathcal{L} \) of legs there are \( \frac{n!}{(n-|S|)!} \) injections \( \phi : S \to \{1, \ldots, n\} \). Thus the above expression may be rewritten

\[
\sum_{n=0}^{\infty} \left( \left( -\frac{1}{2} \right)^n \frac{1}{n!} \right) \sum_{S \subset \mathcal{L}} \frac{n!}{(n-|S|)!} \left( \begin{array}{c} \frac{1}{2} \newline \dot{\ldots} \newline \frac{1}{2} \end{array} \right) \left( \begin{array}{c} a \newline \partial_b \newline \partial_b \newline \partial_b \newline \partial_b \end{array} \right) + \left( \sum_{S \subset \mathcal{L}} \left( -\frac{1}{2} \right)^{|S|} v_S \right).
\]

This is the right hand side of Equation 11.2.1.

11.3. Putting the pieces together. Now we’ll put these pieces together to obtain an expression for the following part of the composition:

\[
\mathcal{B} \xrightarrow{\Upsilon} \mathcal{W} \xrightarrow{\chi_W} \mathcal{W} \xrightarrow{\pi} \mathcal{W}_F \xrightarrow{\lambda} \mathcal{W}_{\lambda}. \]

So let \( v \) be an element of \( \mathcal{B} \). If we substitute \( \Upsilon(v) \) directly into Corollary 11.1.4 then we are given the following expression for \( (B_{\bullet \rightarrow F} \circ \pi \circ \chi_W \circ \Upsilon)(v) \):

\[
B_{\bullet \rightarrow \partial_b} (\Upsilon(v)) \vdash \exp \left( \begin{array}{c} a \newline \partial_b \newline \partial_b \end{array} \right) \left( \begin{array}{c} F \newline \frac{1}{2} \newline a \newline \partial_b \newline \partial_b \end{array} \right) + \left( \begin{array}{c} b \newline \partial_b \newline \partial_b \end{array} \right) + \left( \begin{array}{c} a \newline \partial_b \newline \partial_b \end{array} \right).
\]

We now wish to use Proposition 11.2.1 to re-express this as a direct function of \( v \). To this end, we begin by inserting the missing piece of that proposition into the front of this expression, giving:

\[
\exp \left( \begin{array}{c} a \newline \partial_b \newline \partial_b \end{array} \right) \vdash \exp \left( \begin{array}{c} a \newline \partial_b \newline \partial_b \end{array} \right) \left( B_{\bullet \rightarrow \partial_b} (\Upsilon(v)) \right) + \left( \begin{array}{c} a \newline \partial_b \newline \partial_b \end{array} \right) \left( \begin{array}{c} a \newline \partial_b \newline \partial_b \end{array} \right).
\]

We can do this because all operation products in the resulting expression converge and because, when we set the parameter \( a \) to zero, all the introduced terms will vanish.
Now we perform an associativity rearrangement (a careful reader may wish to read Lemma 10.5.1 and then check that Condition (§) holds before rearranging):

\[
\left( \exp_h \left( -\frac{1}{2} \frac{\partial_b}{\partial_a} \right) \right) \mapsto B \mapsto_{\partial_b} (\Upsilon(v)) \mapsto_{\partial_a} \left( \exp_h \left( -\frac{1}{2} \frac{\partial_b}{\partial_a} \right) \right) \mapsto_{\partial_b} \left( \exp_h \left( -\frac{1}{2} \frac{\partial_b}{\partial_a} \right) \right)
\]

Using Proposition 11.2.1 to replace the bracket, and then doing another associativity rearrangement (checking Condition (§)) we get:

\[
\left( \exp_h \left( -\frac{1}{2} \frac{\partial_b}{\partial_a} \right) \right) \mapsto_{\partial_b} \left( \exp_h \left( -\frac{1}{2} \frac{\partial_b}{\partial_a} \right) \right) \mapsto_{\partial_b} \left( \exp_h \left( -\frac{1}{2} \frac{\partial_b}{\partial_a} \right) \right)
\]

Finally, observe that \( a \)-labelled legs commute with all other types of legs, and \( F \)-legs commute with all other legs (except other \( F \)-legs), so some straightforward rearrangements allow us to write (using \{\} instead of () only to make this equation easier on the eye):

\[
(15) \quad \left[ B \mapsto_{\partial_a} (v) \mapsto_{\partial_a} \left\{ \exp_h \left( \frac{1}{2} \frac{\partial_b}{\partial_a} \right) \right\} \right] \quad a, b, \partial_a, \partial_b = 0
\]

To complete the construction of the composition

\[
B \mapsto_{\partial_a} W \mapsto_{\partial_b} \hat{W} \mapsto_{\lambda} \hat{W}_\lambda
\]

it remains for us to apply \( \lambda \) to the Expression 15. The next section will show that we can commute \( \lambda \) through this expression and begin by applying it to the right-most exponential above.

11.4. Commuting \( \lambda \) through the expression. For the purposes of the discussion to come we’ll refer to legs of the form

\[
\downarrow
\]

as \( \perp \)-legs.
In Section 8.7 we introduced $\lambda$ as a map
$$\lambda : \hat{\mathcal{W}}_F \rightarrow \hat{\mathcal{W}}_\lambda,$$
basedly defined by “gluing $\bot$-legs together in all possible ways (with signs)". In this section we'll need the natural extension of $\lambda$ to the situation where there are some parameter and operator legs present (we’ll make this precise shortly):
$$\lambda : \hat{\mathcal{W}}_F[[a, b]] \rightarrow \hat{\mathcal{W}}_\lambda[[a, b]].$$
The purpose of this section is to prove the following proposition.

**Proposition 11.4.1.** Let $v$ and $w$ be elements of $\hat{\mathcal{W}}_F[[a, b]]$, and assume that $v$ can be expressed without $\bot$-legs. Then:
$$\lambda(v \triangleright w) = v \triangleright \lambda(w).$$

Before turning to the proof, let’s repeat the definition of $\lambda$ (in this more general context). Consider some diagram $w$, a generator of $\hat{\mathcal{W}}_F[[a, b]]$. Let $\mathcal{L}_\bot(w)$ denote the set of $\bot$-legs of $w$. Recall that a pairing of $w$ is a (possibly empty) set of disjoint 2-element subsets of $\mathcal{L}_\bot$. As before, $\mathcal{P}(w)$ denotes the set of pairings of $w$. We then define $\lambda$ by
$$\lambda(w) = \sum_{\varphi \in \mathcal{P}(w)} D_\varphi(w)$$
where the term $D_\varphi(w)$ is constructed by a certain graphical procedure described in detail in Section 8.7.4. The procedure is, recall: draw another orienting line under the existing one, separated by a gap; then join up the $\bot$-legs according to the pairing $\varphi$ using non-self-intersecting full arcs lying entirely in the gap between the two orienting lines; then carry any legs remaining on the original orienting line down to the new orienting line, using full lines for (all) the grade 1 legs and dashed lines for (all) the grade 2 legs. Finally, write down a coefficient of $(-1)^x \left(\frac{1}{2}\right)^y$ where $x$ is the number of self-intersections displayed by full lines between the two orienting lines and $y$ is the number of pairs in $\varphi$.

For example, if
$$w = \begin{array}{c}
\fbox{1} & \fbox{2} & \fbox{3} & \fbox{4} \\
& b & b & b
\end{array},$$
and we wished to construct $D_{\{\{1,4\},\{2,3\}\}}(w)$, then we would draw the diagram
and thus deduce that:

\[ D\{\{1,4\},\{2,3\}\}(w) = (-1)^3 \left(\frac{1}{2}\right)^2 . \]

**Proof.** It suffices to show that

\[ \lambda(v \vdash w) = v \vdash \lambda(w) \]

is true for generators. To perform the product \( \vdash \) we’ll use the graphical approach of Section 10.3. Readers may want to refresh their memories of that approach and accompanying notation before reading this proof.

So let \( v \) and \( w \) be operator Weil diagrams and assume that \( v \) has no \( \perp \)-legs. For the purposes of the discussion below we’ll consider the example where:

\[ v = \]

\[ w = \]

It follows from the construction of the operations \( \lambda \) and \( \vdash \) that the two sides of Equation 16 are sums indexed by the same set. Define that set \( \mathcal{PG} \) to consist of pairs \( (\wp,\sigma) \), where \( \wp \) is a pairing \( \wp \in \mathcal{P}(w) \) and \( \sigma \) is a gluing \( \sigma \in \mathcal{G}(v,w) \). Then:

1. \( \lambda(v \vdash w) = \sum_{(\wp,\sigma) \in \mathcal{PG}} D_{\wp}(t(v,w,\sigma)) \),
2. \( v \vdash \lambda(w) = \sum_{(\wp,\sigma) \in \mathcal{PG}} t(v, D_{\wp}(w), \sigma) \).

