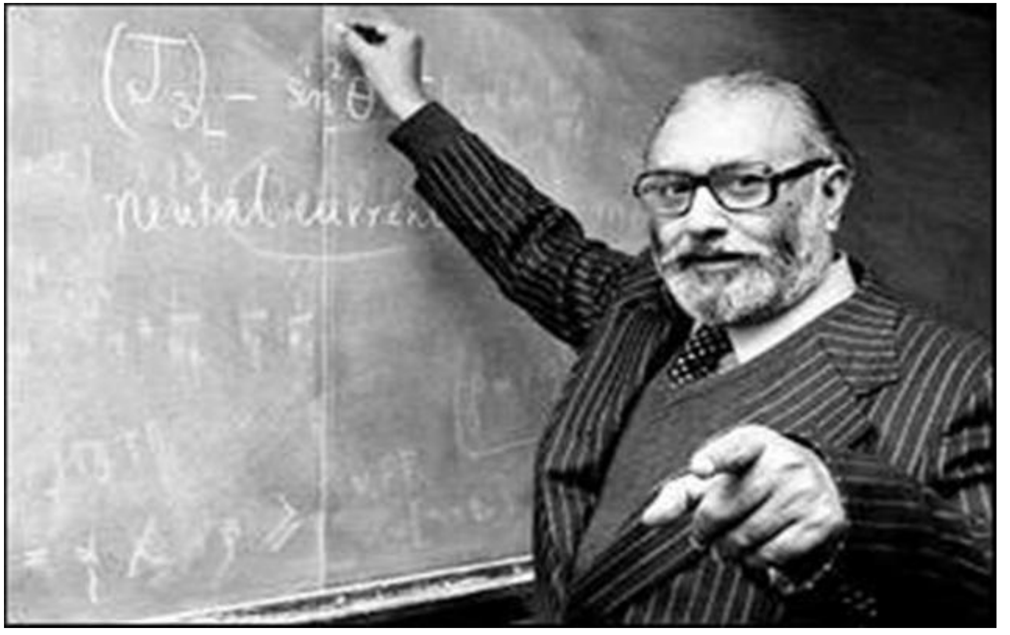
MODULI OF FLAT BUNDLES ON COMPACT RIEMANN SURFACES

M.S. Narasimhan
IISc & TIFR, Bangalore, India

ABDS90, Singapore

Abdus Salam & Mathematics



- I am very thankful to have an opportunity to pay my tributes to Prof. Abdus Salam.
- In particular to acknowledge his support for advancement of mathematics in several developing countries by creating Mathematics section in ICTP.
- Grateful for unreserved support he gave for efforts to build up a high level mathematics group at ICTP.

