Memorial Meeting For Abdus Salam's 90th Birthday

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Chern-Simons Theory with Complex Gauge Group on Seifert Fibred 3-Manifolds.

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What Is The Theory?

The Chern-Simons action is,

$$
\begin{aligned}
I(t,\widehat{t}) &= \frac{t}{8\pi}\int_M \mathrm{Tr}\left(\mathscr{C}\wedge d\mathscr{C}+\frac{2}{3}\mathscr{C}\wedge\mathscr{C}\wedge\mathscr{C}\right) \\
&\quad +\frac{\widehat{t}}{8\pi}\int_M \mathrm{Tr}\left(\overline{\mathscr{C}}\wedge d\overline{\mathscr{C}}+\frac{2}{3}\overline{\mathscr{C}}\wedge\overline{\mathscr{C}}\wedge\overline{\mathscr{C}}\right),
\end{aligned}
$$

with $t,\widehat{t}\in\mathbb{C}$ (but not necessarily complex conjugates).

$$
\mathscr{C}=\mathscr{A}+i\mathscr{B},\quad \mathscr{A},\,\mathscr{B}\in\Omega^1(M,\mathrm{Lie}\,G)
$$

where both $\mathscr{A}$ and $\mathscr{B}$ are anti-Hermitian. Under this split the action becomes

$$
\begin{aligned}
I(k,s) &= \frac{k}{4\pi}\int_M \mathrm{Tr}\left(\mathscr{A}\wedge d\mathscr{A}+\frac{2}{3}\mathscr{A}\wedge\mathscr{A}\wedge\mathscr{A}-\mathscr{B}\wedge d_{\mathscr{A}}\mathscr{B}\right) \\
&\quad -\frac{s}{2\pi}\int_M \mathrm{Tr}\left(\mathscr{B}\wedge F_{\mathscr{A}}-\frac{1}{3}\mathscr{B}\wedge\mathscr{B}\wedge\mathscr{B}\right)
\end{aligned}
$$

where $t=k+is$, $\widehat{t}=k-is$.

What Is The Theory? Continued

- Large gauge transformations the Chern-Simons action for the compact structure group G,

$$I = \frac{1}{4\pi}\int_M \mathrm{Tr}\left(\mathscr{A}\wedge d\mathscr{A} + \frac{2}{3}\mathscr{A}\wedge\mathscr{A}\wedge\mathscr{A}\right)$$

  imply that $k \in \mathbb{Z}$, however, this imposes no constraint on $s \in \mathbb{C}$.

- Unitarity for a Euclidean theory amounts to requiring that the argument of the path integral under a change orientation is the same as complex conjugation. Consequently,

$$-\bar{I}(M, t, \widehat{t}) = I(-M, t, \widehat{t})$$

  There are two choices for s which lead to a unitary theory:
  1. $s \in \mathbb{R}$ with $\mathscr{C} \longrightarrow \mathscr{C}$ under orientation reversal or
  2. $s \in i\mathbb{R}$ and under a reversal of orientation $\mathscr{C} \longrightarrow \overline{\mathscr{C}}$

  The unitary theory that we consider here is the one where $s \in \mathbb{R}$ so that $\widehat{t} = \bar{t}$.

Why Is It Interesting?

- Gauge theories with a complex gauge group $G_{\mathbb{C}}$ are difficult to make sense of.
- There is a 3d-3d correspondence which involves Chern-Simons theory with complex gauge group.
- This is a theory where one can evaluate the path integral exactly and in this way
- Get a better understanding of ‘holomorphic’ factorization.

How Will We Do It?

- Pick a 3-Manifold which is an S^1 bundle so we can Fourier transform the fields, the co-efficicients of which will be fileds on the base of the fibration.
- Pick a gauge so that you only have Gaussian (path) ntegrals to perform, theis breaks the symmetry to the Abelian subgroup (maximal torus)
- Integrate out all charged fields (those living in the root space of the Lie algebra wrt the Cartan subalgebra) and also integrate out all the massive abelian fields.
- Left with the Abelian constant (along the fibre) modes- an Abelian gauge theory on the base. Evaluating this last path integral is straight forward.

