رجسٹرڈ ایل نمبر

ہفتہ وار

اخبار جھنگ سیال

جھنگ جلد نمبر ۲ نمبر ۳۱

چندہ سالانہ مقامی بیرون

۲۰ جون ۱۹۴۰ء

جھنگ مگھیانہ میں اتحاد بورڈ قایم ہو گیا!

[illegible]

ممبر ... پنجاب یونیورسٹی کا ریکارڈ توڑ دیا ... السلام

جس پر ضلع جھنگ کو بجا طور پر ناز ہے!

سیکرٹری ضلع کانگرس کمیٹی واپس آ گئے

[illegible]

عذر داریاں دائر ہو گئیں

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وار فنڈ میں تیس ہزار روپیہ

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سول سرجن صاحب واپس آ گئے

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علیحدہ ہو جاؤ۔ ورنہ

بیگم عین الحق کی ایگزیکٹو آفیسر کو دھمکی

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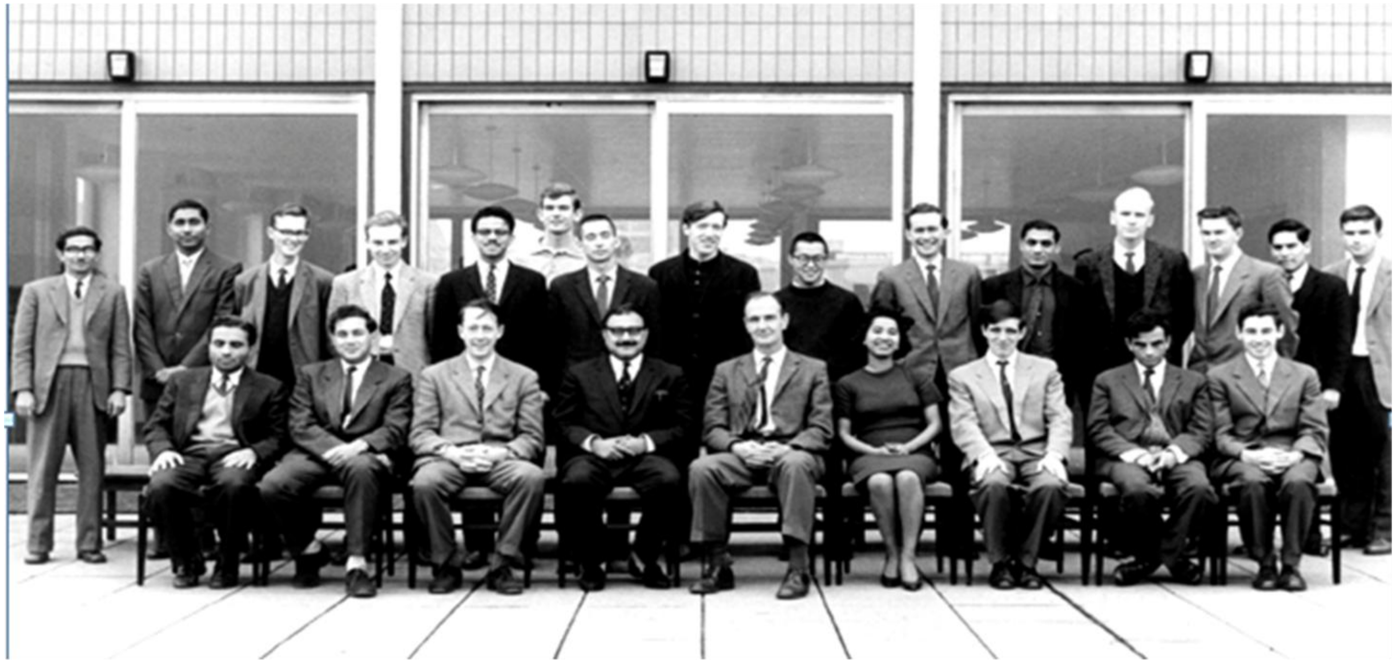
مکرم پروفیسر ڈاکٹر محمد عبدالسلام صاحب
ابن مکرم چوہدری محمد حسین صاحب
PROFESSOR MUHAMMAD ABDUS SALAM
SON OF
CHAUDHRY MUHAMMAD HUSSAIN
AND HAJIRA HUSSAIN
29 JANUARY 1926. 21 NOVEMBER 1996
IN 1979 BECAME THE FIRST MUSLIM NOBEL LAUREATE
FOR HIS WORK IN PHYSICS. IN A SPECIAL TRIBUTE
HAZRAT KHALIFATUL MASIH IV RECOGNISED HIS
EXCELLENT QUALITIES OF SINCERITY, HUMILITY AND
DEVOUT SUBMISSION TO THE ORDER OF THE JAMAAT
WHILST CALLING HIM THE GREATEST SCIENTIST
OF ALL TIME WHO PROVED THE UNITY OF GOD



Theoretical Physics Group, Blacket Laboratories, 1964.
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Ron King.
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Spinor Representation of Finite Rotations of $SO(4)$

Muneer Ahmad Rashid and Ansaruddin Syed

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Abstract

Following the work of C. B. van Wyk on the Lorentz group $SO(3,1)$, we first express a general finite rotation of $SO(4)$ in terms of 2 ordinary (3-dimensional) vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfying certain conditions and then using the homomorphism of $SU(2) \times SU(2)$ onto $SO(4)$, we express the same rotation in terms of a pair of 2×2 matrices, again determined by the same pair of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. This is extremely useful as it allows one to convert the 4×4 matrix multiplication of elements of $SO(4)$ into the 2×2 matrix multiplication of elements of $SU(2)$.

1 Introduction

In a rather old paper, van Wyk [1] considers finite orthochronus Lorentz transformations, and describes them in terms of an antisymmetric 4×4 (complex) matrix U whose elements are determined by 2 ordinary 3-dimensional vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. He then shows that any pair of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfying certain conditions, and a pair of angles θ and ϕ, uniquely determine a Lorentz transformation which he denotes by $\Lambda_1(\boldsymbol{a}, \boldsymbol{b}, \theta, \phi)$. Then using the well known 2 - 1 onto homomorphism of $SL(2, C)$ to $SO(3, 1)$, he obtains the spinor representation of the above Lorentz transformation in terms of the same vectors and angles, i.e., $\boldsymbol{a}, \boldsymbol{b}, \theta, \phi$.

This allows him to calculate the products of Lorentz transformations which are needed in various branches of Physics, and which are obviously products of 4×4 matrices, in terms of their spinor representatives which are, of course, products of 2×2 matrices, and therefore much simpler to evaluate. He then illustrates the utility of this procedure, by a large number of examples. The aim of the present paper is to show that the same procedure can be carried out for $SO(4)$ by using the homomorphism:

$$SU(2) \times SU(2) \to SO(4).$$

We therefore start with the discussion of representation of finite elements of $SO(4)$ in terms of a pair of ordinary vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

2 Finite rotations of $SO(4)$

We consider $R \in SO(4)$ to be an orthogonal transformation in the 4-dimensional space R^4, whose elements will be denoted by

$$x = (x_1, x_2, x_3, x_4)^T \equiv (\boldsymbol{x}, x_4)^T$$

where

$$\boldsymbol{x} = (x_1, x_2, x_3)^T$$

is an ordinary vector in the usual 3-dimensional physical space $\mathbb{R}^3$. We use the standard convention that Greek indices $\lambda, \mu, \nu, ...$, range over 1, 2, 3, 4, while the Latin indices $i, j, k, ...$, range over 1, 2, 3. In analogy with van Wyk [1], given any two real vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfying

$$\boldsymbol{a}^2 + \boldsymbol{b}^2 = 1, \quad \boldsymbol{a}.\boldsymbol{b} = 0,$$

we define a 4×4 real anti-symmetric matrix U and its dual U^D, by

$$U = \begin{bmatrix} 0 & -a_3 & a_2 & b_1 \\ a_3 & 0 & -a_1 & b_2 \\ -a_2 & a_1 & 0 & b_3 \\ -b_1 & -b_2 & -b_3 & 0 \end{bmatrix}, \tag{1a}$$

$$U^D_{\mu\nu} = {}^1\!/\!_2\, \epsilon_{\mu\nu\rho\sigma} U_{\rho\sigma}, \tag{1b}$$

$$\Rightarrow U^D = \begin{bmatrix} 0 & b_3 & -b_2 & -a_1 \\ -b_3 & 0 & b_1 & -a_2 \\ b_2 & -b_1 & 0 & -a_3 \\ a_1 & a_2 & a_3 & 0 \end{bmatrix}. \tag{1c}$$

Then

$$U^2 = \begin{bmatrix} -a_1^2 - a_2^2 - b_1^2 & a_1a_2 - b_1b_2 & a_1a_3 - b_1b_3 & -a_3b_2 + a_2b_3 \\ a_1a_2 - b_1b_2 & -a_3^2 - a_1^2 - b_2^2 & a_2a_3 - b_2b_3 & a_3b_1 - a_1b_3 \\ a_1a_3 - b_1b_3 & a_2as_3 - b_2b_3 & -a_2^2 - a_1^2 - b_3^2 & -a_2b_1 + a_1b_2 \\ -a_3b_2 + a_2b_3 & a_3b_1 - a_1b_3 & -a_2b_1 + a_1b_2 & -b_1^2 - b_2^2 - b_3^2 \end{bmatrix}, \tag{2}$$

and it is easily checked that

$$U^3 = -U, \quad {U^D}^3 = -U^D, \quad UU^D = 0 = U^D U. \tag{3}$$

It follows that

$$e^{U\theta} = 1 + U(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ...) + U^2(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + ...)$$

i.e.,

$$e^{U\theta} = 1 + U\sin\theta + U^2(1-\cos\theta). \tag{4a}$$

$$\text{Similarly} \quad e^{U^D\phi} = 1 + U^D\sin\phi + {U^D}^2(1-\cos\phi). \tag{4b}$$

We now define

$$\Lambda_1 \equiv \Lambda_1(\boldsymbol{a},\boldsymbol{b};\theta,\phi) = e^{U\theta + U^D\phi} \equiv e^{U\theta}.e^{U^D\phi} \tag{5a}$$
$$\equiv \Lambda_1(\boldsymbol{a},\boldsymbol{b};\theta,0)\Lambda_1(\boldsymbol{a},\boldsymbol{b};0,\phi);$$

Clearly

$$\Lambda_1 = \{1 + U\sin\theta + U^2(1-\cos\theta)\}\{1 + U^D\sin\phi + (U^D)^2(1-\cos\phi)\}$$
$$= I + U\sin\theta + U^2(1-\cos\theta) + U^D\sin\phi + (U^D)^2(1-\cos\phi). \tag{5b}$$

As it is easily checked that

$$(e^{U\theta})(e^{U\theta})^T = I = (e^{U^D\phi})(e^{U^D\phi})^T,$$

we conclude that

$$\Lambda_1{\Lambda_1}^T = {\Lambda_1}^T\Lambda_1 = I$$

i.e., Λ_1 is orthogonal; this leads to

$$1 = \det(\Lambda_1{\Lambda_1}^T) = (\det\Lambda_1)^2 \Rightarrow \det\Lambda_1 = \pm 1.$$

Now $\det I = 1$ and Λ_1 is obtained from I by a continuous process $\Rightarrow \det\Lambda_1 = 1$; hence $\Lambda_1 \in SO(4)$ i.e., Λ_1 is a 4-dimensional finite rotation. Then, just as Wyk [1] argues, the fact that Λ_1 depends on 6 independent parameters, means that it can be regarded as the most general finite rotation in 4 dimensions.
We now consider two important special cases:

I: $\Lambda_1(\hat{\boldsymbol{a}},0;\theta,0)$.
If U_b stands for $U|_{\boldsymbol{b}=0}$, then

$$\Lambda_1(\hat{\boldsymbol{a}},0;\theta,0) = e^{U_b\theta} = 1 + U_b\sin\theta + {U_b}^2(1-\cos\theta).$$

But

$$U_b = \begin{bmatrix} 0 & -a_3 & a_2 & 0 \\ a_3 & 0 & -a_1 & 0 \\ -a_2 & a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow {U_b}^2 = -\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} {a_1}^2 & a_1a_2 & a_1a_3 & 0 \\ a_1a_2 & {a_2}^2 & a_2a_3 & 0 \\ a_1a_3 & a_2a_3 & {a_3}^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that

$$
\begin{aligned}
(U_b)_{rs} &= -\epsilon_{rst}a_t && \text{by explicit checking,}\\
(U_b^2)_{rs} &= -\delta_{rs} + a_r a_s && \text{by inspection,}\\
\Rightarrow (\Lambda_1)_{rs} &= \delta_{rs} - \epsilon_{rst}a_t \sin\theta + \{(-\delta_{rs} + a_r a_s)(1-\cos\theta)\}\\
&= \delta_{rs}\cos\theta + a_r a_s(1-\cos\theta) - \epsilon_{rst}a_t\sin\theta.
\end{aligned}
$$

As

$$
(\Lambda_1)_{r4} = (\Lambda_1)_{4r} = 0, \quad (\Lambda_1)_{44} = 1,
$$

it follows that

$$
\Lambda_1(\hat{\boldsymbol{a}}, 0; \theta, 0) = R(\hat{\boldsymbol{a}}, \theta) \tag{6a}
$$

by Carmeli [2], where the RHS is the usual rotation by an angle θ about the axis along $\hat{\boldsymbol{a}}$ in the 3-dimensional physical space spanned by vectors of the form $(x_1, x_2, x_3, 0)^T$.

