

# MAJORANA FERMIONS IN CONDENSED-MATTER PHYSICS

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based in part on joint work with Yiruo Lin

Memorial meeting for Nobel Laureate Professor  
Abdus Salam's 90<sup>th</sup> birthday

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Reminder:

Majorana fermions (M.F.'s) in particle physics (E. Majorana, 1937)

An M.F. is a fermionic particle which is **its own antiparticle**:

$$\psi(\mathbf{r}) \equiv \psi^\dagger(\mathbf{r}) \Rightarrow \{\psi(\mathbf{r}), \psi(\mathbf{r}')\} = \delta(\mathbf{r} - \mathbf{r}')$$

Such a particle must be **massless**.

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How can such a particle arise in condensed matter physics?  
Example: “topological superconductor”.

Mean-field theory of superconductivity (general case):

Consider general Hamiltonian of form  $\hat{H}_0 + \hat{V}$  where

$$\hat{H}_0 \equiv \frac{\hbar^2}{2m} \int \nabla \psi^\dagger(\mathbf{r}) \cdot \nabla \psi(\mathbf{r}) d\mathbf{r} + \sum_{\alpha\beta} U_{\alpha\beta}(\mathbf{r}) \psi_\alpha^\dagger(\mathbf{r}) \psi_\beta(\mathbf{r})$$

$$\hat{V} \equiv \frac{1}{2} \sum_{\alpha\beta\gamma\delta} \iint d\mathbf{r} d\mathbf{r}' V_{\alpha\beta\gamma\delta}(\mathbf{r}, \mathbf{r}') \psi_\alpha^\dagger(\mathbf{r}) \psi_\beta^\dagger(\mathbf{r}') \psi_\gamma(\mathbf{r}') \psi_\delta(\mathbf{r})$$

Introduce notion of **spontaneously broken U(1) symmetry**  $\Rightarrow$  particle number not conserved  $\Rightarrow$  (even-parity) GS of form “SBU(1)S”

$$\Psi_{(\text{even})} = \sum_{N=\text{even}} C_N \Psi_N$$

$\Rightarrow$  quantities such as  $\langle \psi_\alpha(r) \psi_\beta(r) \rangle$  can legitimately be nonzero.

Thus

$\hat{H} \rightarrow \hat{H}_0 + \hat{V}_{mf}$  where (apart from Hartree-Fock terms) mean-field

$$\hat{V}_{mf} \equiv \sum_{\alpha\beta} \iint dr dr' \Delta_{\alpha\beta}(\mathbf{r}, \mathbf{r}') \overbrace{\psi_\alpha^\dagger(\mathbf{r}) \psi_\beta^\dagger(\mathbf{r}')}^{\text{operator}} + H.C.$$

$$\Delta_{\alpha\beta}(\mathbf{r}, \mathbf{r}') \equiv \sum_{\gamma\delta} V_{\alpha\beta\gamma\delta}(\mathbf{r}, \mathbf{r}') \langle \psi_\gamma(\mathbf{r}') \psi_\delta(\mathbf{r}) \rangle \leftarrow \text{c-number}$$

[in BCS case, reduces to

$$\hat{V}_{mf} = \int dr \Delta(r) \psi_\uparrow^\dagger(r) \psi_\downarrow^\dagger(r) + H.C., \quad \Delta(r) \equiv V_0 \langle \psi_\downarrow(r) \psi_\uparrow(r) \rangle]$$

↙ Bogoliubov-de Gennes

Thus, mean-field (BdG) Hamiltonian is schematically of form

$$\hat{H}_{mf} = \sum_{\alpha\beta} \left\{ \underbrace{\int dr K_{\alpha\beta}(r) \psi_{\alpha}^{\dagger}(r) \psi_{\beta}(r) + \frac{1}{2} \iint dr dr' \Delta_{\alpha\beta}(\mathbf{r}, \mathbf{r}') \psi_{\alpha}^{\dagger}(r) \psi_{\beta}^{\dagger}(r')} + HC \right\}$$

bilinear in  $\psi_{\alpha}(r), \psi_{\alpha}^{\dagger}(r)$ :