The Answer For The Partition Function

$$Z_M[t,\, \overline{t}] = \frac{1}{|W|} \sum_{1 \leq \mathbf{n} \leq \mathbf{d}} \int_{\mathfrak{t}_{\mathbb{C}}} d^{2\mathbf{rk}(G)}z \, \exp\left(iI^{\mathbf{n}}(t,\, \overline{t}, z)\right) \; \widehat{\tau}_M^{1/2}(z;\, a_1, \ldots, a_N)$$

- This is an integral over the complex Cartan subalgebra and a summation whose range depends on $|d|$ the order of $\mathrm{H}_1(M, \mathbb{Z})$.
- $\widehat{\tau}_M(z, a_1) = |\prod_{\alpha}(2\sin\alpha(z))^{2-N}\prod_{i=1}^{N}(2\sin\alpha(z)/a_i)|$ is a 'complex'version of the Ray-Singer torsion of M.
- $I^{\mathbf{n}}(t,\, \overline{t}, z) = \mathrm{Tr}\left(i\frac{t}{2}z\mathbf{n} - i\frac{\overline{t}}{2}\overline{z}\mathbf{n}\right) - c_1(\mathcal{L})\,\mathrm{Tr}\left(\frac{t}{8\pi}z^2 + \frac{\overline{t}}{8\pi}\overline{z}^2\right)$
- The circle V-bundle of the line V-bundle $\mathcal{L}$ over S^2 with N orbifold points is the 3-manifold M in question.

The 3-Manifolds

- M is a circle V-bundle over the base $\mathbb{P}^1$ with N orbifold points written as $M[\deg \mathcal{L},\, 0,\, (a_1,\, b_1), \ldots, (a_N,\, b_N)]$ where the a_i are the isotropies of the orbifold points, the b_i are the weights of the line V-bundle at the orbifold points and $\deg \mathcal{L}$ is the degree of that line bundle.
- The local picture at each orbifold point is $(z, w) \simeq (\zeta z, \zeta^b w)$ with $\zeta^a = 1$.
- M is smooth iff $\gcd(a_i,\, b_i) = 1$ for each i.
- It is a $\mathbb{Q}$HS with

$$g = 0, \quad c_1(\mathcal{L}_0^{\otimes d}) = \pm \frac{d}{a_1 \ldots a_N}$$

and with $\gcd(a_i,\, a_j) = 1$ for $i \neq j$.

The Action on M

- κ a connection 1-form and K the vector field of S^1 rotations.
- Decompose connections as $\mathscr{A} = A + \kappa\,\phi, \quad \mathscr{B} = B + \kappa\,\lambda$
- Action becomes

$$\frac{k}{4\pi}\int_M \kappa \wedge \mathrm{Tr}\,\big[-A\wedge L_\phi A + B\wedge L_\phi B - 2\lambda\, d_A B$$
$$+2\phi\, dA + d\kappa\,\big(\phi^2 - \lambda^2\big)\big]$$
$$-\frac{s}{2\pi}\int_M \kappa\wedge \mathrm{Tr}\,\Big[-B\wedge L_\phi\, A + B\wedge d\phi + \lambda\wedge dA$$
$$+\frac{1}{2}B\wedge[\lambda,\, B] - \frac{1}{2}A\wedge[\lambda,\, A] + d\kappa\,\lambda\phi\Big]$$

where the twisted Lie derivative is

$$L_\phi = \iota_K \circ d_\phi + d_\phi \circ \iota_K$$

The Gauge Conditions

- The fields ϕ and λ should be constant along the S^1

$$\iota_K\, d\, \iota_K \mathscr{C} = 0 \Rightarrow \iota_K d\phi = 0, \;\; \iota_K d\lambda = 0$$