II: $\Lambda_1(\hat{\boldsymbol{a}}, 0; 0, \phi)$.
Again, if $U_b^D \equiv U^D|_{\boldsymbol{b}=0}$, we will have

$$
\Lambda_1(\hat{\boldsymbol{a}}, 0; 0, \phi) = e^{U_b^D\phi} = 1 + U_b^D \sin\phi + \left(U_b^D\right)^2(1-\cos\phi),
$$

where

$$
U_b^D = \begin{bmatrix} 0 & 0 & 0 & -a_1 \\ 0 & 0 & 0 & -a_2 \\ 0 & 0 & 0 & -a_3 \\ a_1 & a_2 & a_3 & 0 \end{bmatrix} \Rightarrow (U_b^D)^2 = -\begin{bmatrix} {a_1}^2 & a_1a_2 & a_1a_3 & 0 \\ a_1a_2 & {a_2}^2 & a_2a_3 & 0 \\ a_1a_3 & a_2a_3 & {a_3}^3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

Hence

$$
\begin{aligned}
&(\Lambda_1)_{rs} = \delta_{rs} - a_r a_s(1-\cos\phi)\\
&(\Lambda_1)_{r4} = -a_r\sin\phi, \quad (\Lambda)_{4r} = a_r\sin\phi,\\
&(\Lambda_1)_{44} = 1 - 1.(1-\cos\phi) \;= \cos\phi.
\end{aligned}
$$

As is shown in Appendix A, this means that

$$
\begin{aligned}
\Lambda_1(\hat{\boldsymbol{a}}, 0; 0, \phi) &= \text{a rotation by an angle } \phi \text{ in the}((\hat{\boldsymbol{a}}, 0), i_4) - \text{plane}\\
&\equiv R(((\hat{\boldsymbol{a}}, 0), i_4), \phi), \quad \text{(say)}.
\end{aligned} \tag{6b}
$$

Note from the explicit expressions for U and U^D, that

$$
U^D(\boldsymbol{a}, \boldsymbol{b}) = U(-\boldsymbol{b}, -\boldsymbol{a}) \Rightarrow\; U(\boldsymbol{a}, \boldsymbol{b}) = U^D(-\boldsymbol{b}, -\boldsymbol{a}),
$$

and this leads to

$$
\Lambda_1(\boldsymbol{a}, \boldsymbol{b}; \theta, \phi) = \Lambda_1(-\boldsymbol{b}, -\boldsymbol{a}; \phi, \theta).
$$

3 $\boldsymbol{a}, \boldsymbol{b}, \theta, \phi$ in terms of matrix elements of Λ_1

From Equations (1), (2), and (5b), we have

$$
\begin{aligned}
&\text{Tr } U = \text{Tr } U^D = 0, \qquad \text{Tr } U^2 = \text{Tr } {U^D}^2 = -2.\\
&\text{Tr } \Lambda_1 = 2(\cos\theta + \cos\phi)
\end{aligned} \tag{7a}
$$

Set

$$
M = \frac{1}{2}(\Lambda_1 - {\Lambda_1}^T) = U\sin\theta + U^D\sin\phi \Rightarrow \text{Tr } M = 0, \quad M^2 = U^2\sin^2\theta + {U^D}^2\sin^2\phi,
$$

and

$$\text{Tr } M^2 = -2(\sin^2\theta + \sin^2\phi). \tag{7b}$$

To find θ and ϕ, we note that (7a) and (7b) lead to

$$2\cos\theta\cos\phi = \frac{1}{4}(\text{Tr } \Lambda_1)^2 - \frac{1}{2}\text{Tr } M^2 - 2$$

which, when combined with (7a), gives

$$\cos\theta - \cos\phi = \left\{4 + \text{Tr } M^2 - \frac{1}{4}(\text{Tr } \Lambda_1)^2\right\}^{1/2}. \tag{7c}$$

(7a) and (7c) obviously give us

$$\cos\theta = \frac{1}{4}\text{Tr } \Lambda_1 + \frac{1}{2}\left\{4 + \text{Tr } M^2 - \frac{1}{4}(\text{Tr } \Lambda_1)^2\right\}^{1/2}, \tag{8a}$$

$$\cos\phi = \frac{1}{4}Tr\Lambda_1 - \frac{1}{2}\left\{4 + \text{Tr } M^2 - \frac{1}{4}(\text{Tr } \Lambda_1)^2\right\}^{1/2}. \tag{8b}$$

Next, for $\boldsymbol{a}, \boldsymbol{b}$, we start with

$$\begin{array}{llll} U_{ij} = -\epsilon_{ijk}a_k, & U_{4j} = -b_j, & U_{j4} = b_j, & U_{44} = 0, \\ U^D_{ij} = \epsilon_{ijk}b_k, & U^D_{4j} = a_j, & U^D_{j4} = -a_j, & U^D_{44} = 0, \end{array}$$

which, after a bit of calculations, give

$$\begin{aligned} \frac{1}{2}\epsilon_{jkm}M_{km}\sin\theta &= (-a_j\sin\theta + b_j\sin\phi)\sin\theta, \\ M_{j4}\sin\phi &= (b_j\sin\theta - a_j\sin\phi)\sin\phi, \end{aligned}$$

and so, we get

$$a_j = (\sin^2\theta - \sin^2\phi)^{-1}\left(-\frac{1}{2}\epsilon_{jkm}M_{km}\sin\theta + M_{j4}\sin\phi\right), \tag{9a}$$

$$b_j = (\sin^2\theta - \sin^2\phi)^{-1}\left(-\frac{1}{2}\epsilon_{jkm}M_{km}\sin\phi + M_{j4}\sin\theta\right). \tag{9b}$$

4 Significance of the Commutative Factors $\Lambda_1(\boldsymbol{a}, \boldsymbol{b}; \theta, 0)$ and $\Lambda_1(\boldsymbol{a}, \boldsymbol{b}; 0, \phi)$

To obtain this significance, we define the following four 4-vectors

$$A = \begin{bmatrix} \hat{\boldsymbol{a}} \\ 0 \end{bmatrix}, \qquad B = \begin{bmatrix} \hat{\boldsymbol{b}} \\ 0 \end{bmatrix}$$
$$D = \begin{bmatrix} \hat{\boldsymbol{a}} \times \boldsymbol{b} \\ a \end{bmatrix}, \qquad E = \begin{bmatrix} \boldsymbol{a} \times \hat{\boldsymbol{b}} \\ -b \end{bmatrix}$$

which can be easily verified to satisfy

$$\begin{aligned} & A.A = B.B = D.D = E.E = 1, \\ & A.B = A.D = A.E = B.D = B.E = D.E = 0, \\ \Rightarrow\ & A, B, D, E \text{ form an orthonormal set of vectors of } \mathbb{R}^4, \end{aligned}$$

$$\begin{array}{lll} UA = 0 = UD, & UB = E, & UE = -B, \\ U^D A = D, & U^D D = -A, & U^D B = 0 = U^D E \end{array}$$

Let us find the action of Λ_1 on the vectors A, B, D, and E. As $UA = 0$, we will have

$$\Lambda_1(\boldsymbol{a}, \boldsymbol{b}; \theta, \phi)A = (I + U^D\sin\phi + {U^D}^2(1 - \cos\phi))A = A + D\sin\phi - (1 - \cos\phi)A$$

i.e., $\Lambda_1(\boldsymbol{a}, \boldsymbol{b}; \theta, \phi)A = \Lambda_1(\boldsymbol{a}, \boldsymbol{b}; 0, \phi)A = A\cos\phi + D\sin\phi$;

similarly, we will have:

$$\begin{aligned}
\Lambda_1(\boldsymbol{a},\boldsymbol{b};\theta,\phi)D &= \Lambda_1(\boldsymbol{a},\boldsymbol{b};0.\phi)D = -A\sin\phi + D\cos\phi \\
\Lambda_1(\boldsymbol{a},\boldsymbol{b};\theta,\phi)B &= \Lambda_1(\boldsymbol{a},\boldsymbol{b};\theta,0)B = \quad B\cos\theta + E\sin\theta, \\
\Lambda_1(\boldsymbol{a},\boldsymbol{b};\theta,\phi)E &= \Lambda_1(\boldsymbol{a},\boldsymbol{b};\theta,0)E = -B\sin\theta + E\cos\theta.
\end{aligned}$$

As we easily see that

$$\begin{aligned}
&\Lambda_1(\boldsymbol{a},\boldsymbol{b};\theta,0)A = A, \quad \Lambda_1(\boldsymbol{a},\boldsymbol{b};\theta,0)D = D \\
&\Lambda_1(\boldsymbol{a},\boldsymbol{b};0,\phi)B = B, \quad \Lambda_1(\boldsymbol{a},\boldsymbol{b};0,\phi)E = E
\end{aligned}$$

we conclude that $\Lambda_1(\boldsymbol{a},\boldsymbol{b};\theta,0)$ is a rotation by an angle θ in the 2-plane spanned by $\{B,E\}$ and it keeps the $\{A,D\}$-plane invariant, while $\Lambda_1(\boldsymbol{a},\boldsymbol{b};0,\phi)$ is a rotation by an angle ϕ in the $\{A,D\}$-plane and it keeps $\{B,E\}$-plane invariant, and that $\Lambda_1(\boldsymbol{a},\boldsymbol{b};\theta,\phi)$ is a (commutative) product of these two rotations. Thus, we have re-derived the general result that an element of $SO(4)$ consists of a pair of rotations in two mutually orthogonal 2-planes of $\mathbb{R}^4$; however, we now have far more information than before as the angles of rotation and the configurations of the 2-planes are now immediately given by the parameters $\boldsymbol{a},\boldsymbol{b},\theta,\phi$, while earlier, these had to be obtained by a process involving the solution of the eigenvalue problem of the matrix Λ_1.