(The careful reader will notice that we are making the natural identifications \( \mathcal{P}(w) \simeq \mathcal{P}(t(v,w,\sigma)) \) and \( \mathcal{G}(v,w) \simeq \mathcal{G}(v,D_{\wp}(w)) \) to write these expressions.) The required equality will be established if we can show that for every pair \( (\wp,\sigma) \in \mathcal{PG} \):

\[ D_{\wp}(t(v,w,\sigma)) = t(v, D_{\wp}(w), \sigma) . \]

We’ll begin by illustrating the desired equality for the case of the given example and the pair \( (\wp,\sigma) = (\{\{1,3\},\{2,4\}\}, (4132)) \). A direct application of the definitions tells us that to construct the term \( D_{\wp}(t(v,w,\sigma)) \) we draw the diagram shown in Figure 6. On the other hand: a direct application of the definitions tells us that to construct \( t(v, D_{\wp}(w), \sigma) \) we draw the diagram shown in Figure 7. In both cases the coefficient of the contributing term is given by \((-1)^x \left(\frac{1}{2}\right)^y\), where \( x \) is the number of intersections between full lines displayed inside the total box (by total box we’ll refer to the union of the three displayed dotted boxes), and \( y \) is the number of pairs in \( \wp \) (in this case, 2). The underlying diagrams are obviously the same, so it remains to understand why the two representations introduce the same number of intersections between full lines (mod 2).

To explain why, in generality: First observe that the total box cuts the full lines that go through the total box into a number of arcs. Then notice that there is obvious correspondence between the arcs of the first diagram and the arcs of the second diagram. Finally, notice that the ends of any pair of arcs will have the same
Figure 6. The diagram you draw to build $\mathcal{D}_\nu(t(v, w, \sigma))$.

relative position around the edge of the total box in both diagrams; hence the pair of arcs will have the same number of intersections mod 2 in both diagrams. 

□
12. Computing the operator product I: The inner-most piece.

We’ll begin the computation of the expression in Theorem 11.0.2 with the inner-most piece. The objective of this section is to prove the following theorem.

**Theorem 12.0.2.** The following equality holds in $\hat{W}_n[[a, b]]$.

$$\lambda \left( \exp_# \left( \begin{array}{c}
\begin{array}{c}
\frac{a}{2} \\
\tau
\end{array}
+ \begin{array}{c}
\frac{b}{2}
\end{array}
\end{array} \right) \right) = \exp_# \left( \begin{array}{c}
\frac{1}{2} \ln(\cosh(\frac{a}{2})) + \frac{\tanh(\frac{a}{2})}{\frac{a}{2}} - \frac{\tanh(\frac{a}{2}) - \frac{a}{2}}{(\frac{a}{2})^2}
\end{array} \right).$$

This section consists of two subsections. In Section 12.1 we show that the left-hand side of the above equation can be expressed as an exponential of the series of terms with connected diagrams that arise from the evaluation of $\lambda$. In Section 12.2 we’ll perform a detailed calculation of that series.

12.1. An exponential of connected diagrams.

Consider the terms that arise when you compute the left-hand side of the above equation. We’ll index these terms with a certain set $T$. The set $T$ is defined to be the set of pairs $(w, \varphi)$ consisting of a non-empty word $w$ in the symbols $A$ and $B$ and a disjoint (possibly empty) family $\varphi$ of 2-element subsets of the set $\{1, 2, 3, \ldots, 2\#A + \#B\}$ (where $\#A$ and $\#B$ denote the number of appearances in the word $w$ of the symbols $A$ and $B$ respectively). If $\tau = (w, \varphi)$ we’ll often write $|\tau|$ for $|w|$, the length of the word $w$.

Given such a pair $(w, \varphi)$, the corresponding term $T_{(w, \varphi)}$ is constructed in two steps. The first step is to place a number of copies of

[diagram A]

and

[diagram B]

along an orienting line in the order dictated by $w$. The second step is to pair up (with appropriate signs) the $\perp$-legs of this diagram according to the pairing $\varphi$, multiplying by a factor of $\frac{1}{2}$ for every pair of legs glued together (in other words, by a factor of $\left(\frac{1}{2}\right)^{|\varphi|}$).
For example, to construct $T_{(AABAB,\{(1,5),(2,3),(4,6),(7,8)\})}$, begin by writing down

\[
\begin{array}{c}
\frac{a}{2} \quad \frac{a}{2} \quad \frac{a}{2} \\
\hline
\hline
b \quad b
\end{array}
\]

Then pair up legs according to the pairing information, giving:

\[
(+1) \left(\frac{1}{2}\right)^3
\begin{array}{c}
\frac{a}{2} \quad \frac{a}{2} \quad \frac{a}{2} \\
\hline
\hline
b \quad b
\end{array}
\]

The follow proposition is just a restatement of the definitions.

**Proposition 12.1.1.**

\[
\lambda \left( \exp_\# \left( \frac{a}{2} + \begin{array}{c}
\frac{a}{2} \\
\hline
\hline
b
\end{array} \right) \right) = 1 + \sum_{(w,\wp) \in T} \frac{1}{|w|!} T_{(w,\wp)}.
\]

Now let $T_C$ denote the set of pairs $(w,\wp)$ whose corresponding term $T_{(w,\wp)}$ is **connected**. (By connected we mean that the graph is connected when the orienting line is ignored.) In this discussion we’ll call $T_C$ the set of **connected types**.

**Theorem 12.1.2.** The following equation holds in $\hat{W}_n[[a,b]]$:

\[
\lambda \left( \exp_\# \left( \frac{a}{2} + \begin{array}{c}
\frac{a}{2} \\
\hline
\hline
b
\end{array} \right) \right) = \exp_\# \left( \sum_{\tau \in T_C} \frac{1}{|\tau|!} T\tau \right).
\]

We’ll build up to this theorem with a number of combinatorial lemmas. The computation of $\sum_{\tau \in T_C} \frac{1}{|\tau|!} T\tau$, the series of **connected** diagrams that can arise from the computation, is the subject of the next section. Certain readers may feel that the above theorem is obvious, and we encourage such readers to skip to the Section 12.2. (We remark that the subtlety in this situation is all to do with the signs.)

**12.1.3. The content of a pairing.** Consider some pair $(w,\wp)$. The corresponding diagram $T_{(w,\wp)}$ decomposes into a number a connected components. To each connected component $x$ we can associate some other pair $(w_x,\wp_x)$. Namely, just delete every other component and write down the pair $(w_x,\wp_x)$ which produces the remaining component.

For example, consider the diagram corresponding to

\[(AABAAA,\{(2,5),(4,6),(3,9),(7,11),(8,10)\}).\]

It is:

\[
\begin{array}{c}
\frac{a}{2} \quad \frac{a}{2} \quad \frac{a}{2} \\
\hline
\hline
b
\end{array}
\]
This diagram has 2 connected components. One component corresponds to the pair \((AAAA, \{\{1,6\}, \{2,3\}, \{4,8\}, \{5,7\}\})\):

\[
\begin{array}{c}
\hat{a} \rightarrow \hat{a} \\
\hat{a} \rightarrow \hat{a} \\
\hat{a} \rightarrow \hat{a} \\
\hat{a} \rightarrow \hat{a}
\end{array}
\]

The other component corresponds to the pair \((AB, \{\{2,3\}\})\):

\[
\begin{array}{c}
\hat{a} \rightarrow \hat{a} \\
\hat{a} \rightarrow \hat{a} \\
\hat{a} \rightarrow \hat{a} \\
\hat{a} \rightarrow \hat{a}
\end{array}
\]

Define now the **content** of a pair \((w,\wp)\) to be the map \(c_{(w,\wp)} : T_C \rightarrow \mathbb{N}\) which counts the different types of the connected components. For example, the diagram corresponding to the pair \((w,\wp) = (AAAAABAA, \{\{1,10\}, \{2,9\}, \{3,5\}, \{4,6\}, \{7,13\}, \{8,12\}, \{11,14\}\})\) is

\[
\begin{array}{c}
\hat{a} \rightarrow \hat{a} \\
\hat{a} \rightarrow \hat{a} \\
\hat{a} \rightarrow \hat{a} \\
\hat{a} \rightarrow \hat{a}
\end{array}
\]

and its corresponding content is thus:

\[
c_{(w,\wp)}((\omega,\rho)) = \begin{cases} 
2 & \text{if } (\omega,\rho) = (AA, \{\{1,4\}, \{2,3\}\}), \\
1 & \text{if } (\omega,\rho) = (AA, \{\{1,3\}, \{2,4\}\}), \\
1 & \text{if } (\omega,\rho) = (BA, \{\{1,2\}\}), \\
0 & \text{otherwise.}
\end{cases}
\]

12.1.4. *The “Pairings factorize” lemma.*

**Lemma 12.1.5.** Consider some \(\tau \in T\). Then:

\[
T_\tau = \prod_{\kappa \in T_C} (T_\kappa)^{\#c_\tau(\kappa)}.
\]

The proof of this lemma will appear shortly, in Section 12.1.7. To illustrate it, consider the case that \(\tau = (AAAAAB, \{\{2,9\}, \{4,5\}, \{8,11\}\})\).
In this case, the left hand side of the above equation is equal to

\[(+1) \left( \frac{1}{2} \right)^3 \]

and the right hand side is

\[
\begin{pmatrix}
(+1) \left( \frac{1}{2} \right) \\
\end{pmatrix}
\]

The equality of these two expressions (in \(\hat{\mathcal{W}}_{\lambda}[a, b]\)) may be immediately observed.

