Disorder and Interference: localization phenomena

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Exercise sheet

1 One-dimensional systems

Exercise 1 – Consider an elementary impurity with $V(x) = u_0\delta(x)$. Solve the Schrödinger equation $-\psi'' + (2m/\hbar^2)V\psi = k^2\psi$ at fixed k (use the continuity of the free wave function and compute its derivative discontinuity at $x = 0$). Determine the reflection and transmission coefficients and show that the S-Matrix reads

$$\mathsf{S} = \frac{1}{1+if}\begin{pmatrix} -if & 1 \\ 1 & -if \end{pmatrix} \tag{1}$$

with $f = mu_0/\hbar^2 k$.

Exercise 2 – Check the following interesting properties of the transfer matrix:

(*i*) $\det \mathsf{M} = 1$.

(*ii*) Current conservation (unitarity of S) implies now that $\mathsf{M}\sigma_z\mathsf{M}^\dagger = \sigma_z$, with σ_z the third Pauli matrix.

(*iii*) Equivalently, $\mathsf{M}^{-1} = \sigma_z\mathsf{M}^\dagger\sigma_z$.

(*iv*) $(\mathsf{M}^\dagger\mathsf{M})^{-1} = \sigma_z\mathsf{M}^\dagger\mathsf{M}\sigma_z$. Thus, the hermitian matrices $(\mathsf{M}^\dagger\mathsf{M})^{-1}$ and $\mathsf{M}^\dagger\mathsf{M}$ have the same (real) eigenvalues. Since these eigenvalues must also be each other's inverses, they can only be of the form $\lambda_+ = 1/\lambda_- = \mathrm{e}^{2x}$.

(*v*) $2 + (\mathsf{M}^\dagger\mathsf{M})^{-1} + \mathsf{M}^\dagger\mathsf{M} = 4/T$. Thus, the total transmission probability is $T = 1/(\cosh x)^2$. Verify these properties directly using the transfer matrix obtained for the example (1).

Exercise 3 – We have shown that the phase-averaged transmission across two scatterers reads

$$\langle T_{12}\rangle = \frac{T_1T_2}{1-R_1R_2}. \tag{2}$$

Show that the same result (2) is obtained if one were to use an S-matrix propagating probabilities instead of amplitudes, $\mathring{\mathsf{S}} = \begin{pmatrix} R & T \\ T & R \end{pmatrix}$, by determining the corresponding transfer matrix $\mathring{\mathsf{M}}$ and chaining it.

3 Key experiments

Exercise 4 – Equivalence between dynamical localization and Anderson localization.

The Hamiltonian of the kicked rotor is:

$$H = \frac{p^2}{2} + k\cos\theta \sum_{n=\infty}^{+\infty} \delta(t-nT) \tag{3}$$

Show that the evolution operator over one period is:

$$U = \exp\left(-\frac{i}{\hbar}\frac{p^2T}{2}\right)\exp\left(-\frac{i}{\hbar}k\cos\theta\right). \tag{4}$$

Any unitary operator can be written (why?) as:

$$\exp\left(-iM\right) = \frac{1-i\tan(M/2)}{1+i\tan(M/2)}. \tag{5}$$

What does this expression mean: $(1-i\tan(M/2))(1+i\tan(M/2))^{-1}$ or $(1+i\tan(M/2))^{-1}(1-i\tan(M/2))$?

We look for the eigenstates of the U operator (the so-called Floquet eigenstates). Why are the eigenvalues complex numbers of unit modulus? We write for such an eigenstate:

$$U|\psi\rangle = \exp\left(-\frac{iET}{\hbar}\right) \tag{6}$$

where E is called the quasi-energy.

By using eq. (5), show that the eigen-equation for $|\psi\rangle$ can be rewritten as:

$$\tan\left(\frac{k\cos\theta}{2\hbar}\right)|\phi\rangle = \tan\left(\frac{(p^2/2-E)T}{2\hbar}\right)|\phi\rangle \tag{7}$$

with $|\phi\rangle = \left[1+i\tan\left(\frac{k\cos\theta}{2\hbar}\right)\right]^{-1}|\psi\rangle$.

If one expands $|\phi\rangle$ in the basis of the eigensates of the momentum operator:

$$|\phi\rangle = \sum_{m=-\infty}^{+\infty} c_m|m\rangle, \tag{8}$$

show that the eigen-equation can be reformulated as:

$$\sum_{m=-\infty}^{+\infty} W_r c_{m+r} = \tan\left(\frac{(\frac{m^2\hbar^2}{2}-E)T}{2\hbar}\right)c_m \tag{9}$$

What are the W_r coefficients? Show that this represents an Anderson model with diagonal pseudo-random disorder and hopping to several nearest neighbors. What happens when $\hbar T/\pi$ is a rational number?