5 Representation of Rotations by Unitary Matrices

It is a well known fact that there exists a 2-1 onto homomorphism $SU(2)\times SU(2) \to SO(4)$ which allows one to represent rotations of $SO(4)$ and various algebraic operations on them by pairs of unitary matrices and corresponding operations on these matrices. As multiplying 2×2 matrices is much simpler than doing the same with 4×4 matrices, it appears worthwhile to study the above relationship in detail. A particular concrete homomorphism of $SU(2)\times SU(2)$ onto $SO(4)$ is constructed in Appendix B, according to which corresponding to every pair (V,W) of unitary matrices, there exists a rotation $R \equiv R(V,W) \in SO(4)$, given by

$$R_{\mu\nu} = \frac{1}{2}\,\mathrm{Tr}\,(V\sigma_\nu W^{+}\rho_\mu) \tag{10}$$

where

$$\begin{aligned}
\sigma_i &= i\tau_i, & &\tau_i \text{ are the Pauli matrices,} \\
\sigma_4 &= e, & &\text{the } 2\times2 \text{ unit matrix}
\end{aligned}$$

and

$$\rho_\mu = (-\sigma_i,\ \sigma_4) \equiv (-\boldsymbol{\sigma},\sigma_4) = {\sigma_\mu}^{+};$$

the inverse of (10) are given by

$$\begin{aligned}
\pm V &= \frac{R_{\mu\nu}\sigma_\mu\rho_\nu}{(R_{\mu\nu}R_{\kappa\lambda}\sigma_\mu\rho_\nu\sigma_\lambda\rho_\kappa)^{1/2}} \\
&= \frac{\mathrm{Tr}\ R + (R_{4k} - R_{k4} - R_{ij}\epsilon_{ijk})\rho_k}{[4 + (\mathrm{Tr}\ R)^2 - \mathrm{Tr}\,(RR) - 2(R_{4k} - R_{k4})R_ij\epsilon_{ijk}]^{1/2}},
\end{aligned} \tag{11a}$$

$$\begin{aligned}
\pm W^{+} &= \frac{R_{\mu\nu}\rho_\nu\sigma_\mu}{(R_{\mu\nu}R_{\kappa\lambda}\rho_\mu\sigma_\nu\rho_\lambda\sigma_\kappa)^{1/2}} \\
&= \frac{\mathrm{Tr}\ R + (R_{4k} - R_{k4} - R_{ij}\epsilon_{jik})\rho_k}{[4 + (\mathrm{Tr}\ R)^2 - \mathrm{Tr}\,(RR) + 2(R_{4k} - R_{k4})R_{ij}\epsilon_{ijk}]^{1/2}}.
\end{aligned} \tag{11b}$$

Thus, corresponding to every rotation $\Lambda_1(\boldsymbol{a},\boldsymbol{b};\theta,\phi)$ of $SO(4)$,there will be two elements

$$\pm\Lambda_2(\boldsymbol{a},\boldsymbol{b};\theta,\phi) \equiv \pm(V(\boldsymbol{a},\boldsymbol{b};\theta,\phi), W(\boldsymbol{a},\boldsymbol{b};\theta,\phi))$$

of $SU(2)\times SU(2)$ which represent the above rotation spinorially in the sense that any operation performed on rotations, which takes $\Lambda_1(\boldsymbol{a},\boldsymbol{b};\theta,\phi)$ to $\Lambda_1(\boldsymbol{a}',\boldsymbol{b}';\theta',\phi')$, will also take $\Lambda_2(\boldsymbol{a},\boldsymbol{b};\theta,\phi)$ to $\Lambda_2(\boldsymbol{a}',\boldsymbol{b}';\theta',\phi')$.

Just as we have explicit expression (5b) for $\Lambda_1(\boldsymbol{a},\boldsymbol{b};\theta,\phi)$ in terms of $\boldsymbol{a},\boldsymbol{b},\theta,\phi$, we need explicit expressions for Λ_2 i.e., for $V(\boldsymbol{a},\boldsymbol{b};\theta,\phi)$ and $W(\boldsymbol{a},\boldsymbol{b};\theta,\phi)$ also. To obtain these, we first consider the special cases of the rotations $\Lambda_1(\hat{\boldsymbol{a}},0;\theta,0), \Lambda_1(\hat{\boldsymbol{a}},0;0,\phi)$, whose matrix elements have already been obtained, and then find Λ_2 for another couple of simple rotations which, when taken together, indicate to us the general expressions for $V(\boldsymbol{a},\boldsymbol{b};\theta,\phi), W(\boldsymbol{a},\boldsymbol{b};\theta,\phi)$.

I. $\Lambda_1(\hat{\mathbf{a}},0;\theta,0) \equiv R(\hat{\mathbf{a}},\theta)$:
Here the matrix elements are given by

$$R_{44} = 1, \quad R_{4i} = R_{i4} = 0,$$
$$R_{ij} = \delta_{ij}\cos\theta + (1-\cos\theta)a_i a_j - \epsilon_{ijk}a_k\sin\theta;$$

these lead to

$$\text{Tr } R = 4\cos^2\theta, \quad R_{4k} - R_{k4} = 0,$$
$$R_{ij}\epsilon_{ijk}\rho_k = -4\sin{\theta/2}\cos{\theta/2}\ \hat{\boldsymbol{a}}.\boldsymbol{\rho},$$
$$\text{Tr } (RR) = 4\cos^2\theta,$$

so that we get

$$\pm V = \frac{4\cos^2{\theta/2} + 4\sin{\theta/2}\cos{\theta/2}\ \hat{\boldsymbol{a}}.\boldsymbol{\rho}}{(4+16\cos^4{\theta/2} - 4\cos^2\theta)^{1/2}}$$
$$\text{i.e., } \pm V = \cos{\theta/2} + \sin{\theta/2}\ \hat{\boldsymbol{a}}.\boldsymbol{\rho}.$$

This agrees with Macfarlane's [3] Equation (85) if we note that

$$\boldsymbol{\rho} = -\boldsymbol{\sigma} = -i\boldsymbol{\tau}, \quad \tau_i \equiv \text{Pauli matrices.}$$

As

$$R_{ij}\epsilon_{jik} = -R_{ij}\epsilon_{ijk},$$

these also give

$$\pm W^+ = \frac{4\cos^2{\theta/2} - 4\sin{\theta/2}\cos{\theta/2}\ \hat{\boldsymbol{a}}.\boldsymbol{\rho}}{4\cos{\theta/2}}$$
$$\text{i.e. } W^+ = \cos{\theta/2} - \sin{\theta/2}\ \hat{\boldsymbol{a}}.\boldsymbol{\rho} = V^+ \text{as } \boldsymbol{\rho}^+ = -\boldsymbol{\rho}.$$

Thus in this case

$$V(\hat{\boldsymbol{a}},0;\theta,0) = W(\hat{\boldsymbol{a}},0;\theta,0) = \cos{\theta/2} - i\sin{\theta/2}\ \hat{\boldsymbol{a}}.\boldsymbol{\tau}. \tag{12a}$$

This leads to

$$V(\hat{\boldsymbol{a}},0;\theta,0) = W(\hat{\boldsymbol{a}},0;\theta,0) = e^{-i\theta\ \hat{a}.\tau/2}, \tag{12b}$$

as using the fact that

$$(\hat{\boldsymbol{a}}.\boldsymbol{\tau})^2 = (\hat{\boldsymbol{a}}.\boldsymbol{\tau})(\hat{\boldsymbol{a}}.\boldsymbol{\tau}) = 1,$$

we get

$$e^{-i\frac{\theta}{2}\hat{a}.\tau} = 1 - i\frac{\theta}{2}(\hat{\boldsymbol{a}}.\boldsymbol{\tau}) + \frac{1}{2!}.-\left(\frac{\theta}{2}\right)^2 + \frac{1}{3!}.i\left(\frac{\theta}{2}\right)^3(\hat{\boldsymbol{a}}.\boldsymbol{\tau}) + \frac{1}{4!}\left(\frac{\theta}{2}\right)^4 + \frac{1}{5!}.-i\left(\frac{\theta}{2}\right)^5(\hat{\boldsymbol{a}}.\boldsymbol{\tau})\cdot$$
$$\Rightarrow e^{-i\frac{\theta}{2}\hat{a}.\tau} = \cos{\theta/2} - i\sin{\theta/2}\ \hat{\boldsymbol{a}}.\boldsymbol{\tau}. \tag{13}$$

II. $\Lambda_1(\hat{\boldsymbol{a}},0;0,\phi) \equiv R(((\hat{\boldsymbol{a}},0),i_4),\phi)$
Here, the matrix elements are

$$R_{rs} = \delta_{rs} - a_r a_s(1-\cos\phi),$$
$$R_{r4} = -a_r\sin\phi, \quad R_{4r} = a_r\sin\phi,$$
$$R_{44} = \cos\phi,$$

which lead to

$$\text{Tr } R = 4\cos^2{\phi/2}, \quad \epsilon_{ijk}R_{ij}\rho_k = 0, \quad R_{4k} - R_{k4} = 2a_k\sin\phi,$$
$$\text{Tr } (RR) = 4\cos^2\phi, \quad (R_{4k} - R_{k4})R_{ij}\epsilon_{ijk} = 0,$$

so that

$$\begin{aligned}
\pm V &= \frac{4\cos^2 \phi/2 + 2\sin\phi\ \hat{\boldsymbol{a}}.\boldsymbol{\rho}}{(4+16\cos^4 \phi/2 - 4\cos^2\phi)^{1/2}} \\
&= \cos \phi/2 + \sin \phi/2\ \hat{\boldsymbol{a}}.\boldsymbol{\rho} \\
&= \cos \phi/2 - i\sin \phi/2\ \hat{\boldsymbol{a}}.\boldsymbol{\tau} \\
\pm W^{+} &= \cos \phi/2 + \sin \phi/2\ \hat{\boldsymbol{a}}.\boldsymbol{\rho} \\
&= \cos \phi/2 - i\sin \phi/2\ \hat{\boldsymbol{a}}.\boldsymbol{\tau},
\end{aligned}$$

$$\Rightarrow V(\hat{\boldsymbol{a}},0;0,\phi) = W^{+}(\hat{\boldsymbol{a}},0;0,\phi) = e^{-i\phi\ \hat{a}.\tau/2}. \tag{14}$$

A little bit of additional calculation shows that

$$\begin{aligned}
V\left(\frac{\boldsymbol{i}}{\sqrt{2}},\frac{\boldsymbol{j}}{\sqrt{2}},\theta,0\right) &= \cos \theta/2 - i\sin \theta/2 \frac{\boldsymbol{i}-\boldsymbol{j}}{\sqrt{2}}\cdot\ \boldsymbol{\tau} = e^{-i\frac{\theta}{2}\frac{\boldsymbol{i}-\boldsymbol{j}}{\sqrt{2}}.\boldsymbol{\tau}}, \\
W\left(\frac{\boldsymbol{i}}{\sqrt{2}},\frac{\boldsymbol{j}}{\sqrt{2}},\theta,0\right) &= \cos \theta/2 - i\ \sin \theta/2 \frac{\boldsymbol{i}+\boldsymbol{j}}{\sqrt{2}}.\boldsymbol{\tau} = e^{-i\frac{\theta}{2}\frac{\boldsymbol{i}+\boldsymbol{j}}{\sqrt{2}.\boldsymbol{\tau}}}, \\
V\left(\frac{\boldsymbol{i}}{\sqrt{2}},\frac{\boldsymbol{j}}{\sqrt{2}};0,\phi\right) &= \cos \phi/2 - i\ \sin \phi/2 \frac{\boldsymbol{i}-\boldsymbol{j}}{\sqrt{2}}.\boldsymbol{\tau} = e^{-i\frac{\phi}{2}\frac{\boldsymbol{i}-\boldsymbol{j}}{\sqrt{2}}.\boldsymbol{\tau}}, \\
W\left(\frac{\boldsymbol{i}}{\sqrt{2}},\frac{\boldsymbol{j}}{\sqrt{2}};0,\phi\right) &= \cos \phi/2 + i\ \sin \phi/2 \frac{\boldsymbol{i}+\boldsymbol{j}}{\sqrt{2}}.\boldsymbol{\tau} = e^{i\frac{\phi}{2}\frac{\boldsymbol{i}+\boldsymbol{j}}{\sqrt{2}}.\boldsymbol{\tau}}
\end{aligned}$$

Hence, just as van Wyk does, we generalize these to

$$V(\boldsymbol{a},\boldsymbol{b};\theta,0) = \cos \theta/2 - i\ \sin \theta/2(\boldsymbol{a}-\boldsymbol{b}).\boldsymbol{\tau} \equiv e^{-i\frac{\theta}{2}(\boldsymbol{a}-\boldsymbol{b}).\boldsymbol{\tau}}, \tag{15a}$$

$$W(\boldsymbol{a},\boldsymbol{b};\theta,0) = \cos \theta/2 - i\ \sin \theta/2(\boldsymbol{a}+\boldsymbol{b}).\boldsymbol{\tau} \equiv e^{-i\frac{\theta}{2}(\boldsymbol{a}+\boldsymbol{b}).\boldsymbol{\tau}}, \tag{15b}$$

$$V(\boldsymbol{a},\boldsymbol{b};0,\phi) = \cos \phi/2 - i\ \sin \phi/2(\boldsymbol{a}-\boldsymbol{b}).\boldsymbol{\tau} \equiv e^{-i\frac{\phi}{2}(\boldsymbol{a}-\boldsymbol{b}).\boldsymbol{\tau}}, \tag{15c}$$

$$W(\boldsymbol{a},\boldsymbol{b};0,\phi) = \cos \phi/2 + i\ \sin \phi/2(\boldsymbol{a}+\boldsymbol{b}).\boldsymbol{\tau} \equiv e^{i\frac{\phi}{2}(\boldsymbol{a}+\boldsymbol{b}).\boldsymbol{\tau}}; \tag{15d}$$

these determine $\Lambda_2(\boldsymbol{a},\boldsymbol{b};\theta,0)$, $\Lambda_2(\boldsymbol{a},\boldsymbol{b};0,\phi)$, and hence $\Lambda_2(\boldsymbol{a},\boldsymbol{b};\theta,\phi)$.