Applying this lemma to Proposition 12.1.1 we can write:

\[1 + \sum_{(w, \wp) \in \mathcal{T}} \frac{1}{|w|!} T_{(w, \wp)} = 1 + \sum_{\text{content functions } \tau \in \mathcal{T}} \left( \text{Number of pairs } \tau \in \mathcal{T} \text{ with content } c. \right) \frac{1}{(\sum_{\tau \in \mathcal{T}_c} c(\tau)|\tau|)!} \prod_{\kappa \in \mathcal{T}_c} T_{\kappa}^{\#:c(\kappa)}.\]

It remains for us to count the number of pairs \((w, \wp)\) with some given content \(c : \mathcal{T}_c \rightarrow \mathbb{N}_0\). That is achieved by the next lemma, which is proved in Section 12.1.7.

**Lemma 12.1.6.** Consider some content function \(\tau : \mathcal{T}_c \rightarrow \mathbb{N}_0\). The number of pairs \((w, \wp)\) with this content is

\[
\frac{(\sum_{\tau \in \mathcal{T}_c} c(\tau)|\tau|)!}{\prod_{\kappa \in \mathcal{T}_c} (|\kappa|)!^{c(\kappa)(c(\kappa))!}}.
\]

If we substitute this computation into the right-hand side of the above equation, we get:

\[1 + \sum_{\text{content functions } \tau \in \mathcal{T}_c} \left( \prod_{\kappa \in \mathcal{T}_c} \frac{1}{c(\kappa)!} \left( \frac{T_{\kappa}}{|\kappa|!} \right)^{\#:c(\kappa)}.\right.

This completes the proof of Theorem 12.1.2.

12.1.7. The proofs of the two lemmas. This section contains proofs of the two technical lemmas that were used in the proof of Theorem 12.1.2.

**Proof of Lemma 12.1.5.** We are asked to show that, for \((w, \wp) \in \mathcal{T},

\[T_{(w, \wp)} = \prod_{\kappa \in \mathcal{T}_c} (T_{\kappa})^{\#:c(\kappa)}.\]

The right hand side of this equation is just the left hand side, factored into its connected components. The equality is obvious, except for the possibility that the signs may differ. To establish this equality we'll begin by drawing the diagram representing the left-hand side in a canonical way; then we'll push the legs around (using the signed permutation relations) until the diagram is separated into its
constituent components. Our task is to keep track of what happens to the sign out the front of the term during this process.

We’ll illustrate the following discussion with the example:

\[(w, \wp) = (AAAAAB, \{\{2,9\}, \{4,5\}, \{8,11\}\}) .\]

We begin with the left-hand side. Construct the corresponding term \(T_{(w, \wp)}\) using the graphical approach to \(\lambda\) discussed in Section 8.7.4. In the given example, we would draw:

\[
T_{(w, \wp)} = (-1)^8 \left(\frac{1}{2}\right)^3.
\]

Before we proceed, observe the crucial point: if we draw the diagram in this fashion, then the sign out the front of the term is precisely a product of a \((-1)\) for every intersection displayed by the drawing.

So this is the left hand side, \(T_{(w, \wp)}\). To connect this with the expression on the right-hand side, we will now separate this diagram into its connected components. To do this we have to perform permutations of the legs. Every time we permute a pair of legs we pick up a \((-1)\), but also pick up an extra intersection point in the drawing. So throughout this process the above observation still holds: the sign out the front of the term is precisely \((-1)\) raised to the number of intersections displayed in the drawing. Continuing with our example:

\[
T_{(w, \wp)} = (-1)^9 \left(\frac{1}{2}\right)^3,
\]

\[
= (-1)^{10} \left(\frac{1}{2}\right)^3,
\]

\[
= (-1)^{11} \left(\frac{1}{2}\right)^3.
\]

We have now separated the legs of the connected components into their respective components.
We finish by fully separating the connected components in the drawing. To be precise, we can now do a combination of the following two moves (where the dashed line indicates that there are two connected components involved):

\[ 
\begin{align*}
\text{to separate the connected components in the drawing without affecting the graphical structure of the drawings of the connected components themselves. Note that these moves only change the number of displayed intersections by an even number, so the key property still holds after doing these moves (that the sign out the front of the term is a factor of \((-1\)) for every intersection displayed by the drawing). In our example:}

\begin{align*}
T_{(w,\varphi)} &= (-1)^5 \left(\frac{1}{2}\right)^3, \\
&= (-1)^1 \left(\frac{1}{2}\right)^3.
\end{align*}
\]

After this factorization procedure the sign that we are left with, then, is a \((-1\)) for every intersection displayed by the drawing. Thus the sign is a \((-1\)) for every self-intersection of the connected components, which exactly gives the right-hand side of Equation 18.

\[ \Box \]

**Proof of Lemma 12.1.6.** So, we are given a specific content function

\[ \tau : \mathcal{T}_C \rightarrow \mathbb{N}_0 \]

and asked to count how many pairs \((w,\varphi)\) have this content function. For convenience, just say that the given content consists of \(n_1\) copies of the pair \((w_1,\varphi_1)\), \(n_2\) copies of the pair \((w_2,\varphi_2)\), and so on, up to \(n_m\) copies of the pair \((w_m,\varphi_m)\).

Let \(X \subseteq \mathcal{T}\) denote the set of pairs \((w,\varphi)\) having this content. To count \(X\) we’ll construct a bijection from \(X\) to a certain set \(Y\) of words. Consider the following set of symbols:

\[ \{\zeta_{i,j}, 1 \leq i \leq m, 1 \leq j \leq n_i\}. \]

For every such \(i\) and \(j\), take \(|w_i|\) copies of the symbol \(\zeta_{i,j}\). Let \(Y\) denote the set of words that can be built from this collection of symbols which satisfy the restriction that for every \(i\) and \(j < k\), the first appearance of the symbol \(\zeta_{i,j}\) appears to the left of the first appearance of the symbol \(\zeta_{i,k}\).
The combinatorics of wheeling 89

The map from $X$ to $Y$ is just to scan the corresponding diagram $T(\omega, \rho)$ factor by factor, and for each factor to write down a symbol $\zeta_{i,j}$ if that factor is used by the $j$-th copy of the $i$-th connected type.

For example, consider the following content:

$$c_{(\omega, \rho)}((\omega, \rho)) = \begin{cases} 2 & \text{if } (\omega, \rho) = (AA, \{\{1, 4\}, \{2, 3\}\}), \\ 1 & \text{if } (\omega, \rho) = (AA, \{\{1, 3\}, \{2, 4\}\}), \\ 1 & \text{if } (\omega, \rho) = (BA, \{\{1, 2\}\}), \\ 0 & \text{otherwise.} \end{cases}$$

Here is an example of how a pair $(w, \wp)$ with this content gives a word in these symbols:

![Diagram](image)

This map sets up a bijection between $X$ and $Y$. It is a straightforward combinatorial problem to count $Y$. □

12.2. The computation of $\sum_{\tau \in T_n} \frac{1}{|\tau|} T_\tau$. Our task in this section is to write down the series of all possible terms that can arise by the following procedure:

1. Putting down a number of copies of the diagrams

![Diagram](image)

in some order along an orienting line. If we use $n$ factors in total then we multiply the diagram by $\frac{1}{n!}$.

2. Joining up (with signs) the $\perp$-legs in such a way as to produce a connected diagram (multiplying by a factor of $\frac{1}{2}$ for every pair of legs joined up).

We begin by observing that the connected diagrams that can arise in this way fall into exactly four groups. Below, we’ll refer to a leg of the form $\ldots \uparrow$ as a $\perp$-leg and a leg of the form $\ldots \downarrow\downarrow$ as a $b$-leg. The four possibilities are:

- The resulting diagram has exactly two remaining legs, and they are both $\perp$-legs. For example:

![Diagram](image)
• The resulting diagram has exactly two remaining legs, and they are both $b$-legs. For example:

• The resulting diagram has exactly two remaining legs, one $\perp$-leg and one $b$-leg. For example:

• The resulting diagram has no remaining legs. For example:

Denote these different contributions in the following way:

\[
\sum_{\tau \in \mathcal{T}} \frac{1}{|\tau|!} T_\tau = C_{||} + C_{|b} + C_{bb} + C_{o}.
\]

We’ll compute these different contributions in turn. The computation of the contribution $C_{||}$ will be described in some detail. The other contributions will be computed in much the same way and will be described in less detail. We remark that the subsection describing $C_{||}$ introduces certain definitions used in the other subsections.

12.2.1. The contribution $C_{||}$. The goal of this subsection is the computation that:

\[
C_{||} = \frac{\tanh(\frac{a}{2})}{n!}.
\]

So consider some $n \geq 2$ ($n = 1$ we’ll put in by hand). We wish to compute the contributions from the connected diagrams with 2 legs that we can get by doing signed pairings of the legs of the following term (where the blocks have been numbered for convenience):
To enumerate the possible pairings we’ll construct a certain set $\Gamma_n$. An element of the set $\Gamma_n$ is a word which uses each of the symbols \{1, 2, \ldots, n\} precisely once, where, in addition, each symbol $s$ is decorated by either an arrow pointing to the left $\leftarrow s$ or an arrow pointing to the right $\rightarrow s$. For example, we’ll soon see that $\rightarrow 2 \rightarrow 3 \rightarrow 5 \leftarrow 1 \rightarrow 4 \in \Gamma_5$.