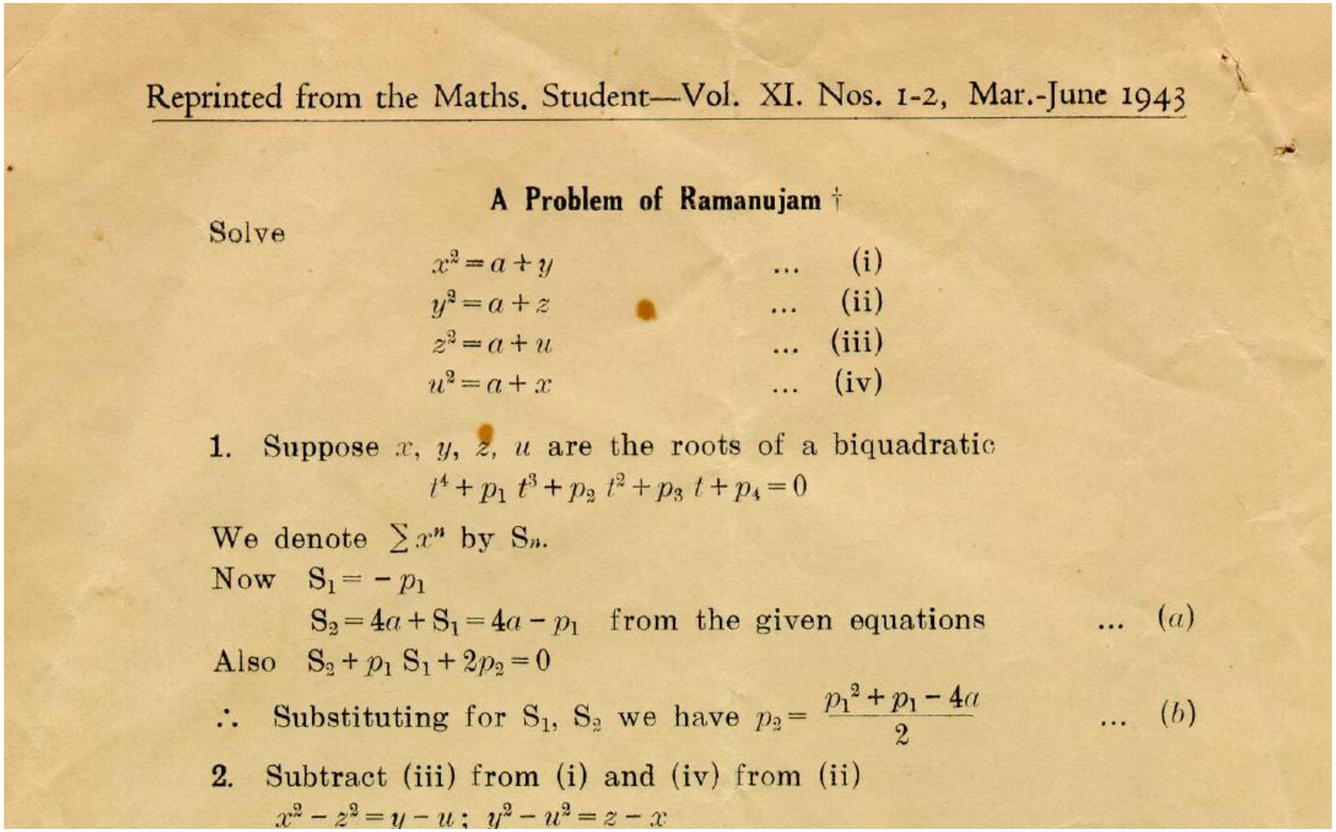
Abdus Salam: Early Interest in Mathematics

Government College, Lahore

A young Abdus Salam

Salam seems to have been interested in mathematics from a very young age.

Salam’s First Paper



Reprinted from the Maths. Student—Vol. XI. Nos. 1-2, Mar.-June 1943

A Problem of Ramanujam †

Solve

$$x^2 = a + y \quad \dots \quad (i)$$

$$y^2 = a + z \quad \dots \quad (ii)$$

$$z^2 = a + u \quad \dots \quad (iii)$$

$$u^2 = a + x \quad \dots \quad (iv)$$

1. Suppose x, y, z, u are the roots of a biquadratic

$$t^4 + p_1 t^3 + p_2 t^2 + p_3 t + p_4 = 0$$

We denote $\sum x^n$ by S_n.

Now $S_1 = -p_1$

$S_2 = 4a + S_1 = 4a - p_1$ from the given equations ... (a)