6 Applications

Following van Wyk, we now illustrate the usefulness of the spinorial representation by considering a number of examples.

I. Product of two ordinary (physical) rotations.
Let us start with two ordinary rotations.

$$R(\hat{\boldsymbol{a}},\theta) \equiv \Lambda_1(\hat{\boldsymbol{a}},0;\theta,0), \qquad R(\hat{\boldsymbol{b}},\phi) \equiv \Lambda_1(\hat{\boldsymbol{b}},0;\phi,0);$$

their product will be

$$\begin{aligned}
\Lambda_1(\hat{\boldsymbol{a}},0;\theta,0)\Lambda_1(\hat{\boldsymbol{b}},0;\phi,0) &= \Lambda_1(\boldsymbol{c},\boldsymbol{d};\xi,\eta), \quad \text{say} \\
&= \Lambda_1(\boldsymbol{c},\boldsymbol{d};\xi,0)\Lambda_1(\boldsymbol{c},\boldsymbol{d};0,\eta) \\
\Rightarrow \Lambda_2(\hat{\boldsymbol{a}},0;\theta,0)\Lambda_2(\hat{\boldsymbol{b}},0;\phi,0) &= \Lambda_2(\boldsymbol{c},\boldsymbol{d};\xi,0)\Lambda_2(\boldsymbol{c},\boldsymbol{d};0,\eta),
\end{aligned}$$

so that in terms of spinor matrices, we will have

$$\big(V(\hat{\boldsymbol{a}},0;\theta,0),W(\hat{\boldsymbol{a}},0;\theta,0)\big).\big(V(\hat{\boldsymbol{b}},0;\phi,0),W(\hat{\boldsymbol{b}},0;\phi,0)\big)=\big(V(\boldsymbol{c},\boldsymbol{d};\xi,0),W(\boldsymbol{c},\boldsymbol{d};\xi,0)\big).\big(V(\boldsymbol{c},\boldsymbol{d};0,\eta),W(\boldsymbol{c},\boldsymbol{d};0,\eta)\big)$$

$$\Rightarrow V(\hat{\boldsymbol{a}},0;\theta,0)V(\hat{\boldsymbol{b}},0;\phi,0)=V(\boldsymbol{c},\boldsymbol{d};\xi,0)V(\boldsymbol{c},\boldsymbol{d};0,\eta), \tag{16a}$$

$$W(\hat{\boldsymbol{a}},0;\theta,0)W(\hat{\boldsymbol{b}},0;\phi,0)=W(\boldsymbol{c},\boldsymbol{d};\xi,0)W(\boldsymbol{c},\boldsymbol{d};0,\eta) \tag{16b}$$

Using the expressions for V and W given by Equations (15), and separating the real and imaginary parts, Equation (16a) will give

$$\cos{}^{(\xi+\eta)}/_2=\cos{}^{\theta}/_2\cos{}^{\phi}/_2-\sin{}^{\theta}/_2\sin{}^{\phi}/_2\,(\hat{\boldsymbol{a}}.\hat{\boldsymbol{b}}),$$
$$\sin{}^{(\xi+\eta)}/_2\,(\boldsymbol{c}-\boldsymbol{d})=\sin{}^{\theta}/_2\sin{}^{\phi}/_2\,(\hat{\boldsymbol{a}}\times\hat{\boldsymbol{b}})+\cos{}^{\theta}/_2\sin{}^{\phi}/_2\,\hat{\boldsymbol{b}}+\sin{}^{\theta}/_2\cos{}^{\phi}/_2\,\hat{\boldsymbol{a}}$$

while Equation (16b) will give

$$\cos{}^{(\xi-\eta)}/_2=\cos{}^{\theta}/_2\cos{}^{\phi}/_2-\sin{}^{\theta}/_2\sin{}^{\phi}/_2\,(\hat{\boldsymbol{a}}.\hat{\boldsymbol{b}}),$$
$$\sin{}^{(\xi-\eta)}/_2\,(\boldsymbol{c}+\boldsymbol{d})=\sin{}^{\theta}/_2\sin{}^{\phi}/_2\,(\hat{\boldsymbol{a}}\times\hat{\boldsymbol{b}})+\cos{}^{\theta}/_2\sin{}^{\phi}/_2\,\hat{\boldsymbol{b}}+\sin{}^{\theta}/_2\cos{}^{\phi}/_2\,\hat{\boldsymbol{a}}$$

These equations imply that

$$\cos{}^{(\xi-\eta)}/_2=\cos{}^{(\xi+\eta)}/_2\ \Rightarrow\ \eta=0 \text{ or } \xi=0.$$

When $\eta=0$, we will have

$$\sin{}^{\xi}/_2(\boldsymbol{c}-\boldsymbol{d})=\sin{}^{\xi}/_2(\boldsymbol{c}+\boldsymbol{d})\ \Rightarrow\ \boldsymbol{d}=0 \text{ or } \boldsymbol{c}=\hat{\boldsymbol{c}}.$$

Thus

$$R(\hat{\boldsymbol{a}},0;\theta,0)R(\hat{\boldsymbol{b}},0;\phi,0)=R(\hat{\boldsymbol{c}},0;\xi,0)$$
$$\text{i.e., } R(\hat{\boldsymbol{a}},\theta)R(\hat{\boldsymbol{b}},\phi)=R(\hat{\boldsymbol{c}},\xi)$$

where

$$\cos{}^{\xi}/_2=\cos{}^{\theta}/_2\cos{}^{\phi}/_2-\sin{}^{\theta}/_2\sin{}^{\phi}/_2\,(\hat{\boldsymbol{a}}.\hat{\boldsymbol{b}}) \tag{16c}$$

$$\sin{}^{\xi}/_2\,\hat{\boldsymbol{c}}=\sin{}^{\theta}/_2\sin{}^{\phi}/_2\,(\hat{\boldsymbol{a}}\times\hat{\boldsymbol{b}})+\cos{}^{\theta}/_2\sin{}^{\phi}/_2\,\hat{\boldsymbol{b}}+\sin{}^{\theta}/_2\cos{}^{\phi}/_2\,\hat{\boldsymbol{a}} \tag{16d}$$

When $\xi=0$, one can show that one gets essentially the same result.

II. Product of two rotations in two 2-planes passing through i_4-axis.
Here, we obviously have to find the product

$$R(\hat{\boldsymbol{a}},0;0,\theta)R(\hat{\boldsymbol{b}},0;0,\phi)=R(\boldsymbol{c},\boldsymbol{d};\xi,\eta)\ \text{(say)};$$

then

$$\big(V(\hat{\boldsymbol{a}},0;0,\theta),W(\hat{\boldsymbol{a}},0;0,\theta)\big).\big(V(\hat{\boldsymbol{b}},0;0,\phi),W(\hat{\boldsymbol{b}},0;0,\phi)\big)=\big(V(\boldsymbol{c},\boldsymbol{d};\xi,0),W(\boldsymbol{c},\boldsymbol{d};\xi,0)\big).\big(V(\boldsymbol{c},\boldsymbol{d};0,\eta),W(\boldsymbol{c},\boldsymbol{d};0,\eta)\big)$$

$$\Rightarrow V(\hat{\boldsymbol{a}},0;0,\theta)V(\hat{\boldsymbol{b}},0;0,\phi)=V(\boldsymbol{c},\boldsymbol{d};\xi,0)V(\boldsymbol{c},\boldsymbol{d};0,\eta), \tag{17a}$$

$$V^{+}(\hat{\boldsymbol{a}},0;0,\theta)V^{+}(\hat{\boldsymbol{b}},0;0,\phi)=W(\boldsymbol{c},\boldsymbol{d};\xi,0)W(\boldsymbol{c},\boldsymbol{d};0,\eta) \tag{17b}$$

As before, these lead to

$$\cos{}^{(\xi+\eta)}/_2=\cos{}^{\theta}/_2\cos{}^{\phi}/_2-\sin{}^{\theta}/_2\sin{}^{\phi}/_2\,(\hat{\boldsymbol{a}}.\hat{\boldsymbol{b}}),$$
$$\sin{}^{(\xi+\eta)}/_2\,(\boldsymbol{c}-\boldsymbol{d})=\sin{}^{\theta}/_2\sin{}^{\phi}/_2\,(\hat{\boldsymbol{a}}\times\hat{\boldsymbol{b}})+\cos{}^{\theta}/_2\sin{}^{\phi}/_2\,\hat{\boldsymbol{b}}+\sin{}^{\theta}/_2\cos{}^{\phi}/_2\,\hat{\boldsymbol{a}}$$
$$\cos{}^{(\xi-\eta)}/_2=\cos{}^{\theta}/_2\cos{}^{\phi}/_2-\sin{}^{\theta}/_2\sin{}^{\phi}/_2\,(\hat{\boldsymbol{a}}.\hat{\boldsymbol{b}}),$$
$$\sin{}^{(\xi-\eta)}/_2\,(\boldsymbol{c}+\boldsymbol{d})=\sin{}^{\theta}/_2\sin{}^{\phi}/_2\,(\hat{\boldsymbol{a}}\times\hat{\boldsymbol{b}})-\cos{}^{\theta}/_2\sin{}^{\phi}/_2\,\hat{\boldsymbol{b}}-\sin{}^{\theta}/_2\cos{}^{\phi}/_2\,\hat{\boldsymbol{a}}$$

which in turn give

$$\xi=0 \tag{17c}$$

$$\cos{}^{\eta}/_2=\cos{}^{\theta}/_2\cos{}^{\phi}/_2-\sin{}^{\theta}/_2\sin{}^{\phi}/_2\,(\hat{\boldsymbol{a}}.\hat{\boldsymbol{b}}), \tag{17d}$$

$$\sin{}^{\eta}/_2\,\boldsymbol{c}=\sin{}^{\theta}/_2\,\hat{\boldsymbol{a}}+\cos{}^{\theta}/_2\sin{}^{\phi}/_2\,\hat{\boldsymbol{b}} \tag{17e}$$

$$\sin{}^{\eta}/_2\,\boldsymbol{d}=-\sin{}^{\theta}/_2\sin{}^{\phi}/_2\,(\hat{\boldsymbol{a}}\times\hat{\boldsymbol{b}}). \tag{17f}$$

Thus

$$R(\hat{\boldsymbol{a}}, 0; 0, \theta) R(\hat{\boldsymbol{b}}, 0; 0, \phi) = R(\boldsymbol{c}, \boldsymbol{d}; 0, \eta)$$

where $\boldsymbol{c}$, $\boldsymbol{d}$, and η are given by the three equations above. (Compare these with Equations (19) of van Wyk). Note that in contrast to the case of product of two ordinary rotations, the product of two rotations in 2-planes through the i_4-axis, is *not* a rotation in a 2-plane through the i_4-axis, although it is still a single rotation as $\xi = 0$. We know that $R(\hat{\boldsymbol{a}}, 0; 0, \theta)$ and $R(\hat{\boldsymbol{b}}, 0; 0, \phi)$ are rotations in the two planes

$$\left\{ \begin{bmatrix} \hat{\boldsymbol{a}} \\ 0 \end{bmatrix}, i_4 \right\}, \left\{ \begin{bmatrix} \hat{\boldsymbol{b}} \\ 0 \end{bmatrix}, i_4 \right\}$$

respectively; the above equation shows that their product is $R(\boldsymbol{c}, \boldsymbol{d}; 0, \eta)$ which is a single rotation in the 2-plane $\{A, D\}$ where

$$A = \begin{bmatrix} \hat{\boldsymbol{c}} \\ 0 \end{bmatrix}, \ \& \ D = \begin{bmatrix} \hat{\boldsymbol{c}} \times \boldsymbol{d} \\ c \end{bmatrix}$$

and which keeps invariant the 2-plane $\{B, E\}$ where

$$B = \begin{bmatrix} \hat{\boldsymbol{d}} \\ 0 \end{bmatrix}, \ \& \ E = \begin{bmatrix} \boldsymbol{c} \times \hat{\boldsymbol{d}} \\ d \end{bmatrix}$$