The set $\Gamma_n$ is defined to be the set of words of this form based on the symbols \{1, 2, \ldots, n\} subject to the single constraint that the first symbol is less than the final symbol. To see that they agree with our definition the reader might like to check that $|\Gamma_n| = n!2^n - 1$.

For example:

\[ \Gamma_3 = \{ \rightarrow 1 \rightarrow 2 \rightarrow 3 , \rightarrow 1 \rightarrow 2 \rightarrow 3 , \rightarrow 1 \rightarrow 2 \rightarrow 3 , \rightarrow 1 \rightarrow 2 \rightarrow 3 , \rightarrow 2 \rightarrow 1 \rightarrow 3 , \rightarrow 2 \rightarrow 1 \rightarrow 3 , \rightarrow 2 \rightarrow 1 \rightarrow 3 , \rightarrow 2 \rightarrow 1 \rightarrow 3 , \rightarrow 1 \rightarrow 3 \rightarrow 2 , \rightarrow 1 \rightarrow 3 \rightarrow 2 , \rightarrow 1 \rightarrow 3 \rightarrow 2 , \rightarrow 1 \rightarrow 3 \rightarrow 2 \} . \]

We’ll demonstrate how the different pairings correspond with the elements of the sets $\Gamma_n$ by means of the following example:

To write down the word corresponding to some gluing, begin at the base of the left-most of the two remaining legs. Now traverse the graph by simply following the edge until you reach the second of the two remaining legs. Write down the symbols \{1, 2, \ldots, n\} in the order in which you visit the different blocks. In this case, you should write down:

41235.

Now decorate this word with arrows to record how you traverse each block (whether from left to right, or from right to left). (Take care not to confuse this arrow with the arrow which locally orients the edges around the $a$ labels.) The decoration in this case is this:

\[ 4 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 5 \ . \]

This word contains sufficient instructions for uniquely reconstructing the pairing, so we get precisely one of the relevant pairings for each element of $\Gamma_n$.

Given some word $w \in \Gamma_n$, let $\gamma_w$ denote the corresponding contribution to $C$ (that is, including the factorial out the front as well as the signs that arise from the
gluing and a factor of \((\frac{1}{2})\) for every joined pair). For example:

\[
\gamma_{\{4\cdots5\}} = (-1)^5 \left(\frac{1}{2}\right)^4 \frac{1}{2^5!} \frac{1}{2^5!} = -\frac{1}{2^5!} \cdot \frac{1}{2^5!} = -\frac{1}{2^5!} \cdot \frac{1}{2^5!}
\]

With this definition we can write the contribution that we are seeking to compute in the following way:

\[
C_{\|} = \frac{a}{2} + \sum_{n=2}^{\infty} \sum_{w \in \Gamma_n} \gamma_w.
\]

Observe that, for some fixed \(n\), all the \(\gamma_w\), for \(w \in \Gamma_n\), are equal, up to sign:

\[
\gamma_w = \pm \frac{1}{2^{n-1}n!} \frac{\left(\frac{1}{2}\right)^n}{2^n}.
\]

The difficulty, then, in computing the sum \(\sum_{w \in \Gamma_n} \gamma_w\), is to determine the signs of the various \(\gamma_w\). This difficulty is dealt with by the next lemma, whose proof is later in this section.

Let \(\Gamma_n\) denote the set of words in the symbols \(\{1, \ldots, n\}\) with the property that the right-most symbol in a word has greater value than the left-most symbol. There is an obvious \(2^n\)-to-1 forgetful map:

\[
f_{\Gamma} : \Gamma_n \to \Gamma_n.
\]

Define the descent of a word \(w \in \Gamma_n\), denoted \(d(w)\), to be the number of times in which the value of the symbol decreases as you scan the word from left to right. For example:

\[
d(41235) = 1,
\]

because the value decreases once (going from 4 to 1).

**Lemma 12.2.2.** Let \(n \geq 2\) and let \(w \in \Gamma_n\). Then:

\[
\gamma_w = (-1)^{d(f_{\Gamma}(w))} \frac{1}{2^{n-1}n!} \frac{\left(\frac{1}{2}\right)^n}{2^n}.
\]
Substituting this computation into Equation 20, we find that:

\[
C_{||} = \frac{a}{2} + \sum_{n=2}^{\infty} \sum_{w \in \Gamma_n^w} \gamma_w ,
\]

\[
= \frac{a}{2} + \sum_{n=2}^{\infty} \sum_{w \in \Gamma_n^w} \left( (-1)^{d(w)} \frac{1}{2^{n-1}n!} \right) ,
\]

\[
= \frac{a}{2} + \sum_{n=2}^{\infty} \left( \frac{2 \sum_{w \in \Gamma_n^w} (-1)^{d(w)}}{n!} \right) .
\]

(In the last equality above the \(2^{n-1}\) is cancelled by a \(2^n\) arising from the fact that there are \(2^n\) words in \(\Gamma_n^w\) for every word in \(\Gamma_n\).) Now note that if \(n\) is even then:

\[
\Psi\left( \frac{a}{2} \right) = 0 .
\]

For example:

\[
\begin{align*}
\frac{a}{2} & = (+1) \quad \frac{a}{2} \quad \frac{a}{2} = (-1) \\
& = (-1) \\
\end{align*}
\]

Thus we may write:

\[
C_{||} = \Psi\left( \frac{a}{2} \right) ,
\]

where \(\Psi(x)\) is the formal power series defined by \(\Psi(x) = \sum_{n=1}^{\infty} \frac{\psi(n)}{n!} x^n\), with

\[
\psi(n) = \begin{cases} 
1 & \text{if } n = 1, \\
2 \sum_{w \in \Gamma_n} (-1)^{d(w)} & \text{if } n > 1 \text{ and } n \text{ is odd,} \\
0 & \text{if } n > 1 \text{ and } n \text{ is even.}
\end{cases}
\]

The required computation, Equation 19, is completed by the following proposition.

**Proposition 12.2.3.**

\[\Psi(x) = \tanh(x) .\]

**Proof.** We’ll begin by replacing \(\psi(n)\) with a function that is easier to use. For every \(n \geq 1\) let \(\Sigma_n\) denote the set of words that can be made using each of the symbols
\{1, \ldots, n\} exactly once (with no restrictions on order) and define
\[ \phi(n) = \sum_{w \in \Sigma_n} (-1)^{d(w)}. \]

Let \( \nu : \Sigma_n \to \Sigma_n \) be the involution of \( \Sigma_n \) which writes a word in its reverse order. Define the descent \( d(w) \) of a word in the obvious way. Notice that:
\begin{equation}
(21)
\begin{align*}
d(\nu(w)) &= +d(w) & \text{if } n \text{ is odd,} \\
&= -d(w) & \text{if } n \text{ is even.}
\end{align*}
\end{equation}

Thus:
\[ \phi(n) = \begin{cases} 
1 & \text{if } n = 1, \\
2 \sum_{w \in \Gamma_n} (-1)^{d(w)} & \text{if } n > 1 \text{ and } n \text{ is odd,} \\
0 & \text{if } n > 1 \text{ and } n \text{ is even.}
\end{cases} \]

In other words, \( \phi(n) = \psi(n) \), and our task is to calculate the power series:
\[ \Psi(x) = \sum_{n=1}^{\infty} \frac{\phi(n)}{n!} x^n. \]

We’ll calculate this power series by writing down a recursion relation which determines the function \( \frac{d(n)}{n} \), and then we’ll identify a power series whose coefficients solve the recursion relation.

To deduce the appropriate recursion relation we’ll partition the set \( \Sigma_n \) according to the position of the symbol \( n \). Let \( \Sigma_n^i \subset \Sigma_n \) denote the subset consisting of the words where the symbol \( n \) appears in position \( i \). Then, for \( n \geq 3 \):
\[ \frac{\phi(n)}{n!} = \frac{1}{n!} \sum_{w \in \Sigma_n} (-1)^{d(w)}, \]
\[ = \frac{1}{n!} \sum_{i=1}^{n} \left( \sum_{w \in \Sigma_n^i} (-1)^{d(w)} \right), \]
\[ = -\frac{1}{n!} \sum_{i=2}^{n-1} \binom{n-1}{i-1} \phi(i-1)\phi(n-i), \]
\[ = -\frac{1}{n} \sum_{i=2}^{n-1} \phi(i-1) \frac{\phi(n-i)}{(i-1)! (n-i)!}. \]

Now observe that this recursion relation, together with the initial conditions \( \phi(1) = 1 \) and \( \phi(2) = 0 \), completely determines the sequence \( \phi(n) \). It follows from this recursion relation that \( \Psi(x) \) is the unique power series satisfying the functional equation:
\[ \frac{d}{dx} [\Psi(x)] = 1 - \Psi(x)^2, \]
with initial terms
\[ \Psi(x) = x + (\text{terms of degree at least 3}). \]

Thus, \( \Psi(x) = \tanh(x) \). \( \square \)
Proof of Lemma 12.2.2. So consider the diagram arising from the pairing corresponding to some word \( w \in \Gamma_n \). Draw the diagram of the pairing canonically. That is, start with:

![Diagram](image)

and, when doing the pairing, introduce only transversal double-point intersections each lying above the orienting line. If you do this then note that the sign of the resulting term is precisely \((-1)^{\text{number of intersections}}\). For example:

\[
\gamma_2 \gamma_1 5 3 4 = \frac{(-1)^4}{5!} \left( \frac{1}{2} \right)^4.
\]

Our problem is to work out what further signs must be introduced to make all the \( \alpha \)-legs lie on the same side of the edge, and to write the final sign as a function of \( w \).

