Quantum Hall Effects – an Introduction

Exercises (Mark O. Goerbig) 16 July 2009

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**Exercice 1 (2D electron in a magnetic field – eigenstates in the symmetric gauge)**

The aim of this exercise is to construct the eigenstates of a 2D electron in a perpendicular magnetic field in the symmetric gauge $\mathbf{A}_S = (-y, x, 0)B/2$. These eigenstates, namely those in the lowest Landau level turn out to be useful in the discussion of trial wave functions *à la Laughlin* describing the fractional quantum Hall effect (second and third lecture).

Remember, from the first lecture, that the system is described by two pairs of ladder operators $a, a^\dagger$ and $b, b^\dagger$. Whereas the first set is related to the gauge-invariant momentum $\mathbf{\Pi} = \mathbf{p} + e\mathbf{A}$ (proportional to the electron velocity) and reveals the dynamical properties, via the Hamiltonian $H = \hbar\omega_C(a^\dagger a + 1/2)$, the second set describes the "mock" momentum $\tilde{\Pi}$, which is a constant of motion related to the centre of the cyclotron motion in a semiclassical picture (see first lecture). The eigenstates are thus described by the quantum numbers $n$ and $m$, with

$$a^\dagger a|n,m\rangle = n|n,m\rangle \qquad \text{and} \qquad b^\dagger b|n,m\rangle = m|n,m\rangle.$$

Remember from basic quantum mechanics (harmonic oscillator) that an arbitrary state may be obtained from the state $|n=0, m=0\rangle$ with the help of the ladder operators $a^\dagger$ and $b^\dagger$,

$$|n,m\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\frac{(b^\dagger)^m}{\sqrt{m!}}|n=0,m=0\rangle. \tag{1}$$

1.1 Lowest-Landau-level condition

A state in the lowest Landau (LL) level ($n = 0$) satisfies the condition

$$a|n=0,m\rangle = 0. \tag{2}$$

Translate this condition into a differential equation for the lowest-LL wave function $\phi_{n=0}(z, z^*)$, where we parametrise the $xy$-plane by complex variables $z = x - iy$. [Hint : remember that $a = \frac{l_B}{\sqrt{2}\hbar}(\Pi_x - i\Pi_y)$ and express $a$ in terms of the variables $x, y$ and their derivatives $\partial_x$ and $\partial_y$ first, $\mathbf{p} = -i\hbar(\partial_x, \partial_y)$. Then use the complex derivatives $\partial \equiv (\partial_x + i\partial_y)/2$ and $\bar{\partial} = (\partial_x - i\partial_y)/2$.] Show that the result is

$$\left(\frac{z}{4l_B} + l_B\bar{\partial}\right)\phi_{n=0}(z, z^*) = 0. \tag{3}$$

1.2 Eigenstates in the lowest LL

Show that the solutions of the differential equation (3) are given by

$$\phi_{n=0}(z, z^*) = f(z)e^{-\frac{|z|^2}{4l_B^2}}, \tag{4}$$

where $f(z)$ is an analytic function, i.e. $\bar{\partial}f(z) = 0$, and $|z|^2 = zz^*$. This means that there is an additional degree of freedom because $f(z)$ may be *any* analytic function. It is not unexpected that this degree of freedom is associated with the second quantum number $m$, as we will now discuss.

Similarly to to the operators $a^{(\dagger)}$, one may express the ladder operators $b$ and $b^\dagger$ in terms of complex variables and derivatives. Show that

$$b = \sqrt{2}\left(\frac{z^*}{4l_B} + l_B\partial\right) \qquad \text{and} \qquad b^\dagger = \sqrt{2}\left(\frac{z}{4l_B} - l_B\bar{\partial}\right).$$

The reference state $|n=0,m=0\rangle$ is, in addition to Eq. (2), characterised by $b|n=0,m=0\rangle=0$. Solve the associated differential equation in the same manner as above. What do you conclude for the wave function $\phi_{n=0,m=0}(z,z^*)$ ? Show that the wave functions $\phi_{n=0,m}(z,z^*)$, resulting from Eq. (1) for arbitrary values of $m$, are given (up to a normalisation) by

$$\phi_{n=0,m}(z,z^*) \propto z^m e^{-\frac{|z|^2}{4l_B^2}} . \tag{5}$$

This is the central result of the exercise which indicates that the lowest-LL wave functions in the symmetric gauge are given by the convenient polynomial basis of analytic functions (multiplied by a Gaussian).

**Exercice 2 (2D electron in a magnetic field – eigenstates in the Landau gauge)**

The Landau gauge, $\mathbf{A}_L = B(-y,0,0)$, is a conveniant choice when describing electrons in a sample of rectangular geometry. Because $x$ is then a cyclic variable, i.e. it does not occur in the Hamiltonian, the system is translationally invariant in the $x$-direction, and the momentum $p_x = \hbar k$ is a good quantum number. This means that the wave functions are plane waves in the $x$-direction and one may choose the ansatz

$$\psi_{n,k}(x,y) = \frac{e^{ikx}}{\sqrt{L}}\chi_{n,k}(y) \; . \tag{6}$$

2.1 Eigenstates

Show that the Hamiltonian may then be rewritten as

$$H = \frac{p_y^2}{2m} + \frac{1}{2}m\omega_C^2(y-y_0)^2, \tag{7}$$

where $y_0 = kl_B^2$. This means that one obtains the Hamiltonian of a 1D harmonic oscillator centred around $y_0$, with the eigenstates

$$\chi_{n,k}(y) = H_n\left(\frac{y-y_0}{l_B}\right) e^{-(y-y_0)^2/4l_B^2},$$

in terms of Hermite polynomials $H_n$. The coordinate $y_0$ now plays the role of the guiding centre component $Y$, the component $X$ being smeared over the whole length $L$ of the sample, as dictated by the Heisenberg uncertainty relation resulting from the commutation relation $[X,Y] = il_B^2$.

2.2 Level degeneracy

Using periodic boundary conditions $k = n \times 2\pi/L$ for the wave vector in the $x$-direction, one may count the number of states in a rectangular surface of length $L$ and width $W$ (in the $y$-direction). Consider the sample to range from $y_{min}=0$ to $y_{max}=W$, the first corresponding (via the above-mentioned condition $y_0 = kl_B^2$) to the wave vector $k=0$ and the latter to a wave vector $k_{max} = N_B \times 2\pi/L$. Show that each state occupies a surface $\sigma = \Delta y \times L = 2\pi l_B^2$ and that the maximal quantum number (= number of states per LL) is indeed given by

$$N_B = LW \times n_B,$$

in terms of the flux density $n_B = 1/2\pi l_B^2 = B/(h/e)$. One thus obtains the same result as from the argument concerning the guiding-centre quantisation (see first lecture), that the number of states per LL is equal to the number of flux quanta threading the surface with area $LW$.