It turns out that expressing B and E in terms of $\boldsymbol{a}, \boldsymbol{b}, \theta, \phi$ is relatively simple and we get

$$kB = (\hat{\boldsymbol{a}} \times \hat{\boldsymbol{b}}, 0), \tag{17g}$$

$$k'E = -\left\{\cos{}^{\theta}\!/_{2} \sin{}^{\phi}\!/_{2} + \sin{}^{\theta}\!/_{2} \cos{}^{\phi}\!/_{2}\ (\hat{\boldsymbol{a}}.\hat{\boldsymbol{b}})\right\}\ \hat{\boldsymbol{a}} + \left\{\sin{}^{\theta}\!/_{2} \cos{}^{\phi}\!/_{2} + \cos{}^{\theta}\!/_{2} \sin{}^{\phi}\!/_{2}\ (\hat{\boldsymbol{a}}.\hat{\boldsymbol{b}})\right\}\ \hat{\boldsymbol{b}} \tag{17h}$$

where

$$k = -|\hat{\boldsymbol{a}} \times \hat{\boldsymbol{b}}| = -\left\{1 - (\hat{\boldsymbol{a}}.\hat{\boldsymbol{b}})^2\right\}^{1/2},$$
$$k' = k \sin{}^{\eta}\!/_{2}.$$

III. Product of an ordinary space rotation and a rotation in a 2-plane passing through i_4-axis.
Here, we have to find $\boldsymbol{c}, \boldsymbol{d}, \xi, \eta$ in terms of $\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}, \theta, \phi$ if

$$\begin{aligned} R(\hat{\boldsymbol{a}}, 0; \theta, 0) R(\hat{\boldsymbol{b}}, 0; 0, \phi) &= R(\boldsymbol{c}, \boldsymbol{d}; \xi, \eta) \\ &= R(\boldsymbol{c}, \boldsymbol{d}; \xi, 0) R(\boldsymbol{c}, \boldsymbol{d}; 0, \eta) \end{aligned}$$

$$\Rightarrow V(\hat{\boldsymbol{a}}, 0; \theta, 0) V(\hat{\boldsymbol{b}}, 0; 0, \phi) = V(\boldsymbol{c}, \boldsymbol{d}; \xi, 0) R(\boldsymbol{c}, \boldsymbol{d}; 0, \eta) \tag{18a}$$

$$V(\hat{\boldsymbol{a}}, 0; \theta, 0) V^{+}(\hat{\boldsymbol{b}}, 0; 0, \phi) = W(\boldsymbol{c}, \boldsymbol{d}; \xi, 0) W(\boldsymbol{c}, \boldsymbol{d}; 0, \eta) \tag{18b}$$

These lead to

$$\begin{aligned} \cos{}^{\xi}\!/_{2} \cos{}^{\eta}\!/_{2} &= \cos{}^{\theta}\!/_{2} \cos{}^{\phi}\!/_{2}, \\ \sin{}^{\xi}\!/_{2} \cos{}^{\eta}\!/_{2} &= \sin{}^{\theta}\!/_{2} \cos{}^{\phi}\!/_{2}\ (\hat{\boldsymbol{a}}.\hat{\boldsymbol{b}}), \\ \boldsymbol{c}\ \sin{}^{\xi}\!/_{2} \cos{}^{\eta}\!/_{2} - \boldsymbol{d}\ \cos{}^{\xi}\!/_{2} \sin{}^{\eta}\!/_{2} &= \sin{}^{\theta}\!/_{2} \cos{}^{\phi}\!/_{2}\ \hat{\boldsymbol{a}}, \\ \boldsymbol{c}\ \cos{}^{\xi}\!/_{2} \sin{}^{\eta}\!/_{2} - \boldsymbol{d}\ \sin{}^{\xi}\!/_{2} \cos{}^{\eta}\!/_{2} &= \sin{}^{\phi}\!/_{2} (\cos{}^{\theta}\!/_{2}\ \hat{\boldsymbol{b}} + \sin{}^{\theta}\!/_{2}\ \hat{\boldsymbol{a}} \times \hat{\boldsymbol{b}}). \end{aligned}$$

These correspond to Equations (22) of van Wyk. Note that as neither ξ nor η is fixed, the above equations can be inverted to give $\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}, \theta, \phi$ in terms of $\boldsymbol{c}, \boldsymbol{d}, \xi, \eta$; this means that an arbitrary rotation $R(\boldsymbol{c}, \boldsymbol{d}; \xi, \eta)$ of $SO(4)$ can always be expressed (at least, in principle) as a product of an ordinary (3-dimensional) rotation and a rotation in a plane through i_4-axis:

$$R(\boldsymbol{c}, \boldsymbol{d}; \xi, \eta) = R(\hat{\boldsymbol{a}}, 0; \theta, 0) R(\hat{\boldsymbol{b}}, 0; 0, \phi)$$

When we try to find out actual values of $\boldsymbol{c}, \boldsymbol{d}, \xi, \eta$ in terms of $\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}, \theta, \phi$, we are led to the equations

$$\sin{}^{\xi}\!/_{2} \pm \sin{}^{\eta}\!/_{2} = \left(\sin^2{}^{\theta}\!/_{2} + \sin^2{}^{\phi}\!/_{2} - \sin^2{}^{\theta}\!/_{2} \sin^2{}^{\phi}\!/_{2} \sin^2 \alpha \pm 2 \sin{}^{\theta}\!/_{2} \sin{}^{\phi}\!/_{2} \cos \alpha\right)^{1/2}$$

In principle, these determine $\sin {}^{\xi}/_2$ and $\sin {}^{\eta}/_2$ explicitly, but the expressions which involve sum and difference of square roots, will be quite messy and will lead to even more messy expressions for $\boldsymbol{c}, \boldsymbol{d}$. As these expressions do not give us any additional insight into the situation, we leave the above four (04) equations as they are do not try to solve them for $\boldsymbol{c}, \boldsymbol{d}, \xi, \eta$. However, it turns out that the inverse problem of solving them for $\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}, \theta, \phi$ in terms of $\boldsymbol{c}, \boldsymbol{d}, \xi, \eta$ is more promising and leads to reasonably simpler expressions; this means that we are able to explicitly express an arbitrary rotation $R(\boldsymbol{c}, \boldsymbol{d}; \xi, \eta)$ of $SO(4)$ as a product of an ordinary (space) rotation $R(\hat{\boldsymbol{a}}, \theta) \equiv R(\hat{\boldsymbol{a}}, 0; \theta, 0)$ and a rotation $R(\hat{\boldsymbol{b}}, 0; 0, \phi)$ in a 2-plane through the i_4-axis. When we actually carry out this inversion, we find that

$$\cos {}^{\phi}/_2 = \left(c^2 \cos^2 {}^{\eta}/_2 + d^2 \cos^2 {}^{\xi}/_2\right)^{1/2}, \tag{18c}$$

$$\cos {}^{\theta}/_2 = \frac{\cos {}^{\xi}/_2 \cos {}^{\eta}/_2}{\cos {}^{\phi}/_2}, \tag{18d}$$

$$\hat{\boldsymbol{a}} = \frac{\boldsymbol{c} \tan {}^{\xi}/_2 - \boldsymbol{d} \tan {}^{\eta}/_2}{\left(c^2 \tan^2 {}^{\xi}/_2 + d^2 \tan^2 {}^{\eta}/_2\right)^{1/2}}, \tag{18e}$$

$$\hat{\boldsymbol{b}}\ \sin {}^{\phi}/_2 \cos {}^{\phi}/_2 = \sin {}^{\eta}/_2 \cos {}^{\eta}/_2\, \boldsymbol{c} - \sin {}^{\xi}/_2 \cos {}^{\xi}/_2\, \boldsymbol{d} - \left(\cos^2 {}^{\xi}/_2 - \cos^2 {}^{\eta}/_2\right) (\boldsymbol{c} \times \boldsymbol{d}). \tag{18f}$$

IV. Equivalence of $R(\boldsymbol{a}, \boldsymbol{b}; \theta, 0)$ to an ordinary space rotation and of $R(\boldsymbol{a}, \boldsymbol{b}; 0, \theta)$ to a rotation in a 2-plane containing i_4-axis.

We now show that there exist similarity transformations by rotations in 2-planes containing i_4-axis, which transform $R(\boldsymbol{a}, \boldsymbol{b}; \theta, 0)$ to $R(\hat{\boldsymbol{n}}, 0; \theta, 0)$, and $R(\boldsymbol{a}, \boldsymbol{b}; 0, \theta)$ to $R(\hat{\boldsymbol{n}}, 0; 0, \theta)$. We do this by finding one set of $\hat{\boldsymbol{u}}, \hat{\boldsymbol{n}}$ and ψ in terms of $\boldsymbol{a}, \boldsymbol{b}, \theta$ such that

$$R^{-1}(\hat{\boldsymbol{u}}, 0; 0, \psi) R(\hat{\boldsymbol{n}}, 0; \theta, 0) R(\hat{\boldsymbol{u}}, 0; 0, \psi) = R(\boldsymbol{a}, \boldsymbol{b}; \theta, 0),$$

and another set such that

$$R^{-1}(\hat{\boldsymbol{u}}, 0; 0, \psi) R(\hat{\boldsymbol{n}}, 0; 0, \theta) R(\hat{\boldsymbol{u}}, 0; 0, \psi) = R(\boldsymbol{a}, \boldsymbol{b}; 0, \theta).$$

In terms of spinors these take the form

$$V(\boldsymbol{a}, \boldsymbol{b}; \theta, 0) = V(\hat{\boldsymbol{u}}, 0; 0, -\psi) V(\hat{\boldsymbol{n}}, 0; \theta, 0) V(\hat{\boldsymbol{u}}, 0; 0, \psi) \tag{19a}$$

$$W(\boldsymbol{a}, \boldsymbol{b}; \theta, 0) = W(\hat{\boldsymbol{u}}, 0; 0, -\psi) W(\hat{\boldsymbol{n}}, 0; \theta, 0) W(\hat{\boldsymbol{u}}, 0; 0, \psi), \tag{19b}$$

and

$$V(\boldsymbol{a}, \boldsymbol{b}; 0, \theta) = V(\hat{\boldsymbol{u}}, 0; 0, -\psi) V(\hat{\boldsymbol{n}}, 0; 0, \theta) V(\hat{\boldsymbol{u}}, 0; 0, \psi) \tag{19c}$$

$$W(\boldsymbol{a}, \boldsymbol{b}; 0, \theta) = W(\hat{\boldsymbol{u}}, 0; 0, -\psi) W(\hat{\boldsymbol{n}}, 0; 0, \theta) W(\hat{\boldsymbol{u}}, 0; 0, \psi), \tag{19d}$$

Equation (19a) gives

$$\cos {}^{\theta}/_2 - i \sin {}^{\theta}/_2\, (\boldsymbol{a} - \boldsymbol{b}).\tau = (\cos {}^{\psi}/_2 + i \sin {}^{\psi}/_2\, \hat{\boldsymbol{u}}.\tau) (\cos {}^{\theta}/_2 - i \sin {}^{\theta}/_2\, \hat{\boldsymbol{n}}.\tau) (\cos {}^{\psi}/_2 - i \sin {}^{\psi}/_2\, \hat{\boldsymbol{u}}.\tau),$$

which after some straight forward calculations, leads to

$$\boldsymbol{a} - \boldsymbol{b} = -2 \sin {}^{\psi}/_2 \cos {}^{\psi}/_2\, \hat{\boldsymbol{u}} \times \hat{\boldsymbol{n}} - \left(\sin^2 {}^{\psi}/_2 - \cos^2 {}^{\psi}/_2\right)\, \hat{\boldsymbol{n}} + 2 \sin^2 {}^{\psi}/_2\, (\hat{\boldsymbol{n}}.\hat{\boldsymbol{u}}) \hat{\boldsymbol{u}}.$$