We’ll put diagrams into a standard form with two steps. The first step will be to add permutations of the following form to the top of the drawing:

![Permutation Diagram](image)

Add permutations so that the tops of the blocks appear in the same order as they appear in the word \( w \). Notice that a single such permutation introduces 4 intersections into the diagram, so it is still true, after such a move, that the sign of the term is precisely \((-1)^{\text{number of intersections}}\). Continuing with our example:

\[
\gamma_2 \gamma_1 5 3 4 = \frac{(-1)^{16}}{5!} \left( \frac{1}{2} \right)^4.
\]

Notice that as you traverse the edge from the base of the left leg to the base of the right leg, then some factors are traversed from left to right (the 4th factor from the left, above), while some factors are traversed from right to left (the other factors). The second and final step in the procedure to put the diagram into standard form
is to add twists of the following form:

\[
\begin{array}{c}
\begin{array}{c}
\text{a} \quad \text{a} \\
\end{array} = (-1)
\end{array}
\]

in order that every factor is traversed from left to right. These diagrams differ by a \((-1)\) which arises from an AS relation which is employed to shift the \(a\)-leg to the other side of the edge. Because this move introduces an extra intersection, it is still true that the sign of the term is just \((-1)\) raised to the number of intersections in the drawing. Continuing our example:

\[
\gamma_{2 \ 1 \ 5 \ 3 \ 4} = (-1)^{20} \left(\frac{1}{2}\right)^{4}
\]

Notice that after these two steps, the initial diagram has been transformed into the following standard form,

where the parts of the edges within the dashed box follow complicated, possibly self-intersecting paths.

On account of our procedure, the sign of the corresponding term is precisely \((-1)\) raised to the number of intersection points displayed in the drawing that we have just obtained. To finish the calculation, then, it remains for us to count the number of intersection points displayed by the diagram within the dashed box.

Notice that the dashed box cuts the edge up into pieces. We’ll call these pieces the arcs. The number of intersections of two different arcs must be even (it may help to think of Reidemeister moves). It remains, then, to count the number of self-intersection points of these arcs. Well, the arcs had no self-intersections at the beginning, so we just have to trace how many self-intersection points were created by our procedure. Self-intersections may be created in the first step of our procedure, when a permutation is added to the top of the diagram, in the following
way:

\[
\begin{array}{c}
\begin{array}{c}
\frac{a}{2} \\
\frac{a}{2}
\end{array}
\begin{array}{c}
\frac{a}{2} \\
\frac{a}{2}
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
\frac{a}{2}
\end{array}
\begin{array}{c}
\frac{a}{2}
\end{array}
\end{array},
\end{array}
\]

There will be one of these for every case of two factors consecutive in \(w\), with the factor appearing later in \(w\) having the smaller value. Thus the sign is \((-1)^{d(w)}\).

\[\square\]

12.2.4. The contribution \(C_o\). The goal of this subsection is the computation that

\[
C_o = \left. \frac{1}{2} \ln(\cosh(\frac{a}{2})) \right|_{\text{sign}}.
\]

So consider some \(n \geq 2\) (the \(n = 1\) case we’ll observe is zero). In this subsection we wish to compute the contributions from the connected diagrams with zero legs that we can get by doing signed pairings of the legs of the following term:

\[
\begin{array}{c}
\begin{array}{c}
\frac{a}{2} \\
\end{array}
\begin{array}{c}
\frac{a}{2}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\frac{a}{2} \\
\end{array}
\begin{array}{c}
\frac{a}{2}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\frac{a}{2} \\
\end{array}
\begin{array}{c}
\frac{a}{2}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\frac{a}{2} \\
\end{array}
\begin{array}{c}
\frac{a}{2}
\end{array}
\end{array}\end{array}
\]

To enumerate these terms we’ll employ a certain set \(\Xi_n\). This set consists of the words that can be made by using each of the symbols \(\{1, \ldots, n\}\) precisely once, such that the left-most symbol is a 1, and where every symbol \(s\) except the initial 1 is decorated by either an arrow pointing to the right \(\rightarrow s\) or an arrow pointing to the left \(\leftarrow s\). For example:

\[
\bar{\Xi}_3 = \{1 \bar{2} 3 \bar{3}, 1 \bar{3} 2 \bar{3}, 1 \bar{2} 3 \bar{2}, 1 \bar{3} 2 \bar{2}, 1 \bar{2} 3 \bar{2}, 1 \bar{3} 2 \bar{2}\}.
\]

Let \(\Xi_n\) denote the set defined in the same way but without the arrow decorations and let

\[
f_\Xi : \bar{\Xi}_n \to \Xi_n
\]

denote the corresponding \(2^{n-1}\)-to-1 forgetful map.

Consider, then, some pairing which uses all the available legs and results in exactly one connected component. For example:

\[
\begin{array}{c}
\begin{array}{c}
\frac{a}{2} \\
\end{array}
\begin{array}{c}
\frac{a}{2}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\frac{a}{2} \\
\end{array}
\begin{array}{c}
\frac{a}{2}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\frac{a}{2} \\
\end{array}
\begin{array}{c}
\frac{a}{2}
\end{array}
\end{array}\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\frac{a}{2} \\
\end{array}
\begin{array}{c}
\frac{a}{2}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\frac{a}{2} \\
\end{array}
\begin{array}{c}
\frac{a}{2}
\end{array}
\end{array}\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\frac{a}{2} \\
\end{array}
\begin{array}{c}
\frac{a}{2}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\frac{a}{2} \\
\end{array}
\begin{array}{c}
\frac{a}{2}
\end{array}
\end{array}\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\frac{a}{2} \\
\end{array}
\begin{array}{c}
\frac{a}{2}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\frac{a}{2} \\
\end{array}
\begin{array}{c}
\frac{a}{2}
\end{array}
\end{array}\end{array}
\]
The word that corresponds with a gluing is determined in the following way. To begin, ignore the arc that terminates on the left-hand side of the left-most block:

Now traverse the graph, starting with the left-most block, writing down the order in which blocks are visited together with the corresponding directions. The example given leads to:

This word contains sufficient information to reconstruct the pairing, so we have just set up a bijection between the set $\Xi_n$ and the set of pairings that we are concerned with presently.

Given some word $w \in \Xi_n$ let $\xi_w$ denote the corresponding contribution.

**Lemma 12.2.5.** Let $n \geq 2$ and let $w \in \Xi_n$. Then:

$$
\xi_w = (-1)^{d(f(w))} \frac{1}{2^n n!} \left( \frac{a}{2} \right)^n .
$$

**Sketch of the proof.** This proof proceeds in much the same way as the proof of Lemma 12.2.2. That is, we start by drawing the diagram of the graph that results from the gluing canonically. Then we put it in standard form in two steps. In the first step we add permutations to the top of the diagram so that the factors appear in the diagram in the same order in which they appear in the word $w$. In the second step we add any twists that are required in order for the diagram to appear in the following standard form:

We can keep track of signs in exactly the same way as we did in the proof of Lemma 12.2.2, and we are led to the given conclusion.

Now we can compute the contribution $C_\circ$. To begin, note that the term that arises in the $n = 1$ case is zero:

$$
\frac{1}{2} \left( \frac{a}{2} \right) = 0 .
$$
Proceeding:

\[ C_\circ = \sum_{n=2}^{\infty} \sum_{w \in \Xi_n} \xi_w, \]

\[ = \sum_{n=2}^{\infty} \sum_{w \in \Xi_n} \left( \frac{1}{2^n n!} (-1)^d(f_\Xi(w)) \right) \left( \frac{d}{2} \right)^n, \]

\[ = \sum_{n=2}^{\infty} \left( \frac{2^{n-1} \sum_{w \in \Xi_n} (-1)^d(w)}{2^n n!} \right) \left( \frac{d}{2} \right)^n, \]

\[ = \frac{1}{2} \sum_{n=2}^{\infty} \left( \frac{1}{n} \frac{\phi(n-1)}{(n-1)!} \right) \left( \frac{d}{2} \right)^n. \]

Thus we have computed that:

\[ C_\circ = \frac{1}{2} \left( \int \Psi \right) \left( \frac{d}{2} \right), \]

where \( \left( \int \Psi \right) \) denotes \( \sum_{n=2}^{\infty} \frac{1}{n} \frac{\phi(n-1)}{(n-1)!} x^n \), the unique power series with zero constant term whose formal derivative is \( \Psi(x) \). That power series is given by:

\[ \left( \int \Psi \right) (x) = \ln \cosh x. \]

12.2.6. The contribution \( C_{bb} \). We now wish to compute the contribution of gluings which lead to connected diagrams with exactly two \( b \)-legs. For example:

The set which enumerates these gluings is the following \( \Xi_n \). The elements of \( \Xi_n \) are certain words that use each of the symbols \( \{1, \ldots, n\} \) precisely once and where each symbol except the first and last symbols in the word is decorated by an arrow. For example, we will see that:

\[ \Xi_5 \]

The set \( \Xi_n \) is defined to be all words of this type with the property that the last symbol is greater than the first symbol.

\[ 2 \xrightarrow{\frac{a}{2}} 3 \xrightarrow{\frac{a}{2}} 5 \xrightarrow{\frac{a}{2}} 1 \xrightarrow{b} 4 \in \Xi_5. \]

The set \( \Xi_n \) is defined to be all words of this type with the property that the last symbol is greater than the first symbol.
To write down the word corresponding to some gluing, traverse the graph, starting at the left-most of the 2 legs, writing down the order that blocks are encountered as you traverse (decorating with the appropriate arrow). For example, the gluing above corresponds with the word:

\[ 3 \xleftarrow{-} 1 \xrightarrow{-} 2 \xrightarrow{-} 4 \xrightarrow{-} 5 \in \Omega_5. \]

Let

\[ f_\Omega : \Omega_n \to \Gamma_n \]

denote the \(2^{n-2}\text{-to-}1\) forgetful map. Let \(\omega_w\) denote the contribution corresponding to some word \(w \in \Omega_n\). We leave the proof of the following lemma as an exercise for the reader.

**Lemma 12.2.7.** Let \(n \geq 2\) and let \(w \in \Omega_n\). Then:

\[ \omega_w = -(-1)^d(f_n(w)) \frac{1}{2^{n-1}n!}. \]

We can now complete the computation of the contribution \(C_{bb}\):

\[
C_{bb} = \sum_{n=2}^{\infty} \sum_{w \in \Omega_n} \omega_w,
\]

\[
= -\sum_{n=2}^{\infty} \sum_{w \in \Omega_n} (-1)^d(\gamma(w)) \frac{1}{2^{n-1}n!},
\]

\[
= -\sum_{n=2}^{\infty} \left( \frac{2^{n-2}}{2^{n-1}n!} \sum_{w \in \Gamma_n} (-1)^d(w) \right),
\]

\[
= -\left( \frac{1}{2} \right)^2 \psi(n) \frac{\psi(n)}{n!},
\]

\[
= -\left( \frac{1}{2} \right)^2 \frac{\psi(\frac{3}{2})}{(\frac{3}{2})^2}.
\]

12.2.8. **The contribution** \(C_{bb}\). To enumerate these terms we’ll employ a set \(\Delta_n\). The elements of this set are words which use each one of the symbols \(\{1, \ldots, n\}\)
exactly once, where every symbol except the last symbol is decorated by an arrow. The set $\overrightarrow{\Delta}_n$ is defined to be the set of all such words. For example:

$$\begin{array}{ccc}
4 & 2 & 3 \\
\downarrow & \downarrow & \downarrow \\
3 & 4 & 2
\end{array} \in \overrightarrow{\Delta}_5.$$

To write down the word that corresponds with a given pairing, traverse the resulting diagram, starting at the $\perp$-leg and proceeding until you reach the $b$-leg. As you traverse, record the order in which you visit the different blocks and the directions in which you travel as you visit the blocks. For example, the pairing

![Diagram of a pairing](image)

corresponds with the word

$$\begin{array}{ccc}
1 & 2 & 3 \\
\downarrow & \downarrow & \downarrow \\
2 & 1 & 3
\end{array} \in \overrightarrow{\Delta}_5.$$

For some $w \in \overrightarrow{\Delta}_n$, let $\delta_w$ denote the corresponding term. Let

$$f_\Delta : \overrightarrow{\Delta}_n \to \Sigma_n$$

denote the $2^{n-1}$-to-1 forgetful map.

**Lemma 12.2.9.** Let $n \geq 2$ and let $w \in \overrightarrow{\Delta}_n$. Then:

$$\delta_w = -(-1)^d(f_\Delta(w)) \frac{1}{2^{n-1}n!}.$$

Thus we can calculate:

$$C_{\nabla b} = \sum_{n=1}^{\infty} \sum_{w \in \overrightarrow{\Delta}_n} \delta_w ,$$

$$= -\sum_{n=1}^{\infty} \sum_{w \in \overrightarrow{\Delta}_n} (-1)^d(\delta(w)) \frac{1}{2^{n-1}n!} ,$$

$$= -\sum_{n=1}^{\infty} \sum_{w \in \Sigma_n} (-1)^d(w) \frac{\psi(\frac{2}{n})}{n!} ,$$

$$= -\sum_{n=1}^{\infty} \frac{\psi(\frac{2}{n})}{n!} ,$$

where $\psi$ is the digamma function.
13. Computing the operator product II.

In this section we’ll take the computation of the last section (Theorem 12.0.2) and use it to compute the series \([\mathcal{X}]_{b, \partial_b=0}\); this will give us Theorem 11.0.3. A direct substitution of the previous section’s result into \(\mathcal{X}\) (taking terms without \(b\)-legs out the front of the expression) yields that \([\mathcal{X}]_{b, \partial_b=0}\) is equal to:

\[
\exp\left(\frac{1}{2} \begin{array}{c} \ln(\cosh(\frac{a}{2})) \\ \tanh(\frac{a}{2}) \end{array} \right) \neq Y
\]

where \(Y\) is equal to

\[
\left[ \exp\left( -\frac{1}{2} \frac{\tanh(\frac{a}{2})}{b} \right) \right] \cdot \exp\left( \frac{\tanh(\frac{a}{2})}{b} - \left(\frac{1}{2}\right)^2 \right) \]

This series \(Y\) will be computed with the following lemma, whose proof is the subject of this section.

**Lemma 13.0.10.** Let \(Y(a)\) be a power series containing only even powers of \(a\) and let \(Z(a)\) be a power series containing only odd powers of \(a\). Then:

\[
\left[ \exp\left( \frac{\tanh(\frac{a}{2})}{\partial_b} \right) \right] \cdot \exp\left( \frac{Z(a)}{2} \right) = \exp\left( -\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \left( 1 - \tanh\left(\frac{a}{2}\right) \right)^n \right) \cdot \exp\left( \frac{Y(a)(aZ(a))}{a\partial_b} \right)
\]

To apply this lemma to Equation 22 we set \(Y(a) = -\frac{\tanh(\frac{a}{2})}{\frac{1}{2}}\) and \(Z(a) = -\frac{\tanh(\frac{a}{2})}{\left(\frac{1}{2}\right)^2}\). With these assignments:

\[
\sum_{n=1}^{\infty} \frac{1}{n} (Z(a)a)^n = \sum_{n=1}^{\infty} \frac{1}{n} \left( 1 - \tanh\left(\frac{a}{2}\right) \right)^n = -\ln\left(\tanh\left(\frac{a}{2}\right)\right).
\]
Furthermore:

\[
\frac{1}{2} \sum_{n=0}^{\infty} Y(a) (aZ(a))^n aY(a) = \sum_{n=0}^{\infty} \frac{\tanh(\frac{a}{2} \cdot \frac{1}{2})}{\frac{2}{2} - \tanh(\frac{a}{2})} \left(1 - \frac{\tanh(\frac{a}{2})}{\frac{2}{2}}\right)^n a \tanh(\frac{a}{2}),
\]

\[
= \frac{\tanh(\frac{a}{2})}{\frac{2}{2}} \cdot \frac{1}{1 - \left(1 - \tanh(\frac{a}{2})\right)} = \tanh(\frac{a}{2}).
\]

With these calculations in hand, a direct application of Lemma 13.0.10 to Equation 22 yields that \( Y \) is equal to

\[
\exp\# \left( \frac{1}{2} \ \ln \left( \frac{\tanh(\frac{a}{2})}{\frac{2}{2}} \right) \ - \ tanh(\frac{a}{2}) \right).
\]

Thus \([\mathcal{X}]_{b, \partial b=0}\) is equal to

\[
\exp\# \left( \frac{1}{2} \ \ln \left( \frac{\sinh(\frac{a}{2})}{\frac{2}{2}} \right) \right),
\]

as required. This completes the proof of Theorem 11.0.3.