(with a term  $\mu\delta_{\alpha\beta}$  included in  $K_{\alpha\beta}(r)$  to fix average particle number  $\langle \hat{N} \rangle$ .)  $\hat{H}_{mf}$  does not conserve particle number, but does conserve particle number **parity**, so consider even parity. (Then can minimize  $\hat{H}_{mf}$  to find even-parity CS, but) in our context, interesting problem is to find simplest fermionic (odd-parity) states (“Bogoliubov quasiparticles”). For this purpose write schematically (ignoring (real) spin degree of freedom)

$$\hat{\gamma}_i^{\dagger} = \int \{u_i(r) \hat{\psi}^{\dagger}(r) + v_i(r) \hat{\psi}(r)\} dr \quad \left( \equiv \begin{pmatrix} u(r) \\ v(r) \end{pmatrix} \right) \leftarrow \text{“Nambu spinor”}$$

and determine the coefficients  $u_i(r), v_i(r)$  by solving the Bogoliubov-de Gennes equations

$$[\hat{H}_{mf}, \hat{\gamma}_i^{\dagger}] = E_i \hat{\gamma}_i^{\dagger}$$

so that

$$\hat{H}_{mf} = \sum_i E_i \gamma_i^{\dagger} \gamma_i + \text{const.}$$

(All this is standard textbook stuff...)

Note crucial point: In mean-field treatment, fermionic quasiparticles are **quantum superpositions of particle and hole**  $\Rightarrow$  do not correspond to definite particle number (justified by appeal to SBU(1)S). This “particle-hole mixing” is sometimes (misleadingly) regarded as analogous to the mixing of different bands in an insulator by spin-orbit coupling. (hence, analogy “topological insulator”  $\Leftrightarrow$  topological superconductor.)



## Majoranas

Recap: fermionic (Bogoliubov) quasiparticles created by operators

$$\gamma_i^\dagger = \int dr \{u_i(r)\psi^\dagger(r) + v_i(r)\psi(r)\}$$

with the coefficients  $u_i(r), v_i(r)$  given by solution of the BdG equations

$$[\hat{H}_{mf}, \gamma_i^\dagger] = E_i \gamma_i$$

Question: Do there exist solutions of the BdG equations such that

$$\gamma_i^\dagger = \gamma_i \quad (\text{and thus } E_i = 0)?$$

This requires (at least)

1. Spin structure of  $u(r), v(r)$  the same  $\Rightarrow$  pairing of parallel spins (spinless or spin triplet, not BCS s-wave)
2.  $u(r) = v^*(r)$
3. “interesting” structure of  $\Delta_{\alpha\beta}(\mathbf{r}, \mathbf{r}')$ , e.g. “ $p + ip$ ” ( $\Delta(\mathbf{r}, \mathbf{r}') \equiv \Delta(\mathbf{R}, p) \sim \Delta(R)(p_x + ip_y)$ )

Case of particular interest: “half-quantum vortices” (HQV’s) in  $\text{Sr}_2\text{RuO}_4$  (widely believed to be  $(p + ip)$  superconductor). In this case a M.F. predicted to occur in (say)  $\uparrow\uparrow$  component, (which sustains vortex), not in  $\downarrow\downarrow$  (which does not). Not that vortices always come in pairs (or second MF solution exists on boundary)

Why the special interest for topological quantum computing?

- (1) Because MF is exactly equal superposition of particle and hole, it should be **undetectable by any local probe**.
- (2) MF’s should behave under braiding as **Ising anyons\***: if 2 HQV’s, each carrying a M.F., interchanged, phase of MBWF changed by  $\pi/2$  (note not  $\pi$  as for real fermions!)

So in principle‡:

- (1) create pairs of HQV’s with and without MF’s
  - (2) braid adiabatically
  - (3) recombine and “measure” result
- ⇓
- (partially) topologically protected quantum computer!

\* D. A. Ivanov, PRL **86**, 268 (2001)

‡ Stone & Chung, Phys. Rev. B **73**, 014505 (2006)

## Comments on Majarama fermions (within the standard “mean-field” approach)

(1) What is a M.F. anyway?

Recall: it has energy exactly zero, that is its creation operator  $\gamma_i^\dagger$  satisfies the equation

$$[H, \gamma_i^\dagger] = 0$$

But this equation has two possible interpretations:

- (a)  $\gamma_i^\dagger$  creates a fermionic quasiparticle with exactly zero energy (i.e. the odd- and even-number-parity GS's are exactly degenerate)
- (b)  $\gamma_i^\dagger$  annihilates the (even-parity) groundstate (“pure annihilator”)

However, it is easy to show that in neither case do we have  $\gamma_i^\dagger = \gamma_i$ . To get this we must superpose the cases (a) and (b), i.e.

a Majorana fermion is simply a quantum superposition of a real Bogoliubov quasiparticle and a pure annihilator.