Equation (19b) similarly leads to

$$\boldsymbol{a} + \boldsymbol{b} = 2 \sin {}^{\psi}/_2 \cos {}^{\psi}/_2\, \hat{\boldsymbol{u}} \times \hat{\boldsymbol{n}} - 9 \left(\sin^2 {}^{\psi}/_2 - \cos^2 {}^{\psi}/_2\right)\, \hat{\boldsymbol{n}} + 2 \sin^2 {}^{\psi}/_2\, (\hat{\boldsymbol{n}}.\hat{\boldsymbol{u}}) \hat{\boldsymbol{u}}.$$

and so, we get

$$\boldsymbol{a} = \left(\cos^2 {}^{\psi}/_2 - \sin^2 {}^{\psi}/_2\right)\, \hat{\boldsymbol{n}} + 2 \sin^2 {}^{\psi}/_2\, (\hat{\boldsymbol{n}}.\hat{\boldsymbol{u}}) \tag{20a}$$

$$\boldsymbol{b} = -2 \cos {}^{\psi}/_2 \sin {}^{\psi}/_2\, (\hat{\boldsymbol{n}} \times \hat{\boldsymbol{u}}). \tag{20b}$$

When we consider Equations (19c), (19d), they again lead to the same pair of equations i.e., we have the important result that the similarity transformation which takes $R(\boldsymbol{a}, \boldsymbol{b}; 0, \theta)$ to $R(\hat{\boldsymbol{n}}, 0; 0, \theta)$ is the same as the one which takes $R(\boldsymbol{a}, \boldsymbol{b}; \theta, 0)$ to $R(\hat{\boldsymbol{n}}, 0; \theta, 0)$. Now this pair of equations corresponds to Equations (28), pp-1300, of van Wyk. As our aim is to find $\hat{\boldsymbol{n}}, \hat{\boldsymbol{u}}, \psi$ in terms of $\boldsymbol{a}, \boldsymbol{b}, \theta$, we must invert the above equations. Although van Wyk does not attempt to carry out this inversion for his Equations (28), saying that it is a formidable nonlinear problem, recently, Amir and Rashid [4] have been able to carry out this inversion; we therefore closely follow their method and invert our Equations (20a) (20b). The first point to be noted is that as $\boldsymbol{a}, \boldsymbol{b}$ satisfy the two relations, viz, $\boldsymbol{a}^2 + \boldsymbol{b}^2 = 1$, and $\boldsymbol{a}.\boldsymbol{b} = 0$, they will have only four (04) independent components, so that (20a), (20b) is a system of only four (04) independent equations. As the two unit vectors $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{u}}$ also have four (04) independent components, the four (04) equations will determine these two unit vectors so that ψ will remain undetermined. This means that *psi* acts as a parameter in the sense that corresponding to each value of ψ, there will be a similarity transformation which converts $R(\boldsymbol{a}, \boldsymbol{b}; \theta, 0)$ into $R(\hat{\boldsymbol{n}}, 0; \theta, 0)$ and $R(\boldsymbol{a}, \boldsymbol{b}; 0, \theta)$ into $R(\hat{\boldsymbol{n}}, 0; 0, \theta)$.

The trick used by Amir and Rashid is to convert the nonlinear system (28) of van Wyk, in $\hat{\boldsymbol{v}}, \hat{\boldsymbol{n}}$, into a linear system by some simple manipulations. Let us do the same for our system (20a), (20b). Equation (20b) gives

$$\boldsymbol{b}^2 = 4\cos^2 \psi/2 \sin^2 \psi/2\, |\hat{\boldsymbol{n}} \times \hat{\boldsymbol{u}}|^2 = 4\cos^2 \psi/2 \sin^2 \psi/2 \left\{1 - (\hat{\boldsymbol{n}}.\hat{\boldsymbol{u}})^2\right\}$$

$$\Rightarrow\ (\hat{\boldsymbol{n}}.\hat{\boldsymbol{u}})^2 = 1 - \frac{\boldsymbol{b}^2}{4\cos^2 \psi/2 \sin^2 \psi/2} \tag{21}$$

While taking the cross product of the two equations, we get

$$\begin{aligned}
\frac{\boldsymbol{a} \times \boldsymbol{b}}{2\sin \psi/2 \cos \psi/2} &= \left(\cos^2 \psi/2 - \sin^2 \psi/2\right)\, \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \hat{\boldsymbol{u}}) + 2\sin^2 \psi/2\, (\hat{\boldsymbol{n}}.\hat{\boldsymbol{u}})\hat{\boldsymbol{u}} \times (\hat{\boldsymbol{n}} \times \hat{\boldsymbol{u}}) \\
&= \left(\cos^2 \psi/2 - \sin^2 \psi/2\right) \{(\hat{\boldsymbol{n}}.\hat{\boldsymbol{u}})\hat{\boldsymbol{n}} - \hat{\boldsymbol{u}}\} + 2\sin^2 \psi/2 (\hat{\boldsymbol{n}}.\hat{\boldsymbol{u}})\, \{\hat{\boldsymbol{n}} - (\hat{\boldsymbol{u}}.\hat{\boldsymbol{n}})\hat{\boldsymbol{u}}\} \\
&= (\hat{\boldsymbol{n}}.\hat{\boldsymbol{u}})\hat{\boldsymbol{n}} - \left\{\left(\cos^2 \psi/2 - \sin^2 \psi/2\right) + 2\sin^2 \psi/2 (\hat{\boldsymbol{n}}.\hat{\boldsymbol{u}})^2\right\} \hat{\boldsymbol{u}} \\
&= (\hat{\boldsymbol{n}}.\hat{\boldsymbol{u}})\hat{\boldsymbol{n}} - \left\{1 - \frac{\boldsymbol{b}^2}{2\cos^2 \psi/2}\right\} \hat{\boldsymbol{u}},
\end{aligned}$$

using Equation (21). Thus we get a linear system in $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{u}}$

$$\left(1 - 2\sin^2 \psi/2\right) \hat{\boldsymbol{n}} + 2\sin^2 \psi/2 (\hat{\boldsymbol{n}}.\hat{\boldsymbol{u}})\hat{\boldsymbol{u}} = \boldsymbol{a} \tag{22a}$$

$$(\hat{\boldsymbol{n}}.\hat{\boldsymbol{u}})\hat{\boldsymbol{n}} - \left(1 - \frac{\boldsymbol{b}^2}{2\cos^2 \psi/2} \hat{\boldsymbol{u}}\right) = \frac{\boldsymbol{a} \times \boldsymbol{b}}{2\sin \psi/2 \cos \psi/2}, \tag{22b}$$

this means that the system of Equations (20a) (20b), which is nonlinear when expressed in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$, becomes linear when expressed in terms of $\boldsymbol{a}$ and $\boldsymbol{a} \times \boldsymbol{b}$.

The determinant of coefficients is

$$\begin{vmatrix} 1 - 2\sin^2 \psi/2 & 2\sin^2 \psi/2 (\hat{\boldsymbol{n}}.\hat{\boldsymbol{u}}) \\ \hat{\boldsymbol{n}}.\hat{\boldsymbol{u}} & -\left(1 - \frac{\boldsymbol{b}^2}{2\cos^2 \psi/2}\right) \end{vmatrix}$$

which simplifies to $-\boldsymbol{a}^2$, so that the solutions for $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{u}}$ will be

$$\hat{\boldsymbol{n}} = \frac{1}{\boldsymbol{a}^2} \left\{\left(1 - \frac{\boldsymbol{b}^2}{2\sin^2 \psi/2}\right) \boldsymbol{a} + \frac{\sin \psi/2}{\cos \psi/2} (\hat{\boldsymbol{n}}.\hat{\boldsymbol{u}})(\boldsymbol{a} \times \boldsymbol{b})\right\} \tag{23a}$$

$$\hat{\boldsymbol{u}} = \frac{1}{\boldsymbol{a}^2} \left\{(\hat{\boldsymbol{n}}.\hat{\boldsymbol{u}})\boldsymbol{a} - \frac{1 - 2\sin^2 \psi/2}{2\sin \psi/2 \cos \psi/2} (\boldsymbol{a} \times \boldsymbol{b})\right\} \tag{23b}$$

where $\hat{\boldsymbol{n}}.\hat{\boldsymbol{u}}$ is given by Equation (21).

Consider now an arbitrary $R(\boldsymbol{a}, \boldsymbol{b}; \theta, \phi)$; we will have

$$\begin{aligned}
R^{-1}(\hat{\boldsymbol{u}}, 0; 0, \psi) R(\boldsymbol{a}, \boldsymbol{b}; \theta, \phi) R(\hat{\boldsymbol{u}}, 0; 0, \psi) &= R^{-1}(\hat{\boldsymbol{u}}, 0; 0, \psi) R(\boldsymbol{a}, \boldsymbol{b}; \theta, 0) R(\hat{\boldsymbol{u}}, 0; 0, \psi) R^{-1}(\hat{\boldsymbol{u}}, 0; 0, \psi) R(\boldsymbol{a}, \boldsymbol{b}; 0, \phi) R(\hat{\boldsymbol{u}}, 0; 0, \psi) \\
&= R(\hat{\boldsymbol{n}}, 0; \theta, 0) R(\hat{\boldsymbol{n}}, 0; 0, \phi)
\end{aligned}$$

i.e., there exists a rotation in a 2-plane containing i_4-axis which transforms, by similarity transformation, an arbitrary element $R(\boldsymbol{a}, \boldsymbol{b}; \theta, \phi) \in SO(4)$ into a product of a pure (ordinary) rotation by an angle θ and a pure rotation by an angle ϕ in a 2-plane containing i_4-axis.

7 Conclusion

We have proved in this paper that the theory developed by van Wyk for the representation of Lorentz transformation in terms of a 4×4 antisymmetric matrix determined by 2 ordinary (3-dimensional) vectors satisfying pair of relations, and its use to obtain an elegant spinorial representation of these transformations by 2×2 matrices, which are obviously much easier to deal with than the 4×4 Lorentz transformation matrices, can be extended in toto, to the 4-dimensional rotations of $SO(4)$. In the process, we are able to obtain explicitly, the matrix elements of a rotation in a 2-plane which passes through the i_4-axis. In addition, we are also able to extend the abstractly known fact that an element of $SO(4)$ consists of a pair of rotations in a pair of mutually orthogonal 2-planes, by trivially obtaining the angles of rotation and the configuration of the 2-planes in which these rotations take place. Finally, we have obtained a concrete 2–1 homomorphism of $SU(2)\times SU(2)$ onto $SO(4)$. As a future plan, we hope to extend the whole theory to the non-compact group $SO(2,2)$.

Appendix A

In this appendix, we obtain an explicit expression for the matrix elements of $R((\boldsymbol{a},0),i_4)$ where $\boldsymbol{a}$ is a unit 3-vector. If the vectors $l,m \in \mathbb{R}^4$ are orthogonal and normalized, a rotation R in the $(1,m)$-plane by an angle θ will take l,m to

$$\begin{aligned} Rl &= l\cos\theta + m\sin\theta, \\ Rm &= -l\sin\theta + m\cos\theta. \end{aligned}$$

Writing an arbitrary vector x as

$$x = \Big\{x - (x.l)l - (x.m)m + (x.l)l + (x.m)m\Big\},$$

we note that

$$x - (x.l)l - (x.m)m$$

is orthogonal to both l and m and hence to the (l,m)-plane, and so, R will leave it unchanged. Hence, we will get

$$\begin{aligned} Rx &= x - (x.l)l - (x.m)m + (x.l)Rl + (x.m)Rm \\ &= x - (x.l)l - (x.m)m + (x.l)(l\cos\theta + m\sin\theta) + (x.m)(-l\sin\theta + m\cos\theta) \\ &= x - \Big\{(x.l)(1-\cos\theta) + (x.m)\sin\theta\Big\}l - \Big\{(x.m) - (x.l)\sin\theta - (x.m)\cos\theta\Big\}m \\ \Rightarrow\ R_{\mu\nu}x_\nu &= \delta_{\mu\nu} - \Big\{x_\nu l_\nu(1-\cos\theta) + x_\nu\, m_\nu \sin\theta\Big\}l_\mu - \Big\{x_\nu\, m_\nu - x_\nu\, l_\nu \sin\theta - x_\nu\, m_\nu \cos\theta\Big\}n \\ \Rightarrow\ R_{\mu\nu} &= \delta_{\mu\nu} - \Big\{l_\nu(1-\cos\theta) + m_\nu \sin\theta\Big\}l_\mu - \Big\{m_\nu - l_\nu \sin\theta - m_\nu\cos\theta\Big\}m_\mu \\ \Rightarrow\ R_{\mu\nu} &= \delta_{\mu\nu} - (1-\cos\theta)(l_\mu l_\nu + m_\mu m_\nu) + (-l_\mu m_\nu + l_\nu m_\mu)\sin\theta \end{aligned}$$

Choose now

$$m = (0,0,0,1) \quad \Rightarrow l = (a_1,a_2,a_3,0), \quad \boldsymbol{a} \text{ a unit 3-vector}$$

so that R becomes rotation $R((\boldsymbol{a},0),i_4)$, and we get

$$\begin{aligned} R_{ij} &= \delta_{ij} - (1-\cos\theta)a_i a_j, \\ R_{i4} &= -a_i\sin\theta, \qquad R_{4i} = a_i\sin\theta, \\ R_{44} &= \cos\theta, \end{aligned}$$

as the required matrix elements.