13.1. Proof of Lemma 13.0.10. Our task in this section is the combinatorial computation of the following expression:

\[
(23) \quad \left[ \exp_b \left( \begin{array}{c}
\frac{a}{2} \\
\delta_b
\end{array} \right) + \exp\# \left( \begin{array}{c}
Y(a) \\
\delta_a
\end{array} \right) + \frac{Z(a)}{2} \right]_{b, \partial b=0}.
\]

We’ll begin with some general observations. First of all note that, because we set \( b \) and \( \partial b \) to zero at the end of the calculation, there will only be contributions from those terms arising from the expansion of the exponentials with the property that the number of \( \partial b \) legs in the first factor is equal to the number of \( b \) legs in the
second factor. A typical contributing term is:

\[ \frac{1}{4!} \]

\[ \partial b \partial b \partial b \partial b \]

\[ \frac{1}{5!} \]

\[ Y(a) \frac{Z(a)}{2} \]

Let’s briefly recall, then, how to compute such an operation product using the graphical method described in Section 10.3. We draw a grid over an orienting line, placing the legs of the first factor in order up the left-hand side of the grid, and the legs of the second factor in order along the top of the grid:

Because we are going to set \( b \) and \( \partial b \) to zero at the end of the calculation, we will get exactly one contribution for every different way of wiring all the \( \partial b \)-legs to all the \( b \)-legs. In other words, we get exactly one contribution for every bijection

\[ \phi : \{1, \ldots, 8\} \rightarrow \{1, \ldots, 8\}. \]
E.g., to construct the contribution corresponding to the bijection \((12345678, 15274683)\) we, first of all, wire up the grid using this bijection:

\[
\begin{array}{cccc}
Y(a) & Z(a) & Y(a) & Z(a) \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{array}
\]

And then the contribution is this diagram multiplied by \(\frac{1}{4!} \frac{1}{5!} (-1)^x\), where \(x\) denotes the number of intersections displayed within the grid. In this example, then, the contribution is:

\[
Y(a) \left( -\frac{3}{2} \right) \frac{Z(a)}{2} \left( \frac{3}{2} \right) Y(-a) \quad \left( -\frac{3}{2} \right) \frac{Z(-a)}{2} \left( \frac{3}{2} \right) Z(a)
\]

\[
= (+1) \frac{1}{4!} \frac{1}{5!} (-1)^{32}
\]

Here we have replaced \(a\) by \(-a\) whenever we have had to use AS relations to make \(a\)-legs lie on the correct side of the (oriented) edge.

13.1.1. The formal development. Now let’s set this up more formally. We’ll begin by defining a set \(\nabla\) which will index the different terms that arise when Expression 23 is evaluated. The elements of \(\nabla\) are triples \((n, w, \sigma)\), where \(n\) is a positive integer, where \(w\) is a word in the symbols \(Y\) and \(Z\) such that \((\#Y) + 2(\#Z) = 2n\), and where \(\sigma\) is a bijection \(\sigma: \{1, 2, \ldots, 2n\} \to \{1, 2, \ldots, 2n\}\).

Given some triple \((n, w, \sigma) \in \nabla\), let \(\tilde{T}_{(n,w,\sigma)}\) denote the corresponding term (where the \(\tilde{T}\) has a tilde to logically distinguish it from the \(T\)’s of Section 12). For example, we just showed that

\[
\tilde{T}_{(4, YZYZZ, (12345678, 15274683))} = (+1) \frac{1}{4!} \frac{1}{5!} \]

\[
Y(a) \left( -\frac{3}{2} \right) \frac{Z(a)}{2} \left( \frac{3}{2} \right) Y(-a) \quad \left( -\frac{3}{2} \right) \frac{Z(-a)}{2} \left( \frac{3}{2} \right) Z(a)
\]
Let $\nabla_C \subset \nabla$ denote the elements of $\nabla$ whose corresponding diagram is \textbf{connected}. (For example, the case just treated is \textit{not} an element of this subset.) We’ll omit the proof of the following proposition, which is a tedious combinatorial argument closely analogous to the proof of Proposition 12.1.2.

\textbf{Proposition 13.1.2.} \textit{The expression to be computed, \textit{Expression 23}, is equal to}

\[ \exp \# \left( \sum_{\tau \in \nabla_C} \tilde{T}_{\tau} \right). \]

\textbf{13.1.3. The computation of $\sum_{\tau \in \nabla_C} \tilde{T}_{\tau}$}. We can group this sum into two contributions:

\[ \sum_{\tau \in \nabla_C} \tilde{T}_{\tau} = C_0 + C_2, \]

where:

- $C_0$ denotes the series of terms $\tilde{T}_{\tau}$ whose underlying diagrams are connected and have no legs. An example is $\tilde{T}_{(4, zzzz, (12345678 41685372))}$, which is:

\[ \frac{1}{4!4!} (-1)^{13} \]

- $C_2$ denotes the series of terms $\tilde{T}_{\tau}$ whose underlying diagrams are connected and have 2 legs, such as:

\[ \tilde{T}_{(2, yzy, (1234 2314))} = \frac{1}{2!3!} (-1)^{6} \frac{4}{2} \frac{4}{2} \]

These classes are the only combinatorial possibilities. We’ll compute these two contributions in turn.

\textbf{13.1.4. The contribution $C_0$}. The goal of this subsection is the computation that

\[ C_0 = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \left( Z(a) \right)^n \]
We'll begin by defining a set $\Phi_n$ which will index, in a convenient manner, the various terms which contribute to $C_0$. The elements of $\Phi_n$ will be pairs $(w_1, w_2)$ of words, the first word $w_1$ using each of the symbols $\{1, \ldots, n\}$ exactly once, and the second word $w_2$ using each of the symbols $\{2, \ldots, n\}$ exactly once. In addition, every symbol $s$ of the first word $w_1$ is decorated by either an arrow pointing to the left $\leftarrow s$ or an arrow pointing to the right $\rightarrow s$, and every symbol $s$ of the second word $w_2$ is decorated by either an arrow pointing up $s \uparrow$ or an arrow pointing down $s \downarrow$. The set $\Phi_n$ is defined to be the set of all such pairs. For example: $(\rightarrow 4 \leftarrow 1 \leftarrow 3 \rightarrow 2, 2 \uparrow 3 \downarrow 4 \downarrow) \in \Phi_4$. To write down the pair of words corresponding to some gluing giving a connected diagram with no legs, start at the top-most factor $Z(a)$.

Now traverse the diagram until you return to where you started (the arrow in the example indicates how to begin this traverse). The first word records the order and direction in which you encounter the factors written along the top line as you traverse (in this example the corresponding word is $\rightarrow 4 \leftarrow 1 \leftarrow 3 \rightarrow 2$), and the second word records the order in which you encounter the factors written down the left-hand side (in this case, $2 \uparrow 3 \downarrow 4 \downarrow$).

Given some element $(w_1, w_2) \in \Phi_n$, let $\phi_{(w_1,w_2)}$ denote the corresponding term. In the example at hand:

$$\phi\left(\rightarrow 4 \leftarrow 1 \leftarrow 3 \rightarrow 2, 2 \uparrow 3 \downarrow 4 \downarrow\right) = (-1)^{17} \frac{1}{4!4!}$$
The series to be calculated can now be written
\[
C_0 = \sum_{n=1}^{\infty} \sum_{(w_1, w_2) \in \Phi_n} \phi_{(w_1, w_2)}.
\]
This calculation is greatly simplified by the observation that, for a fixed \(n\), all the \(\phi_{(w_1, w_2)}\), for \((w_1, w_2) \in \Phi_n\), are precisely equal. This fact is part of the following lemma, whose proof appears later in this section.

Lemma 13.1.5. Consider some \(n \geq 1\) and some \((w_1, w_2) \in \Phi_n\). Then:
\[
\phi_{(w_1, w_2)} = -\frac{1}{n!n!} \left( \frac{Z(a)}{2} \right)^n.
\]

With this information in hand, \(C_0\) is easily computed:
\[
C_0 = \sum_{n=1}^{\infty} \sum_{(w_1, w_2) \in \Phi_n} \phi_{(w_1, w_2)}
\]
\[
= (-1) \sum_{n=1}^{\infty} \left| \Phi_n \right| \frac{1}{n!n!} \left( \frac{Z(a)}{2} \right)^n
\]
\[
= (-1) \sum_{n=1}^{\infty} \frac{2^n n! 2^{n-1} (n - 1)!}{n! n!} \left( \frac{Z(a)}{2} \right)^n
\]
\[
= -\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{Z(a)}{a} \right)^n,
\]
as required.

Proof of Lemma 13.1.5. We’ll begin by introducing some notation. Given a gluing datum \((w_1, w_2) \in \Phi_n\), let \(D_{(w_1, w_2)}\) denote the series of diagrams (in \(\hat{W}_\lambda[a, b]\)) represented by the drawing you get when you wire up a grid according to \((w_1, w_2)\). Let \(x(w_1, w_2)\) denote the number of intersections displayed by that drawing. According to these definitions, the corresponding contribution is written:
\[
\phi_{(w_1, w_2)} = \frac{1}{n!n!} (-1)^{x_{(w_1, w_2)}} D_{(w_1, w_2)}.
\]
This proof is based on two moves, which we’ll call R-moves (for tRansposition) and W-moves (for tWist), that we can perform on gluing data:
\[
(w_1, w_2) \xrightarrow{\text{R-move}} (w_1', w_2') \quad \text{and} \quad (w_1, w_2) \xrightarrow{\text{W-move}} (w_1', w_2'),
\]
whose key property is that:
\[
\frac{1}{n!n!} (-1)^{x_{(w_1, w_2)}} D_{(w_1, w_2)} = \frac{1}{n!n!} (-1)^{x_{(w_1', w_2')}} D_{(w_1', w_2')},
\]
We’ll begin by introducing these two moves and establishing that the key property holds for them.