But Majorana solutions always come in pairs  $\Rightarrow$  by superposing two MF's we can make a **real zero-energy fermionic quasiparticle**

HQV1



$\gamma_1^\dagger$

HQV2



$\gamma_2^\dagger$

Bog. qp.  $\longrightarrow$   $\alpha^\dagger \equiv \gamma_1^\dagger + i\gamma_2^\dagger$

The curious point: the extra fermion is “split” between two regions which may be **arbitrarily far apart!** (hence, usefulness for TQC)

Thus, e.g. interchange of 2 vortices each carrying an MF  $\sim$  rotation of zero-energy fermion by  $\pi$ . (note predicted behavior (phase change of  $\pi/2$ ) is “average” of usual symmetric (0) and antisymmetric ( $\pi$ ) states)

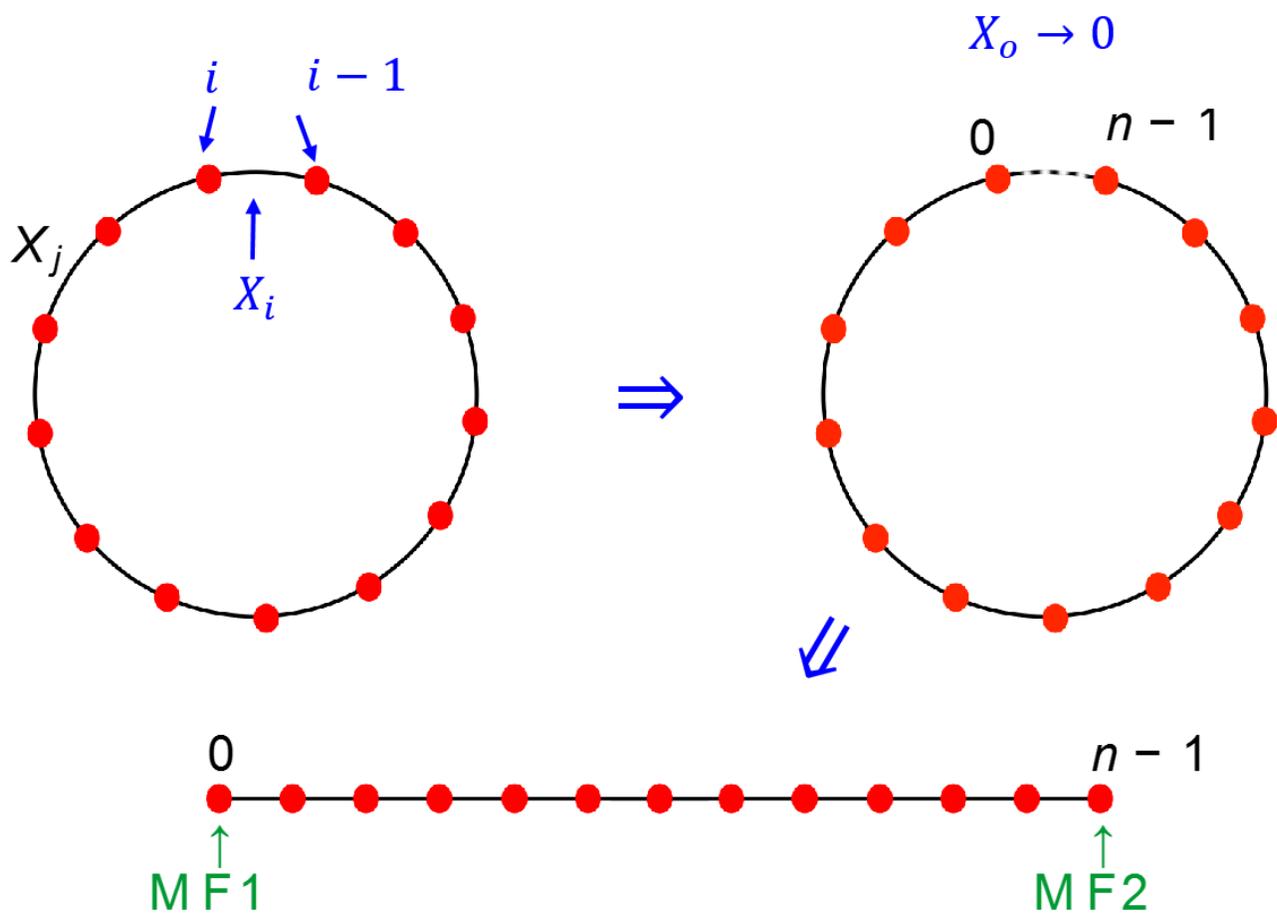
An intuitive way of generating MF's in the KQW:

Kitaev quantum wire 

For this problem, fermionic excitations have form

$$\alpha_i^\dagger = (a_i^\dagger + ia_i) + (a_{i-1}^\dagger + ia_{i-1})$$

so localized on links not sites. Energy for link  $(i, i - 1)$  is  $X_i$



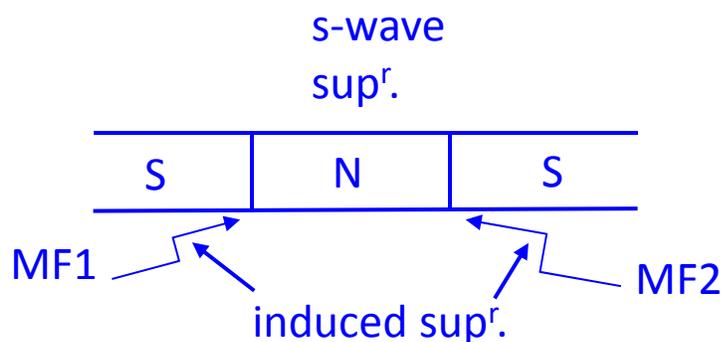
## Comments on M.F.'s (within standard mean-field approach) (cont.)

### (2) The experimental situation

$\text{Sr}_2\text{RuO}_4$ : so far, evidence for HQV's, none for MF's.

$^3\text{He-B}$ : circumstantial evidence from ultrasound attenuation

Alternative proposed setup (very schematic)



← zero-bias anomaly

Detection: ZBA in I-V characteristics

(Mourik et al., 2012, and several subsequent experiments)

dependence on magnetic field, s-wave gap, temperature...  
roughly right

“What else could it be?”

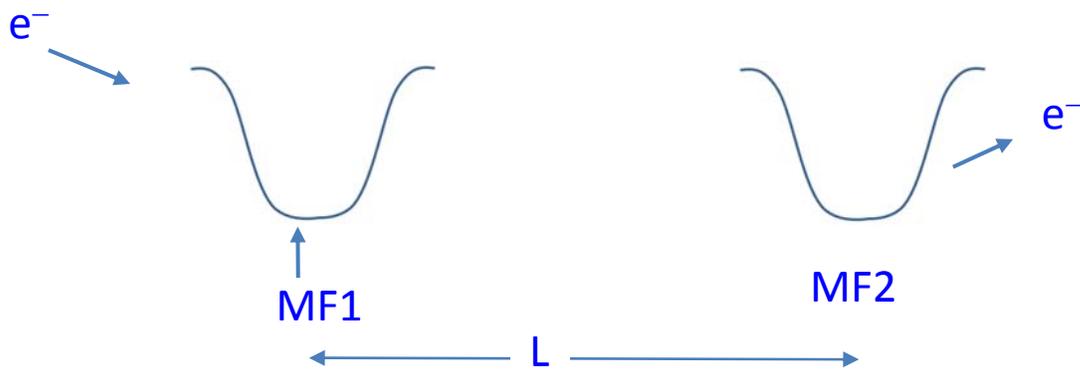
Answer: quite a few things!

Second possibility: Josephson circuit involving induced (p-wave-like) sup $\gamma$ .

Theoretical prediction: “ $4\pi$ -periodicity” in current-phase relation.

Problem: parasitic one-particle effects can mimic.

One possible smoking gun: teleportation!



$$\Delta T \ll L/v_F ?$$

↙ Fermi velocity

Problem: theorists can't agree on whether teleportation is for real!

## Majorana fermions: beyond the mean-field approach

Problem: The whole apparatus of mean-field theory rests fundamentally on the notion of SBU(1)S ← spontaneously broken U(1) gauge symmetry:

$$\Psi_{\text{even}} \sim \sum_{\substack{N= \\ \text{even}}} C_N \Psi_N \quad (C_N \sim |C_N| e^{iN\varphi})$$

$$\Psi_{\text{odd}}^{(c)} \sim \int dr \{u(r) \hat{\psi}^\dagger(r) + v(r) \hat{\psi}(r)\} |\Psi_{\text{even}}\rangle \quad (\equiv \hat{\gamma}_i^\dagger |\Psi_{\text{even}}\rangle)^*$$

But in real life condensed-matter physics,

**SB U(1)S IS A MYTH!!**

This doesn't matter for the even-parity GS, because of "Anderson trick":

$$\Psi_{2N} \sim \int \Psi_{\text{even}}(\varphi) \exp -iN\varphi d\varphi$$

But for odd-parity states equation ( \* ) is fatal! Examples:

(1) Galilean invariance

(2) NMR of surface MF in  $^3\text{He-B}$

We must replace ( \* ) by

$$\hat{\gamma}_i^\dagger = \int dr \{u(r)\hat{\psi}^\dagger(r) + v(r)\hat{\psi}c^\dagger\}$$

creates extra Cooper pairs  
↓

This doesn't matter, so long as Cooper pairs have no "interesting" properties (momentum, angular momentum, partial localization...)

But to generate MF's, pairs **must** have "interesting" properties!

⇒ doesn't change arguments about existence of MF's, but **completely changes arguments** about their braiding, undetectability etc.

Need completely new approach!