- Conjugate ϕ and λ into the Cartan subalgebra

$$\iota_\kappa \mathscr{C}^{\mathbf{k}} = 0 \;\Rightarrow \phi^{\mathbf{k}} = 0, \quad \lambda^{\mathbf{k}} = 0$$

Calculation of the Determinants

The ratio of determinants to evaluate is, $\Phi = \phi + i\lambda$,

$$\frac{\mathrm{Det}\left(i\widetilde{L}_{(\phi,\,\lambda)}\right)_{\Omega^0(M,\mathfrak{k})\otimes\Omega^0(M,\mathfrak{k})}}{\sqrt{\mathrm{Det}\left(*\kappa\wedge i\widehat{L}_{(\phi,\,\lambda)}\right)_{\Omega^1_H(M,\mathfrak{k})\otimes\Omega^1_H(M,\mathfrak{k})}}}$$

$$= \left|\prod_{\alpha}(2\sin\alpha(\Phi))^{2-N}\prod_{i=1}^{N}2\sin\left(\alpha(\Phi)/a_i\right)\right|$$

where

$$\Omega^1_H(M,\,\mathfrak{k}) = \bigoplus_n \Omega^1(S^2,\,\mathcal{L}^{\otimes n}\otimes V_{\mathfrak{k}}),$$

$$\Omega^0(M,\,\mathfrak{k}) = \bigoplus_n \Omega^0(S^2,\,\mathcal{L}^{\otimes n}\otimes V_{\mathfrak{k}})$$

2-Dimensional Path Integral

- Having integrated out all the non-zero modes in the S^1 direction of M and all those zero modes in the $\mathfrak{k}_{\mathbb{C}}$ part of the Lie algebra, we are left with an Abelian theory on the orbifold base of the fibration, $\mathbb{A} = A + iB$,

$$Z_M[t,\,\overline{t}] = \sum_{\mathbf{n}} \int D\Phi\, D\overline{\Phi}\, D\mathbb{A}\, D\overline{\mathbb{A}}\; \mathrm{e}^{\left(iI^{\mathbf{n}}_{\Sigma}(t,\,\overline{t})\right)}\;\; \widehat{\tau}^{1/2}_{M}(\Phi;\, a_1, \ldots, a_N)$$

$$\begin{aligned} I^{\mathbf{n}}_{\Sigma}(t,\,\overline{t}) \;&=\; \frac{t}{4\pi}\int_{\Sigma}\mathrm{Tr}\left(\Phi\wedge\left(F_{\mathbb{A}} + i2\pi\mathbf{n}\,\omega\right)\right) + \frac{t}{8\pi}\int_{\Sigma}\mathrm{Tr}\left(\Phi^2\right)\wedge\omega \\ &+\; \frac{\overline{t}}{4\pi}\int_{\Sigma}\mathrm{Tr}\left(\overline{\Phi}\wedge\left(F_{\overline{\mathbb{A}}} - i2\pi\mathbf{n}\,\omega\right)\right) + \frac{\overline{t}}{8\pi}\int_{\Sigma}\mathrm{Tr}\left(\overline{\Phi}^2\right)\wedge\omega \end{aligned}$$

- Notice the sum over $\mathbf{rank}(G)$ $U(1)$ bundles.
- The range of the summation depends on the 3-manifold.
- Notice that the form of partition function still suggests a 'holomorphic' factorization of the path integral