(T-moves.) This move is: transposition of adjacent columns or adjacent rows. Here is an example of a T-move, (where the arcs which are unaltered by the move have not been drawn in):

\[
\begin{array}{c}
\begin{array}{c}
Z(a) \quad Z(a) \\
2 \quad 2
\end{array}
\end{array}
\xrightarrow{\text{T-move}}
\begin{array}{c}
\begin{array}{c}
Z(a) \quad Z(a) \\
2 \quad 2
\end{array}
\end{array}
\]

This move can only change the number of displayed intersections by an even number (indeed, observe that the relative positions of the ends of all arcs down the left-hand edge of the grid is unaltered by this move). Thus:

\[
\frac{1}{n!} (-1)^{x(w_1, w_2)} D_{(w_1, w_2)} = \frac{1}{n!} (-1)^{x'(w_1', w_2')} D_{(w_1', w_2')} \quad \text{(As } D_{(w_1', w_2')} = D_{(w_1, w_2)}),
\]

\[
\frac{1}{n!} (-1)^{x'(w_1', w_2')} D_{(w_1', w_2')} = \frac{1}{n!} (-1)^{x(w_1, w_2)} D_{(w_1, w_2)} \quad \text{(As } x'(w_1', w_2') = x(w_1, w_2) + \text{an even number}).
\]

(W-moves.) This move is a ‘half-twist’ of a single column or row:

Note that in this case, \(D_{(w_1', w_2')} = (-1)D_{(w_1, w_2)},\) because

\[
\begin{array}{c}
\begin{array}{c}
Z(a) \quad Z(a) \\
2 \quad 2
\end{array}
\end{array}
\xrightarrow{\text{W-move}}
\begin{array}{c}
\begin{array}{c}
Z(a) \quad Z(a) \\
2 \quad 2
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
Z(a) \quad Z(a) \\
2 \quad 2
\end{array}
\end{array}
\xrightarrow{\text{W-move}}
\begin{array}{c}
\begin{array}{c}
Z(a) \quad Z(a) \\
2 \quad 2
\end{array}
\end{array}
\]
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So:

\[ \frac{1}{n!} (-1)^{x(w_1,w_2)} D_{(w_1,w_2)} \]

\[ = \frac{1}{n!} (-1)^{x(w'_1,w'_2)} D_{(w'_1,w'_2)} \]

\[ = \frac{1}{n!} (-1)^{x(w'_1,w'_2)} D_{(w'_1,w'_2)} \]

(As \( D_{(w'_1,w'_2)} = (-1)D_{(w_1,w_2)} \)),

as required.

With these moves in hand, we can turn to the general argument. The simple idea is to show that we can transform any gluing \((w_1, w_2)\) that we are given, via a sequence of R- and W-moves, into a standard gluing:

\((w_1, w_2) \rightarrow (w'_1, w'_2) \rightarrow (w''_1, w''_2) \rightarrow \ldots \rightarrow (1 \ldots n, 2 \ldots n)\).

For example, the standard gluing in the case \(n = 4\) is:

\[ D_{(1 \ldots 3 \ldots 1)} \]

We'll now explain how to transform any gluing \((w_1, w_2)\) into the standard gluing using R- and W-moves. The explanation will be illustrated by the example of the gluing \((1 \ 2 \ 3 \ 4, 4 \ 2 \ 3)\):

\[ (26) \]

There are two steps in the procedure.

(The first step.) Begin by doing R-moves to put the factors along the top and down the side into the order in which they appear in the words \(w_1\) and \(w_2\). Let's go through this in the case of the example shown in line (26). We'll start by swapping...
row 3 and row 4, to get:

The we’ll swap row 2 and row 3 (i.e. this swaps factor 2 and factor 4), giving:

This finishes the first step of the procedure (notice that after this step the factors along the top are visited in order from left to right, and the factors down the side are visited in order from top to bottom).

(The second step.) The second step of the procedure is to employ W-moves to arrange it so that, as the drawing is traversed (as indicated in Line 24), the factors along the top are traversed from left-to-right, and the factors up the side (except the top-most) are traversed from top-to-bottom.

Our example requires four such twists. Continuing from line 27:
And:

\[
\begin{array}{c}
Z(a) & Z(a) & Z(a) & Z(a) \\
2 & 2 & 2 & 2 \\
\end{array}
\]

These two steps will transform any given \((w_1, w_2)\) into the standard gluing, and so, by repeated application of Equation 25, for any \((w_1, w_2) \in \Phi_n\),

\[
\phi(w_1, w_2) = \frac{1}{n!n!} (-1)^{x(w_1, w_2)} D_{(w_1, w_2)} = \frac{1}{n!n!} (-1)^{x(w_1^*, w_2^*)} D_{(w_1^*, w_2^*)},
\]

where \((w_1^*, w_2^*) = (\frac{1}{2} \ldots \frac{n}{2}, 2 \downarrow 3 \downarrow \ldots n \downarrow)\).

Now let’s work out what term that standard gluing represents. We can put it in a simplified form in the following way (taking the \(n = 4\) case as a representative example):

\[
\phi(w_1^*, w_2^*) = \frac{1}{4!4!} (+1)
\]

as required.
13.1.6. The contribution $C_2$.

The computation of $C_2$ is closely analogous to the computation of $C_0$, so we will only provide a sketch of it.

Consider some integer $n \geq 1$. The contributions to $C_2$ from terms which use $n$ copies of $\frac{\partial}{\partial a}$ can be indexed by a certain set $\Theta_n$. An element of the set $\Theta_n$ is a pair of words $(w_1, w_2)$ where:

- $w_1$ is a word which uses each of the symbols $\{1, \ldots, n+1\}$ precisely once. The last symbol of $w_1$ has greater value than the first symbol of $w_1$.
- Every symbol $s$ of $w_1$, except the first and last symbol, is decorated by either an arrow pointing to the right $\rightarrow s$ or an arrow pointing to the left $\leftarrow s$.
- The word $w_2$ is a word using each of the symbols $\{1, 2, \ldots, n\}$ exactly once.
- Every symbol $s$ of $w_2$ is decorated by either an arrow pointing up $s \uparrow$ or an arrow pointing down $s \downarrow$.

To every element $(w_1, w_2)$ of $\Theta_n$ there correspond a contribution $\theta_{(w_1, w_2)}$ to $C_2$.

Consider the following example:

\[
\begin{array}{cccc}
Y(a) & Z(a) & Y(a) & Z(a) \\
\frac{a}{2} & & & \\
Y(a) & & & \\
\frac{a}{2} & & & \\
\frac{a}{2} & & & \\
\frac{a}{2} & & & \\
\end{array}
\]

To write down the pair $(w_1, w_2)$ that corresponds with this gluing start at the base of the left-most of the 2 legs. Then trace the diagram. The word $w_1$ records the order in which you encounter the factors along the top of the diagram; the word $w_2$ records the order in which you encounter the factors written down the left-hand side of the diagram. Thus:

\[
\theta_{(1, 3, 2 | 1, 3)} = (-1)^{17} \frac{1}{3!4!} Y(a)^2 \frac{a}{2} Y(a)^n \frac{a}{2} Y(a)^{n-1} \frac{a}{2} Y(a)
\]

The following lemma says that every contribution $\theta_{(w_1, w_2)}$, for $(w_1, w_2) \in \Theta_n$, is equal.

**Lemma 13.1.7.** Let $n$ be an integer $n \geq 1$ and let $(w_1, w_2) \in \Theta_n$. Then:

\[
\theta_{(w_1, w_2)} = -\frac{1}{n!(n+1)!} Y(a) \left( \frac{a}{2} \frac{Z(a)}{2} \right)^{n+1} \frac{a}{2} Y(a)
\]
Comments on the proof. This proof is analogous to the proof of Lemma 13.1.5. In the present case, the “standard form” we wish to put the diagram into by means of column/row transpositions (R-moves) and twists (W-moves) is

$$(1 \rightarrow \ldots \rightarrow n+1, 1 \downarrow 2 \downarrow \ldots \downarrow n \downarrow).$$

For example, the standard contribution for $n = 3$ is:

$$\theta_{(1 \rightarrow 3, 4, 1 \downarrow 2 \downarrow 3)} = \frac{1}{3!4!} (-1)^7 \cdot \frac{1}{3!4!} = \frac{1}{3!4!} (-1)^7.$$

which is equal to

$$\frac{1}{3!4!} (-1)^7 = \frac{1}{3!4!} (-1)^7,$$

as required. The final equality above used the fact that $Y(a)$ was only assumed to have even powers of $a$.

Finally, we can compute:

$$C_2 = \sum_{n=1}^{\infty} \sum_{(w_1, w_2) \in \Theta_n} \theta_{(w_1, w_2)}$$

$$= -\sum_{n=1}^{\infty} \left| \Theta_n \right| \frac{1}{n!(n+1)!} Y(a) \left( \frac{2}{Z(a)} \right)^{n-1} \frac{2}{Z(a)} Y(a)$$

$$= -\sum_{n=1}^{\infty} n!2^n (n+1)! \frac{1}{2} \frac{2}{2} \frac{2}{2} n-1 \frac{1}{n!(n+1)!} Y(a) \left( \frac{2}{Z(a)} \right)^{n-1} \frac{2}{Z(a)} Y(a)$$

$$= -\sum_{n=1}^{\infty} \frac{Y(a)(aZ(a))^{n}}{2} aY(a)$$

The ends the computation of $C_2$ and the proof of Proposition 13.0.10.
References


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