Appendix B

In this appendix, we construct a concrete 2-1 homomorphism of $SU(2) \times SU(2)$ onto $SO(4)$. Several steps are needed for this construction, which we discuss one by one.

I. The matrices σ_μ and ρ_ν:

We start with the Pauli matrices which we denote by τ_i, so that

$$\tau_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tau_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \tag{B.1a}$$

these satisfy

$$\tau_i \text{ are Hermitian }, \quad \tau_i^2 = 1, \quad \text{Tr } \tau_i = 0. \tag{B.1b}$$

We set

$$\sigma_i = i\tau_i, \quad \Leftrightarrow \quad \tau_i = -i\sigma_i,$$

so that

$$\sigma_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \tag{B.2a}$$

and these satisfy

$$\tau_i \text{ are unitary }, \quad \sigma_i^2 = -1, \quad \sigma_i^+ = -\sigma_i, \text{Tr } \sigma_i = 0, \tag{B.2b}$$

$$\sigma_i\sigma_j = -\delta_{ij} - \epsilon_{ijk}\sigma_k, \tag{B.3a}$$
$$\sigma_i\sigma_j \pm \sigma_j\sigma_i = -2\delta_{ij}, \quad -2\epsilon_{ijk}\sigma_k, \tag{B.3b}$$
$$\sigma_i\sigma_j\sigma_k = -\delta_{ij}\sigma_k + \delta_{ik}\sigma_j - \delta_{jk}\sigma_i + \epsilon_{ijk}, \tag{B.3c}$$
$$\text{Tr } (\sigma_i\sigma_j) = -2\delta_{ij}, \tag{B.3d}$$
$$\text{Tr } (\sigma_i\sigma_j\sigma_k) = 2\epsilon_{ijk}. \tag{B.3e}$$

Set now

$$\sigma_4 = e \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$\sigma_\mu \equiv (\sigma_1, \sigma_2, \sigma_3, \sigma_4) \equiv (\sigma_i, \sigma_4) \equiv (\boldsymbol{\sigma}, \sigma_4),$$

and then define

$$\rho_\mu = \zeta(\sigma_\mu^T)\zeta^{-1} = (-\boldsymbol{\sigma}, \sigma_4) \equiv \sigma_\mu^+, \tag{B.4}$$

where

$$\zeta = i\sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \Rightarrow \zeta^{-1} = \zeta,$$

so that

$$\zeta^\times = -\zeta, \quad (\zeta^{-1})^\times = -\zeta^{-1}, \quad \zeta^T = -\zeta, \quad (\zeta^{-1})^T = -\zeta^{-1}.$$

It is easy to check that

$$\mathrm{Tr}\,(\sigma_\mu \rho_\nu) = 2\delta_{\mu\nu}, \tag{B.5a}$$

$$\sigma_\mu \rho_\nu = \delta_{\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}\sigma_\kappa \rho_\lambda, \tag{B.5b}$$

$$\rho_\mu \sigma_\nu = \delta_{\mu\nu} - \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}\rho_\kappa \sigma_\lambda \tag{B.5c}$$

$$\sigma_\mu \rho_\nu \pm \sigma_\nu \rho_\mu = 2\delta_{\mu\nu}, \quad \epsilon_{\mu\nu\kappa\lambda}\sigma_\kappa \rho_\lambda, \tag{B.5d}$$

$$\rho_\mu \sigma_\nu \pm \rho_\nu \sigma_\mu = 2\delta_{\mu\nu}, \quad \epsilon_{\mu\nu\kappa\lambda}\rho_\kappa \sigma_\lambda, \tag{B.5e}$$

$$\sigma_\mu \rho_\nu \sigma_\kappa = \delta_{\mu\nu}\sigma_\kappa - \delta_{\mu\kappa}\sigma_\nu + \delta_{\nu\kappa}\sigma_\mu + \epsilon_{\mu\nu\kappa\lambda}\sigma_\lambda, \tag{B.5f}$$

$$\rho_\mu \sigma_\nu \rho_\kappa = \delta_{\mu\nu}\rho_\kappa - \delta_{\mu\kappa}\rho_\nu + \delta_{\nu\kappa}\rho_\mu + \epsilon_{\mu\nu\kappa\lambda}\rho_\lambda, \tag{B.5g}$$

$$\begin{aligned}\sigma_\mu \rho_\nu \sigma_\kappa \rho_\lambda = {} & \delta_{\mu\nu}\delta_{\kappa\lambda} - \delta_{\mu\kappa}\delta_{\nu\lambda} + \delta_{\mu\lambda}\delta_{\nu\kappa} - \frac{1}{2}(\delta_{\mu\nu}\epsilon_{\kappa\lambda\alpha\beta} - \delta_{\mu\kappa}\epsilon_{\nu\lambda\alpha\beta} + \delta_{\mu\lambda}\epsilon_{\nu\kappa\alpha\beta} + \delta_{\nu\kappa}\epsilon_{\mu\lambda\alpha\beta} - \delta_{\nu\lambda}\epsilon_{\mu\kappa\alpha\beta} \\ & + \delta_{\kappa\lambda}\epsilon_{\mu\nu\alpha\beta})\sigma_\alpha \rho_\beta + \epsilon_{\mu\nu\kappa\lambda},\end{aligned} \tag{B.5h}$$

$$\begin{aligned}\rho_\mu \sigma_\nu \rho_\kappa \sigma_\lambda = {} & \delta_{\mu\nu}\delta_{\kappa\lambda} - \delta_{\mu\kappa}\delta_{\nu\lambda} + \delta_{\mu\lambda}\delta_{\nu\kappa} + \frac{1}{2}(\delta_{\mu\nu}\epsilon_{\kappa\lambda\alpha\beta} - \delta_{\mu\kappa}\epsilon_{\nu\lambda\alpha\beta} + \delta_{\mu\lambda}\epsilon_{\nu\kappa\alpha\beta} + \delta_{\nu\kappa}\epsilon_{\mu\lambda\alpha\beta} - \delta_{\nu\lambda}\epsilon_{\mu\kappa\alpha\beta} \\ & + \delta_{\kappa\lambda}\epsilon_{\mu\nu\alpha\beta})\rho_\alpha \sigma_\beta - \epsilon_{\mu\nu\kappa\lambda},\end{aligned} \tag{B.5i}$$

$$(\sigma_\mu)_{ab}\ (\rho_\mu)_{cd} = (\rho_\mu)_{ab}\ (\sigma_\mu)_{cd} = 2\ \delta_{ad}\ \delta_{bc}. \tag{B.5j}$$

II. A concrete homomorphism of $SU(2) \times SU(2)$ onto $SO(4)$.

For any nonzero $x \equiv x_\mu \in \mathbb{R}^4$, we set

$$X = x_\mu \sigma_\mu \equiv \begin{bmatrix} ix_3 + x_4 & ix_1 + x_2 \\ ix_1 - x_2 & -ix_3 + x_4 \end{bmatrix};$$

and X is of the form

$$\begin{bmatrix} a & b \\ -b^\times & a^\times \end{bmatrix}$$

so that

$$\frac{X}{\sqrt{(\det X)}} \in SU(2)$$

If we set

$$\widetilde{X} = \begin{bmatrix} X_{11} \\ X_{12} \\ X_{21} \\ X_{22} \end{bmatrix},$$

then

$$\widetilde{X} = Ax \qquad \text{where } A = \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & i & 1 & 0 \\ 0 & i & -1 & 0 \\ 1 & 0 & 0 & -i \end{bmatrix}$$

But then A^{-1} exists and is given by

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -i & -i & 0 \\ 0 & 1 & -1 & 0 \\ -i & 0 & 0 & i \end{bmatrix}$$

so that

$$x = A^{-1}\widetilde{X}$$

Note that if Y is an arbitrary nonzero 2×2 matrix such that $\det Y$ is real and $\det Y > 0$, and

$$\frac{Y}{\sqrt{\det Y}} \in SU(2)$$

then there exist a real $\{y_\mu\}$ such that $Y = y_\mu \sigma_\mu$. For

$$\frac{Y}{\sqrt{\det Y}} \in SU(2) \;\Rightarrow\; \frac{Y}{\sqrt{\det Y}} = \begin{bmatrix} p & q \\ -q^\times & p^\times \end{bmatrix} \quad \text{with } |p|^2 + |q|^2 = 1,$$

$$\Rightarrow \;\; Y = \begin{bmatrix} a & b \\ -b^\times & a^\times \end{bmatrix}, \quad a = p\sqrt{\det Y}, \quad b = q\sqrt{\det Y},$$

$$= \begin{bmatrix} iy_3 + y_4 & iy_1 + y_2 \\ iy_1 - y_2 & -iy_3 + y_4 \end{bmatrix} = y_\mu \sigma_\mu.$$

Given any pair of elements $V, W \in SU(2)$, we define a mapping $R \equiv R(V, W)$ which takes

$$X \to \widehat{R}X \equiv X' = VXW^+. \tag{B.6}$$

Clearly

$$\det X' = \det V. \det X. \det W^+ = \det X > 0$$

so that

$$\frac{X'}{\det X'} = V\frac{X}{\det X}W^+ \in SU(2).$$

$\Rightarrow$ there exist real x'_μ such that

$$X' = x'_\mu \sigma_\mu \quad \Rightarrow x' = A^{-1}\widetilde{X}'.$$

Now (B.6) gives

$$\widetilde{X}' = B\widetilde{X}.$$

where

$$B = \begin{bmatrix} V_{11}W^+_{11} & V_{11}W^+_{21} & V_{12}W^+_{11} & V_{12}W^+_{12} \\ V_{11}W^+_{12} & V_{11}W^+_{22} & V_{12}W^+_{12} & V_{12}W^+_{22} \\ V_{21}W^+_{11} & V_{21}W^+_{21} & V_{22}W^+_{11} & V_{22}W^+_{21} \\ V_{21}W^+_{12} & V_{21}W^+_{22} & V_{22}W^+_{12} & V_{22}W^+_{22} \end{bmatrix}$$

so that

$$x' = A^{-1}BAx$$

$$\textit{i.e.,}\; x' = Rx, \quad R = A^{-1}BA.$$

Although we started with nonzero $x \in \mathbb{R}^4$ so that this equation has been proved only for such x, but as it is trivially true for $x = 0$ also, we conclude that it is valid for all $R \in \mathbb{R}^4 \;\Rightarrow\; R \in L(4, \mathbb{C})$. Let us obtain the properties of R:

i) R is real. For

$$x'_\mu = (Rx)_\mu = R_{\mu\lambda}x_\lambda;$$

take now $x = i_\nu$ so that $x_\lambda = \delta_{\lambda\nu}$ and we get

$$x'_\mu = R_{\mu\lambda}\delta_{\lambda\nu} = R_{\mu\nu} \;\Rightarrow\; R_{\mu\nu} \text{ is real, as asserted.}$$

ii) $\det R = 1$ as

$$\det R = \det A^{-1} \det B \det A = \det B = 1$$

as can be verified by evaluating $\det B$ with the help of Laplace Theorem.

iii) $\det X' = \det X$.

$$\Rightarrow\ x_1'^2 + x_2'^2 + x_3'^2 + x_4'^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

i.e., R preserves the length of vectors of $\mathbb{R}^4$. (i)–(iii) obviously $\Rightarrow R \in SO(4)$. Thus corresponding to each pair (V, W) of elements of $SU(2)$, we have defined an element $R \equiv R(V, W) \in SO(4)$. We now show that this correspondence is actually a (group) homomorphism. For if

$$(V_1, W_1), (V_2, W_2) \in SU(2) \times SU(2)$$

and

$$X' = V_1 X W_1^+, \quad X'' = V_2 X' W_2^+,$$

then

$$X'' = V_2 V_1 X W_1^+ W_2^+ = (V_2 V_1) X (W_2 W_1)^+$$
$$\Rightarrow x'' = R(V_2 V_1, W_2 W_1) x.$$

But we also have

$$x'' = R(V_2, W_2) x' = R(V_2, W_2) R(V_1, W_1) x$$

so that

$$R\{(V_2, W_2)(V_1, W_1)\} = R(V_2, V_1, W_2 W_1) = R(V_2, W_2) R(V_1, W_1)$$
$\Rightarrow$ the correspondence
$$(V, W) \leftrightarrow R(V, W)$$

is indeed a homomorphism.