Finite Dimensional Integrals

- Integration over $\mathbb{A}$ and $\overline{\mathbb{A}}$ imply that $d\Phi = 0$
- Which leads us to,

$$Z_M[t, \bar{t}] \simeq \sum_{\mathbf{n}} \int_{\mathfrak{t}_{\mathbb{C}}} \exp\left(i I^{\mathbf{n}}(t, \bar{t})\right) \; \widehat{\tau}_M^{1/2}(\Phi; a_1, \ldots, a_N)$$

with

$$I^{\mathbf{n}}(t, \bar{t}) = \mathrm{Tr}\left(i\frac{t}{2}\Phi\mathbf{n} - i\frac{\bar{t}}{2}\overline{\Phi}\mathbf{n} \right) - c_1(\mathcal{L})\,\mathrm{Tr}\left(\frac{t}{8\pi}\Phi^2 + \frac{\bar{t}}{8\pi}\overline{\Phi}^2 \right)$$

- $c_1(\mathcal{L}) \in \mathbb{Q}$ indeed $c_1(\mathcal{L}) = d/P$, $P = a_1 \ldots a_N$

A Symmetry And Some Factors

- The finite dimensional 'action' $I^{\mathbf{n}}(t,\,\bar{t})$ transforms as

$$I^{\mathbf{n}}(t,\,\bar{t}) \longrightarrow I^{\mathbf{n}}(t,\,\bar{t}) - 2\pi kP\,\mathrm{Tr}\,(\mathbf{n}\,\mathbf{r}) - \pi kPd\,\mathrm{Tr}\,(\mathbf{r}\,\mathbf{r})$$

  under

$$\Phi \longrightarrow \Phi + 2\pi i\mathbf{r}P, \quad \mathbf{n} \longrightarrow \mathbf{n} + d\mathbf{r} \tag{0.1}$$

- As $\mathrm{Tr}\,(\mathbf{r}\,\mathbf{r}) \in 2\mathbb{Z}$ then $\exp\,(iI^{\mathbf{n}}(t,\,\bar{t}))$ is invariant. So too is the Ray-Singer torsion $\widehat{\tau}_M$. These transformations correspond to shifts by the integral lattice I of $\mathfrak{t}$.
- Should also factor out the Weyl group which acts naturally on $\mathfrak{t}$ as well as on $\mathfrak{t}_{\mathbb{C}}$. This is a symmetry of finite dimensional theory as it is part of the (ungauged) gauge group.

Two Equivalent Formulae

On quotienting out by these residual invariances we have, either of two formulae. One by using the symmetry to reduce the integrals to $\mathfrak{t}_{\mathbb{C}}/I \rtimes W$ the other to restrict the range of the summation. Consequently,

$$Z_M[t, \bar{t}] = \sum_{\mathbf{n}} \int_{\mathfrak{t}_{\mathbb{C}}/I \rtimes W} \exp\left(iI^{\mathbf{n}}(t, \bar{t})\right) \; \widehat{\tau}_M^{1/2}(\Phi;\, a_1, \ldots, a_N)$$

or

$$Z_M[t, \bar{t}] = \frac{1}{|W|} \cdot \sum_{1 \leq \mathbf{n} \leq \mathbf{d}} \int_{\mathfrak{t}_{\mathbb{C}}} \exp\left(iI^{\mathbf{n}}(t, \bar{t})\right) \; \widehat{\tau}_M^{1/2}(\Phi;\, a_1, \ldots, a_N)$$

- The Ray-Singer Torsion diverges at the zeros of the sine functions when the number of orbifold points is such that $N > 2$. This situation can be regularised by introducing small masses which do not break the Abelian invariance.

Summing Over Flat Connections

- The summation in the 'answer' for the path integral is over integers which, presently, appear to have no geometric significance from the point of view of the original theory.
- However, they represent a sum over a certain class of flat connections on the 3-manifold (at least in the large s limit).
- One can generate the background by adding an explicit connection

$$A = 2\pi \mathbf{n} \frac{P}{d} \kappa, \quad F_A = 2\pi \mathbf{n} \omega \ \text{and} \ d_A * F_A = 0$$

.

- Yang-Mills connections on Σ correspond to having a central extension of the fundamental group, as

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(M) \longrightarrow \pi_1(\Sigma) \longrightarrow 1$$

that generator is precisely the holonomy of this connection.