Also $S_2 + p_1 S_1 + 2p_2 = 0$

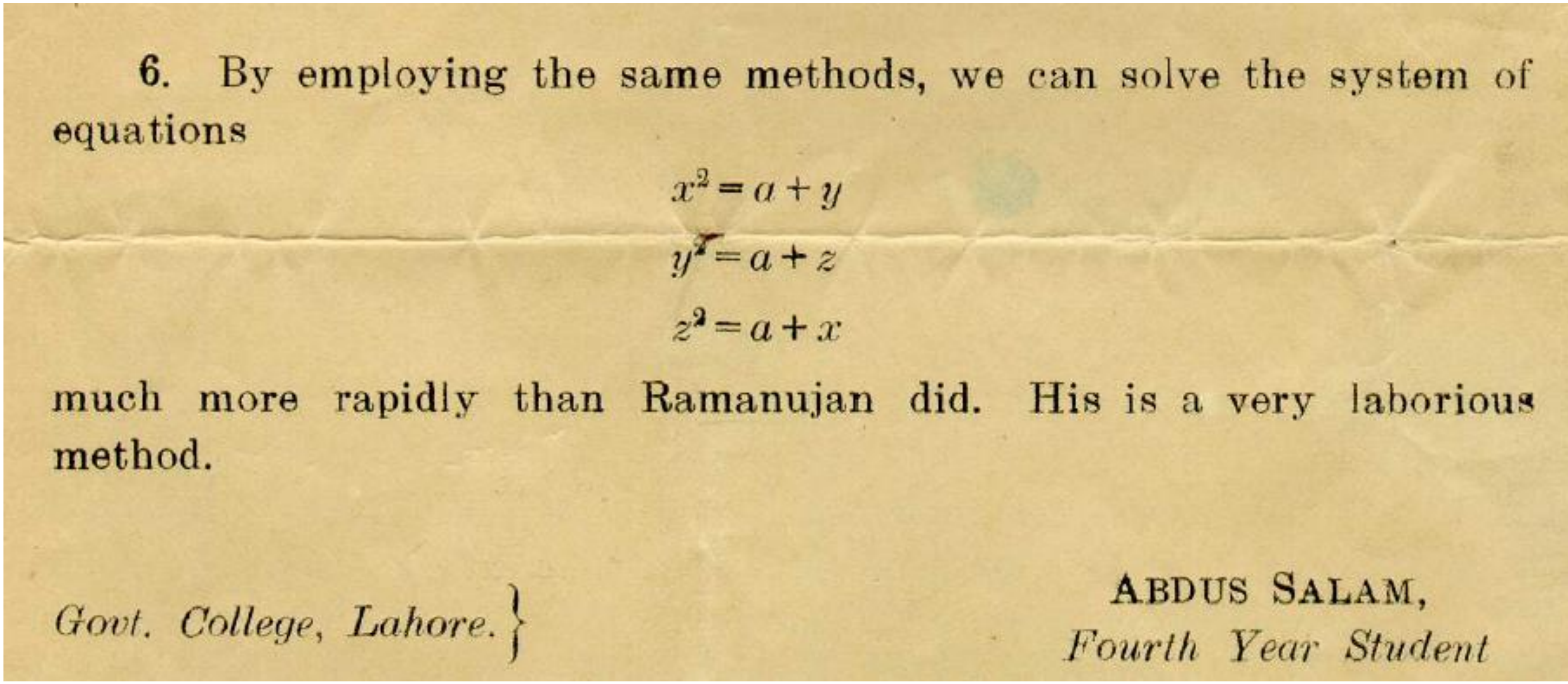
$\therefore$ Substituting for S_1, S_2 we have $p_2 = \dfrac{p_1^2 + p_1 - 4a}{2}$ ... (b)

2. Subtract (iii) from (i) and (iv) from (ii)

$$x^2 - z^2 = y - u\,; \quad y^2 - u^2 = z - x$$

Salam’s first paper is in mathematics, published in 1943 when he was a fourth year student in Lahore, and he was then just 17 years old. The title of the paper is " A problem of Ramanujam" and published in the journal "Mathematics Student".

Salam’s First Paper



6. By employing the same methods, we can solve the system of equations

$$x^2 = a + y$$
$$y^2 = a + z$$
$$z^2 = a + x$$

much more rapidly than Ramanujan did. His is a very laborious method.

Govt. College, Lahore.

ABDUS SALAM,
Fourth Year Student

It seems that Salam had already the confident personality that we know of : at the end of the paper he remarks: "By employing the same methods we can solve [a system of equations] much more rapidly than Ramanujan did. His is a very laborious method."

St. John’s College
Cambridge

He obtained his MA in Mathematics from Lahore, and double first class honors in physics and mathematics from St. John’s College, Cambridge.

Theoretical Physics & Mathematics

Bust of the mathematician Ramanujan in the ICTP Library

Salam retained his interest in mathematics as he became a theoretical physicist, and realized it was important for physicists to interact with mathematicians, and therefore established the Mathematics Section at ICTP.

Mathematics & Physics

"The interaction between mathematics and physics is a two-way process, with each of the two subjects drawing from and inspiring the other." [1]

Will now speak about such an interaction between some parts of algebraic & differential geometry on one side, and gauge theory and conformal field theory on the other.

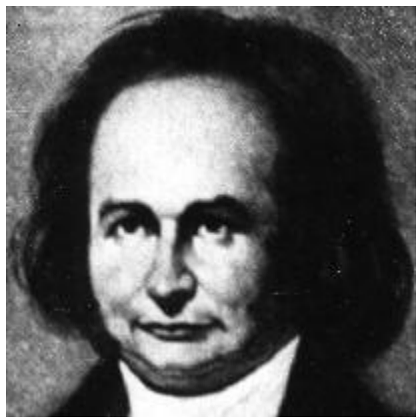
Also related to number theory, Langlands program.