III. Relations between R and V, W.
It turns out that the expression for R in terms of V and W is obtained rather easily, but the inversion of this relation i.e., obtaining V and W in terms of R, is found to be quite complicated. So we start with the simpler problem. We have

$$x' = Rx \ \Rightarrow\ x'_\mu = R_{\mu\nu} x_\nu$$
$$\Rightarrow\ R_{\mu\nu} x_\nu \sigma_\mu = x'_\mu \sigma_\mu = X' = V X W^+ = V x_\nu \sigma_\nu W^+$$
$$\Rightarrow\ R_{\mu\nu} \sigma_\mu = V \sigma_\nu W^+ \tag{B.7}$$
$$\Rightarrow\ R_{\mu\nu} \sigma_\mu \rho_\lambda = V \sigma_\nu W^+ \rho_\lambda$$
$$\mathrm{Tr}\,(V \sigma_\nu W^+ \rho_\lambda) = R_{\mu\nu}\,\mathrm{Tr}\,(\sigma_\mu \rho_\lambda = 2 R_{\mu\nu} \lambda_{\mu\lambda}$$
$$\Rightarrow\ R_{\mu\nu} = \frac{1}{2}\,\mathrm{Tr}\,(V \sigma_\nu W^+ \rho_\mu), \tag{B.8}$$

which is the required expression for R in terms of V and W. In order to invert this equation, we write (B.7) as

$$\sigma_\nu = R_{\mu\nu} V^{-1} \sigma_\lambda (W^+)^{-1}$$

and this leads to

$$R_{\mu\nu} = V^{-1} \sigma_\mu (W^+)^{-1} \tag{B.9}$$

Now (with $a, b = 1, 2$)

$$(V \sigma_\nu W^+ \rho_\nu)_{ab} = V_{ac} (\sigma_\nu)_{cd} (W^+)_{de} (\rho_\nu)_{eb} = 2 V_{ac} \delta_{cb} \delta_{de} (W^+)_{de} = 2 V_{ab} (W^+)_{dd} = 2\,\mathrm{Tr}\,(W+) V_{ab}$$
$$\Rightarrow V \sigma_\nu W^+ \rho_{nu} = 2\,\mathrm{Tr}\,(W^+) V,$$

so that (B.7) gives

$$R_{\mu\nu} \sigma_\mu \rho_\nu = 2\,\mathrm{Tr}\,(W^+) V. \tag{B.10a}$$

Similarly, (B.9) leads to

$$R_{\mu\nu}\sigma_\nu\rho_\mu = 2\,\mathrm{Tr}\,(W^+)^{-1}V^{-1}, \tag{B.10b}$$

and so, we get

$$R_{\mu\nu}R_{\kappa\lambda}\sigma_\mu\rho_\nu\sigma_\lambda\rho_\kappa = 4\,\mathrm{Tr}\;W^+\,\mathrm{Tr}\,(W+)^{-1} = 4(\mathrm{Tr}\;W^+)^2$$
$$\text{as Tr } W^+ = \mathrm{Tr}\,(W^+)^{-1} \quad \text{for } W^+ \in SU(2).$$

Thus

$$\mathrm{Tr}\;W^+ = \pm\frac{1}{2}(R_{\mu\nu}R_{\kappa\lambda}\sigma_\mu\rho_\nu\sigma_\lambda\rho_\kappa)^{1/2},$$

and so (B.10) gives

$$\pm V = \frac{R_{\mu\nu}\sigma_\mu\rho_\nu}{(R_{\mu\nu}R_{\kappa\lambda}\sigma_\mu\rho_\nu\sigma_\lambda\rho_\kappa)^{1/2}} \tag{B.11a}$$

Similarly, multiplying (B.7) and (B.9) by ρ_ν and ρ_μ respectively, on the left, we will be led to

$$\pm W^+ = \frac{R_{\mu\nu}\rho_\mu\sigma_\nu}{(R_{\mu\nu}R_{\kappa\lambda}\rho_\mu\sigma_\nu\rho_\lambda\sigma_\kappa)^{1/2}} \tag{B.11b}$$

These are the required inverses of (B.8).

Certain alternative expressions for V and W, which are more useful practically, are obtained as follows. We have

$$R_{\mu\nu}\sigma_\mu\rho_\nu = R_{\mu\mu} + R_{4k}\rho_k + R_{k4} - \rho_k + R_{ij}\sigma_i\rho_j, \quad i \neq j,$$

so that as

$$R_{ij}\sigma_i\rho_j = -R_{ij}\sigma_i\sigma_j = R_{ij}\epsilon_{ijk}\sigma_k = -R_{ij}\epsilon_{ijk}\rho_k,$$

we get

$$R_{\mu\nu}\sigma_\mu\rho_\nu = \mathrm{Tr}\;R + (R_{4k} - R_{k4} - R_{ij}\epsilon_{ijk})\rho_k. \tag{B.12a}$$

On the other hand, equation (B.5h) gives

$$\begin{aligned}
&R_{\mu\nu}R_{\kappa\lambda}\sigma_\mu\rho_\nu\sigma_\lambda\rho_\kappa\\
&= R_{\mu\nu}R_{\kappa\lambda}(\delta_{\mu\nu}\delta_{\lambda\kappa} - \delta_{\mu\lambda}\delta_{\nu\kappa} + \delta_{\mu\kappa}\delta_{\nu\lambda}) - \frac{1}{2}R_{\mu\nu}R_{\kappa\lambda}(\delta_{\mu\nu}\epsilon_{\lambda\kappa\alpha\beta} - \delta_{\mu\lambda}\epsilon_{\nu\kappa\alpha\beta} + \delta_{\mu\kappa}\epsilon_{\nu\lambda\alpha\beta} + \delta_{\nu\lambda}\epsilon_{\mu\kappa\alpha\beta} - \delta_{\nu\kappa}\\
&\epsilon_{\mu\lambda\alpha\beta} + \delta_{\lambda\kappa}\epsilon_{\mu\nu\alpha\beta})\sigma_\alpha\rho_\beta + R_{\mu\nu}R_{\kappa\lambda}\epsilon_{\mu\nu\lambda\kappa}\\
&= R_{\mu\mu}R_{\kappa\kappa} - R_{\mu\nu}R_{\nu\mu} + R_{\mu\nu}R_{\mu\nu} + R_{\mu\nu}R_{\kappa\lambda}\epsilon_{\mu\nu\lambda\kappa} - \frac{1}{2}(R_{\mu\mu}R_{\kappa\lambda}\epsilon_{\lambda\kappa\alpha\beta} - R_{\mu\nu}R_{\kappa\mu}\epsilon_{\nu\kappa\alpha\beta} + R_{\mu\nu}R_{\mu\lambda}\epsilon_{\nu\lambda\alpha\beta} + R_{\mu\nu}R_{\kappa\nu}\epsilon_{\mu\kappa\alpha\beta}\\
&- R_{\mu\nu}R_{\nu\lambda}\epsilon_{\mu\lambda\alpha\beta} + R_{\mu\nu}R_{\kappa\kappa}\epsilon_{\mu\nu\alpha\beta})\sigma_\alpha\rho_\beta\\
&= (Tr\;R)^2 - Tr\;R^2 + R_{\mu\nu}R_{\mu\nu} + R_{\mu\nu}R_{\kappa\lambda}\epsilon_{\mu\nu\lambda\kappa} - \frac{1}{2}[(Tr\;R)R_{\kappa\lambda}\epsilon_{\lambda\kappa\alpha\beta}\sigma_\alpha\rho_\beta - (R^2)_{\kappa\nu}\epsilon_{\nu\kappa\alpha\beta}\sigma_\alpha\rho_\beta - (R^2)_{\mu\lambda}\epsilon_{\mu\lambda\alpha\beta}\sigma_\alpha\rho_\beta\\
&+ (Tr\;R)R_{\mu\nu}\epsilon_{\mu\nu\alpha\beta}\sigma_\alpha\rho_\beta]\\
&= (Tr\;R)^2 - Tr\;R^2 + 4 + R_{\mu\nu}R_{\kappa\lambda}\epsilon_{\mu\nu\lambda\kappa}
\end{aligned}$$

as

$$R_{\mu\nu}R_{\mu\nu} = R_{\mu\nu}R_{\mu\kappa}\delta_{\nu\kappa} = \delta_{\nu\kappa}\delta_{\nu\kappa} = \delta_{\nu\nu} = 4.$$

Also

$$\begin{aligned}
&R_{\mu\nu}R_{\kappa\lambda}\epsilon_{\mu\nu\lambda\kappa}\\
&= R_{4k}R_{ij}\epsilon_{4kji} + R_{k4}R_{ij}\epsilon_{k4ji} + R_{ij}R_{k4}\epsilon_{ij4k} + R_{ij}R_{4k}\epsilon_{ijk4}\\
&= (R_{ij}R_{4k} - R_{ij}R_{k4} - R_{ij}R_{k4} + R_{ij}R_{4k})\epsilon_{ijk}\\
&= 2(R_{4k} - R_{k4})R_{ij}\epsilon_{ijk}.
\end{aligned}$$

It follows that

$$\pm V = \frac{Tr\,R + (R_{4k} - R_{k4} - R_{ij}\epsilon_{ijk})\rho_k}{[4 + (Tr\,R)^2 - TR\,R^2 + R_{\mu\nu}R_{\kappa\lambda}\epsilon_{\mu\nu\lambda\kappa}]^{\frac{1}{2}}} \tag{B.13a}$$

$$= \frac{Tr\,R + (R_{4k} - R_{k4} - R_{ij}\epsilon_{ijk})\rho_k}{[4 + (Tr\,R)^2 - Tr\,R^2 + 2(R_{4k} - R_{k4})R_{ij}\epsilon_{ijk}]^{\frac{1}{2}}}. \tag{B.13b}$$

It can similarly be proved that

$$\pm W^+ = \frac{\text{Tr}\ R + (R_{4k} - R_{k4} + R_{ij}\epsilon_{ijk})\rho_k}{\left\{(\text{Tr}\ R)^2 + 4 - \text{Tr}\ R^2 - \epsilon_{\mu\nu\kappa\lambda}R_{\mu\nu}R_{\kappa\lambda}\right\}^{1/2}} \tag{B.14a}$$

$$= \frac{\text{Tr}\ R + (R_{4k} - R_{k4} + R_{ij}\epsilon_{ijk})}{\left\{(\text{Tr}\ R)^2 + 4 - \text{Tr}\ R^2 - 2(R_{4k} - R_{k4})R_{ijk}\epsilon_{ijk}\right\}^{1/2}} \tag{B.14b}$$

as required.

References

[1] C. B. van Wyk. General Lorentz transformations and applications. *Journal of Mathematical Physics*, 27:1306–1310, 1986.

[2] M. Carmeli. Representation of the three dimensional rotation group in terms of direction and angle of rotation. *Journal of Mathematical Physics*, 9:1987–1994, 1968.

[3] A. J. Macfarlane. On restricted Lorentz group and groups homomorphically related to it. *Journal of Mathematical Physics*, 3:1116–1129, 1962.

[4] N. Amir and M. A. Rashid. General Lorentz transformations and applications. *VFAST Transactions on Mathematics*, 3(1):01–08, 2014.