‘Holomorphic’ Factorization

Another motivation for this work is to understand the holomorphic factorization of $G_{\mathbb{C}}$ Chern-Simons theory. We have

$$Z[M,\, t,\, \overline{t}] = \int D\mathscr{C}\, D\overline{\mathscr{C}}\ \mathrm{e}^{\left(itI(\mathscr{C}) + i\overline{t}I(\overline{\mathscr{C}})\right)}$$

The natural question is how close are we to being able to make sense of a factorization of the path integral into holomorphic and anti-holomorphic parts

$$Z[M,\, t,\, \overline{t}] \stackrel{?}{=} Z[M,\, t]\,.\,\overline{Z[M,\, t]}$$

where

$$Z[M,\, t] = \int D\mathscr{C}\ \mathrm{e}^{\left(itI(\mathscr{C})\right)} \quad \text{and} \quad \overline{Z[M,\, t]} = \int D\overline{\mathscr{C}}\ \mathrm{e}^{\left(i\overline{t}I(\overline{\mathscr{C}})\right)}$$

and if such partition functions are to make sense what ‘contour’ path integrals are being performed?

'Holomorphic' Factorization Continued

One expects that a correct formula is

$$Z[M,\ t,\ \overline{t}] = \sum_{\rho,\overline{\rho}} n_{(\rho,\overline{\rho})}\ Z^{\rho}[M,\ t]\,.\,\overline{Z^{\rho}[M,\ t]}$$

where ρ labels the solution to the flatness equation, and for particular contours.

All of our formulae factorize (at least for $N \leq 2$) since the following integral

$$\int d^2z\ \exp\left(az^2 + b\overline{z}^2 + cz + d\overline{z}\right) = \frac{\pi}{\sqrt{-ab}}\ \exp\left(-\frac{c^2}{4a} - \frac{d^2}{4b}\right)$$

may be expressed as the product of two contour integrals (not closed contours)

$$e^{(-i\pi/2)}\,.\,\int_{\Gamma} dz\ \exp\left(az^2 + cz\right)\,.\,\int_{\Gamma'} d\overline{z}\ \exp\left(b\overline{z}^2 + d\overline{z}\right)$$

for example with both Γ and Γ' being the real axis.

‘Holomorphic’ Factorization Continued

These last formulae immediately imply that the ‘holomorphic’ factor of the $G_{\mathbb{C}}$ theory is just the partition function of the G theory (at least up to framing and a change of level) once we sum over $\mathbf{n}$.

Clearly our formulae appear to have the holomorphic decomposition

$$Z_M[t,\, \overline{t}] \stackrel{?}{=} \frac{\exp\left(-i\pi/2\right)}{|W|} \sum_{\mathbf{n}} Z^{\mathbf{n}}_{\mathfrak{t}_\Gamma}[t] \, . \, \overline{Z^{\mathbf{n}}_{\mathfrak{t}_\Gamma}[t]}$$

with

$$Z^{\mathbf{n}}_{\mathfrak{t}_\Gamma}[t] = \int_{\mathfrak{t}_\Gamma} \exp\left(i I^{\mathbf{n}}_{\mathfrak{t}_{\mathbb{C}}}[t]\right) \tau_M^{1/2}(\Phi;\, a_1, \ldots, a_N)$$

where

$$I^{\mathbf{n}}_{\mathfrak{t}_{\mathbb{C}}}[t] = \mathrm{Tr}\left(i\frac{t}{2}\Phi\mathbf{n} - c_1(\mathcal{L})\frac{t}{8\pi}\Phi^2\right)$$

and there is either a restriction on the range of summation or on the integration. We could have also introduced a phase into the definition, but prefer not to. The contour $\mathfrak{t}_\Gamma$ needs to be defined. In the case of Lens spaces one may take it to be $\mathfrak{t}_\Gamma = \mathfrak{t}$.