1. E. Frenkel

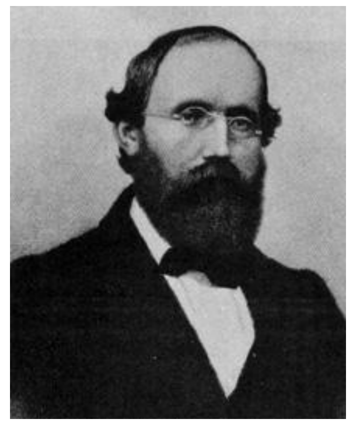
Flat Unitary Bundles on a Compact Riemann Surface

Vast “non-abelian” generalization of work of Abel, Jacobi & Riemann on a compact Riemann surface, envisaged by Andre Weil.

Abel



Jacobi



Riemann

Weil

Classical theory:

Constructed Jacobian, a complex torus, using periods of abelian differentials.

Character group of the first homology group of the surface.

Non-abelian Generalization

- Consider unitary representations of the fundamental group of the surface (instead of the character group).
- These give rise to flat (holomorphic) bundles on the Riemann surface.
- This space of n-dimensional unitary flat bundles $M(n)$, generalizes the Jacobian. (For $n=1$ we get the character group of the homology group).
- M is a compact topological space and any such representation gives a holomorphic vector bundle (of Chern class zero) on the Riemann surface; such a bundle is called a flat unitary bundle.
- Using the complex structure on the Riemann surface we can put a complex structure (or even an algebraic structure) on $M(n)$.
- This algebraic variety will be called the moduli space of flat bundles. ($M(1)$ is the Jacobian.)

Stable Bundles & Unitary Flat Bundles

- To construct the algebraic structure one has to have an algebraic characterisation of unitary flat bundles (which are "transcendental objects“).

- This was given by a theorem by Seshadri and myself.

DEFINITION: A holomorphic vector bundle of Chern class zero on a compact Riemann surface is said to be stable if the Chern class of every proper holomorphic subbundle has strictly negative Chern class. (Semistablity is defined by replacing strictly negative by ≤ 0 (Informally every subbundle is less positive than the original bundle).

THEOREM: A vector bundle of degree zero is stable if and only if it is a flat unitary bundle arising from an irreducible unitary representation.

Projectively Flat Bundles; Higgs Bundles

• We can also define semistable and stable bundles with arbitrary Chern class and moduli space of projectively flat unitary bundles can be constructed.

• There is a vast literature studying properties of these moduli spaces.

•If we look at representations into $GL(n,C)$ (instead of $U(n)$),it was discovered by Hitchin that the corresponding algebro-geometric counterpart is a "stable" pair (E,f) where E is a holomorphic vector bundle and f is "Higgs Field".

•These Higgs moduli spaces have been found to be very useful in number theory.

Gauge Theory

- 2-d gauge theory and the associated solutions of Yang-Mills equations (with the unitary group as gauge group) are closely related to the moduli of bundles on Riemann surfaces.

- Atiyah and Bott made a detailed study of Yang-Mills on Riemann surfaces and were instrumental in popularising moduli of stable vector bundles on Riemann surfaces among physicists.

Conformal Field Theory

• Just as classical theta functions are holomorphic sections of line bundles on the Jacobian, it is reasonable to expect a theory of generalised "non-abelian" theta functions, which would be sections of line bundles on moduli spaces of bundles.

• Such a theory was developed by algebraic geometers.

• It turns out that these generalised theta functions are the same as conformal blocks defined by physicists using representations of Kac-Moody algebras.

• The famous Verlinde formula for the dimension of conformal blocks ,yields the dimension of linear systems on the moduli space of flat bundles.

Derived Categories, Stability & Branes

- At present I am studying derived categories (of coherent sheaves) on projective varieties.

- Physicists are interested in these derived categories and stabilty conditions in them, apparently since "this category could be obtained as a category of boundary conditions in the B-type topologically twisted sigma model on the variety“.

- Be that as it may ,the derived category of coherent sheaves is an interesting object for mathematicians .

- Let me end with a recent theorem of mine:
The bounded derived category of a compact Riemann surface of genus ≥ 4 can be embedded in that of moduli space of rank 2 bundles with fixed determinant of odd degree.

- (In fact the Fourier-Mukai transform
defined by a Poincare bundle gives a fully faithful embedding).